

# Two-Loop Effective Potential of $O(N)$ -Symmetric Scalar QED in $4 - \varepsilon$ Dimensions

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The effective potential of scalar QED is computed analytically up to two loops in the Landau gauge. The result is given in  $4 - \varepsilon$  dimensions using minimal subtraction and  $\varepsilon$ -expansions. In three dimensions ( $\varepsilon = 1$ ), our calculation is intended to help throw light on unsolved problems of the superconducting phase transition, where critical exponents and the position of the tricritical point have not yet found a satisfactory explanation within the renormalization group approach.

## I. INTRODUCTION

Recently, important progress has been made to the evaluation of Feynman diagrams in  $4 - \varepsilon$  dimensions. In particular, two-loop Feynman diagrams with unequal masses of the internal lines are now available analytically [1,2]. Moreover, the full  $\varepsilon$ -expansion of the sunset diagram is known [3]. This is important for critical phenomena in  $4 - \varepsilon$  dimensions since, for instance, a three-loop calculation requires the knowledge of all terms of order  $\varepsilon$  of the two-loop diagrams.

With the help of these results, we investigate in the present paper an  $O(N)$ -symmetric version of scalar QED containing  $N/2$  complex fields in  $4 - \varepsilon$  dimensions, which, for  $N = 2$ , reduces to the Ginzburg-Landau model of superconductivity. Our results are intended to shed some light on the open problems of the superconducting phase transition in three dimensions. From a dual formulation of the Ginzburg-Landau theory we know that there must exist a regime of the parameter  $\kappa$ , the ratio between scalar and vector masses, where the transition changes from first to second order [4]. This scenario has been confirmed by several Monte-Carlo simulations, see [5].

In contrast to this, the traditional  $4 - \varepsilon$ -dimensional renormalization group analysis of the Ginzburg-Landau model has produced more puzzles than answers. At the one-loop order, a second-order transition is obtained only for  $N \geq N_c = 366$ , much larger than the  $N = 2$ -value of a superconductor [6,7]. The situation does not improve by going to two loops [8]. Only a somewhat contrived resummation procedure leads to a desired fixed point [9]. Only in direct three-dimensional calculations has a second-order transition been obtained for  $N = 2$ , see for example [10]. The absence of a charged fixed point for  $N = 2$  in  $4 - \varepsilon$  dimensions seems thus to be a specific weakness of this approach.

At present it is hoped that variational perturbation theory may be able to locate a fixed point of the Ginzburg-Landau model, thus allowing to extract physical values independently of Ref. [9]. This theory developed in Ref. [11] has proven to be a powerful tool for determining critical exponents in three [12-14] as well as in  $4 - \varepsilon$  dimensions [15,16]. Recently, we have applied it to the determination of amplitude ratios in three dimensions of the  $O(N)$ -model [17]. We did these calculations using a method proposed by the Aachen group [18], where analytic renormalization is applied in the form of minimal subtraction although working directly in  $D = 3$  dimensions, without any  $\varepsilon$ -expansion. A similar determination of amplitude ratios in  $4 - \varepsilon$  dimensions with  $\varepsilon$ -expansion is still missing. For determining the associated amplitude ratios, we need the effective potential in  $4 - \varepsilon$  dimensions. This is what we want to calculate here.

As a cross check we shall often set the electric charge equal to zero and check that we recover the results for the usual  $O(N)$ -symmetric  $\phi^4$ -theory.

The renormalization constants and various beta functions of scalar QED have been calculated up to two loops in the early work on this subject [8,19]. It is not our purpose to rederive these results. However, in the process of obtaining the effective potential, i.e., of calculating the various Feynman diagrams in the symmetry-broken phase, we obtain not only the effective potential but also the renormalization constants of mass  $Z_{m^2}$  and  $\phi^4$ -coupling constant  $Z_g$  to up to two loops. The renormalization constants of the scalar and vector fields  $Z_\phi$  and  $Z_A$ , on the other hand, enters only with the first loop order. The fact that the charge coupling renormalization constant  $Z_e$  is not obtained to the same order as the  $\phi^4$ -coupling constant one has its origin in the fact that, due to a Ward identity,  $Z_e = Z_\phi$ , see Eq. (10).

Before closing the introduction, we point out that the effective potential of the electroweak part of the standard model has already been calculated to the two-loop order [1]. However, our work is more than the abelian  $U(1)$  subset of this reference for two reasons: first, we work with  $N/2$ -complex scalar fields ( $N=2$  in [1]); second, in view of the application to critical phenomena in three dimensions, the  $\varepsilon$ -expansion is used. At the two-loop order, this means

that the one-loop part of the effective potential is expanded up to the order  $\varepsilon$ . This term is of course not there when being interested in going in four dimensions, as is the case of Ref. [1].

Finally, we mention that our work can be seen as the extension to a two coupling constants problem of the work of Brézin et al. [20] (see also [21]), which consider the  $\varepsilon$ -expansion of the equation of state of the  $N$ -components  $\phi^4$  theory to the two-loop order<sup>1</sup>. Without the gauge field, the effective potential can be used to determine this equation of state.

The paper is organised as follow: in Section II, we specify the model and our conventions. In Section III, we present the various intermediate steps for the obtaining of the effective potential. The individual results are combined in Section IV, and the conclusions drawn in Section V.

## II. MODEL

The Lagrangian density to be studied contains  $N/2$  complex scalar fields  $\phi$  coupled to the abelian fields  $A_\mu$  and reads, with a covariant gauge fixing,

$$\mathcal{L} = |D\phi|^2 + m^2\phi^2 + \frac{g}{3!}|\phi|^4 + \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2, \quad (1)$$

where  $D_\mu = \partial_\mu - ieA_\mu$  is the covariant derivative,  $F_{\mu\nu}$  is the usual field-strength tensor and  $\alpha$  is a gauge parameter. There is no need to introduce ghost fields to guarantee gauge invariance since these decouple from the system in an abelian gauge theory.

The effective potential will be obtained using the so-called *background-field method* of DeWitt [23]. We shift the scalar field by an unknown constant  $\Phi$ :  $\phi \rightarrow \Phi + \phi$ . This generates new vertices. To simplify the calculation, we shall use throughout the Landau gauge  $\alpha \rightarrow 0$ . This reduces the number of Feynman diagrams and, since  $\alpha = 0$  enforces  $\partial_\mu A_\mu \equiv 0$  at the Lagrangian level, removes a possible mixing of  $A_\mu\phi$  and  $A_\mu\phi^\dagger$  terms, thus decoupling scalar and gauge propagators. It is further advantageous to use real fields, defining

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2). \quad (2)$$

Then the Lagrangian has the expansion around the background field:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_0 &+ \frac{1}{2}\phi[G^T P^T + G^L P^L]\phi + \frac{1}{2}A_\mu D^T P_{\mu\nu}^T A_\nu \\ &+ \frac{g}{4!}[(\phi^2)^2 + 4\Phi\phi(\phi^2)] + e^2 A^2 \Phi\phi + \frac{1}{2}e^2 A_\mu^2 \phi^2 + eA_\mu(\phi_2\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_2), \end{aligned} \quad (3)$$

where  $\phi$  and  $\Phi$  are now  $N$  components real fields written as two-dimensional iso-vectors  $\phi = (\phi_1, \phi_2)$  and  $\Phi = (\Phi_1, \Phi_2)$ . The notation is:

$$\mathcal{L}_0 = \frac{1}{2}m^2\Phi^2 + \frac{g}{4!}\Phi^4 \quad (4)$$

$$G^T \equiv -\partial^2 + m_T^2 = -\partial^2 + m^2 + \frac{g}{3!}\Phi^2, \quad G^L \equiv -\partial^2 + m_L^2 = -\partial^2 + m^2 + \frac{g}{2}\Phi^2, \quad (5)$$

$$D^T \equiv -\partial^2 + m_\gamma^2 = -\partial^2 + e^2\Phi^2, \quad (6)$$

$$P_{ij}^T = \delta_{ij} - \frac{\Phi_i\Phi_j}{\Phi^2}, \quad P_{ij}^L = \frac{\Phi_i\Phi_j}{\Phi^2}, \quad P_{\mu\nu}^T = \delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}, \quad (7)$$

with  $G^T, G^L$  being the transverse and longitudinal inverse propagators of the scalar field, and  $D_{\mu\nu}^T$  the inverse transverse propagator of the photon field. The transversality of the latter is due to the Landau gauge. Note that there is no term  $eA_\mu\partial_\mu(\Phi_2\phi_1 - \Phi_1\phi_2)$  which would mix vector and scalar propagators.

Compared to the complex-field representation, real fields have one small complication: the gauge-scalar-scalar vertex involves a vector product since it mixes the real and imaginary parts  $\phi_1$  and  $\phi_2$ . For complex fields it is diagonal in the scalar field space.

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<sup>1</sup>For the Ising model, corresponding to  $N = 1$ , the authors of [22] have succeeded going to the three-loop order of the  $\varepsilon$ -expansion of the equation of state.

Up to two loops, the fastest way of determining the Feynman diagrams with their proper weight is to use the Wick theorem. In higher loops, it might be interesting to use a more efficient algorithm to generate the diagrams and their weight, in the same spirit as, for instance, [24,25]. Using a similar procedure, we have derived the diagrams of scalar QED up to the fourth order. Their calculation will form the subject of a separate publication [26].

We shall calculate the diagrams in the following expansion, in which  $\hbar$  counts the number of loops:

$$\begin{aligned}
V_{\text{eff. pot.}} = & V_{\text{classical}} + \frac{\hbar}{2} \left[ (N-1) \text{circle}^T + \text{circle}^L + (D-1) \text{blob} \right] \\
& \hbar^2 \left\{ \frac{g}{4!} \left[ (N^2-1) \text{two-circles}^{TT} + 2(N-1) \text{two-circles}^{TL} + 3 \text{two-circles}^{LL} \right] + \frac{e^2}{2} (D-1) \left[ (N-1) \text{circle}^T \text{blob} + \text{circle}^L \text{blob} \right] \right. \\
& \left. - \left( \frac{g}{3!} \right)^2 \left[ (N-1) \text{circle}^T \text{circle}^T + 3 \text{circle}^L \text{circle}^L \right] - 4 \left( \frac{e^2}{2} \right)^2 \Phi^2 \text{blob} - e^2 \left[ (N-2) \text{circle}^T \text{blob} + 2 \text{circle}^L \text{blob} \right] \right\}. \quad (8)
\end{aligned}$$

Explicitly, this equation reads in a unit  $D$ -dimensional volume:

$$\begin{aligned}
V = & \frac{1}{2} m^2 \Phi^2 + \frac{g}{4!} \Phi^4 + \frac{\hbar}{2} \left[ (N-1) \text{Tr ln}(G^T) + \text{Tr ln}(G^L) + (D-1) \text{Tr ln}(D^T) \right] \\
& + \hbar^2 \left\{ \frac{g}{4!} \left[ \left[ (N-1) \int \bar{d}p \Delta(p, m_T) + \int \bar{d}p \Delta(p, m_L) \right]^2 + 2(N-1) \left[ \int \bar{d}p \Delta(p, m_T) \right]^2 + 2 \left[ \int \bar{d}p \Delta(p, m_L) \right]^2 \right] \right. \\
& + \frac{e^2}{2} (D-1) \left[ \int \bar{d}p \Delta(p, m_\gamma) \right] \left[ (N-1) \int \bar{d}p \Delta(p, m_T) + \int \bar{d}p \Delta(p, m_L) \right] \\
& - \left( \frac{g}{3!} \right)^2 \Phi^2 \left[ (N-1) \int \bar{d}p \bar{d}q \Delta(p, m_T) \Delta(q, m_T) \Delta(p+q, m_L) + 3 \int \bar{d}p \bar{d}q \Delta(p, m_L) \Delta(q, m_L) \Delta(p+q, m_L) \right] \\
& - 4 \left( \frac{e^2}{2} \right)^2 \Phi^2 \left[ \int \bar{d}p \bar{d}q P_{\mu\nu}^T(p) P_{\mu\nu}^T(q) \Delta(p, m_\gamma) \Delta(q, m_\gamma) \Delta(p+q, m_L) \right] \\
& \left. - e^2 \left\{ \int \bar{d}p \bar{d}q P_{\mu\nu}^T(p) q_\mu q_\nu \Delta(p, m_\gamma) \left[ (N-2) \Delta(q, m_T) \Delta(p+q, m_T) + 2 \Delta(q, m_L) \Delta(p+q, m_T) \right] \right\} \right\}, \quad (9)
\end{aligned}$$

where  $\bar{d}p \equiv d^D p / (2\pi)^D$ , and  $\Delta(p, m) = 1/(p^2 + m^2)$ .

All quantities in these expressions (fields, coupling constants, and masses) are bare quantities. Up to the second order in the loop expansion, the divergences show up as poles in  $\varepsilon$  up to the order  $1/\varepsilon^2$ . They have to be removed to have a finite limit  $\varepsilon \rightarrow 0$ . They may either be removed by adding counter-terms to the initial Lagrangian, and calculating the associated Feynman diagrams in addition to those in (8). Alternatively, and this is how we shall proceed, we may use (8) and (9) and include renormalization constants which are reexpanded up to any given order in the loop parameter, i.e., up to the order  $\hbar^2$  in our case. These renormalization constants are defined as

$$\phi = \phi_r \sqrt{Z_\phi}, \quad A_\mu = A_{\mu r} \sqrt{Z_A}, \quad m^2 = m_r^2 \frac{Z_{m^2}}{Z_\phi}, \quad g = g_r \mu^\varepsilon \frac{Z_g}{Z_\phi^2}, \quad e = e_r \mu^{\varepsilon/2} \frac{Z_e}{Z_\phi \sqrt{Z_A}} = \frac{e_r}{\sqrt{Z_A}} \mu^{\varepsilon/2} \quad (10)$$

where, in the last equation, we have taken into account the relation  $Z_e = Z_\phi$ , which is a consequence of a Ward identity. Intuitively, it comes from the requirement  $D_\mu \phi \rightarrow \sqrt{Z_\phi} D_\mu \phi_r$  for the covariant derivative. In the above equations, the bare quantities are on the left-hand-side and the renormalized ones are on the right-hand-side, indicated by the subscript ‘‘r’’.

We also note that the vacuum requires a special treatment [27]. For  $\Phi = 0$ , the dimensionality requires  $V \propto m^D$ . We therefore add a term  $m^4 h/g$  to the Lagrangian. With this we define a new renormalization constant absorbing the vacuum divergencies by

$$\frac{m^4}{g} h = \frac{m_r^4}{g_r \mu^\varepsilon} Z_v \quad (11)$$

In the next Section, we calculate the different loop orders, based on (9).

### III. EVALUATION OF THE DIAGRAMS

In the following, the subscript indicating the renormalized quantities will be omitted, for brevity of notation. The different renormalization constants are expanded with respect to the loop-parameter:

$$Z_j = 1 + \sum_{l=1}^L \hbar^l Z_j^{(l)}. \quad (12)$$

The results are then identified order by order in  $\hbar$ . To fix the ideas, we give here to procedure for the gauge field trace-ln: The bare diagram has to be modified according to:  $e^2 \Phi^2 \rightarrow e^2 \Phi^2 Z_\phi / Z_A$ , and the result reads, to the first order in  $\hbar$

$$\text{Tr ln}(D^T) = \text{Tr ln} \left( -\partial^2 + e^2 \Phi^2 \frac{Z_\phi}{Z_A} \right) \approx \text{Tr ln}(-\partial^2 + e^2 \Phi^2) + \hbar e^2 \Phi^2 (Z_\phi^{(1)} - Z_A^{(1)}) \text{Tr} \left( \frac{1}{-\partial^2 + e^2 \Phi^2} \right). \quad (13)$$

The last term will contribute to the two-loop result, removing parts of its  $\varepsilon$ -poles.

#### A. Renormalized zero- and one-loop order

The renormalized zero-loop order is trivial

$$V(l=0) = \frac{1}{2} m^2 \Phi^2 + \frac{g}{4!} \mu^\varepsilon \Phi^4 + \frac{m^4}{g \mu^\varepsilon}, \quad (14)$$

while the renormalized one is simply a combination of the previous bare zero-order and the trace-ln terms:

$$V(l=1) = \frac{1}{2} m^2 \Phi^2 Z_{m^2}^{(1)} + \frac{g}{4!} \mu^\varepsilon \Phi^4 Z_g^{(1)} + \frac{m^4}{g \mu^\varepsilon} Z_v^{(1)} + \frac{\Gamma(1-D/2)}{(4\pi)^{D/2}} \frac{1}{D} \mu^{-\varepsilon} \left[ (N-1)(m_T^2)^2 \left( \frac{\mu^2}{m_T^2} \right)^{\varepsilon/2} + (m_L^2)^2 \left( \frac{\mu^2}{m_L^2} \right)^{\varepsilon/2} + (D-1)(m_\gamma^2)^2 \left( \frac{\mu^2}{m_\gamma^2} \right)^{\varepsilon/2} \right]. \quad (15)$$

The constants  $Z_j^{(1)}$  are determined in order to remove the  $\varepsilon$ -poles at  $D=4$  in the Euler  $\Gamma(1-D/2)$  function. In four dimensions, it would be sufficient to make the  $\varepsilon$ -expansion of (15) up to the order  $\varepsilon^0$ . However, we are also interested in going to dimension  $D=3$ . For this reason, it will be necessary to perform the  $\varepsilon$ -expansion up to the order  $\varepsilon$ .

#### B. Renormalized two-loop order

The two-loop order contains contributions from the zero- and one-loop bare diagrams, which will cancel the two-loop poles from the bare two-loop diagrams. These contributions read ("ct" is for counter-terms)

$$V(l=2)^{\text{ct}} = \frac{1}{2} m^2 \Phi^2 Z_{m^2}^{(1)} + \frac{g}{4!} \mu^\varepsilon \Phi^4 Z_g^{(1)} + \frac{m^4}{g \mu^\varepsilon} Z_v^{(1)} + \frac{\Gamma(1-D/2)}{(4\pi)^{D/2}} \frac{1}{2} \mu^{-\varepsilon} \left\{ (N-1)(m_T^2)^2 \left( \frac{\mu^2}{m_T^2} \right)^{\varepsilon/2} \left[ \frac{1}{2} (3m_T^2 - m_L^2) Z_{m^2}^{(1)} + \frac{1}{2} (m_L^2 - m_T^2) Z_g^{(1)} - m_T^2 Z_\phi^{(1)} \right] + (m_L^2)^2 \left( \frac{\mu^2}{m_L^2} \right)^{\varepsilon/2} \left[ \frac{1}{2} (3m_T^2 - m_L^2) Z_{m^2}^{(1)} + \frac{3}{2} (m_L^2 - m_T^2) Z_g^{(1)} - m_L^2 Z_\phi^{(1)} \right] + (D-1)(m_\gamma^2)^2 \left( \frac{\mu^2}{m_\gamma^2} \right)^{\varepsilon/2} \left[ m_\gamma^2 (Z_\phi^{(1)} - Z_A^{(1)}) \right] \right\}. \quad (16)$$

To simplify the evaluation of the two-loop diagrams, we introduce the functions  $J(m)$  and  $I(m_1, m_2, m_3)$ , where we have kept the same notation as in [1]:

$$J(m) = \int \bar{d}p \Delta(p, m), \quad (17)$$

$$I(m_1, m_2, m_3) = \int \bar{d}p \bar{d}q \Delta(p, m_1) \Delta(q, m_2) \Delta(p+q, m_3). \quad (18)$$

The function  $J(m)$  is trivial to determine, and its result has in fact been used in (16). For  $I(m_1, m_2, m_3)$ , we use the result obtained in [1], which are in accordance with [2] and the recent work [3]. The latter reference gives an expansion for all order in  $\varepsilon$  but, since we need in the present work only the term of order  $\varepsilon^0$ , we stop at this order:

$$J(m_1) = \frac{\Gamma(1 - D/2)}{(4\pi)^{D/2}} (m^2)^{D/2-1}, \quad (19)$$

$$\begin{aligned} (\mu^\varepsilon)^2 (4\pi)^4 I(m_1, m_2, m_3) = & -\frac{2}{\varepsilon^2} (m_1^2 + m_2^2 + m_3^2) - \frac{2}{\varepsilon} \left[ \frac{3}{2} (m_1^2 + m_2^2 + m_3^2) - L_1(m_1, m_2, m_3) \right] \\ & - \frac{1}{2} \left\{ L_2(m_1, m_2, m_3) - 6L_1(m_1, m_2, m_3) + (m_2^2 + m_3^2 - m_1^2) \overline{\ln}(m_2^2) \overline{\ln}(m_3^2) + (m_3^2 + m_1^2 - m_2^2) \overline{\ln}(m_3^2) \overline{\ln}(m_1^2) \right. \\ & \left. + (m_1^2 + m_2^2 - m_3^2) \overline{\ln}(m_1^2) \overline{\ln}(m_2^2) + \xi(m_1, m_2, m_3) + (m_1^2 + m_2^2 + m_3^2) [7 + \zeta(2)] \right\} \end{aligned} \quad (20)$$

where we have defined

$$\overline{\ln}(m^2) = \ln \left( \frac{m^2}{\mu^2} \right) + \gamma - \ln(4\pi), \quad (21)$$

$$L_1(m_1, m_2, m_3) = m_1^2 \overline{\ln}(m_1^2) + m_2^2 \overline{\ln}(m_2^2) + m_3^2 \overline{\ln}(m_3^2), \quad (22)$$

$$L_2(m_1, m_2, m_3) = m_1^2 [\overline{\ln}(m_1^2)]^2 + m_2^2 [\overline{\ln}(m_2^2)]^2 + m_3^2 [\overline{\ln}(m_3^2)]^2, \quad (23)$$

$$\xi(m_1, m_2, m_3) = 4(2m_1^2 m_2^2 + 2m_1^2 m_3^2 + 2m_2^2 m_3^2 - m_1^4 - m_2^4 - m_3^4)^{1/2} \left[ L(\theta_1) + L(\theta_2) + L(\theta_3) - \frac{\pi}{2} \ln(2) \right]. \quad (24)$$

In the latter expression,  $L(t)$  is the Lobachevsky function, defined as

$$L(t) = - \int_0^t dx \ln \cos x, \quad (25)$$

and the angles are given by

$$\theta_j = \arctan \left[ \frac{(m_1^2 + m_2^2 + m_3^2) - 2m_j^2}{(2m_1^2 m_2^2 + 2m_1^2 m_3^2 + 2m_2^2 m_3^2 - m_1^4 - m_2^4 - m_3^4)^{1/2}} \right]. \quad (26)$$

These expressions, as well as the function  $\xi(m_1, m_2, m_3)$ , are valid for a positive argument of the square root. This depends on the value of the masses. For a negative value, one has to substitute

$$\xi(m_1, m_2, m_3) = 4(m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2)^{1/2} [-M(-\theta_1) + M(\theta_2) + M(\theta_3)], \quad (27)$$

$$M(t) = - \int_0^t dx \ln \sinh x, \quad (28)$$

$$\theta_j = \coth^{-1} \left[ \frac{(m_1^2 + m_2^2 + m_3^2) - 2m_j^2}{(m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2)^{1/2}} \right]. \quad (29)$$

In the following, we shall always keep the symbolic notation  $\xi(m_1, m_2, m_3)$ . This will avoid to check the sign of the argument of the square root. It is not necessary to do so in the pure  $\phi^4$ -case where we know that  $m_L > m_T$ , and, then, where the representation (24) is valid. In the present case, because we do not know the fixed points, it is not possible to specify the value of the photon mass vs. the transverse or longitudinal mass. We however mention that several simplifications arise when two masses are equal, or when one, or two masses, are vanishing. For two equal masses, say  $m_1 = m_2$ , the argument of the square root becomes  $m_3^4(1 - 4m_1^2/m_3^2)$ . This case arises when investigating the  $O(N)$ -theory. The ratio of the masses is the ratio  $4m_T^2/m_L^2$  which is always smaller than unity. The relevant equations are then (24), (25) and (26). The case of vanishing masses is a little bit problematic for the extraction of the power  $\varepsilon^0$  because of a cancellation of a diverging part of  $M(t)$  with a corresponding  $\overline{\ln}_j$  in (20). For this reason, we also give the following integrals:

$$(\mu^\varepsilon)^2 I(m_1, 0, 0) = \frac{1}{(4\pi)^D} \frac{\Gamma(2 - D/2)\Gamma(3 - D)\Gamma(D/2 - 1)^2}{\Gamma(D/2)} m_1^2 \left(\frac{m_1^2}{\mu^2}\right)^{-\varepsilon}, \quad (30)$$

$$(\mu^\varepsilon)^2 I(m_1, m_1, 0) = \frac{1}{(4\pi)^D} \frac{\Gamma(2 - D/2)\Gamma(1 - D/2)}{D - 3} m_1^2 \left(\frac{m_1^2}{\mu^2}\right)^{-\varepsilon}, \quad (31)$$

$$\begin{aligned} (\mu^\varepsilon)^2 (4\pi)^4 I(m_1, m_2, 0) &= -\frac{2}{\varepsilon^2} (m_1^2 + m_2^2) - \frac{2}{\varepsilon} \left[ \frac{3}{2} (m_1^2 + m_2^2) - L_1(m_1, m_2, 0) \right] \\ &- \frac{1}{2} \left\{ L_2(m_1, m_2, 0) - 6L_1(m_1, m_2, 0) + 2m_1^2 \overline{\ln}(m_1^2) \overline{\ln}(m_2^2) + [\overline{\ln}(m_1^2 - m_2^2)]^2 (m_1^2 - m_2^2) \right. \\ &\left. - 2\overline{\ln}(m_1^2 - m_2^2) \overline{\ln}(m_2^2) (m_1^2 - m_2^2) + 2(m_1^2 - m_2^2) \text{Li}_2\left(\frac{m_2^2}{m_2^2 - m_1^2}\right) + (m_1^2 + m_2^2)[7 + \zeta(2)] + \frac{\pi^2}{3} (m_1^2 - m_2^2) \right\}, \quad (32) \end{aligned}$$

where  $\text{Li}_2(z) = \sum_{i=1}^{\infty} z^i/i^2$  is the dilogarithm. It is a simple exercise to check that, using  $m_1 = m_2$  in (32), we recover the  $\varepsilon$ -expansion of (31), while using  $m_3 = 0$ , we recover the  $\varepsilon$ -expansion of (30). In the previous determination of (32), we have assumed that  $m_1 > m_2$ . The case  $m_2 > m_1$  is obtained from the former by the replacement  $m_1 \leftrightarrow m_2$ . This only affects the term of order  $\varepsilon^0$ . Comparing Eq. (32) with Eq. (20), we can also extract the way  $\xi(m_1, m_2, m_3)$  diverges as one of the masses, say  $m_3$ , goes to zero. For  $m_1 > m_2$ , we have

$$\begin{aligned} \lim_{m_3 \rightarrow 0} \{ \xi(m_1, m_2, m_3) + (m_1^2 - m_2^2) [\overline{\ln}(m_1^2) - \overline{\ln}(m_2^2)] \overline{\ln}(m_3^2) \} &= \\ (m_1^2 - m_2^2) \left\{ \overline{\ln}(m_1^2) \overline{\ln}(m_2^2) + [\overline{\ln}(m_1^2 - m_2^2)]^2 - 2\overline{\ln}(m_1^2 - m_2^2) \overline{\ln}(m_2^2) + 2\text{Li}_2\left(\frac{m_2^2}{m_2^2 - m_1^2}\right) + \frac{\pi^2}{3} \right\}, \quad (33) \end{aligned}$$

while the opposite case  $m_1 < m_2$  is obtained by permuting  $m_1$  and  $m_2$  in this equation. From (33) we can also determine the case of two simultaneously vanishing masses:

$$\lim_{m_3 \rightarrow 0} \{ \xi(m_1, m_3, m_3) + 2m_1^2 \overline{\ln}(m_1^2) \overline{\ln}(m_3^2) - m_1^2 [\overline{\ln}(m_3^2)]^2 \} = m_1^2 \left\{ \overline{\ln}(m_1^2) \right\}^2 + \frac{\pi^2}{3}. \quad (34)$$

With all these definitions, we are now armed to compute the two-loop diagrams. They are given by

$$\text{Diagram 1} = \frac{g}{4!} \mu^\varepsilon \left\{ [(N-1)J(m_T) + J(m_L)]^2 + 2[(N-1)J(m_T)^2 + J(m_L)^2] \right\}, \quad (35)$$

$$\text{Diagram 2} = \frac{e^2}{2} \mu^\varepsilon (D-1) J(m_\gamma) [(N-1)J(m_T) + J(m_L)], \quad (36)$$

$$\text{Diagram 3} = \left(\frac{g\mu^\varepsilon}{3!}\right)^2 \Phi^2 [(N-1)I(m_T, m_T, m_L) + 3I(m_L, m_L, m_L)], \quad (37)$$

$$\begin{aligned} \text{Diagram 4} &= 4 \left(\frac{e^2 \mu^\varepsilon}{2}\right)^2 \Phi^2 \left\{ I(m_\gamma, m_\gamma, m_L) \left[ (D-2) + \frac{1}{4m_\gamma^4} (m_L^2 - 2m_\gamma^2)^2 \right] \right. \\ &\left. - \frac{1}{4m_\gamma^4} [J(m_\gamma)^2 (m_L^2 - 2m_\gamma^2) + 2m_\gamma^2 J(m_\gamma) J(m_L) + 2(m_L^2 - m_\gamma^2)^2 I(m_\gamma, m_L, 0) - m_L^4 I(m_L, 0, 0)] \right\}, \quad (38) \end{aligned}$$

$$\begin{aligned} \text{Diagram 5} &= e^2 \mu^\varepsilon \frac{1}{4} \left\{ (N-2) [2J(m_\gamma) J(m_T) - J(m_T)^2 + I(m_\gamma, m_T, m_T) (m_\gamma^2 - 4m_T^2)] \right. \\ &\left. + 2 \left[ \frac{(m_\gamma^2 + m_L^2 - m_T^2)}{m_\gamma^2} J(m_\gamma) J(m_L) + \frac{(m_\gamma^2 - m_L^2 + m_T^2)}{m_\gamma^2} J(m_\gamma) J(m_T) - J(m_L) J(m_T) \right] \right\} \end{aligned}$$

$$-I(m_T, m_L, 0) \frac{(m_T^2 - m_L^2)^2}{m_\gamma^2} + I(m_\gamma, m_T, m_L) \frac{(m_T^2 - m_L^2)^2 + m_\gamma^2(m_\gamma^2 - 2m_T^2 - 2m_L^2)}{m_\gamma^2} \Big] \Big\} \quad (39)$$

where, compared to Eq. (8), we have included in the same diagram the different transverse and longitudinal components, and where the vertices and corresponding multiplicities have been once again specified.

#### IV. EFFECTIVE POTENTIAL

Collecting the results from the previous section, the renormalization coupling constants are tuned to cancel the poles in  $1/\varepsilon^2$  and  $1/\varepsilon$ . Working with the effective potential, as we do here, logarithms are appearing. Some of them have poles in  $\varepsilon$  coefficients. These logarithms with pole coefficients have been named dangerous poles by Chung and Chung [28]. The cancellation of these dangerous poles is a non trivial consequence of the renormalizability of the theory. We can see it as follow: Asking for the cancellation of one-loop poles, we identify straightforwardly

$$Z_{m^2}^{(1)} = g \frac{(N+2)}{3\varepsilon}, \quad (40)$$

$$gZ_g^{(1)} = \frac{g^2(N+8) + 108e^4}{3\varepsilon}, \quad (41)$$

$$Z_v^{(1)} = g \frac{N}{2\varepsilon}, \quad (42)$$

where a factor  $1/(4\pi)^2$  has been absorbed in the definition of  $\hbar$  (see the expansion (12)).

The cancellation of the poles at the two-loop order gives renormalization coefficients  $Z_{m^2}^{(2)}, Z_v^{(2)}$  which depend on  $\overline{\ln}(m_L), \overline{\ln}(m_T)$  and  $Z_\phi^{(1)}$  and a renormalization coefficient  $Z_g^{(2)}$  which depends on  $\overline{\ln}(m_L), \overline{\ln}(m_T), \overline{\ln}(m_\gamma)$  and  $Z_\phi^{(1)}, Z_A^{(1)}$ . We find that, asking for the independence with respect to these dangerous logarithms, fixes  $Z_\phi^{(1)}, Z_A^{(1)}$ . This is non trivial since there are more conditions than renormalization coefficients, and this is the mentioned consequence of the renormalization. Everything together, the cancellation of the poles at the two-loop order gives then

$$Z_{m^2}^{(2)} = \frac{(N+2)}{9\varepsilon^2} [g^2(N+5) - 18ge^2 + 54e^4] - \frac{1}{6\varepsilon} [g^2(N+2) - 8ge^2(N+2) - 6e^4(5N+1)] \quad (43)$$

$$gZ_g^{(2)} = \frac{1}{9\varepsilon^2} [g^3(N+8)^2 - 18g^2e^2(N+8) + 108ge^4(N+8)^2 + 108e^6(N+18)] - \frac{1}{9\varepsilon} [g^3(5N+22) - 12g^2e^2(N+5) - 18ge^4(5N+13) + 18e^6(7N+90)], \quad (44)$$

$$Z_v^{(2)} = g \frac{N}{6\varepsilon^2} [-18e^2 + g(N+2)] + ge^2 N \frac{2}{\varepsilon}. \quad (45)$$

$$Z_\phi^{(1)} = e^2 \frac{6}{\varepsilon}, \quad (46)$$

$$Z_A^{(1)} = -e^2 \frac{N}{3\varepsilon}, \quad (47)$$

where, as for the one-loop renormalization constants, a factor  $1/(4\pi)^2$  has also been absorbed in the definition of  $\hbar$ .

With these one-loop and two-loop renormalization constants, all the poles disappear from the  $\varepsilon$ -expansion of the theory. Up to this order (two loops), and, in view of applications of this theory to phase transitions in three dimensions, the one-loop order term has to be developed to the order  $\varepsilon$ . The two-loop effective potential in the Landau gauge can then be written as

$$V = V^{(0)} + \frac{\hbar}{(4\pi)^2} [V^{(1,0)} + \varepsilon V^{(1,\varepsilon)}] + \left[ \frac{\hbar}{(4\pi)^2} \right]^2 V^{(2)}, \quad (48)$$

where we have specified that a factor  $(4\pi)^2$  is absorbed in the definition of  $\hbar$ , and with

$$V^{(0)} = \frac{1}{2} m^2 \Phi^2 + \frac{g}{4!} \mu^\varepsilon \Phi^4 + \frac{m^4}{g\mu^\varepsilon}, \quad (49)$$

$$V^{(1,0)} = \frac{\mu^{-\varepsilon}}{8} \{ (N-1)m_T^4 [-3 + 2\overline{\ln}(m_T^2)] + m_L^4 (-3 + 2\overline{\ln}(m_L^2)) + m_\gamma^4 [-5 + 6\overline{\ln}(m_\gamma^2)] \}, \quad (50)$$

$$V^{(1,\varepsilon)} = -\frac{\mu^{-\varepsilon}}{96} \left( (N-1)m_T^4 \{21 - 18\overline{\ln}(m_T^2) + 6[\overline{\ln}(m_T^2)]^2 + \pi^2\} + m_L^4 \{21 - 18\overline{\ln}(m_L^2) + 6[\overline{\ln}(m_L^2)]^2 + \pi^2\} \right. \\ \left. + 3m_\gamma^4 \{9 - 10\overline{\ln}(m_T^2) + 6[\overline{\ln}(m_T^2)]^2 + \pi^2\} \right). \quad (51)$$

$$(52)$$

The term  $V^{(2)}$  is much longer to write. For this reason, we give its expansion on the  $\overline{\ln}$ -bases. With the definition

$$V^{(2)} = \mu^{-\varepsilon} \sum_{i=0}^2 \sum_{j=0}^2 \sum_{k=0}^2 V_{i,j,k}^{(2)} [\overline{\ln}(m_T^2)]^i [\overline{\ln}(m_L^2)]^j [\overline{\ln}(m_\gamma^2)]^k, \quad (53)$$

we have (see however the remark after these equations)

$$V_{0,0,0}^{(2)} = \frac{1}{432gm_\gamma^4} \left[ 18m_\gamma^2 \left( g^2m_\gamma^2 [(N-1)(N+5)m_T^4 + 18m_L^2m_T^2 - (2N+13)m_L^4] \right. \right. \\ \left. \left. + ge^2 \{42(m_L^2 - m_T^2)^2(m_L^2 + m_T^2) - 6m_\gamma^2 [(19N-31)m_T^4 + 24m_T^2m_L^2 + 7m_L^4] \right. \right. \\ \left. \left. - 18m_\gamma^4 [(N-1)m_T^2 + m_L^2] + (19N-36)m_\gamma^6 \} - 108e^4m_\gamma^2(m_L^2 - m_T^2) [(N-1)m_T^2 + 3m_L^2] \right) \\ \left. + 6g\Phi^2 \{7g^2m_\gamma^4 [2(N-1)m_T^2 + (N+8)m_L^2] + 18e^4(7m_L^6 - 7m_L^4m_\gamma^2 + 11m_L^2m_\gamma^4 + 54m_\gamma^6)\} \right. \\ \left. + \pi^2 \left( g^2m_\gamma^4 [2(N-1)m_T^2 + (N+8)m_L^2] [3(m_T^2 - m_L^2) + g\Phi^2] \right. \right. \\ \left. \left. + 18ge^2m_\gamma^2 \{ (m_L^2 - m_T^2)^2(m_L^2 + m_T^2) + 3m_\gamma^4 [(N-1)m_T^2 + m_L^2] - 9m_\gamma^6 \} \right. \right. \\ \left. \left. + 18e^4 \{ -9m_\gamma^4(m_L^2 - m_T^2) [(N-1)m_T^2 + 3m_L^2] + g\Phi^2(2m_L^6 - m_L^4m_\gamma^2 + 5m_L^2m_\gamma^4 + 10m_\gamma^6) \} \right) \\ \left. + 216(4\pi)^4 ge^2 \left[ m_\gamma^2(m_L^2 - m_T^2)^2 I^{(0)}(m_L, m_T, 0) + e^2\Phi^2(m_\gamma^2 - m_L^2)^2 I^{(0)}(m_\gamma, m_L, 0) \right] \right. \\ \left. + 54ge^2 \left( m_\gamma^2 \{ (N-2)(m_\gamma^2 - 4m_T^2)m_\gamma^2 \xi(m_\gamma, m_T, m_T) \right. \right. \\ \left. \left. + 2 [m_T^4 - 2m_T^2(m_\gamma^2 + m_L^2) + (m_\gamma^2 - m_L^2)^2] \xi(m_\gamma, m_T, m_L) \} \right. \right. \\ \left. \left. + e^2\Phi^2(m_L^4 - 4m_L^2m_\gamma^2 + 12m_\gamma^4) \xi(m_\gamma, m_\gamma, m_L) \right) \right. \\ \left. + 6g^3\Phi^2m_\gamma^4 [(N-1)\xi(m_T, m_T, m_L) + 3\xi(m_L, m_L, m_L)] \right], \quad (54)$$

$$V_{1,0,0}^{(2)} = -\frac{m_T^2}{12gm_\gamma^2} \left( g^2(N-1)m_\gamma^2[-m_L^2 + (N+3)m_T^2 + 2g\Phi^2] \right. \\ \left. + 6ge^2 \{3(m_L^2 - m_T^2)^2 - m_\gamma^2[2m_L^2 + m_T^2(7N-9)] + 5(N-1)m_\gamma^4\} - 54e^4(N-1)m_\gamma^2(m_L^2 - m_T^2) \right), \quad (55)$$

$$V_{2,0,0}^{(2)} = \frac{1}{72gm_\gamma^2} \left( g^2(N-1)m_\gamma^2 \{ -m_L^2(6m_T^2 + g\Phi^2) + m_T^2[3m_T^2(N+3) + 4g\Phi^2] \} \right. \\ \left. - 9ge^2 [-2(m_L^2 - m_T^2)^2m_T^2 + m_\gamma^2(N-2)(m_\gamma^4 + 6m_T^4) - 6(2N-3)m_\gamma^4m_T^2] \right. \\ \left. - 162e^4(N-1)m_\gamma^2(m_L^2 - m_T^2)m_T^2 \right), \quad (56)$$

$$V_{0,1,0}^{(2)} = -\frac{m_L^2}{12gm_\gamma^4} \left( g^2m_\gamma^4 [-(N+5)m_L^2 + (2N+7)m_T^2 + (N+8)g\Phi^2] \right. \\ \left. + 6ge^2m_\gamma^2 \{3(m_L^2 - m_T^2)^2 - m_\gamma^2[5m_L^2 + 2m_T^2] + 5m_\gamma^4\} \right. \\ \left. - 6e^4 \{m_\gamma^4 [27(m_L^2 - m_T^2) - 16g\Phi^2] + 3g\Phi^2m_L^2(2m_\gamma^2 - m_L^2)\} \right), \quad (57)$$

$$V_{0,2,0}^{(2)} = \frac{m_L^2}{72gm_\gamma^4} \left( g^2m_\gamma^4 [-3(N+5)m_L^2 + 3(N+8)m_T^2 + (N+17)g\Phi^2] \right.$$



$$\begin{aligned}
& + 18g e^2 m_\gamma^2 [(m_L^2 - m_T^2)^2 + 3m_\gamma^4] \\
& - 9e^4 \{ 2m_\gamma^4 [27(m_L^2 - m_T^2) - 7g\Phi^2] + g\Phi^2 m_L^2 (4m_\gamma^2 - 3m_L^2) \},
\end{aligned} \tag{58}$$

$$\begin{aligned}
V_{0,0,1}^{(2)} &= \frac{e^2}{6m_\gamma^2} \left\{ -9e^2 m_L^4 \Phi^2 - 3m_\gamma^2 [m_L^4 + m_T^4 - 2m_L^2(m_T^2 + 4e^2\Phi^2)] \right. \\
& \left. + 12m_\gamma^4 [m_L^2 + (N-1)m_T^2 - 6e^2\Phi^2] - m_\gamma^6(4N-9) \right\},
\end{aligned} \tag{59}$$

$$V_{0,0,2}^{(2)} = \frac{e^2}{8m_\gamma^4} [-18m_\gamma^8 + e^2\Phi^2 (-m_L^6 + 8m_L^4 m_\gamma^2 - 22m_L^2 m_\gamma^4 + 40m_\gamma^6)], \tag{60}$$

$$\begin{aligned}
V_{1,1,0}^{(2)} &= \frac{1}{36m_\gamma^2} \left\{ g(N-1)m_L^2 m_\gamma^2 (3m_T^2 + g\Phi^2) \right. \\
& \left. - 9e^2 [-(m_L^2 - m_T^2)^2 (m_L^2 + m_T^2) + 3m_\gamma^2 (m_L^4 + m_T^4) - 3m_\gamma^4 (m_L^2 + m_T^2) + m_\gamma^6] \right\},
\end{aligned} \tag{61}$$

$$V_{1,0,1}^{(2)} = \frac{e^2}{4m_\gamma^2} [-(m_L^2 - m_T^2)^3 + 3m_\gamma^2 (m_L^4 - m_T^4) - 3m_\gamma^4 (m_L^2 - m_T^2) + (N-1)m_\gamma^6], \tag{62}$$

$$\begin{aligned}
V_{0,1,1}^{(2)} &= \frac{e^2}{4m_\gamma^4} \left\{ e^2 m_L^6 \Phi^2 + m_\gamma^2 [m_L^6 - m_L^4 (3m_T^2 + 4e^2\Phi^2) + 3m_L^2 m_T^4 - m_T^6] \right. \\
& \left. + m_\gamma^4 [m_L^2 (-3m_L^2 + 14e^2\Phi^2) + 3m_T^4] + 3m_\gamma^6 (m_L^2 - m_T^2) + m_\gamma^8 \right\},
\end{aligned} \tag{63}$$

where the function  $I^{(0)}(m_1, m_2, 0)$  denotes the  $\varepsilon \rightarrow 0$  non-diverging piece of (32), without the coefficient  $(\mu^\varepsilon)^2$ . Some part of this function may be put in the other terms of the expansion (53), because they lead, for example, to  $\overline{\ln}(m_1)\overline{\ln}(m_2)$ . We have not proceed in this way because, in the scalar QED case, it is not yet known if  $m_\gamma > m_T$ . One has to study first the fixed points of the system. At the critical point, one knows that, in the  $4 - \varepsilon$  formalism with  $\varepsilon$ -expansion,  $m \rightarrow 0$ . At the critical point, one would then need to compare  $e^2$  to  $g/6$ . If  $m_\gamma > m_T$ , one is allowed to use (32) with  $m_1 = m_\gamma, m_2 = m_T$ . In the opposite case  $m_\gamma < m_T$ , one has to use (32) with  $m_2 = m_\gamma, m_1 = m_T$ . We also note that  $I^{(0)}(m_1, m_2, 0)$  generate terms containing  $\overline{\ln}(m_1^2 - m_2^2)$ . These terms can never be put on the basis defined in (53).

To see the coherence of our effective potential, we shall look at the  $\phi^4$ -limit, which can be reached using  $e^2 \rightarrow 0$ .

### A. $\phi^4$ limit

In the limit  $e^2 \rightarrow 0$ , which gives the  $\phi^4$  model, the renormalized coupling constants and the effective potential are much simplified. The renormalization constants can be readily read off Eqs.(40)–(47) with  $e^2 = 0$ . For the effective potential, it is necessary to clean the results given in (49)–(51) and (54)–(63). Taking the limit  $e^2 \rightarrow 0$ , we obtain

$$V^{(0)} = \frac{1}{2}m^2\Phi^2 + \frac{g}{4!}\mu^\varepsilon\Phi^4 + \frac{m^4}{g\mu^\varepsilon}, \tag{64}$$

$$V^{(1,0)} = \frac{\mu^{-\varepsilon}}{8} \{ (N-1)m_T^4 [-3 + 2\overline{\ln}(m_T^2)] + m_L^4 (-3 + 2\overline{\ln}(m_L^2)) \}, \tag{65}$$

$$\begin{aligned}
V^{(1,\varepsilon)} &= -\frac{\mu^{-\varepsilon}}{96} \left( (N-1)m_T^4 \{ 21 - 18\overline{\ln}(m_T^2) + 6[\overline{\ln}(m_T^2)]^2 + \pi^2 \} + m_L^4 \{ 21 - 18\overline{\ln}(m_L^2) + 6[\overline{\ln}(m_L^2)]^2 + \pi^2 \} \right).
\end{aligned} \tag{66}$$

$$\tag{67}$$

The term  $V^{(2)}$  is considerably simplified compared to the full QED model. It has the expansion

$$V^{(2)} = \mu^{-\varepsilon} \sum_{i=0}^2 \sum_{j=0}^2 V_{i,j}^{(2)} [\overline{\ln}(m_T^2)]^i [\overline{\ln}(m_L^2)]^j, \tag{68}$$

with the following coefficients:

$$V_{0,0}^{(2)} = \frac{g}{432} \left( 6 \{ 54m_L^2 m_T^2 - 3(2N+13)m_L^4 + 7(N+8)m_L^2 g\Phi^2 + (N-1)m_T^2 [3(N+5)m_T^2 + 14g\Phi^2] \} \right)$$

$$\begin{aligned}
& + [2(N-1)m_T^2 + (N+8)m_L^2] [3(m_T^2 - m_L^2) + g\Phi^2] \pi^2 + 6g\Phi^2 [(N-1)\xi(m_T, m_T, m_L) + 3\xi(m_L, m_L, m_L)] \\
& = \frac{g}{24} \{ m_T^4(N-1)(N-9) + m_L^4(5N+43) + m_L^2 m_T^2(7N-52) \\
& \quad + (m_L^2 - m_T^2) [(N-1)\xi(m_T, m_T, m_L) + 3\xi(m_L, m_L, m_L)] \}
\end{aligned} \tag{69}$$

$$\begin{aligned}
V_{1,0}^{(2)} & = -\frac{g}{12}(N-1)m_T^2 [-m_L^2 + (N+3)m_T^2 + 2g\Phi^2] \\
& = -\frac{g}{12}(N-1)m_T^2 [5m_L^2 + (N-3)m_T^2]
\end{aligned} \tag{70}$$

$$\begin{aligned}
V_{2,0}^{(2)} & = -\frac{g}{72}(N-1) \{ m_L^2(6m_T^2 + g\Phi^2) - m_T^2[3(N+3)m_T^2 + 4g\Phi^2] \} \\
& = -\frac{g}{24}(N-1) [m_L^4 - 3m_L^2 m_T^2 - (N-1)m_T^4]
\end{aligned} \tag{71}$$

$$\begin{aligned}
V_{0,1}^{(2)} & = -\frac{g}{12}m_L^2 [-(N+5)m_L^2 + (2N+7)m_T^2 + (N+8)g\Phi^2] \\
& = \frac{g}{12}m_L^2 [(N+17)m_T^2 - (2N+19)m_L^2]
\end{aligned} \tag{72}$$

$$\begin{aligned}
V_{0,2}^{(2)} & = \frac{g}{72}m_L^2 [-3(N+5)m_L^2 + 3(N+8)m_T^2 + (N+17)g\Phi^2] \\
& = \frac{g}{8}m_L^2(4m_L^2 - 3m_T^2)
\end{aligned} \tag{73}$$

$$\begin{aligned}
V_{1,1}^{(2)} & = \frac{g}{36}(N-1)m_L^2(3m_T^2 + g\Phi^2) \\
& = \frac{g}{12}(N-1)m_L^4,
\end{aligned} \tag{74}$$

$$\tag{75}$$

where, in the second equality, we have replaced  $g\Phi^2$  by  $3(m_L^2 - m_T^2)$ . We have checked that Eqs. (69)–(74) reproduce the effective potential given in [1]. The correct limit of the pure  $\phi^4$ -theory is then recovered.

#### *Remark*

In the previous subsection, we have obtained the effective potential of  $\phi^4$ -theory by taking the limit  $e^2 \rightarrow 0$  for fixed  $m_\gamma^2$ , then taking the limit  $m_\gamma^2 \rightarrow 0$ . This is the fastest way to obtain the correct limit, but it is quite *cavalier* to proceed so. We have checked, replacing everywhere  $m_\gamma^2$  by its value  $m_\gamma^2 = e^2\Phi^2$ , that it was justified to do so: all the  $m_\gamma^2$ -terms in the denominator of Eqs. (54)–(63) are properly compensated to give the two-loop result of Eqs (69)–(74).

## V. CONCLUSION

In this paper, we have determined the effective potential of scalar QED in  $4 - \varepsilon$  dimensions in an  $\varepsilon$ -expansion. For zero charge, we recover the well-known result for the pure  $O(N)$ -symmetric  $\phi^4$ -theory. The full effective potential will be used in order to determine various amplitude ratios in the context of a scalar field in the presence of a gauge field. This comprises the superconducting case, for which  $N = 2$ . In this respect, it might be interesting to work in an arbitrary gauge and to show that these amplitude ratios are gauge independent. This can be done as in Ref. [29].

We shall also, in a subsequent paper, recast our result in the scaling form of Widom which shows the critical behavior of the free energy explicitly.

An other interesting work will be the application of variational perturbation theory to derive the various critical exponents in scalar QED, including the amplitude ratios along the lines of Ref. [17]. In  $4 - \varepsilon$  dimensions with  $\varepsilon$ -expansion, this has not even been done for the simpler  $O(N)$ -symmetric  $\phi^4$ -model without gauge field.

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- [1] C. Ford, I. Jack, and D. R. T. Jones, Nucl. Phys. B **387**, 373 (1992).
- [2] A. I. Davydychev, and J. B. Tausk, Nucl. Phys. B **397**, 123 (1993).
- [3] A. I. Davydychev, Phys. Rev. D **61**, 087701 (2000).
- [4] H. Kleinert, Lett. Nuovo Cimento **35**, 405 (1982) (<http://www.physik.fu-berlin.de/~kleinert/97>);  
H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. I Superflow and Vortex Lines, World Scientific Publishing Co., Singapore 1989, pp. 735-742 (<http://www.physik.fu-berlin.de/~kleinert/b1>).
- [5] J. Bartholomew, Phys. Rev. B **28**, 5378 (1983); Y. Muneshisa, Phys. Lett. B **155**, 159 (1985).
- [6] B. I. Halperin, T. C. Lubensky, and S.-K. Ma, Phys. Rev. Lett. **32**, 292 (1974);  
J.-H. Chen, T. C. Lubensky, and D. R. Nelson, Phys. Rev. B **17**, 4274 (1978).
- [7] I. A. Lawrie, Nucl. Phys. B **200**, 1 (1982).
- [8] J.-P. Tessman, *Two-loop Renormierung der skalaren Elektrodynamik*, Diplomarbeit, Freie Universität Berlin, written under the supervision of one of the authors (H.K.)
- [9] R. Folk, and Yu. Holovatch, J. Phys. A **29**, 3409 (1996);  
R. Folk, and Yu. Holovatch, *Critical fluctuations in normal-to-superconducting transition*, Lecture at the 1st Winter Workshop "Cooperative Phenomena in Condensed Matter", March, 1998, Pamporovo, Bulgaria ; cond-mat/9807421.
- [10] M. Kiometzis, H. Kleinert, and A. M. J. Schakel, Phys. Rev. **73**, 1975 (1994);  
I. F. Herbut, and Z. Tesanović, Phys. Rev. Lett. **76**, 4588 (1996);  
C. de Calan, A. P. C. Malbuisson, F. S. Nogueira, and N. F. Svaiter, Phys. Rev. B **59**, 554 (1999);  
C. de Calan, and F. S. Nogueira, Phys. Rev. B **60**, 4255 (1999) ; B. Bergerhoff, F. Freire, D. F. Litim, S. Lola, and C. Wetterich, Phys. Rev. B **53**, 5734 (1996).
- [11] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, World Scientific. Variational perturbation theory is developed in chapters 5 and 17.
- [12] H. Kleinert, Phys. Rev. D **57**, 2264 (1998) ; Addendum **58** 107702 (1998).
- [13] H. Kleinert, Phys. Rev. D **60**, 085001 (1999).
- [14] F. Jasch, and H. Kleinert, J. Math. Phys. **42**, 52 (2001).
- [15] H. Kleinert, Phys. Lett. B **434**, 74 (1998); **463**, 69 (1999).
- [16] H. Kleinert, and V. Schulte-Frohlinde, J. Phys. A **34**, 1037 (2001).
- [17] H. Kleinert, and B. Van den Bossche, to be published in Phys. Rev. E; *Three-loop critical exponents, amplitude functions and amplitude ratios from variational perturbation theory*, cond-mat/0011329
- [18] M. Strösser, L. A. Larin, and V. Dohm, Nucl. Phys. B **540**, 654 (1999).
- [19] S. Kolnberger, and R. Folk, Phys. Rev. B **41**, 4083 (1992).
- [20] E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. Lett. **29**, 591 (1972); Phys. Rev. B **7**, 232 (1973).
- [21] E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase transition and critical phenomena*, edited by C. Domb and M. S. Green (Academic Press, New-York, 1976), Vol. 6., p. 125.
- [22] D. J. Wallace, and R. K. P. Zia, Phys. Lett. A **46**, 261 (1973); J. Phys. C **7**, 3480 (1974).
- [23] B. De Witt, Phys. Rev. **162**, 1195 (1967).
- [24] B. Kastening, Phys. Rev. E **61**, 3501 (2000).
- [25] M. Bachman, A. Kleinert, and A. Pelster, Phys. Rev. D **61**, 085017 (2000);  
H. Kleinert, A. Pelster, B. Kastening, and M. Bachman, Phys. Rev. E **62**, 1537 (2000).
- [26] H. Kleinert, A. Pelster, and B. Van den Bossche, in preparation.
- [27] B. Kastening, Phys. Rev. D **54**, 3965 (1996);  
B. Kastening, Phys. Rev. D **57**, 3567 (1998).
- [28] J. M. Chung, and B. K. Chung, Phys. Rev. D **56**, 6508 (1997);  
J. M. Chung, and B. K. Chung, J. Korean Phys. Soc. **33**, 643 (1998).
- [29] J. S. Kang, Phys. Rev. D **10**, 3455 (1974).