

Charged fixed point in the Ginzburg-Landau superconductor and the role of the Ginzburg parameter κ

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Abstract

We present a semi-perturbative approach which yields an infrared-stable fixed point in the Ginzburg-Landau for $N = 2$, where $N/2$ is the number of complex components. The calculations are done in $d = 3$ dimensions and below T_c , where the renormalization group functions can be expressed directly as functions of the Ginzburg parameter κ which is the ratio between the two fundamental scales of the problem, the penetration depth λ and the correlation length ξ . We find a charged fixed point for $\kappa > 1/\sqrt{2}$, that is, in the type II regime, where $\Delta\kappa \equiv \kappa - 1/\sqrt{2}$ is shown to be a natural expansion parameter. This parameter controls a momentum space instability in the two-point correlation function of the order field. This instability appears at a nonzero wave-vector \mathbf{p}_0 whose magnitude scales like $\sim \Delta\kappa^{\bar{\beta}}$, with a critical exponent $\bar{\beta} = 1/2$ in the one-loop approximation, a behavior known from magnetic systems with a Lifshitz point in the phase diagram. This momentum space instability is argued to be the origin of the negative η -exponent of the order field.

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I. INTRODUCTION

Whereas the field-theoretic understanding of the universal critical properties of the superfluid phase transition has become quite satisfactory [1, 2], the renormalization group (RG) have so far not been completely successful for the superconducting phase transition. The standard field theory investigated in this context is the Ginzburg-Landau (GL) model, initially in $4 - \epsilon$ dimensions by Halperin, Lubensky, and Ma (HLM)[3], who generalized a calculation of Coleman and Weinberg [4] in four dimensions, and found no infrared stable fixed point at the one-loop level. The renormalization group (RG) flow was shown to run off to infinity in the $g - f$ -plane if $N < 365.9$, with $N/2$ being the number of complex components of the scalar order parameter field. Here g and f are the dimensionless versions of the quartic self-coupling u of the order parameter and the square charge e^2 , respectively. HLM interpreted this result as indicating that the phase transition in a superconductor - which corresponds to a order parameter with $N = 2$ - is of first-order. They supported this scenario by a mean-field estimate of the effective potential obtained by integrating out the vector potential, which led to a term [5] $\sim |\psi|^{4-\epsilon}$, which leads to first-order phase transition. This was in contrast to Monte Carlo simulations of lattice superconductors in 1981 by Dasgupta and Halperin [6] where a clear second-order phase transition was found. It also contradicted work by Lawrie and Athorne [7] who performed a RG study in a non-linear realization of the GL model and found no indication of a first-order phase transition. Moreover, the first-order scenario was also contradicted by experiments on the smectic-A to nematic phase transition in liquid crystals [8] which according to de Gennes [9] can also be described by a GL model.

For small e^2 or large Ginzburg parameter κ , the GL-model lies in the deep type II regime where it goes over into a pure complex $|\psi|^4$ -model which describes the superfluid phase transition, which is the best-understood second-order transition experimentally and theoretically. We may therefore expect a large- κ portion of the type II regime to undergo a second-order transition. Indeed, the Monte Carlo simulations of Ref. 6 and the non-linear realization of Ref. 7 were both in this regime.

The arguments leading to a first-order transition could therefore at best be reliable in the large- e^2 regime where the superconductor is of type I and the Ginzburg parameter κ is small. Only in this regime can the ϵ -expansion be trusted and the important problem arose how to find a theory explaining both type I and type II regimes.

One solution to this problem was proposed in 1982 by one of us [10, 11] by deriving a disorder field theory dual to the original GL model. The RG analysis of the disorder field theory shows very clearly that a charged fixed point exist in the original GL model [12]. The disorder field theory exhibited a first- and a second-order regime with a tricritical point at [10] $\kappa_t = (3/\pi)\sqrt{3/2}[1 - (4/9)^4] \equiv 0.79/\sqrt{2}$, a result confirmed only recently with excellent agreement by elaborate Monte Carlo simulations [14] which gave $\kappa_t = (0.76 \pm 0.04)/\sqrt{2}$. Since the disorder theory is in pure $|\phi|^4$ universality class, the critical exponent ν of the disorder field in the second-order regime has the XY -value $\nu \approx 0.67$ [12], a result confirmed by Monte Carlo simulations [15].

The RG situation of the original GL model, however, remained completely unclear. No fixed point at $N = 2$ appeared in an extension to two-loops by Tessmann [16], a result confirmed later by Kolnberger and Folk using a more thorough analysis [17, 18]. Folk and Holovatch [19] managed to find a charged fixed point only by setting up a suitable Padé-Borel resummation of the ϵ -expansion. A three-loop calculation using the ϵ -expansion is presently under progress. As a first step, the ground state energy of the GL model was computed recently up to three-loop by Kastening, Kleinert, and Van den Bossche [20].

Another method that leads to a charged fixed point at $N = 2$ was used by Bergerhof *et al.* [21]. Their method is based in the so called *exact renormalization group* [22]. In this approach, exact renormalization group equations are solved approximately using a truncated form for a scale-dependent effective action. The problem with this type of approach is that it seems to be uncontrolled, that is, there is no obvious expansion parameter. But this seems to be a common feature of most fixed dimension approaches. However, as we shall see later on in this paper, fixed dimension approaches are the best candidates to solve this old problem.

At fixed dimension $d = 3$ there is the approach by Herbut and Tesanovic [23] who performed a perturbative calculation directly at the critical point. These authors obtained a charged fixed point at one loop and $N = 2$ by using two different momentum scales and different renormalization points for the two dimensionless couplings, $f \equiv e^2/\mu'^{4-d}$ and $g \equiv u/\mu^{4-d}$. This introduces a new parameter $c \equiv \mu/\mu'$. They find charged fixed points if $c > 5.16$. Their approach has the problem that the parameter c must be fixed from the outside. They do this by demanding that the value of the ratio $\kappa = \sqrt{g/2f}$ matches at the tricritical point with Kleinert's value [10] $\kappa_t \simeq 0.79/\sqrt{2}$, and found a critical exponent

$\nu \approx 0.53$. The authors used also an older Monte Carlo estimate[24] $\kappa_t \simeq 0.42/\sqrt{2}$ to fix c which led to a ν closer to the the XY value. In the light of the recent simulation work [14] obtaining the value $\kappa_t \simeq 0.77/\sqrt{2}$, the result $\kappa_t \simeq 0.42/\sqrt{2}$ becomes obsolete.

There are two important universal observations made in Ref. 23: first, if a charged fixed point exists, then the anomalous dimension of the vector potential is *exactly* given by $\eta_A = 4 - d$ for $d \in (2, 4]$. Second, this anomalous dimension implies the existence of [15, 23] a *dimension-independent* scaling relation $\lambda \propto \xi$ between the magnetic penetration depth λ and the coherence length ξ . Subsequently, the AC conductivity $\sigma(\omega)$ was shown[25, 26] to have the scaling behavior $\sigma(\omega) \sim \xi^{z-2}$, where z is the dynamic exponent. These results are in contrast to the *dimension-dependent* scaling relations $\lambda \sim \xi^{(d-2)/2}$ and $\sigma(\omega) \sim \xi^{2-d+z}$ derived by D.S. Fisher *et al.*[27] by *neglecting* thermal fluctuations of the magnetic field, leading to $\eta_A = 0$.

It is important to emphasize that in Ref. 23 the two renormalization scales μ and μ' play a decisive role for the emergence of the charged fixed point. However, since the correlation functions were calculated *at* the critical point, there was no direct relation between μ and μ' and the two physical length scales of the GL model. One of this lengths, the penetration depth, is observable only in the *ordered*, since the vector potential becomes massive only below T_c through the Anderson-Higgs mechanism. It is therefore desirable to study the RG flow in such a way that T_c is approached from below, and this is what we shall do in this paper by employing a semi-perturbative approach.

Since below T_c the vector potential is a massive field, it provides us with the second physical mass scale of the system, and thus with the Ginzburg parameter κ on which the RG functions depend explicitly. We shall exhibit a charged fixed point for $N = 2$ and $\kappa > 1/\sqrt{2}$, that is, inside the type II regime. In our analysis, $\Delta\kappa \equiv \kappa - 1/\sqrt{2}$ appears as a natural expansion parameter when T_c is approached from below. This is related to the existence of a singular behavior at $\kappa = 1/\sqrt{2}$. This singular behavior is already apparent in the mean field solution in a uniform external magnetic field by Abrikosov [42]. For example, the magnetization diverges at $\kappa = 1/\sqrt{2}$. To our knowledge, in the RG context this singular behavior in κ has never been explored before. We shall show that the existence of the charged fixed point is related to a singular behavior at $\kappa = 1/\sqrt{2}$ occurring in the vector potential correlation function. This singularity has the physical meaning that it represents the point of separation between type I and type II superconductivity. As discussed in Ref.

14, the value of κ separating these two regimes should coincide with κ_t , being therefore lower than $1/\sqrt{2}$. The approximation we consider here is not able to correct upon the value $\kappa = 1/\sqrt{2}$, but higher order corrections will do. At this point it is worth to explain what we mean by higher order. Our method is semi-perturbative. By this we mean that we start by assuming that a perturbative expansion in f and g is possible. However, we parametrize our theory by f and κ such that all RG functions depend on these parameters. This is possible because $\kappa^2 = m^2/m_A^2 = g/2f$, where $m = \xi^{-1}$ is the Higgs mass and $m_A = \lambda^{-1}$ is the vector potential mass. We use m as the running scale for the RG. Thus, κ arises in the RG functions from two different sources: the coupling g , which is rewritten as $g = 2f\kappa^2$, and the loop integrals, from where the masses are combined in such a way as to produce functions of κ . The RG functions obtained in this way are polynomials in f with coefficients depending on κ . Thus, it is not really a standard perturbative series. This procedure will be explained in Sections III and IV. The key point is that for κ sufficiently close to $\kappa = 1/\sqrt{2}$ from above, f is small enough such that a charged fixed point can be obtained at $N = 2$.

An interesting point revealed by our analysis is the role of κ in controlling the appearance of momentum space instabilities in the correlation functions. For $\kappa > 1/\sqrt{2}$, the two-point bare correlation function is maximized for a nonzero wave vector, similar to the roton peak in the correlation function of superfluid helium. This behavior seems to be the origin of the negative sign of the η -exponent and implies a critical behavior[26] known from the theory of Lifshitz points. In contrast to scalar models of Lifshitz point, where the momentum space instability is already present at the tree level, the instability in the GL model is induced by magnetic fluctuations. Interestingly, such fluctuation-induced Lifshitz points also occur in the non-commutative ϕ^4 -theory, which also possesses a negative η exponent[28].

The plan of the paper is the following. In Section II we discuss the advantages of a RG approach to the GL model in $d = 3$ dimensions in the disordered phase over the usual $4 - \epsilon$ -dimensional calculations. This approach is a generalization to the GL model of Parisi's three-dimensional RG technique [29]. In particular, we show that this approach, being sensitive to the infrared divergences of the vector potential, yields at the one-loop order a more reliable result than the one-loop ϵ -expansion. In Sections III we renormalize the ordered phase followed by a detailed renormalization group analysis in Section IV which culminates in the desired infrared-stable fixed point which could not be found in $4 - \epsilon$ dimensions. Some useful integrals used in the paper compiled in the Appendix.

II. THREE-DIMENSIONAL RENORMALIZATION GROUP IN GL MODEL AND ITS ADVANTAGES

The bare GL hamiltonian considered in this paper is given by

$$\mathcal{H} = \frac{1}{2}(\nabla \times \mathbf{A}_0)^2 + |(\nabla - ie_0 \mathbf{A}_0)\psi_0|^2 + m_0^2 |\psi_0|^2 + \frac{u_0}{2} |\psi_0|^4 + \mathcal{H}_{\text{gf}}, \quad (1)$$

where

$$\mathcal{H}_{\text{gf}} = \frac{1}{2\alpha_0} (\nabla \cdot \mathbf{A}_0)^2 \quad (2)$$

fixes the gauge. It will be convenient to express the complex fields in terms of real fields as

$$\psi_0 = \frac{1}{\sqrt{2}} (\psi_0^{(1)} + i\psi_0^{(2)}). \quad (3)$$

The propagators are given in momentum space by

$$G_{ij}^{(0)}(\mathbf{p}) \equiv \langle \psi_0^{(i)}(\mathbf{p}) \psi_0^{(j)}(-\mathbf{p}) \rangle = \frac{\delta_{ij}}{\mathbf{p}^2 + m_0^2}, \quad (4)$$

$$D_{\mu\nu}^{(0)}(\mathbf{p}) \equiv \langle A_0^\mu(\mathbf{p}) A_0^\nu(-\mathbf{p}) \rangle = \frac{1}{\mathbf{p}^2} \left[\delta_{\mu\nu} + (\alpha_0 - 1) \frac{p_\mu p_\nu}{\mathbf{p}^2} \right]. \quad (5)$$

Since the free vector field is massless, perturbative calculation run into infrared problems. For instance, the Feynman graph in Fig. 1 contributing to the four-point function is infrared divergent at zero external momenta for any dimension $d \in (2, 4)$. Indeed, the loop integral in this graph is proportional to $(\mathbf{p}^2)^{(d-4)/2}$, so that the Feynman integral yields in dimensional regularization [17, 19, 30] a pole term $1/\epsilon$ ($\epsilon = 4 - d$). However, this pole reflects an *ultraviolet* divergence for $d = 4$, not an *infrared* divergence, which is physical source of critical properties. Recall that the reason why ultraviolet divergences give nevertheless information on the infrared behavior is that a $1/\epsilon$ pole in $d = 4 - \epsilon$ is equivalent to a logarithmic divergence of the form $\ln(\mathbf{p}^2/\Lambda^2)$ at $d = 4$, where Λ is the ultraviolet cutoff, and the ultraviolet limit $\Lambda \rightarrow \infty$ at fixed $|\mathbf{p}|$ is equivalent to taking the infrared limit $|\mathbf{p}| \rightarrow 0$ at fixed Λ .

Alternatively, we can choose $|\mathbf{p}| \equiv \mu \neq 0$ as a scale parameter of the problem, confining the analysis to $d \in (2, 4)$ to avoid the logarithmic divergence at $d = 4$. This procedure is applicable in a RG analysis *at* the critical point performed by Herbut and Tesanovic[23].

However, as we have discussed in the introduction, this procedure needs a second scale parameter μ' which has to be determined from the outside.

Another way of regulating the infrared divergences of the GL model is to introduce a Proca term in the Hamiltonian, $M_0^2 \mathbf{A}_0^2/2$, which explicitly breaks gauge invariance. Then we may perform the calculations of correlation functions using the infrared-finite propagator

$$D_{\mu\nu}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + M_0^2} \left[\delta_{\mu\nu} + (\alpha_0 - 1) \frac{p_\mu p_\nu}{\mathbf{p}^2 + \alpha_0 M_0^2} \right], \quad (6)$$

and taking $M_0 \rightarrow 0$ at the end. However, if dimensional regularization is used to evaluate the massive Feynman integrals, final results are the same as in the dimensional regularization of the massless case, since the $1/\epsilon$ pole is the same for all M_0 :

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(\mathbf{p}^2 + M_0^2)^2} = \frac{1}{8\pi^2 \epsilon} + \mathcal{O}(1). \quad (7)$$

Thus, the β -functions obtained with this procedure will have the same ϵ -expansion as in the original work[3], where a charged fixed point exists only for a number of $\psi_0^{(i)}$ field components $N > 365$.

A completely different result is obtained if we study the system directly in three dimensions. We specify the renormalization constants by rewriting the Hamiltonian (1) in terms of renormalized quantities as

$$\mathcal{H} = \frac{Z_A}{2} (\nabla \times \mathbf{A})^2 + \frac{M^2}{2} \mathbf{A}^2 + Z_\psi |(\nabla - ie\mathbf{A})\psi|^2 + Z_m m^2 |\psi|^2 + Z_g \frac{gm}{2} |\psi|^4 + \mathcal{H}_{\text{gf}}, \quad (8)$$

where the renormalized fields are defined by $\psi = Z_\psi^{-1/2} \psi_0$ and $\mathbf{A} = Z_A^{-1/2} \mathbf{A}_0$. We shall work in the Landau gauge where $\alpha_0 = 0$ in \mathcal{H}_{gf} . The term $e\mathbf{A}$ does not need a renormalization as a consequence of the Ward identity [1] which implies that $e^2 = Z_A e_0^2$. The renormalization constants Z_ψ, Z_A, Z_m , and Z_g and the renormalized Proca mass M are fixed using the normalization conditions for the one-particle irreducible renormalized n -point functions[1, 2] (no sum over repeated i indices):

$$\frac{\partial \Gamma_{ii}^{(2)}(\mathbf{p})}{\partial \mathbf{p}^2} \Big|_{\mathbf{p}=0} = 1, \quad (9)$$

$$\Gamma_{ii}^{(2)}(0) = m^2, \quad (10)$$

$$\Gamma_{iiii}^{(4)}(0, 0, 0, 0) = 3mg, \quad (11)$$

$$\frac{\partial \Gamma_{\mu\mu}^{(2)}(\mathbf{p})}{\partial \mathbf{p}^2} \Big|_{\mathbf{p}=0} = 2, \quad (12)$$

$$\Gamma_{\mu\mu}^{(2)}(0) = 3M^2. \quad (13)$$

It is straightforward to calculate the following results at the one-loop level:

$$Z_\psi = 1 + \frac{2}{3\pi} \frac{e_0^2}{m+M}, \quad (14)$$

$$Z_A = 1 - \frac{e_0^2}{24\pi m}, \quad (15)$$

$$g = \frac{u_0}{m} + \frac{4}{3\pi} \frac{e_0^2 u_0}{m(m+M)} - \frac{5u_0^2}{8\pi m^2} - \frac{e_0^4}{2\pi m M}. \quad (16)$$

We define a dimensionless renormalized square charge by $f \equiv e^2/m$. Using $e^2 = Z_A e_0^2$ we find the one-loop equation

$$f = \frac{e_0^2}{m} - \frac{e_0^4}{24\pi m^2}. \quad (17)$$

The β -functions are defined in general by the logarithmic derivatives

$$\beta_f \equiv \lim_{M \rightarrow 0} m \frac{\partial f}{\partial m}, \quad (18)$$

$$\beta_g \equiv \lim_{M \rightarrow 0} m \frac{\partial g}{\partial m}. \quad (19)$$

The derivatives with respect to m have to be performed at fixed bare couplings. At this point we make a crucial observation. In the derivatives, we reexpress the bare couplings in terms of the renormalization f and g , while discarding terms beyond one-loop. However, the term proportional to e_0^4 in Eq. (16) contains only a single power of m in the denominator. *Therefore this term will contribute only to the term $-g$ in β_g and no term proportional to f^2 will be present.* The result is

$$\begin{aligned} m \frac{\partial g}{\partial m} &= -\frac{u_0}{m} - \frac{4}{3\pi} \frac{e_0^2 u_0}{m(m+M)} - \frac{4}{3\pi} \frac{e_0^2 u_0}{(m+M)^2} + \frac{5u_0^2}{4\pi m^2} + \frac{e_0^4}{2\pi m M} \\ &= -\left[\frac{u_0}{m} + \frac{4}{3\pi} \frac{e_0^2 u_0}{m(m+M)} - \frac{5u_0^2}{8\pi m^2} - \frac{e_0^4}{2\pi m M} \right] - \frac{4}{3\pi} \frac{e_0^2 u_0}{(m+M)^2} + \frac{5u_0^2}{8\pi m^2} \\ &= -g - \frac{4}{3\pi} \frac{e_0^2 u_0}{(m+M)^2} + \frac{5u_0^2}{8\pi m^2}. \end{aligned} \quad (20)$$

The derivatives of M with respect to m do not contribute since they are of higher than one-loop order. Taking the physically relevant limit $M \rightarrow 0$, we obtain

$$\begin{aligned} \beta_g &= -g - \frac{4}{3\pi} \frac{e_0^2 u_0}{m^2} + \frac{5u_0^2}{8\pi m^2} \\ &\simeq -g - \frac{4}{3\pi} f g + \frac{5g^2}{8\pi}, \end{aligned} \quad (21)$$

where in the second line we have replaced $u_0/m \rightarrow g$ and $e_0^2/m \rightarrow f$, the error committed with this substitution is beyond the one-loop order under consideration. It is this result where the three-dimensional calculation in this paper differs essentially from the dimensionally regularized one in $d = 4 - \epsilon$ dimensions in which the β -function β_g contains additional f^2 term. Precisely this term is the culprit for the nonexistence of a charged fixed point in HLM.

The second β -function is

$$\beta_f = -f + \frac{f^2}{24\pi}, \quad (22)$$

From Eqs. (21) and (22) we obtain the infrared stable fixed point $f_* = 24\pi$ and $g_* = 264\pi/5$.

At this point we may think that the problem is solved, and that the absence of a charged fixed point was merely an artifact of the ϵ -expansion. Unfortunately this is not true since at the fixed point, *the critical exponents have completely unphysical values*. For example, the critical exponent η , which is defined by the fixed point value of the RG function

$$\gamma_\psi \equiv \lim_{M \rightarrow 0} m \frac{\partial \ln Z_\psi}{\partial m}, \quad (23)$$

has the value $\eta = -16$, which is unphysical since it does not respect the bound $\eta > -1$. The reason for this is that the fixed-point value f_* is quite large, and this leads to the unphysical value of η .

Does this mean that the present approach must be discarded as well? Not completely. We are still much better off than HLM. As mentioned before, they judged how far their result was from nature, by changing the number of complex fields from 1 to $N/2$ with an $O(N)$ symmetry among the real components. This served to suppress the f^2 term, since the graph of Fig. 1 is of order[36] $1/N^2$. Our calculation has the advantage that the f^2 term is absent for any N . Suppose we go through our analysis for $N > 2$. Then we find

$$\beta_f = -f + \frac{Nf^2}{48\pi}, \quad (24)$$

$$\beta_g = -g - \frac{4fg}{3\pi} + \frac{(N+8)g^2}{16\pi}. \quad (25)$$

The critical exponent has now the value $\eta = -32/N$, which is physically meaningful provided $N > N_c = 32$. Thus our three-dimensional approach yields a critical N value $N_c = 32$, where the result becomes physical. This lies much lower than the critical HLM one-loop value [3] $N_c^{\text{HLM}} = 365$. If our one-loop value $N_c = 32$ is lowered by the two-loop corrections, there is

a good chance of getting a physical exponent η for a single complex order field ψ . Such a calculation remains to be done.

Further insight can be obtained by calculating the critical exponent ν . This is given by

$$\frac{1}{\nu} = 2 + \gamma_m^* - \eta, \quad (26)$$

where γ_m^* is the fixed point value of the RG function

$$\gamma_m \equiv m \frac{\partial \ln Z_m}{\partial m}. \quad (27)$$

At one-loop order it is given by

$$\gamma_m = -\frac{(N+2)g}{16\pi}. \quad (28)$$

Thus,

$$\frac{1}{\nu} = 2 - \frac{(N+2)(N+64)}{N(N+8)} + \frac{16}{N}. \quad (29)$$

Positive values of ν are obtained for $N > 34$. However, for any finite value of N we have $\nu > 1$. When $N \rightarrow \infty$ we have $\nu = 1$.

As a last remark in this section, let us comment about the absence of a tricritical fixed point in our one-loop calculation. The tricritical fixed point is absent because there is no f^2 in the β -functions (21) and (25). With this respect, our approach gives a result similar to the $1/N$ expansion at $\mathcal{O}(1/N)$, where there is only a second-order fixed point and no tricritical fixed point. It is evident, however, that a tricritical fixed point will be generated at two-loops, but in this case resummation is necessary. The same result is to be expected in the $1/N$ expansion at $\mathcal{O}(1/N^2)$.

III. RENORMALIZATION CONSTANTS IN THE ORDERED PHASE

A. Problems with unitary gauge

We have seen in the preceding section that in three dimensions the situation improves by using a massive vector field to avoid infrared divergences in the Feynman integrals. Above T_c , this mass is set equal to zero at the end. In the ordered phase, such a mass is automatically

present as a consequence of the Meissner effect. In fact, there are no massless modes at all in the ordered phase, the Goldstone boson supplying the longitudinal component to the massive vector field. The simplest way of representing this effect is in the unitary gauge. At the mean-field level, this amounts to the field parametrizations (dropping the subscripts 0 indicating bare quantities for simplicity)

$$\psi = \frac{1}{\sqrt{2}}(v + \rho)e^{i\theta}, \quad (30)$$

$$\mathbf{B} = \mathbf{A} + \frac{1}{e}\nabla\theta. \quad (31)$$

The above parametrization allows for a nonzero expectation value of the order field, that is, by setting $\langle\psi\rangle = v$. At the tree level this happens for $m^2 < 0$. To avoid proliferating minus signs, it will be more convenient in this section to replace the term $m^2|\psi|^2$ in the Hamiltonian by $-m^2|\psi|^2$, such that a nonzero expectation value of the order field exists for $m^2 > 0$. In the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ the partition function is calculated from the following functional integral

$$Z = \int \mathcal{D}\mathbf{A}\mathcal{D}\rho\mathcal{D}\theta \det(-\partial^2)\delta(\nabla \cdot \mathbf{A}) e^{-\int d^3r\mathcal{H}}, \quad (32)$$

the last factor in the measure being the Faddeev-Popov determinant. This is a convenient formulation of the field system since by performing the change of variables (30) and (31) the functional integral can immediately be simplified by integrating out the angular field θ . Its value is fixed by the delta function enforcing the Coulomb gauge. The result of this integral precisely cancel the Faddeev-Popov determinant.

In terms of the fields (30), the Hamiltonian becomes

$$\mathcal{H} = V_0(v) + \frac{1}{2}(\nabla \times \mathbf{B})^2 + \frac{m_A^2}{2}\mathbf{B}^2 + \frac{1}{2}(\nabla\rho)^2 + \frac{e^2}{2}\rho^2\mathbf{B}^2 + e^2v\rho\mathbf{B}^2 + m^2\rho^2 + \frac{uv}{2}\rho^3 + \frac{u}{8}\rho^4, \quad (33)$$

where $V_0(v) = -m^2v^2/2 + uv^4/8$, $m_A^2 = e^2v^2$, and $m^2 = uv^2/2$. In this well known tree level analysis the Goldstone has disappeared and the vector potential has becomes massive.

The free propagator of the vector field is given by

$$D_{\mu\nu}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + m_A^2} \left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{m_A^2} \right), \quad (34)$$

which corresponds to take the gauge-fixing parameter $\alpha \rightarrow \infty$. Now, at large $|\mathbf{p}|$ the above propagator behaves like a constant. Thus, the free scaling dimension of \mathbf{B} does not have

the typical canonical scaling dimension $1/2$ of a vector field, but $3/2$. But then power counting tells us that the interaction $e^2 \rho^2 \mathbf{B}^2/2$ is not renormalizable in $d = 3$. Thus, in the unitary gauge, the absorption of the Goldstone mode in the longitudinal part of the massive vector meson seems to destroy renormalizability. The only way to obtain a renormalizable theory out of the unitary gauge is by keeping the gauge-fixing parameter α nonzero during the Feynman integrals calculation and take the limit $\alpha \rightarrow \infty$ at the end [54]. In order to implement such a program, we should not use the unitary gauge parametrization given by Eqs. (30) and (31). This means that the Goldstone boson will still be present in the theory. The pole coming from the Goldstone boson propagator will be, however, canceled from physical quantities.

Beside the renormalizability problem, there is another difficulty with the unitary gauge which is related to the vortex content of the theory. The point is that the covariant derivative term $|(\nabla - ie\mathbf{A})\psi|^2$ in the GL Hamiltonian (1) goes over, in the parametrizations (30) and (31), into $|(\nabla - ie\mathbf{B} - 2\pi\boldsymbol{\theta}_v)\rho|^2$, where $\boldsymbol{\theta}_v$ is the *vortex gauge field*, a vector field describing the vortices in the superfluid [11, 13]. This field arises from the fact that θ is not a single-valued field, but a cyclic field with the property $\int d\theta(\mathbf{r}) = 2\pi n_v(\mathbf{r})$, where n_v is the winding number or vorticity. For this reason, the gradient $\nabla e^{i\theta}$ is not simply equal to $i(\nabla\theta)e^{i\theta}$, but must be supplemented by such a vortex gauge field yielding $i(\nabla\theta - 2\pi\boldsymbol{\theta}_v)e^{i\theta}$, and ensuring the periodicity under $\theta \rightarrow \theta + 2\pi$. The vector gauge field removes possible delta functions arising from the gradient of jumps by 2π in the cyclic field which exist around a vortex line in the superconducting phase [11, 13]. In fact, these vortices are crucial for driving the phase transition and can therefore not be neglected. It is, however, difficult to treat the vortex gauge field by conventional perturbation theory using Feynman diagrammatic techniques. If we want to take vortices into account in perturbation expansions we must therefore avoid multivalued fields.

B. The Landau gauge

A field parametrization which avoids the problems of the unitary gauge and has more desirable properties with respect to power counting is

$$\psi_0 = \frac{1}{\sqrt{2}}(v_0 + \sigma_0 + i\pi_0). \quad (35)$$

Here the bare Hamiltonian becomes

$$\begin{aligned} \mathcal{H} = & \mathcal{H}_{\text{free}} + e_0 \mathbf{A}_0 \cdot (\sigma_0 \nabla \pi_0 - \pi_0 \nabla \sigma_0) + e_0^2 v_0 \sigma_0 \mathbf{A}_0^2 + \frac{e_0^2}{2} (\sigma_0^2 + \pi_0^2) \mathbf{A}_0^2 \\ & + \frac{u_0 v_0}{2} \sigma_0^3 + \frac{u_0 v_0}{2} \sigma_0 \pi_0^2 + \frac{u_0}{8} (\sigma_0^2 + \pi_0^2)^2, \end{aligned} \quad (36)$$

where $\mathcal{H}_{\text{free}}$ denotes the free part of the Hamiltonian:

$$\begin{aligned} \mathcal{H}_{\text{free}} = & \frac{1}{2} (\nabla \times \mathbf{A}_0)^2 + \frac{m_{A,0}^2}{2} \mathbf{A}_0^2 + \frac{1}{2} [(\nabla \sigma_0)^2 + (\nabla \pi_0)^2] + \frac{1}{2} (-\bar{m}_0^2 + 3m_0^2) \sigma_0^2 \\ & + \frac{1}{2} (-\bar{m}_0^2 + m_0^2) \pi_0^2 + J_0 \sigma_0 + \mathcal{H}_{\text{gf}}. \end{aligned} \quad (37)$$

The Coulomb gauge $\nabla \cdot \mathbf{A}_0 = 0$ is fixed by letting $\alpha_0 \rightarrow 0$, which corresponds to the so called Landau gauge. We have therefore omitted a crossed term $e_0 \mathbf{A}_0 \cdot \nabla \pi_0$. The bare masses are $m_0^2 = u_0 v_0^2/2$ and $m_{A,0}^2 = e_0^2 v_0^2$. We have introduced a source term for the longitudinal field σ_0 . In the equation of motion, a zero source corresponds to the minimum of the effective action. At the tree level we have $J_0 = v_0(-\bar{m}_0^2 + m_0^2)$. Thus, the tree level minimum implies $\bar{m}_0^2 = m_0^2$.

From $\mathcal{H}_{\text{free}}$ we obtain the propagators:

$$G_{\sigma\sigma}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + 3m_0^2 - \bar{m}_0^2}, \quad (38)$$

$$G_{\pi\pi}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + m_0^2 - \bar{m}_0^2}, \quad (39)$$

$$D_{\mu\nu}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + m_{A,0}^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{\mathbf{p}^2} \right). \quad (40)$$

Now power counting gives the desired dimensions in the ultraviolet since the vector propagator behaves like $\sim 1/\mathbf{p}^2$ for large $|\mathbf{p}|$. Moreover, the vector field is massive and the graph of Fig. 1 is convergent in $d = 3$. However, we still have a massless mode, the π -field. Interestingly, in the one-loop calculation of the vector potential and π two-point functions, $\Gamma_{\mu\nu}^{(2)}$ and $\Gamma_{\pi\pi}^{(2)}$, respectively, the infrared divergences from the would-be Goldstone boson do not play any role. We shall calculate here the bare two-point functions. The Feynman graphs contributing to these two-point functions are shown in Fig. 2. The only ultraviolet divergences come from the tadpoles and they are all proportional to the ultraviolet cutoff

Λ . These divergences are absorbed into the bare masses m_0^2 and $m_{A,0}^2$. Also, in the loop integrals we shall replace the bare masses by the renormalized ones, since the error involved in this replacement is of higher order.

The renormalized two-point functions are given in terms of the bare ones by $\Gamma_{\sigma\sigma}^{(2)} = Z_\sigma \Gamma_{0,\sigma\sigma}^{(2)}$, $\Gamma_{\pi\pi}^{(2)} = Z_\pi \Gamma_{0,\pi\pi}^{(2)}$, and $\Gamma_{\mu\nu}^{(2)} = Z_A \Gamma_{0,\mu\nu}^{(2)}$. In order to determine the renormalization constants Z_π and Z_A we employ the normalization conditions

$$\left. \frac{\partial \Gamma_{\pi\pi}^{(2)}(\mathbf{p})}{\partial \mathbf{p}^2} \right|_{\mathbf{p}=0} = 1, \quad (41)$$

$$\left. \frac{\partial \Gamma_{\mu\mu}^{(2)}(\mathbf{p})}{\partial \mathbf{p}^2} \right|_{\mathbf{p}=0} = 2. \quad (42)$$

Since π is a would-be Goldstone boson, we require also the normalization conditions for renormalized source $J = Z_\sigma^{1/2} J_0 = 0$:

$$\Gamma_{\sigma\sigma}^{(2)}(0) = uv^2 \equiv 2m^2, \quad (43)$$

$$\Gamma_{\pi\pi}^{(2)}(0) = 0, \quad (44)$$

such that all π -propagators in loop integrals are massless.

The following normalization condition defines the renormalized mass of the vector field

$$\Gamma_{\mu\mu}^{(2)}(0) = 3e^2 v^2 \equiv 3m_A^2. \quad (45)$$

Note that, as usual, the renormalization conditions are chosen in such a way as to be consistent with the tree-level and the Ward identities [37]. Note that the above conditions determine completely the renormalization constants. Since we have five renormalization constants, the above five conditions determine them completely. Actually, the Ward identities imply that $Z_\sigma = Z_\pi$ if the Coulomb gauge is chosen, a result consistent with the gauge invariance of the Hamiltonian. The renormalization conditions imply that the renormalized Ginzburg parameter is given by $\kappa^2 = m^2/m_A^2 = u/2e^2$, that is, it has the same form as at the tree-level but with the bare couplings replaced by the renormalized ones.

After some work we obtain

$$\begin{aligned}
\Gamma_{0;\pi\pi}^{(2)}(\mathbf{p}) &= \mathbf{p}^2 - \bar{m}_0^2 + m_0^2 - \frac{u_0\sqrt{2}m}{8\pi} - \frac{e_0^2 m_A}{2\pi} \\
&- \frac{u_0 m^2}{4\pi|\mathbf{p}|} b(\mathbf{p}, \sqrt{2}m) - e_0^2 \left\{ \frac{1}{8\pi m_A^2 |\mathbf{p}|} [4m_A^2 \mathbf{p}^2 + (\mathbf{p}^2 + 2m^2 - m_A^2)^2] a(\mathbf{p}, \sqrt{2}m, m_A) \right. \\
&\left. + \frac{\mathbf{p}^2}{4\pi m_A} - \frac{1}{8\pi m_A^2 |\mathbf{p}|} (\mathbf{p}^2 + 2m^2)^2 b(\mathbf{p}, \sqrt{2}m) \right\}, \tag{46}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{0;\mu\nu}^{(2)}(\mathbf{p}) &= \delta_{\mu\nu} \left\{ \mathbf{p}^2 + m_{A,0}^2 - \frac{e_0^2 m_A}{8\pi} (\sqrt{2}\kappa - 1) - \frac{e_0^2 m_A^3}{8\pi \mathbf{p}^2} (1 - 2\kappa^2 - \sqrt{2}\kappa) - \frac{e_0^2 \sqrt{2} m^3}{4\pi \mathbf{p}^2} \right. \\
&\left. + \frac{e_0^2}{16\pi} \frac{[\mathbf{p}^2 + m_A^2 (\sqrt{2}\kappa + 1)^2][\mathbf{p}^2 + m_A^2 (\sqrt{2}\kappa - 1)^2] - 8m_A^2 \mathbf{p}^2}{|\mathbf{p}|^3} a(\mathbf{p}, \sqrt{2}m, m_A) \right\} \\
&+ \frac{p_\mu p_\nu}{\mathbf{p}^2} \left\{ -\mathbf{p}^2 + \frac{e_0^2 m_A}{8\pi} (\sqrt{2}\kappa - 3) + \frac{3e_0^2 \sqrt{2} m^3}{4\pi \mathbf{p}^2} + \frac{3e_0^2 m_A^3}{8\pi \mathbf{p}^2} (1 - 2\kappa^2 - \sqrt{2}\kappa) \right. \\
&\left. - \frac{e_0^2 m^4}{2\pi |\mathbf{p}|^3} b(\mathbf{p}, \sqrt{2}m) - \frac{e_0^2}{16\pi |\mathbf{p}|^3} [3(\mathbf{p}^2 + 2m^2)^2 - 12m^2 m_A^2] \right. \\
&\left. + 3m_A^4 - 2m_A^2 \mathbf{p}^2 \right\} a(\mathbf{p}, \sqrt{2}m, m_A) \tag{47}
\end{aligned}$$

with the functions

$$a(\mathbf{p}, m_1, m_2) = \arctan\left(\frac{\mathbf{p}^2 + m_1^2 - m_2^2}{2m_2|\mathbf{p}|}\right) + \arctan\left(\frac{\mathbf{p}^2 + m_2^2 - m_1^2}{2m_1|\mathbf{p}|}\right), \tag{48}$$

$$\begin{aligned}
b(\mathbf{p}, m_1) &= \lim_{m_2 \rightarrow 0} a(\mathbf{p}, m_1, m_2) \\
&= \frac{\pi}{2} + \arctan\left(\frac{\mathbf{p}^2 - m_1^2}{2m_1|\mathbf{p}|}\right), \tag{49}
\end{aligned}$$

where $\kappa \equiv m/m_A = \lambda/\xi$ is the renormalized Ginzburg parameter. The bare Ginzburg parameter is given by $\kappa_0 = \sqrt{u_0/2e_0^2}$, since $m_0^2 = u_0 v_0^2/2$ and $m_{0,A}^2 = e_0^2 v_0^2$.

On account of the normalization condition (44), we obtain the one-loop correction to \bar{m}_0^2 :

$$\bar{m}^2 = m_0^2 - \frac{3u_0\sqrt{2}m}{8\pi} - \frac{e_0^2 m_A}{2\pi}. \tag{50}$$

In the following, we shall use extensively the replacement $\kappa_0^2 \rightarrow \kappa^2 = m^2/m_A^2 = u^2/2e^2 = g/2f$, where $f = e^2/m$ and $g = u/m$ are dimensionless renormalized couplings, neglecting all errors of higher than one-loop order. The parameter κ arises in two different contexts: once from the renormalized mass ratio m/m_A , and once from the ratio of couplings as $\sqrt{u/2e^2}$.

We shall then parametrize all RG functions in terms of κ and f instead of g and f . Our expansion will be controlled initially by powers of f , but once g is eliminated in favor of κ , then κ will no longer be assumed to be small, such that the RG functions are not polynomials in f and κ as in the previous expansions in powers of f and g in Section II.

Below we shall see that in this formulation there exists a natural expansion parameter related to κ , namely $\Delta\kappa \equiv \kappa - 1/\sqrt{2}$. Remarkably, the present description of the fixed-point structure requires only the knowledge of the two-point functions, since from Eq. (50) $m^2 = Z_\pi \bar{m}^2$ and $e^2 = Z_A e_0^2$, with Z_A being determined from the renormalization condition (42) (see below). This represents an immense advantage since the four-point functions contain severe infrared singularities coming from the would-be Goldstone boson. Of course, we expect these singularities to cancel at the end, but this is a difficult issue in fixed dimensions. In the $4 - \epsilon$ -dimensional regularization scheme this issue is avoided since the counterterms required for a renormalization in the ordered phase are identical to those in the disordered phase where they can be calculated without Goldstone-bosons [1].

From the normalization condition (41) we derive

$$Z_\pi = 1 - \frac{f\kappa(2\kappa^2 + \sqrt{2}\kappa - 8)}{12\pi(\sqrt{2}\kappa + 1)}, \quad (51)$$

which satisfies $Z_\pi > 1$ for $\kappa < \hat{\kappa} \equiv (\sqrt{33} - 1)/2\sqrt{2} \approx 1.6774$. Only for $\kappa > \hat{\kappa}$ will the wave function renormalization satisfy the usual inequality $0 \leq Z_\pi < 1$ found in the absence of gauge fields.

Let us now calculate m_A^2 at one-loop level using the normalization condition (45). We find

$$\begin{aligned} \frac{1}{3}\Gamma_{0,\mu\mu}^{(2)} &= m_{A,0}^2 + \frac{e_0^2 m_A}{12\pi(\sqrt{2}\kappa + 1)}(2\kappa^2 + \sqrt{2}\kappa - 8) \\ &\simeq Z_\pi^{-1} m_{A,0}^2. \end{aligned} \quad (52)$$

Since $\Gamma_{\mu\nu}^{(2)} = Z_A \Gamma_{0,\mu\nu}^{(2)}$, we obtain

$$m_A^2 = Z_A Z_\pi^{-1} m_{A,0}^2. \quad (53)$$

Thus Z_π appears in the renormalization of the mass m_A of the vector field. This result is expected from the Ward identities. It provides us with a good check of the consistency of our calculations. Physically it shows clearly how the fluctuations of the Goldstone boson renormalize the mass of the vector field, which has been exploited in the theory of electroweak interactions to build a renormalizable theory of massive vector mesons and Higgs fields. Thus, although the Goldstone boson is present in the calculations, the Higgs mechanism takes care of absorbing its fluctuating degrees of freedom to build a fluctuating longitudinal degree of freedom for the vector potential.

The superfluid density is intimately related to the penetration depth $\lambda = 1/m_A$. It is given by

$$\rho_s = Z_\pi^{-1} v_0^2, \quad (54)$$

which reflects a relation due to Josephson[38]. On account of (53) and the relation $e^2 = Z_A e_0^2$ we can write $m_A^2 = e^2 \rho_s$.

It remains to calculate Z_A . This is done using the normalization condition (42), yielding

$$Z_A = 1 - \frac{\sqrt{2}C(\kappa)f}{24\pi(2\kappa^2 - 1)^3}, \quad (55)$$

where

$$C(\kappa) = 4\kappa^6 + 10\kappa^4 - 24\sqrt{2}\kappa^3 + 27\kappa^2 + 4\sqrt{2}\kappa - 1/2. \quad (56)$$

We note that the second term in Eq. (55) is singular at $\kappa = 1/\sqrt{2}$. This singularity is analogous to the $1/\epsilon$ singularity in dimensionally regularized theories. The important role of this singularity will be discussed in more detail in the next Section.

IV. RENORMALIZATION GROUP ANALYSIS IN THE ORDERED PHASE. ROLE OF THE GINZBURG PARAMETER κ

A. Renormalization group functions

We are now prepared to calculate the RG functions. Let us define the RG functions:

$$\gamma_\pi \equiv m \frac{\partial \ln Z_\pi}{\partial m}, \quad (57)$$

$$\gamma_A \equiv m \frac{\partial \ln Z_A}{\partial m}. \quad (58)$$

The RG functions γ_π and γ_A are given explicitly at one-loop order as

$$\gamma_\pi = \frac{\kappa f}{12\pi} \frac{2\kappa^2 + \sqrt{2}\kappa - 8}{\sqrt{2}\kappa + 1}, \quad (59)$$

$$\gamma_A = \frac{\sqrt{2}C(\kappa)f}{24\pi(2\kappa^2 - 1)^3}. \quad (60)$$

Note that while deriving Eqs. (59) and (60) we have not differentiated κ , since this would lead to higher powers of f beyond the one-loop approximation. We observe also that the singularity at $\kappa = 1/\sqrt{2}$ present at Z_A is also present in γ_A .

The β -function of f is then given by

$$\beta_f = (\gamma_A - 1)f. \quad (61)$$

To obtain the β -function of κ^2 we need to know the evolution equation for m_A^2 . From Eq. (53) we obtain

$$m \frac{\partial m_A^2}{\partial m} = (\gamma_A - \gamma_\pi)m_A^2 + Z_A Z_\pi^{-1} m \frac{\partial m_{A,0}^2}{\partial m}, \quad (62)$$

thus reducing the problem to a calculation of $m \partial m_{A,0}^2 / \partial m$. It must be emphasized that in our approach this is not zero. Our differentiations are performed at fixed $\kappa_0^2 = m_0^2 / m_{A,0}^2$, such that $m \partial \kappa_0^2 / \partial m = 0$ whereas both m_0 and $m_{A,0}$ remain functions of m . Due to the fixed κ_0 , their derivatives $m \partial m_{A,0}^2 / \partial m = \zeta m_{A,0}^2$ and $m \partial m_0^2 / \partial m = \zeta m_0^2$, are governed by a common RG function ζ . In order to obtain $m \partial m_0^2 / \partial m$ we use Eq. (50) to write

$$m_0^2 = Z_\pi^{-1} m^2 + \frac{3u_0 \sqrt{2} m}{8\pi} + \frac{e_0^2 m_A}{2\pi}, \quad (63)$$

where we have inserted $m^2 = Z_\pi \bar{m}^2$. The derivative yields

$$m \frac{\partial m_0^2}{\partial m} = (2 + \zeta_\pi - \gamma_\pi) m_0^2, \quad (64)$$

where up to one loop

$$\zeta_\pi = -\frac{\sqrt{2}}{4\pi} f \left(\frac{3\kappa^2}{2} + \frac{1}{\sqrt{2}\kappa} \right). \quad (65)$$

From Eqs. (62) and (64) we derive

$$m \frac{\partial m_A^2}{\partial m} = (2 + \zeta_\pi + \gamma_A - 2\gamma_\pi) m_A^2. \quad (66)$$

It is now straightforward to obtain the β -function for κ^2 :

$$\beta_{\kappa^2} \equiv m \frac{\partial \kappa^2}{\partial m} = m \frac{\partial m^2/m_A^2}{\partial m} = \kappa^2 \left(2 - \frac{m}{m_A^2} \frac{\partial m_A^2}{\partial m} \right) = \kappa^2 (2\gamma_\pi - \gamma_A - \zeta_\pi). \quad (67)$$

From Eqs. (61) and (67) we see that charged fixed points satisfy the equations

$$\eta_A \equiv \gamma_A(f_*, \kappa_*) = 1 \quad (68)$$

and

$$2\eta - 1 = \zeta_\pi(f_*, \kappa_*), \quad \text{where} \quad \eta \equiv \gamma_\pi(f_*, \kappa_*). \quad (69)$$

The exponents η_A and η determine the anomalous scaling dimensions of the fields \mathbf{A} and π , respectively. Our equations yield fixed points at

$$f_* \approx 0.3, \quad \kappa_* \approx 1.17/\sqrt{2}. \quad (70)$$

At this fixed point the value of the η -exponent is $\eta \approx -0.02$, which fulfills the inequality $\eta > -1$, showing that our fixed point in the ordered phase is completely physical, in contrast to the $N = 2$ case in the disordered phase calculation of Section II.

Our η -exponent is less negative than most values found in the literature. There is, however, no consensus about this value. For instance, a recent Monte Carlo simulation [34] give $\eta \approx -0.24$, disagreeing with another Monte Carlo value $\eta \approx -0.79$ published recently [35], which is similar to $\eta = -0.74$ found by Herbut and Tesanovic by adjusting their parameter c to fit the Kleinert's value of the tricritical Ginzburg parameter, $\kappa_t = 0.77/\sqrt{2}$. The value $\eta = -0.2$ is obtained when the c -parameter is fitted to the now less precise (see Ref. 14) Monte Carlo value [24] $\kappa_t = 0.42/\sqrt{2}$. There are other values of η reported in the literature [21, 43], exhibiting a lack of consensus on its numerical value.

B. Validity of the method

We now want to explain what makes our three-dimensional procedure so successful in finding the charged fixed points. The main obstacle in our calculation in the disordered phase in Section II was the too large value of f_* . In previous approaches in $4 - \epsilon$ dimensions, this prevented a fixed point to exist. The size of f_* was diminished by a factor $2/N$ via the artificial extension to $N/2$ complex fields, and this led to a fixed point after all for $N > N_c = 365$. The three-dimensional calculation in the disordered phase is far less unphysical since it always give a fixed point. Still, there was the problem that the associated critical exponent η was too negative, violating the bound $\eta > -1$, again because of a too large f_* , unless $N > N_c = 32$.

These difficulties are absent in three dimensions below T_c . To understand this let us define an *effective square charge* $\bar{f}(\kappa)$ depending on κ by the equation

$$\gamma_A(\bar{f}, \kappa) = 1, \quad (71)$$

meaning that we go to the “fixed line” in the two-dimensional space of coupling constants defined by the vanishing of the β -function β_f in Eq. (61), but not at the fixed point for Eq. (67). The κ -dependence of $\bar{f}(\kappa)$ is

$$\bar{f}(\kappa) = \frac{24\pi(2\kappa^2 - 1)^3}{\sqrt{2}C(\kappa)}. \quad (72)$$

We see that $\bar{f}(\kappa) = 0$ for $\kappa = 1/\sqrt{2}$, which is just the value separating the type I from the type II regime at the mean-field level. The κ -dependence of $\bar{f}(\kappa)$ is plotted in Fig. 3. The effective square charge is negative if $0.096/\sqrt{2} < \kappa < 1/\sqrt{2}$ and very large positive for $0 \leq \kappa < 0.096/\sqrt{2}$. This makes it *impossible* to have an infrared-stable positive charge in the type-I regime $\kappa < 1/\sqrt{2}$, and we conclude that the phase transition is of first-order in the type-I regime. Moreover, we observe that for κ slightly above $1/\sqrt{2}$, the effective critical charge is small.

An important remark is in order here. In Fig. 3 there is an asymptote which separates two distinct regions: one where the charged fixed point is inaccessible (the left side of the asymptote), and another one where the charge fixed point is accessible. This asymptote reveals a Landau-ghost-like behavior. The so called “runaway” of the RG flow[3] corresponds to the left side of the asymptote. It is worth mentioning that a similar feature can be

expected in the β -function of QCD[39]. There the asymptote is supposed to separate the asymptotically free regime from the regime containing an infrared stable fixed point which governs quark confinement.

Consider now the large- κ limit which corresponds to an extreme type II regime. There are two ways of taking $\kappa \rightarrow \infty$: either by letting $g \rightarrow \infty$ at f fixed, or by letting $f \rightarrow 0$ at g fixed. In the first case we obtain from Eq. (58) $\gamma_A|_{\kappa=\infty, f} = f/(24\pi\sqrt{2})$, while in the second: $\gamma_A|_{\kappa=\infty, g} = 0$. In the first case the magnetic fluctuations are still relevant and the $|\psi|^4$ interaction may be replaced by a pure phase field with fixed absolute value of the order parameter, i.e., the GL model be approximated by its London limit. In the second case the magnetic fluctuations disappear completely from the system and the universality class is that of a pure $|\psi|^4$ -theory which is equal to that of the XY -model.

By substituting $\bar{f}(\kappa)$ of Eq. (72) into (59), we obtain the effective exponent $\bar{\eta}(\kappa)$. For $\kappa = 1/\sqrt{2}$, this has the mean-field value $\eta = \bar{\eta}(1/\sqrt{2}) = 0$. In the physically interesting interval $1/\sqrt{2} < \kappa < \hat{\kappa} \equiv (\sqrt{33} - 1)/2\sqrt{2} \approx 1.67746$, the effective exponent $\bar{\eta}(\kappa)$ is *negative*, as shown in Fig. 4. Above $\hat{\kappa}$, it is positive. Remarkably, it lies above the physical lower bound -1 for all $\kappa > 1/\sqrt{2}$, i.e., in the entire type-II regime in the mean-field approximation.

Thus, our RG approach in the ordered phase gives effectively an expansion around $\kappa = 1/\sqrt{2}$. It is therefore convenient to introduce the expansion parameter $\Delta\kappa \equiv \kappa - 1/\sqrt{2}$. Then we can write the leading expansion term in $\Delta\kappa$ for the effective square charge $\bar{f}(\kappa)$ and the effective exponent $\bar{\eta}(\kappa)$ as

$$\bar{f}(\kappa) = 48\pi\Delta\kappa^3, \quad (73)$$

$$\bar{\eta}(\kappa) = -6\sqrt{2}\Delta\kappa^3. \quad (74)$$

Similarly, we expand

$$\beta_{\kappa^2}(\kappa^2, \bar{f}(\kappa)) = -\frac{1}{2} - \sqrt{2}\Delta\kappa - \Delta\kappa^2 + 162\sqrt{2}\Delta\kappa^3. \quad (75)$$

We now discuss the other important critical exponent ν . Recall that it is defined in general by the scaling relation

$$\nu = \frac{1}{2 - \eta_m}, \quad (76)$$

where η_m is the fixed-point value of the logarithmic derivative

$$\gamma_m \equiv \frac{m}{m_0^2} \frac{\partial m_0^2}{\partial m} = \gamma_\pi - \zeta_\pi, \quad (77)$$

the right-hand side following from (64). By analogy with the other effective quantities we define the effective exponent

$$\bar{\nu}(\kappa) = \frac{1}{2 + \bar{\zeta}_\pi(\kappa) - \bar{\eta}(\kappa)}, \quad (78)$$

where $\bar{\zeta}_\pi(\kappa) = \zeta_\pi(\bar{f}(\kappa))$. The effective exponent $\bar{\nu}(\kappa)$ gives the critical exponent ν at the fixed point $\kappa = \kappa_*$. To leading order in $\Delta\kappa$ we obtain

$$\bar{\nu}(\kappa) = \frac{1}{2} + \frac{165}{\sqrt{2}} \Delta\kappa^3. \quad (79)$$

By substituting the previously found fixed-point value $\kappa_* = 1.17/\sqrt{2}$ of Eq. (70) into (78), we obtain $\nu \approx 1.02$. To estimate the systematic error, we calculate the fixed-point value κ^* from the expanded β -function (75), which yields $\kappa_* = 1.22/\sqrt{2}$. Inserting this into Eq. (79) gives $\nu \approx 0.92$. The expanded Eq. (74) yields $\eta \approx -0.03$.

The critical values of ν are quite far from the expected XY-model value $\nu_{XY} \approx 0.67$, but at the one-loop level we should not expect a higher accuracy.

Systematic improvements in our approach is considerably more difficult than in other more conventional approaches. The fundamental difference between our method and other methods lies in the fact that we perturbatively expand in powers of f only. The essence of the method lies in the computation of γ_A and further determination of \bar{f} using Eq. (71). Generally we have that γ_A is given by the series:

$$\gamma_A(f, \kappa) = \sum_{l=1}^{\infty} c_l(\kappa) f^l. \quad (80)$$

The powers of f correspond to the number of loops and the coefficients $c_l(\kappa)$ are not polynomials in κ and should diverge at some value $\kappa = \kappa_t$ separating type I from type II regimes. At one-loop order $\kappa_t = 1/\sqrt{2}$, receiving no corrections with respect to the mean-field value.

C. Scaling behavior as T_c is approached from below

Let us discuss the scaling behavior as T_c is approached from below from a perspective independent of perturbation theory. It is convenient to write the formulas for any dimension

$d \in (2, 4)$. Thus, the β -function for the gauge coupling is given by

$$\beta_f = (\gamma_A + d - 4)f. \quad (81)$$

The β -function β_{κ^2} can be written as

$$\beta_{\kappa^2} = \left(4 - d - \gamma_A + \frac{\beta_g}{g}\right) \kappa^2. \quad (82)$$

Since $m_A^2 = m^2/\kappa^2$, we obtain

$$m \frac{\partial m_A^2}{\partial m} = \left(d - 2 + \gamma_A - \frac{\beta_g}{g}\right) m_A^2. \quad (83)$$

The existence of a charged fixed point implies $\eta_A \equiv \gamma_A^* = 4 - d$ and $\beta_g^* = 0$. Thus, in the neighborhood of this charged fixed point Eq. (83) becomes $m \partial m_A^2 / \partial m \approx 2m_A^2$, implying that $m_A^2 \sim m^2$. Therefore, λ and ξ diverge with the same critical exponent [15, 23]. This is of course obvious from the definition of the Ginzburg parameter: $\kappa^2 = g/2f = \lambda^2/\xi^2$. If g_* and f_* are different from zero the ratio λ/ξ is a constant at the critical point. This can only happen if they have the same exponent. At the XY fixed point, however, $\eta_A = 0$ because $f_* = 0$ and $m \partial m_A^2 / \partial m \approx (d - 2)m_A^2$ near the fixed point. In this case $m_A^2 \sim m^{d-2}$ and the penetration depth exponent is given by $\nu' = \nu(d - 2)/2$. This is the so called XY superconducting universality class [27]. The universality class governed by the charged fixed point is fundamentally different from the XY universality class. Note that the above argument works only if we approach T_c from below, since $m_A^2 \neq 0$ only for $T < T_c$.

From Eqs. (53) and (54) we can write $m_A^2 = e^2 \rho_s$. Due to Eq. (81), $e^2 \sim m^{4-d}$ near the critical point governed by the charged fixed point. Since there $m_A^2 \sim m^2$, we obtain the Josephson relation [38] $\rho_s \sim m^{d-2}$.

V. MOMENTUM SPACE INSTABILITIES. ORIGIN OF THE NEGATIVE SIGN OF THE η -EXPONENT

A much debated property of the superconducting phase transition is the sign of the η -exponent [26, 31–34]. Recently, it was pointed out by one of us[26] (F.S.N.) that the origin of the negative sign of η lies on momentum space instabilities of the order-parameter correlation function arising from magnetic field fluctuations. Such momentum space instabilities are

very similar to those occurring in scalar models of Lifshitz points[40]. In these models, the η -exponent (which in this context is usually denoted by η_{14}) is also found to be negative. The only difference with respect to the GL model is that in the scalar models for Lifshitz points the momentum space instabilities are included from the beginning explicitly into the Hamiltonian.

Inspired by works on lattice spin models of Lifshitz points[41], we expand $\Gamma_{0,\pi\pi}^{(2)}$ up to order $(\mathbf{p}^2)^3$:

$$\Gamma_{0,\pi\pi}^{(2)}(\mathbf{p}) \simeq \mathbf{p}^2 \left[Z_\pi^{-1} + \frac{Af}{\pi} \frac{\mathbf{p}^2}{m_A^2} + \frac{Bf}{\pi} \left(\frac{\mathbf{p}^2}{m_A^2} \right)^2 \right], \quad (84)$$

where

$$A = \frac{\sqrt{2}}{4} (1 - \sqrt{2}\kappa)(\sqrt{2}\kappa + 1)^{-3} \left(\frac{1}{6} + \frac{2}{5}\sqrt{2}\kappa + \frac{\kappa^2}{5} \right), \quad (85)$$

$$B = \frac{1}{56\sqrt{2}} - \frac{1}{105\sqrt{2}} \frac{1 + 5\sqrt{2}\kappa + 20\kappa^2 + 18\sqrt{2}\kappa^3}{(\sqrt{2}\kappa + 1)^5}. \quad (86)$$

The above expansion is like a Landau expansion where \mathbf{p} play the role of an “order parameter”. Remarkably, the coefficient A vanishes for $\kappa = 1/\sqrt{2}$. In the type I regime A has a positive sign while in the type II regime A is negative. The coefficient B is always positive, ensuring the stability. Close to $\kappa = 1/\sqrt{2}$ we find

$$A \simeq -\frac{1}{24} \left(\kappa - \frac{1}{\sqrt{2}} \right) = -\frac{\Delta\kappa}{24}, \quad (87)$$

$$B \simeq \frac{1}{96\sqrt{2}}. \quad (88)$$

For $\kappa \gtrsim 1/\sqrt{2}$, $\Gamma_{0,\pi\pi}^{(2)}(\mathbf{p})$ has a zero at

$$|\mathbf{p}_0| \simeq 2^{3/4} m_A \Delta \kappa^{\bar{\beta}}, \quad (89)$$

with the exponent $\bar{\beta} = 1/2$. Note that $\Gamma_{0,\pi\pi}^{(2)}(\mathbf{p})$ is just the inverse of the transverse susceptibility $\chi_T(\mathbf{p})$. Thus, the transverse susceptibility is maximized at $|\mathbf{p}_0|$. This is the behavior assumed so far in all generic models exhibiting a Lifshitz point[40]. It implies the existence of a modulated regime for the order parameter. On the basis of these considerations we conclude that the transition from type I to type II behavior happens at a Lifshitz point.

This conclusion is, in fact, in accordance with another physical property of the transition from type I to type II superconductors. In the second, fluctuating vortex lines play an important role. This is precisely the reason why the fluctuation term $\sim |\psi|^{4-\epsilon}$ calculated under the assumption of a constant $|\psi|$ is unreliable. At the core of each vortex line, $|\psi|$ has to vanish, such that in the type II regime the order field is perforated by lines of zeros [53].

In our one-loop calculation the κ -value where a type I superconductor crosses over to type II has the mean-field values $\kappa_L = 1/\sqrt{2}$. This value is expected to decrease in higher-loop calculations towards the tricritical value $\kappa_t \approx 0.77/\sqrt{2}$ where we expect the cross over to take place. Indeed, in the dual theory [10, 11], the tricritical point comes about by the sign change of the quartic term in the disorder field which corresponds to a change of the average interaction between vortex lines changing from repulsion to attraction. This sign is also what distinguishes the type I from the type II regime experimentally.

It remains to be checked by going to higher loops, that the fluctuation corrected value of κ_L does indeed coincide with the Ginzburg parameter at the tricritical point, that is, $\kappa_L \simeq 0.77/\sqrt{2}$ [10, 14].

The exact value of the exponent $\bar{\beta}$ controlling the vanishing of $|\mathbf{p}_0|$ at the type-I-type II boundary in Eq. (89) is expected to increase slightly above $1/2$.

An interesting open problem is to prove the conjecture of Ref. 26 that the Lifshitz point coincides with the tricritical point obtained from the disorder field theory [10].

VI. CONCLUSION

In this paper we have initiated a new RG approach to the GL model. Since we work directly in $d = 3$ dimensions and in the ordered phase, we were able to express the RG functions as functions of the Ginzburg parameter $\kappa = m/m_A$. This replaces the coupling constant g used above T_c , since we can also write $\kappa^2 = g/2f$. The RG functions have no longer a polynomial form in κ , being polynomial only in the gauge coupling constant f . The effective gauge coupling $\bar{f}(\kappa)$ determined by the condition $\beta_f = 0$ shows that the charge is small for κ sufficiently close to $\kappa = 1/\sqrt{2}$ from above. Interestingly, the fixed point κ_* lies close to $\kappa = 1/\sqrt{2}$.

We have compared our values for the critical exponents with those found in the literature. Our one-loop value of ν is larger than the XY value obtained from Monte Carlo simulations

[15] and from RG analysis of the disorder field theory [12]. Using the expansion in powers of $\Delta\kappa$ of Section IV we have obtained $\eta \approx -0.03$ and $\nu \approx 0.92$. With these values of η and ν we obtain a critical exponent $\gamma = \nu(2 - \eta) \approx 1.87$. This agrees with the Padé approximant analysis of the two-loop ϵ -expansion obtained by Folk and Holovatch [19]. On the basis of the Padé approximants, these authors obtained the value $\nu \approx 0.857$, showing a $\sim 7\%$ numerical difference from our result $\nu \approx 0.92$. The exponent γ obtained from recent Monte Carlo simulations is $\gamma \approx 1.45$ [33].

One question that immediately arises is how the critical behavior obtained with our method is related to approaches where T_c is approached from above. In the pure $O(N)$ symmetric ϕ^4 theory the critical singularities above and below T_c are known to be the same. This result is obtained by analysing the Ward identities of the theory. For a dimensionally regularized theory, the same counterterms used in the disordered phase also renormalize the theory in the ordered phase [1]. Since the critical exponents in the ϵ -expansion are a direct consequence of the $1/\epsilon$ singularities included in the counterterms, we conclude that the critical exponents obtained by approaching T_c from below are the same as when T_c is approached from above. This should be true even if we work in fixed dimension. If we use the ϵ -expansion in the GL model, the same argument holds. The problem in this case is that the superconducting fixed point appears only for sufficiently large N . We have seen that the value of N can be dramatically reduced if we consider a fixed dimension approach for $T \geq T_c$. Unfortunately this is not enough to obtain a charged fixed point for $N = 2$. Then we have shown that by approaching T_c from below we reach a charged fixed point for the physically interesting $N = 2$ case. It is not obvious which renormalization scheme above T_c corresponds to our scheme below T_c . Clearly, by working above T_c we would never find the same renormalization constants. The point is that κ , which appears explicitly below T_c in a non-polynomial form, can be only defined above T_c through the ratio between the coupling constants. The vector potential is massive only below T_c .

We have discussed in detail the agreement of our theoretical η -exponent with recent Monte Carlo simulations. It is now definite that it is negative. The only possible way to enforce a positive $\eta > 0$ in the GL model is by adding an appropriate mass term, like the Proca term considered in Ref. 31. This should not be confused with our calculation of Section II where $M \rightarrow 0$ at the end. A negative η is caused by the attraction of the vortex lines on the average which causes a fluctuating vortex globule to have a smaller size than a

free random chain. This point of view is related to the geometric interpretation considered in Ref. 51. The addition of a Chern-Simons term which changes the size of vortex globules by a topological interaction seems incapable of producing a positive η , since entanglement of vortex lines also tends to contract the globules. Indeed, one-loop calculation gives a negative η for all values of the topological coupling [44, 45].

There is, however, the possibility that for larger topological coupling η could become positive, since for infinite topological coupling $\eta \rightarrow 0$. This is suggested when calculating η in a $1/N$ expansion [52].

We hope that the new approach introduced in this paper will stimulate further discussions of the nature of the superconducting phase transition. In particular, we call for two- and higher-loop calculations to substantiate our claim that $\kappa_L = \kappa_t$, and that the type-I-type-II crossover point is a Lifshitz point.

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APPENDIX A: USEFUL INTEGRALS

In this appendix we shall write some basic momentum space integrals in three dimensions that are used to obtain the results in the text.

The simplest integral, arising when computing tadpoles, is

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2 + m^2} = \frac{\Lambda}{2\pi^2} - \frac{m}{4\pi} + \mathcal{O}(m/\Lambda), \quad (\text{A1})$$

where the ultraviolet cutoff is assumed to be very large, $\Lambda \gg m$.

The following type of integral appears in the calculation of the two-point functions:

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{[(\mathbf{p} - \mathbf{k})^2 + m_1^2](\mathbf{k}^2 + m_2^2)} = \frac{1}{8\pi|\mathbf{p}|} a(\mathbf{p}, m_1, m_2), \quad (\text{A2})$$

where $a(\mathbf{p}, m_1, m_2)$ is defined in Eq. (48). The following particular cases of the above integral are relevant:

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{[(\mathbf{p}-\mathbf{k})^2 + m^2](\mathbf{k}^2 + m^2)} = \frac{1}{8\pi|\mathbf{p}|} b(\mathbf{p}, m), \quad (\text{A3})$$

where $b(\mathbf{p}, m)$ is defined in Eq. (49), and

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{(\mathbf{k}^2 + m_1^2)(\mathbf{k}^2 + m_2^2)} = \frac{1}{4\pi(m_1 + m_2)}. \quad (\text{A4})$$

When computing loop integrals involving the vector potential propagator we need the integrals:

$$\int \frac{d^3k}{(2\pi)^3} \frac{k_\mu}{[(\mathbf{p}-\mathbf{k})^2 + m_1^2](\mathbf{k}^2 + m_2^2)} = \frac{p_\mu}{16\pi|\mathbf{p}|^3} [2|\mathbf{p}|(m_2 - m_1) + (\mathbf{p}^2 + m_1^2 - m_2^2)a(\mathbf{p}, m_1, m_2)], \quad (\text{A5})$$

$$\begin{aligned} & \int \frac{d^3k}{(2\pi)^3} \frac{k_\mu k_\nu}{[(\mathbf{p}-\mathbf{k})^2 + m_1^2](\mathbf{k}^2 + m_2^2)} = -\frac{\delta_{\mu\nu}}{32\pi} \left\{ m_1 + m_2 - \frac{(m_1 + m_2)(m_1 - m_2)^2}{\mathbf{p}^2} \right. \\ & + \left. \frac{[\mathbf{p}^2 + (m_1 + m_2)^2][\mathbf{p}^2 + (m_1 - m_2)^2]}{2|\mathbf{p}|^3} a(\mathbf{p}, m_1, m_2) \right\} + \frac{p_\mu p_\nu}{32\pi\mathbf{p}^2} \{3m_2 - 5m_1 \\ & - \frac{3(m_1 + m_2)(m_1 - m_2)^2}{\mathbf{p}^2} + \frac{3(\mathbf{p}^2 + m_1^2)^2 - 6m_1^2 m_2^2 + 3m_2^2 - 2m_2^2 \mathbf{p}^2}{2|\mathbf{p}|^3} a(\mathbf{p}, m_1, m_2)\} \end{aligned} \quad (\text{A6})$$

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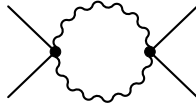


FIG. 1: Feynman graph contributing to the four-point function. The external lines correspond to the scalar fields, while the wiggles are the vector potential propagators.

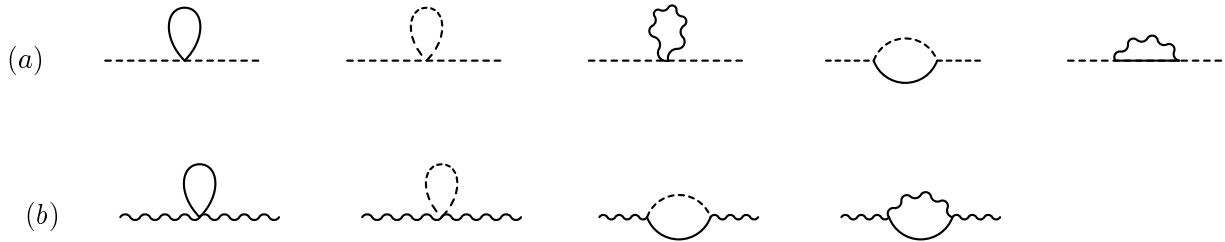


FIG. 2: Graphs contributing to the evaluation of the π (a) and vector potential (b) two-point functions. The solid lines indicate the σ -propagators, the dashed ones π -propagators. The wiggles represent the vector potential.

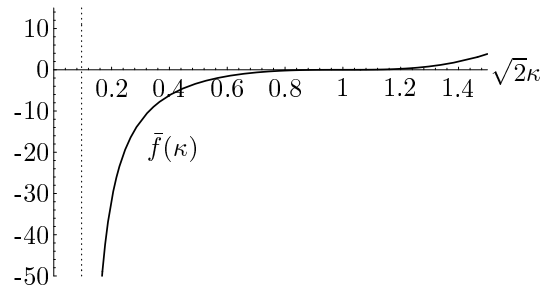


FIG. 3: Behavior of effective square charge $\bar{f}(\kappa)$. To the left of the vertical dashed line marking the the interval $0 \leq \sqrt{2}\kappa < 0.096$, $\bar{f}(\kappa)$ is very large and positive, starting out from $\bar{f}(0) \approx 53.3$ and going to infinity for $\kappa \rightarrow 0.096/\sqrt{2}$.

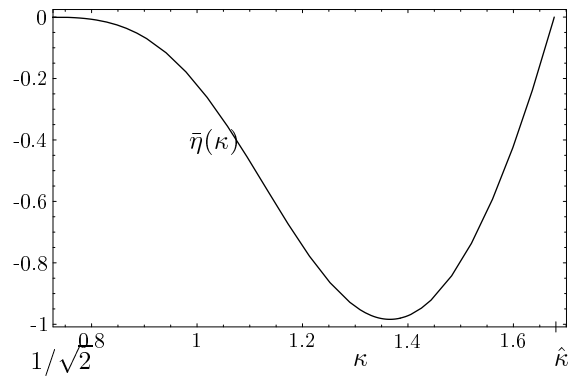


FIG. 4: Plot of $\bar{\eta}$ as a function of κ in the interval $1/\sqrt{2} < \kappa < \hat{\kappa} \equiv (\sqrt{33} - 1)/2\sqrt{2} \approx 1.67746$.