Schrödinger Wave Functions from Classical Trajectories

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We introduce a two-dimensional classical stochastic differential equation for a field $\mathbf{u}(\mathbf{x};t)$ constructed from all possible deterministic trajectories of a point particle in two dimensions, and show that its components define real and imaginary parts of a complex field satisfying the Schrödinger equation in any desired potential. In this way we can derive arbitrary quantum-mechanical spectra from classical dynamics.

1. In a recent analysis of quantum mechanics from the point of view of information processing, it was s pointed out [1] that decoherence will become an insurmountable obstacle for the practical construction of quantum computers. It was suggested that instead of relying on quantum behavior of microparticles it seems more promising to simulate quantum behavior with the help of fast classical systems. As a step towards such a goal we construct a simple classical model which allows us to simulate the quantum behavior of a harmonic oscillator. In particular we show that the discrete energy spectrum with a definite ground state energy can be obtained in a classical model. In the latter respect we go beyond an earlier model in Ref. [2] whose spectrum had the defect of being unbounded from below. In a second step, our construction is generalized to an arbitrary potential.

2. Let $\mathbf{u}(\mathbf{x}) = \left(u^1(\mathbf{x}), u^2(\mathbf{x})\right)$ be a time-independent field in two dimensions to be called *mother field*. The reparametrization freedom of the spatial coordinates is fixed by choosing harmonic coordinates in which

$$\boldsymbol{\nabla}^2 u(\mathbf{x}) = 0, \tag{1}$$

where $\boldsymbol{\nabla}^2$ is the Laplace operator. Equivalently, we may require the components $u^1(\mathbf{x})$ and $u^2(\mathbf{x})$ to satisfy the Cauchy-Riemann equations

$$\partial_\mu u^\nu = \epsilon_\mu{}^\rho \epsilon^\nu{}_\sigma \partial_\rho u^\sigma, \quad (\mu, \nu, \ldots = 1, 2), \tag{2}$$

where $\epsilon_{\mu\nu}$ is the antisymmetric Levi-Civita pseudotensor. The metric is $\delta_{\mu\nu}$, so that indices can be sub- or superscripts.

Consider now a point particle in contact with a heat bath of "temperature" $\hbar$. Its classical orbit $\mathbf{x}(t)$ is assumed to follow a stochastic differential equation consisting of a fixed rotation and a random translation in the diagonal direction $\mathbf{n} \equiv (1,1)/\sqrt{2}$:

$$\dot{\mathbf{x}}(t) = \boldsymbol{\omega} \times \mathbf{x}(t) + \mathbf{n}\, \eta(t), \tag{3}$$

where $\boldsymbol{\omega}$ is the rotation vector of length ω pointing orthogonal to the plane, and $\eta(t)$ a white-noise variable with zero expectation and the correlation function

$$\langle \eta(t)\eta(t') \rangle = \hbar\, \delta(t - t'). \tag{4}$$

For a particle whose trajectory ends at a point $\mathbf{x}(t) = \mathbf{x}$, the position $\mathbf{x}(t')$ at an earlier time t' is a function of $\mathbf{x}$ and a *functional* of the the noise variable $\eta(t'')$ for $t < t'' < t'$:

$$\mathbf{x}(t') = \mathbf{X}_{t'}[t, \mathbf{x}; \eta]. \tag{5}$$

To simplify the notation we indicate the t'-dependence of functionals of η by a subscript t'.

We now use the orbits ending at all possible final points $\mathbf{x} = \mathbf{x}(t)$ to define a time-dependent field $\mathbf{u}(\mathbf{x};t)$ which is equal to $\mathbf{u}(\mathbf{x})$ at $t = 0$, and evolves with time as follows:

$$\mathbf{u}(\mathbf{x};t) = \mathbf{u}_t[\mathbf{x};\eta] \equiv \mathbf{u}\left(\mathbf{X}_0[t, \mathbf{x}; \eta]\right), \tag{6}$$

where the notation $\mathbf{u}_t[\mathbf{x};\eta]$ indicates the variables as in (5).

As a consequence of the dynamic equation (3), the change of the field $\mathbf{u}(\mathbf{x};t)$ in a small time interval from $t = 0$ to $t = \Delta t$ has the expansion

$$\begin{aligned}\Delta\mathbf{u}_0[\mathbf{x};\eta] &= \Delta t\, [\boldsymbol{\omega} \times \mathbf{x}] \cdot \boldsymbol{\nabla}\, \mathbf{u}_0[\mathbf{x};\eta] \\ &+ \int_0^{\Delta t} dt'\, \eta(t')\, (\mathbf{n} \cdot \boldsymbol{\nabla})\, \mathbf{u}_0[\mathbf{x};\eta] \\ &+ \frac{1}{2}\int_0^{\Delta t} dt' \int_0^{\Delta t} dt''\, \eta(t')\eta(t'')\, (\mathbf{n} \cdot \boldsymbol{\nabla})^2\, \mathbf{u}_0[\mathbf{x};\eta] + \ldots\ ..\end{aligned} \tag{7}$$

The omitted terms are of order $\Delta t^{3/2}$.

We now perform a noise average in Eq. (7), defining the average field

$$\bar{\mathbf{u}}(\mathbf{x};t) \equiv \langle \mathbf{u}_t[\mathbf{x};\eta] \rangle. \tag{8}$$

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Using the vanishing average of $\eta(t)$ and the correlation function (4), we obtain in the limit $\Delta t \to 0$ the time derivative

$$\partial_t \bar{\mathbf{u}}(\mathbf{x};t) = \hat{\mathcal{H}}\,\bar{\mathbf{u}}(\mathbf{x};t), \quad \text{at} \ \ t=0. \tag{9}$$

with the time evolution operator

$$\hat{\mathcal{H}} \equiv [\boldsymbol{\omega}\times\mathbf{x}]\cdot\boldsymbol{\nabla} + \frac{\hbar}{2}(\mathbf{n}\cdot\boldsymbol{\nabla})^2.. \tag{10}$$

The average field $\bar{\mathbf{u}}(\mathbf{x};t)$ at an arbitrary time t is obtained by the operation

$$\bar{\mathbf{u}}(\mathbf{x};t) = \hat{\mathcal{U}}(t)\bar{\mathbf{u}}(\mathbf{x};0) \equiv e^{\hat{\mathcal{H}}t}\bar{\mathbf{u}}(\mathbf{x};0). \tag{11}$$

Note that the average over η has made the operator $\hat{\mathcal{H}}$ time-independent: $\hat{\mathcal{H}}\hat{\mathcal{U}}(t) = \hat{\mathcal{U}}(t)\hat{\mathcal{H}}$. Moreover, the operator $\hat{\mathcal{H}}$ commutes with the Laplace operator $\boldsymbol{\nabla}^2$, thus ensuring that the harmonic property (1) of $\mathbf{u}(\mathbf{x})$ remains true for all times, i.e.,

$$\boldsymbol{\nabla}^2\bar{\mathbf{u}}(\mathbf{x},t) \equiv 0 \tag{12}$$

3. We now show that Eq. (9) describes the quantum mechanics of a harmonic oscillator. Let us restrict our attention to the line with arbitrary $x_1 \equiv x$ and $x_2 = 0$. Applying the Cauchy-Riemann equations (2), we can rewrite Eq. (9) in the pure x-form

$$\partial_t \bar{u}^1_t = \ \ \omega\, x\, \partial_x \bar{u}^2_t - \frac{\hbar}{2}\partial_x^2\, \bar{u}^2_t, \tag{13}$$

$$\partial_t \bar{u}^2_t = -\omega\, x\, \partial_x \bar{u}^1_t + \frac{\hbar}{2}\partial_x^2\, \bar{u}^1_t. \tag{14}$$

Now we introduce a complex field

$$\psi(x;t) \equiv e^{-\omega x^2/2\hbar}\left[\bar{u}^1(x,t) + i\bar{u}^2(x,t)\right], \tag{15}$$

where we have written $\bar{u}^\mu_t(x)$ for $\bar{u}^\mu_t(\mathbf{x})\,|_{x_1=x,x_2=0}$. This satisfies the differential equation

$$i\hbar\partial_t\psi(x;t) = \left(-\frac{\hbar^2}{2}\partial_x^2 + \frac{\omega^2}{2}x^2 - \frac{\hbar\omega}{2}\right)\psi(x;t), \tag{16}$$

which is the Schrödinger equation of a harmonic oscillator with the discrete energy spectrum $E_n = (n+1/2)\hbar\omega,\ n = 0,1,2,\dots$. The exponential prefactor in the wave function (15) is the ground state wave function of the oscillator. We have no a priori physical interpretation for its presence except that it is needed to arrive at the Schrödinger equation (16).

4. The method can easily be generalized to an arbitrary potential. We simply replace (3) by

$$\begin{aligned} \dot{x}^1(t) &= -\partial_2 S^1(\mathbf{x}(t)) + n^1\,\eta(t), \\ \dot{x}^2(t) &= -\partial_1 S^1(\mathbf{x}(t)) + n^2\,\eta(t), \end{aligned} \tag{17}$$

where $\mathbf{S}(\mathbf{x})$ shares with $\mathbf{u}(\mathbf{x})$ the harmonic property (1):

$$\boldsymbol{\nabla}^2\mathbf{S}(\mathbf{x}) = 0, \tag{18}$$

i.e., the functions $S^\mu(\mathbf{x})$ with $\mu = 1,2$ fulfill Cauchy-Riemann equations like $u^\mu(\mathbf{x})$ in (2). Repeating the above steps we find, instead of the operator (10),

$$\hat{\mathcal{H}} \equiv -(\partial_2 S^1)\partial_1 - (\partial_1 S^1)\partial_2 + \frac{\hbar}{2}(\mathbf{n}\cdot\boldsymbol{\nabla})^2, \tag{19}$$

and Eqs. (13) and (14) become:

$$\partial_t \bar{u}^1_t = \ \ (\partial_x S^1)\partial_x \bar{u}^2_t - \frac{\hbar}{2}\partial_x^2\, \bar{u}^2_t, \tag{20}$$

$$\partial_t \bar{u}^2_t = -(\partial_x S^1)\,\partial_x \bar{u}^1_t + \frac{\hbar}{2}\partial_x^2\, \bar{u}^1_t. \tag{21}$$

This time evolution preserves the harmonic nature of $\mathbf{u}(\mathbf{x})$. Indeed, using the harmonic property $\boldsymbol{\nabla}^2\mathbf{S}(\mathbf{x}) = 0$ we can easily derive the following time dependence of the Cauchy-Riemann combinations in Eq. (2):

$$\begin{aligned} &\partial_t(\partial_1 u_1 - \partial_2 u_2) = \hat{\mathcal{H}}(\partial_1 u_1 - \partial_2 u_2) \\ &\quad - \partial_2\partial_1 S^1(\partial_1 u_1 - \partial_2 u_2) + \partial_2^2 S^1(\partial_2 u_1 + \partial_1 u_2), \\ &\partial_t(\partial_2 u_1 + \partial_1 u_2) = \hat{\mathcal{H}}(\partial_2 u_1 + \partial_1 u_2) \\ &\quad - \partial_2\partial_1 S^1(\partial_2 u_1 + \partial_1 u_2) - \partial_2^2 S^1(\partial_1 u_1 - \partial_2 u_2). \end{aligned} \tag{22}$$

Thus $\partial_1 u_1 - \partial_2 u_2$ and $\partial_2 u_1 + \partial_1 u_2$ which are zero at any time remain zero at all times.

On account of Eqs. (21), the combination

$$\psi(x;t) \equiv e^{-S^1(x)/\hbar}\left[\bar{u}^1(x;t) + i\bar{u}^2(x;t)\right]. \tag{23}$$

satisfies the Schrödinger equation

$$i\hbar\partial_t\psi(x;t) = \left[-\frac{\hbar^2}{2}\partial_x^2 + V(x)\right]\psi(x;t), \tag{24}$$

where the potential is related to $S^1(x)$ by the Riccati differential equation

$$V(x) = \frac{1}{2}[\partial_x\, S^1(x)]^2 - \frac{\hbar}{2}\partial_x^2\, S^1(x). \tag{25}$$

The harmonic oscillator is recovered for the pair of functions $S^1(\mathbf{x}) + iS^2(\mathbf{x}) = \omega(x^1 + ix^2)^2/2$.

5. The noise $\eta(t)$ in the stochastic differential equation Eq. (17) can also be replaced by a source composed of deterministic classical oscillators $q_k(t)$, $k = 1,2,\dots$ with the equations of motion

$$\dot{q}_k = p_k, \qquad \dot{p}_k = -\omega_k^2 q_k, \tag{26}$$

as

$$\eta(t) \equiv \sum_k \dot{q}_k(t), \tag{27}$$

The initial positions $q_k(0)$ and momenta $p_k(0)$ are assumed to be randomly distributed with a Boltzmann factor $e^{-\beta H_{\rm osc}/\hbar}$, such that

$$\langle q_k(0) q_k(0) \rangle = \hbar/\omega_k^2, \quad \langle p_k(0) p_k(0) \rangle = \hbar. \tag{28}$$

Using the equation of motion

$$\dot{q}_k(t) = \omega_k q_k(0) \sin \omega_k t + p_k(0) \sin \omega_k t, \tag{29}$$

we find the correlation function

$$\begin{aligned} \langle \dot{q}_k(t) \dot{q}_k(t') \rangle &= \omega_k^2 \cos \omega_k t \cos \omega_k t' \langle q_k(0) q_k(0) \rangle \\ &\quad + \sin \omega_k t \sin \omega_k t \langle p_k(0) p_k(0) \rangle \\ &= \cos \omega_k (t - t'). \end{aligned} \tag{30}$$

We may now assume that the oscillators $q_k(t)$ are the Fourier components of a massless field, for instance the gravitational field whose frequencies are $\omega_k = k$, and whose random intial conditions are caused by the big bang. If the sum over k is simply a momentum integral $\int_{-\infty}^{\infty} dk$, then (30) yields a white-noise correlation function (4) for $\eta(t)$.

6. We have shown that it is possible to simulate the quantum-mechanical wave functions $\psi(x,t)$ and the energy spectrum of an arbitrary potential problem by classical stochastic equations of motion, or by deterministic equations with random initial conditions.

It remains to solve the open problem of finding a classical origin of the second important ingredient of quantum theory: the theory of quantum measurement associated with a stochastically evolving wave function $\psi(x,t)$ [3]. Only then shall we understand how God throws his dice [4].

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