Integrals over Products of Distributions from Perturbation Expansions of Path Integrals in Curved Space

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Abstract. – We show that the requirement of coordinate invariance of perturbatively defined quantum-mechanical path integrals in curved space leads to an extension of the theory of distributions by specifying unique rules for integrating products of distributions. The rules are derived by using equations of motion and partial integration, while keeping track of certain minimal features stemming from the unique definition of all singular integrals in $1-\epsilon$ dimensions. Our rules guarantee complete agreement with much more cumbersome calculations in $1-\epsilon$ dimensions where the limit $\epsilon \to 0$ is taken at the end. In contrast to our previous papers where we solved the same problem for an infinite time interval or zero temperature, we consider here the more involved case of finite-time or temperature amplitudes.

Introduction. – Until recently, a coordinate-independent definition of quantum mechanical path integrals in curved space existed only in the time-sliced formulation [1]. This is in contrast to field-theoretic path integrals between two and four spacetime dimensions which are well-defined in continuous spacetimes by perturbation expansions. Initial difficulties in guaranteeing coordinate independence were solved by 't Hooft and Veltman [5] using dimensional regularization with minimal subtractions (for a detailed description of this method see the textbook [6]). Coordinate independence emerges after calculating all Feynman integrals in an arbitrary number of dimensions d, and continuing the results to the desired physical integer value. Infinities occuring in the limit are absorbed into parameters of the action.

In contrast, and surprisingly, numerous attempts [7–15] to define the simpler *quantum mechanical* path integrals in curved space by perturbation expansions encountered problems in evaluating the Feynman integrals. Although all final results are finite and uniquely given by Schrödinger theory, the Feynman integrals in the expansions are highly singular and mathematically undefined. When evaluated in momentum space, they yield different results depending on the order of integration. Various definitions chosen by the authors of [7–15] were

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not coordinate-independent, and this could only be cured by adding coordinate-dependent "correction terms" to the classical action—a highly unsatisfactory procedure violating the basic Feynman rule that physical amplitudes should consist of a sum over all paths of phase factors $e^{i\mathcal{A}}$ whose exponents contains only the classical action along the paths.

The first satisfactory perturbative definition of path integrals in curved space was found only recently by us [2–4]. The results enabled us to set up simple rules for treating integrals over products of distributions in one dimensions to ensure coordinate invariance [4]. These rules were given for path integrals on an infinite time interval or zero temperature, where we could apply most directly the dimensionally continued integration rules of 't Hooft and Veltman [5] in momentum space.

In a recent paper [16], the authors of [10] and [14] have adapted our methods developed in [2, 3], applying them to the calculation of finite-time amplitudes (see also [17, 18]). In doing this, they have not taken advantage of the great simplifications brought about by the developments in our paper [4] which make explicit evaluations of Feynman integrals in $d = 1-\epsilon$ dimensions superfluous. The purpose of the present work is to show how this happens. We shall derive rules for calculating integrals over products of distributions which automatically guarantee coordinate independence. All integrals will be evaluated in one dimension, after having been brought to a regular form by some trivial manipulations which require only a small residual information on the initial $1-\epsilon$ -dimensional nature of the Feynman integrals.

Consider the short-time amplitude of a particle in curvilinear coordinates

$$(q_b\tau_b|q_a\tau_a) = \int \mathcal{D}q(\tau)\sqrt{g}\, e^{-\mathcal{A}[q]}, \tag{1}$$

where $\mathcal{A}[q]$ is the euclidean action of the form

$$\mathcal{A}[q] = \int_{\tau_a}^{\tau_b} d\tau \left[\frac{1}{2}g_{ij}(q(\tau))\dot{q}^i(\tau)\dot{q}^j(\tau) + V(q(\tau))\right]. \tag{2}$$

The dots denote τ-derivatives, $g_{ij}(q)$ is a metric, and $g = \det g$ its determinant. The path integral may formally be defined perturbatively as follows: The metric $g_{ij}(q)$ and the potential $V(q)$ are expanded around some point q_0^i near q_a and q_b in powers of $\delta q^i \equiv q^i - q_0^i$. After this, the action $\mathcal{A}[q]$ is separated into a free part $\mathcal{A}_0[q_0;\delta q] \equiv (1/2)\int_{\tau_a}^{\tau_b} d\tau\, g_{ij}(q_0)\dot{q}^i\dot{q}^j$, and an interacting part $\mathcal{A}_{\rm int}[q_0;\delta q] \equiv \mathcal{A}[q] - \mathcal{A}_0[q_0;\delta q]$.

A simply curable ultraviolet (UV) divergence problem is encountered in the square root in the measure of functional integration in (1). Taking it into the exponent and expanding in powers of δq, it corresponds to an effective action

$$\mathcal{A}_{\sqrt{g}} = -\frac{1}{2}\delta(0)\int_{\tau_a}^{\tau_b} d\tau \log[g(q_0+\delta q)/g(q_0)], \tag{3}$$

which contains the δ-function at the origin $\delta(0)$. This infinite quantity represents formally the inverse infinitesimal lattice spacing on the time axis, and is equal to the momentum integral $\delta(0) \equiv \int dp/(2\pi)$. With (3), the path integral (1) takes the form

$$(q_b\tau_b|q_a\tau_a) = \int \mathcal{D}q(\tau)\, e^{-\mathcal{A}[q]-\mathcal{A}_{\sqrt{g}}[q]} = \int \mathcal{D}\delta q(\tau)\, e^{-\mathcal{A}[q_0+\delta q]-\mathcal{A}_{\sqrt{g}}[q_0+\delta q]}\,. \tag{4}$$

The main problem arises in the expansion of the amplitude in powers of the interaction. For simplicity, we shall set $\tau_a = 0$, $\tau_b = \beta$, as in thermodynamics and assume $q_b = q_a = q_0 = 0$. Performing all Wick contractions, the origin to origin amplitude $(0\,\beta|0\,0)$ is expressed as a sum of loop diagrams. There are interaction terms involving $\delta\dot{q}^2\delta q^n$ which lead to Feynman integrals over products of distributions. The diagrams contain four types of lines representing the correlation functions

$$
\begin{aligned}
\Delta(\tau,\tau') &\equiv \langle\delta q(\tau)\delta q(\tau')\rangle = \text{———}\,, \qquad & \Delta^{\cdot}(\tau,\tau') \equiv \langle\delta q(\tau)\delta\dot{q}(\tau')\rangle = \text{—- - - -}\,,\\
{}^{\cdot}\Delta(\tau,\tau') &\equiv \langle\delta\dot{q}(\tau)\delta q(\tau')\rangle = \text{- - -—}\,, \qquad & {}^{\cdot}\Delta^{\cdot}(\tau,\tau') \equiv \langle\delta\dot{q}(\tau)\delta\dot{q}(\tau')\rangle = \text{- - - - - - -}\,.
\end{aligned}
\tag{5}
$$

The right-hand sides show the line symbols to be used in Feynman diagrams.

The first correlation function $\Delta(\tau,\tau') = \Delta(\tau',\tau)$ is determined by the free part $\mathcal{A}_0[q_0;\delta q]$ of the action. It is the Green function of the equation of motion

$$
{}^{\cdot\cdot}\Delta(\tau,\tau') = \Delta^{\cdot\cdot}(\tau,\tau') = -\delta(\tau-\tau'),
\tag{6}
$$

satisfying the Dirichlet boundary conditions

$$
\Delta(0,\tau') = \Delta(\beta,\tau') = 0, \qquad \Delta(\tau,0) = \Delta(\tau,\beta) = 0.
\tag{7}
$$

Explicitly, it reads

$$
\Delta(\tau,\tau') = \Delta(\tau',\tau) = \frac{1}{2}\left[-\epsilon(\tau-\tau')(\tau-\tau') + \tau + \tau'\right] - \frac{\tau\tau'}{\beta},
\tag{8}
$$

where $\epsilon(\tau-\tau')$ is the antisymmetric distribution which is equal to ± 1 for $\tau \gtrless \tau'$ and vanishes at the origin.

The second and third correlation functions ${}^{\cdot}\Delta(\tau,\tau')$ and $\Delta^{\cdot}(\tau,\tau')$ are

$$
{}^{\cdot}\Delta(\tau,\tau') = -\frac{1}{2}\epsilon(\tau-\tau') + \frac{1}{2} - \frac{\tau'}{\beta}, \quad \Delta^{\cdot}(\tau,\tau') = \frac{1}{2}\epsilon(\tau-\tau') + \frac{1}{2} - \frac{\tau}{\beta} = {}^{\cdot}\Delta(\tau',\tau)\,,
\tag{9}
$$

with a discontinuity at $\tau = \tau'$. Here and in the following, dots on the right and left of $\Delta(\tau,\tau')$ denote time derivatives with respect to τ and τ', respectively.

The fourth correlation function ${}^{\cdot}\Delta^{\cdot}(\tau,\tau')$ is simply

$$
{}^{\cdot}\Delta^{\cdot}(\tau,\tau') = \delta(\tau-\tau') - 1/\beta.
\tag{10}
$$

The δ-function arises from the derivative $\delta(\tau-\tau') = \dot{\epsilon}(\tau-\tau')/2$. Its value at the origin must be equal to the prefactor $\delta(0)$ of the effective action (3) of the measure to cancel all ultraviolet (UV) infinities. Note the close similarity of (10) to the equation of motion (6).

The difficulty in calculating the loop integrals over products of such distributions is best illustrated by observing the lack of reparametrization invariance of the path integral of a free particle in n-dimensional curvilinear coordinates first done by Gervais and Jevicki [7], Salomonson [8], and recently also by Bastianelli, van Nieuwenhuizen, and collaborators [9–15]. The basic ambiguous integrals causing problems arise from the two-loop diagrams

 :

$$
I_{14} = \int_0^\beta \int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau')\, \Delta^{\cdot}(\tau,\tau')\, {}^{\cdot}\Delta^{\cdot}(\tau,\tau'),
\tag{11}
$$

 :

$$
I_{15} = \int_0^\beta \int_0^\beta d\tau\, d\tau' \Delta(\tau,\tau')\, {}^{\cdot}\Delta^{\cdot\,2}(\tau,\tau').
\tag{12}
$$

It is shown in Appendix A that the requirement of coordinate independence implies that these integrals have the values

$$
I_{14} = \beta/24, \qquad I_{15}^R = -\beta/8,
\tag{13}
$$

where the superscript R denotes the finite part of an integral.

Let us demonstrate that these values are incompatible with partial integration and the equation of motion (6). In the integral (11), we use the symmetry ${}^{\cdot\cdot}\Delta(\tau,\tau') = \Delta^{\cdot\cdot}(\tau,\tau')$, apply partial integration twice taking care of nonzero boundary terms, and obtain on the one hand

$$\begin{aligned} I_{14} &= \frac{1}{2}\int_0^\beta \int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau')\frac{d}{d\tau}\left[\Delta^{\cdot 2}(\tau,\tau')\right] = -\frac{1}{2}\int_0^\beta \int_0^\beta d\tau\, d\tau'\, \Delta^{\cdot 2}(\tau.\tau')\, {}^{\cdot\cdot}\Delta(\tau,\tau') \\ &= -\frac{1}{6}\int_0^\beta \int_0^\beta d\tau\, d\tau' \frac{d}{d\tau'}\left[\Delta^{\cdot 3}(\tau,\tau')\right] = \frac{1}{6}\int_0^\beta d\tau\left[\Delta^{\cdot 3}(\tau,0) - \Delta^{\cdot 3}(\tau,\beta)\right] = \frac{\beta}{12}. \end{aligned} \tag{14}$$

On the other hand, we apply Eq. (10) and perform two regular integrals, reducing I_{14} to a form containing an undefined integral over a product of distributions:

$$\begin{aligned} I_{14} &= \int_0^\beta \int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau')\, \Delta^{\cdot}(\tau,\tau')\delta(\tau-\tau') - \frac{1}{\beta}\int_0^\beta \int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau')\, \Delta^{\cdot}(\tau,\tau') \\ &= \int_0^\beta \int_0^\beta d\tau\, d\tau'\left[-\frac{1}{4}\epsilon^2(\tau-\tau')\delta(\tau-\tau')\right] + \int_0^\beta d\tau\, {}^{\cdot}\Delta^{2}(\tau,\tau) + \frac{\beta}{12} \\ &= \beta\left[-\frac{1}{4}\int d\tau\, \epsilon^2(\tau)\delta(\tau) + \frac{1}{6}\right]. \end{aligned} \tag{15}$$

A third, mixed way of evaluating I_{14} employs one partial integration as in the first line of Eq. (14), then the equation of motion (6) to reduce I_{14} to yet another form

$$\begin{aligned} I_{14} &= \frac{1}{2}\int_0^\beta \int_0^\beta d\tau\, d\tau'\, \Delta^{\cdot 2}(\tau,\tau')\delta(\tau-\tau') = \\ &= \frac{1}{8}\int_0^\beta \int_0^\beta d\tau\, d\tau' \epsilon^2(\tau-\tau')\delta(\tau-\tau') + \frac{1}{2}\int_0^\beta d\tau\, {}^{\cdot}\Delta^{2}(\tau,\tau) = \\ &= \beta\left[\frac{1}{8}\int d\tau\, \epsilon^2(\tau)\delta(\tau) + \frac{1}{24}\right]. \end{aligned} \tag{16}$$

We now see that if we set

$$\int d\tau [\epsilon(\tau)]^2\, \delta(\tau) \equiv \frac{1}{3} \tag{17}$$

the last two results (16) and (15) coincide with the first in Eq. (14). The definition (17) is obviously consistent with partial integration if we insert $\delta(\tau) = \dot\epsilon(\tau)/2$:

$$\int d\tau [\epsilon(\tau)]^2\, \delta(\tau) = \frac{1}{2}\int d\tau [\epsilon(\tau)]^2\, \dot\epsilon(\tau) = \frac{1}{6}\int d\tau \frac{d}{d\tau}[\epsilon(\tau)]^3 = \frac{1}{3}. \tag{18}$$

While the integration rule (17) is consistent with partial integration and equation of motion, it is incompatible with the requirement of coordinate independence. This can be seen from the discrepancy between the resulting value $I_{14} = \beta/12$ and the necessary (13). This discrepancy was compensated in Refs. [7–15] by adding the above-mentioned noncovariant term to the classical action.

A similar problem appears with the other Feynman integral (12). Applying first Eq. (10) we obtain

$$I_{15} = \int_0^\beta \int_0^\beta d\tau\, d\tau' \Delta(\tau,\tau')\delta^2(\tau-\tau') - \frac{2}{\beta}\int_0^\beta d\tau \Delta(\tau,\tau) + \frac{1}{\beta^2}\int_0^\beta \int_0^\beta d\tau d\tau' \Delta(\tau,\tau'). \tag{19}$$

For the integral containing the square of the δ-function we must postulate the integration rule

$$\int d\tau \, [\delta(\tau)]^2 \, f(\tau) \equiv \delta(0) f(0) \tag{20}$$

to obtain a divergent term

$$I_{15}^{\rm div} = \delta(0) \int_0^\beta d\tau \Delta(\tau, \tau) = \delta(0) \frac{\beta^2}{6}. \tag{21}$$

proportional to $\delta(0)$ compensating a similar term from the measure. The remaining integrals in (19) are finite and yield the regular part of I_{15}

$$I_{15}^R = -\frac{\beta}{4}. \tag{22}$$

In another calculation of I_{15}, we first add and subtract the UV divergent term, writing

$$I_{15} = \int_0^\beta \int_0^\beta d\tau \, d\tau' \Delta(\tau, \tau') \left[{}^{\cdot}\Delta^{\cdot\,2}(\tau, \tau') - \delta^2(\tau - \tau') \right] + \delta(0) \frac{\beta^2}{6}. \tag{23}$$

Replacing $\delta^2(\tau - \tau')$ by the square of the left-hand side of the equation of motion (6), and integrating the terms in brackets by parts, we obtain

$$\begin{aligned}
I_{15}^R &= \int_0^\beta \int_0^\beta d\tau \, d\tau' \Delta(\tau, \tau') \left[{}^{\cdot}\Delta^{\cdot\,2}(\tau, \tau') - \, \Delta^{\cdot\cdot\,2}(\tau, \tau') \right] \\
&= \int_0^\beta \int_0^\beta d\tau \, d\tau' \left[- {}^{\cdot}\Delta \, (\tau, \tau') \, \Delta^{\cdot}(\tau, \tau') \, {}^{\cdot}\Delta^{\cdot}(\tau, \tau') - \Delta(\tau, \tau') \, \Delta^{\cdot}(\tau, \tau') \, {}^{\cdot\cdot}\Delta^{\cdot}(\tau, \tau') \right] \\
&- \int_0^\beta \int_0^\beta d\tau \, d\tau' \left[- \Delta^{\cdot\,2}(\tau, \tau') \, \Delta^{\cdot\cdot}(\tau, \tau') - \Delta(\tau, \tau') \, \Delta^{\cdot}(\tau, \tau') \, \Delta^{\cdot\cdot\cdot}(\tau, \tau') \right] \\
&= -I_{14} + \int_0^\beta \int_0^\beta d\tau \, d\tau' \, \Delta^{\cdot\,2}(\tau, \tau') \, \Delta^{\cdot\cdot}(\tau, \tau') = -I_{14} - \beta/6.
\end{aligned} \tag{24}$$

The value of the last integral follows from partial integration.

For a third evaluation of I_{15} we insert the equation of motion (6) and bring the last integral in the fourth line of (24) to

$$- \int_0^\beta \int_0^\beta d\tau \, d\tau' \, \Delta^{\cdot\,2}(\tau, \tau') \delta(\tau - \tau') = -\beta \left[\frac{1}{4} \int d\tau \, \epsilon^2(\tau) \delta(\tau) + \frac{1}{12} \right]. \tag{25}$$

All three ways of calculation lead to the same result $I_{15}^R = -\beta/4$ using the rule (17). This, however, is again in disagreement with the coordinate-invariant value in Eq. (13). Note that both integrals I_{14} and I_{15}^R are too large by a factor 2 with respect to the necessary (13) for coordinate invariance.

How can we save coordinate invariance while maintaining the equation of motion and partial integration? The direction in which the answer lies is suggested by the last line of Eq. (16): we must find a consistent way to have an integral

$$\int d\tau \, [\epsilon(\tau)]^2 \, \delta(\tau) = 0, \tag{26}$$

instead of (17), which means that we need a reason for forbidding the application of partial integration to this singular integral. For the calculation at the infinite time interval, this problem was solved in our previous papers [2–4] with the help of the dimensional regularization, carried to higher orders in Refs. [16, 17]). The extension of our rules to the short-time amplitude considered here is straightforward. It can be done without performing any of the cumbersome calculations in $1-\varepsilon$-dimension. We must only keep track of the essential features of the structure of the Feynman integrals in arbitrary dimensions. For this we continue the imaginary time coordinate τ to a d-dimensional spacetime vector $\tau \to x^\mu = (\tau, x^1, \dots, x^{d-1})$. In $d = 1 - \varepsilon-$ dimensions, the correlation function reads

$$\Delta(\tau, \mathbf{x};\, \tau', \mathbf{x}') = \int \frac{d^\varepsilon k}{(2\pi)^\varepsilon} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \Delta_\omega(\tau, \tau'). \tag{27}$$

Here the extra ε-dimensional coordinates $\mathbf{x}$ are assumed to live on infinite axes to have translational invariance along all $\mathbf{x}$-directions, with only the τ-coordinate lying in a finite interval $0 \le \tau \le \beta$, with Dirichlet boundary conditions for (27). The one-dimensional correlation function $\Delta_\omega(\tau, \tau')$ in the integrand has a mass $\omega = k = \sqrt{\mathbf{k}^2}$. It is the Green function on the finite τ-interval

$$-\ddot{\Delta}_\omega(\tau, \tau') + \omega^2 \Delta_\omega(\tau, \tau') = \delta(\tau - \tau'), \tag{28}$$

satisfying the Dirichlet boundary conditions

$$\Delta_\omega(0, \tau) = \Delta_\omega(\beta, \tau) = 0. \tag{29}$$

Explicitly, it reads [1]

$$\Delta_\omega(\tau, \tau') = \frac{\sinh \omega(\beta - \tau_>) \sinh \omega \tau_<}{\omega \sinh \omega\beta}\,, \tag{30}$$

where $\tau_>$ and $\tau_<$ denote the larger and smaller of the imaginary times τ and τ', respectively.

In d dimensions, the equation of motion (6) becomes a scalar field equation of the Klein-Gordon type. Using Eq. (28), we obtain

$$\begin{aligned}
{}_{\mu\mu}\Delta(\tau, \mathbf{x};\, \tau', \mathbf{x}') &= \Delta_{\mu\mu}(\tau, \mathbf{x};\, \tau', \mathbf{x}') = \ddot{\Delta}(\tau, \mathbf{x};\, \tau', \mathbf{x}') + {}_{\mathbf{xx}}\Delta(\tau, \mathbf{x};\, \tau', \mathbf{x}') \\
&= \int \frac{d^\varepsilon k}{(2\pi)^\varepsilon} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \left[\ddot{\Delta}_\omega(\tau, \tau') - \omega^2 \Delta_\omega(\tau, \tau')\right] = \\
&= -\,\delta(\tau - \tau')\, \delta^{(\varepsilon)}(\mathbf{x} - \mathbf{x}') = -\,\delta^{(d)}(x - x').
\end{aligned} \tag{31}$$

The important observation is now that for d-spacetime dimensions, perturbation expansion of the path integral yields for the second correlation function $\dot{\Delta}\dot{}(\tau, \tau')$ in Eqs. (11) and (12) the extension ${}_\mu\Delta_\nu(x, x')$. This function differs from the contracted function ${}_\mu\Delta_\mu(x, x')$, and from ${}_{\mu\mu}\Delta(x, x')$ which satisfies the field equation (31). In fact, all correlation functions $\dot{\Delta}\dot{}(\tau, \tau')$ encountered in the diagrammatic expansion which have different time arguments always have the d-dimensional extension ${}_\mu\Delta_\nu(x, x')$. An important exception is the correlation functions at *equal* times $\dot{\Delta}\dot{}(\tau, \tau)$ whose d-dimensional extension is always ${}_\mu\Delta_\mu(x, x)$, which satisfies the equation (10) in the $\varepsilon \to 0$-limit. Indeed, it follows from Eq. (27) that

$${}_\mu\Delta_\mu(x, x) = \int \frac{d^\varepsilon k}{(2\pi)^\varepsilon} \left[\dot{\Delta}\dot{}_\omega(\tau, \tau) + \omega^2\, \Delta_\omega(\tau, \tau)\right]. \tag{32}$$

With the help of Eq. (30), the integrand in Eq. (32) can be brought to

$$\dot{\Delta}\dot{}_\omega(\tau, \tau) + \omega^2 \Delta_\omega(\tau, \tau) = \delta(0) - \frac{\omega \cosh \omega(2\tau - \beta)}{\sinh \omega\beta}\,. \tag{33}$$

Substituting this into Eq. (32), we obtain

$$ {}_{\mu}\Delta_{\mu}(x,x) = \delta^{(d)}(x,x) - I^{\varepsilon}\,. \tag{34} $$

The integral I^{ε} is calculated as follows

$$ \begin{aligned} I^{\varepsilon} &= \int \frac{d^{\varepsilon}k}{(2\pi)^{\varepsilon}} \frac{\omega\cosh\omega(2\tau-\beta)}{\sinh\omega\beta} = \frac{1}{\beta}\frac{S_{\varepsilon}}{(2\pi\beta)^{\varepsilon}} \int_0^{\infty} dz z^{\varepsilon} \frac{\cosh z(1-2\tau/\beta)}{\sinh z} \\ &= \frac{1}{\beta}\frac{S_{\varepsilon}}{(2\pi\beta)^{\varepsilon}} \frac{\Gamma(\varepsilon+1)}{2^{\varepsilon+1}} [\zeta(\varepsilon+1, 1-\frac{\tau}{\beta}) + \zeta(\varepsilon+1, \frac{\tau}{\beta})], \end{aligned} \tag{35} $$

where $S_{\varepsilon} = 2\pi^{\varepsilon/2}/\Gamma(\varepsilon/2)$ is the surface of unit sphere in ε dimension, and $\Gamma(z)$ and $\zeta(z,q)$ are gamma and zeta functions, respectively. For small $\varepsilon \to 0$, they have the limits $\zeta(\varepsilon+1,q) \to 1/\varepsilon - \psi(q)$, and $\Gamma(\varepsilon/2) \to 2/\varepsilon$. It then follows from Eq. (35) that $I^{\varepsilon} \to 1/\beta$ in this limit, proving that the d-dimensional equation (34) reduces indeed to Eq.(10). The explicit d-dimensional form will never be needed, since we can simply treat all ${}_{\mu}\Delta_{\mu}(x,x)$s as purely one-dimensional objects $\dot{}\Delta\dot{}(\tau,\tau)$, which can in turn be replaced everywhere by the right-hand side $\delta(0) - 1/\beta$ of (10).

We now show that by carefully keeping track of the different contractions of the derivatives, we obtain a consistent calculation scheme which yields results equivalent to assuming integration rule (26) in the calculation of I_{14} and I_{15}, thus ensuring coordinate independence. The integral (11) for I_{14} is extended to

$$ I_{14}^{d} = \int\!\!\int d^d x\, d^d x'\, {}_{\mu}\Delta(x,x')\Delta_{\nu}(x,x')\, {}_{\mu}\Delta_{\nu}(x,x'), \tag{36} $$

and the different derivatives on ${}_{\mu}\Delta_{\nu}(x,x')$ prevent us from applying the field equation (31), in contrast to the one-dimensional calculation. We can, however, apply partial integration as in the first line of Eq. (14), and arrive at

$$ I_{14}^{d} = -\frac{1}{2}\int\!\!\int d^d x\, d^d x' \Delta_{\nu}^{2}(x,x')\Delta_{\mu\mu}(x,x'). \tag{37} $$

In contrast to the one-dimensional expression (14), a further partial integration is impossible. Instead, we apply the field equation (31), go back to one dimension, and apply the integration rule (26) as in Eq. (16) to obtain the correct result $I_{14} = \beta/24$ guaranteeing coordinate invariance.

The Feynman integral (12) for I_{15} is treated likewise. Its d-dimensional extension is

$$ I_{15}^{d} = \int\!\!\int d^d x\, d^d x' \Delta(x,x')\, [{}_{\mu}\Delta_{\nu}(x,x')]^2\,. \tag{38} $$

The different derivatives on ${}_{\mu}\Delta_{\nu}(x,x')$ make it impossible to apply a dimensionally extended version of equation (10) as in Eq. (19). We can, however, extract the UV divergence as in Eq. (23), and perform a partial integration on the finite part which brings it to a dimensionally extended version of Eq. (24):

$$ I_{15}^{R} = -I_{14} + \int d^d x\, d^d x' \Delta_{\nu}^{2}(x,x')\Delta_{\mu\mu}(x,x'). \tag{39} $$

On the right-hand side we use the field equation (31), as in Eq. (25), return to $d=1$, and use the rule (26) to obtain the result $I_{15}^{R} = -I_{14} - \beta/12 = -\beta/8$, again guaranteeing coordinate independence.

Thus, by keeping only track of a few essential properties of the theory in d dimensions we indeed obtain a simple consistent procedure for calculating singular Feynman integrals. All results obtained in this way ensure coordinate independence. They agree with what we would obtain using the one-dimensional integration rule (26) for the product of two ϵ- and one δ-distribution.

Our procedure gives us unique rules telling us where we are allowed to apply partial integration and the equation of motion in one-dimensional expressions. Ultimately, all integrals are brought to a regular form, which can be continued back to one time dimension for a direct evaluation. This procedure is obviously much simpler than the previous explicit calculations in d-dimension with the limit $d \to 1$ taken at the end.

We now apply this procedure to the perturbation expansion of the short-time amplitude of a free particle in curvilinear coordinates.

Perturbation Expansion. – A free point particle of unit mass has the action

$$\mathcal{A}_0[x] = \frac{1}{2}\int_0^\beta d\tau\, \dot{x}^2(\tau). \tag{40}$$

The amplitude $(0\,\beta|0\,0)_0$ is given by the Gaussian path integral

$$(0\,\beta|0\,0)_0 = \int \mathcal{D}x(\tau)\, e^{-\mathcal{A}_0[x]} = e^{-(1/2)\mathrm{Tr}\log(-\partial^2)} = [2\pi\beta]^{-1/2}. \tag{41}$$

A coordinate transformation $x(\tau) = f(q(\tau))$ brings the action (40) to the form

$$\mathcal{A}[q] = \frac{1}{2}\int_0^\beta d\tau\, g(q(\tau))\dot{q}^2(\tau), \tag{42}$$

where $g(q) = f'^2(q)$. The measure $\mathcal{D}x(\tau) \equiv \prod_\tau dx(\tau)$ transforms as follows:

$$\mathcal{D}x(\tau) \equiv \prod_\tau dx(\tau) = J\prod_\tau dq(\tau) \equiv J\,\mathcal{D}q(\tau), \tag{43}$$

where J is the Jacobian of the coordinate transformation

$$J = e^{(1/2)\delta(0)\int_0^\beta d\tau \log g(q(\tau))}. \tag{44}$$

Thus the transformed path integral (41) takes precisely the form (4), with the total action in the exponent

$$\mathcal{A}_{\mathrm{tot}}[q] = \int_0^\beta d\tau\left[\frac{1}{2}g(q(\tau))\dot{q}^2(\tau) - \frac{1}{2}\delta(0)\log g(q(\tau))\right]. \tag{45}$$

This is decomposed into a free part

$$\mathcal{A}_0[q] = \frac{1}{2}\int_0^\beta d\tau\, \dot{q}^2(\tau) \tag{46}$$

and an interacting part

$$\mathcal{A}_{\mathrm{int}}[q] = \int_0^\beta d\tau \frac{1}{2}[g(q)-1]\dot{q}^2 - \int_0^\beta d\tau\frac{1}{2}\delta(0)\left\{[g(q)-1] - \frac{1}{2}[g(q)-1]^2 + \ldots\right\}. \tag{47}$$

The path integral (41) is now formally defined by the perturbation expansion

$$
\begin{aligned}
\langle 0,\beta|0,0\rangle &= \int \mathcal{D}q(\tau)e^{\mathcal{A}_0[q]-\mathcal{A}_{\rm int}[q]} = \int \mathcal{D}q(\tau)e^{-\mathcal{A}_0[q]}\left(1-\mathcal{A}_{\rm int}+\frac{1}{2}\mathcal{A}_{\rm int}^2-\dots\right)\\
&= (2\pi\beta)^{-1/2}\left[1-\langle\mathcal{A}_{\rm int}\rangle+\frac{1}{2}\langle\mathcal{A}_{\rm int}^2\rangle-\dots\right],\\
&= (2\pi\beta)^{-1/2}e^{-\langle\mathcal{A}_{\rm int}\rangle_c+\frac{1}{2}\langle\mathcal{A}_{\rm int}^2\rangle_c-\dots},
\end{aligned}
\tag{48}
$$

with the harmonic expectation values

$$
\langle\dots\rangle = (2\pi\beta)^{1/2}\int \mathcal{D}q(\tau)(\dots)e^{-\mathcal{A}_0[q]}, \tag{49}
$$

and their cumulants $\langle\mathcal{A}_{\rm int}^2\rangle_c = \langle\mathcal{A}_{\rm int}^2\rangle - \langle\mathcal{A}_{\rm int}\rangle^2,\dots$ containing only connected diagrams. If our calculation procedure respects coordinate independence, all expansion terms must vanish to yield the trivial exact results (41). As an example we shall consider the coordinate transformation

$$
x = f(q) = q - \frac{1}{3}\varepsilon q^3 + \frac{1}{5}\varepsilon^2 q^5 - \dots, \tag{50}
$$

such that

$$
g(q) = f'^2(q) = 1 - 2\varepsilon^2 q^2 + 3\varepsilon^2 q^4 - 4\varepsilon^4 q^6 + \dots, \tag{51}
$$

where ε is a smallness parameter. Substituting Eq. (51) into Eq. (47) yields up to second order in ε:

$$
\mathcal{A}_{\rm int}[q] = \int_0^\beta d\tau\left\{\left[-\varepsilon q^2(\tau)+\frac{3}{2}\varepsilon^2 q^4(\tau)\right]\dot q^2(\tau) - \delta(0)\left[-\varepsilon q^2(\tau)+\frac{1}{2}\varepsilon^2 q^4(\tau)\right]\right\}. \tag{52}
$$

We shall now calculate order by order in ε the expansion terms contributing to the square bracket in the second line of Eq. (48).

Diagrams. – To first order in ε, the square bracket in the second line of Eq. (48) receives a contribution from the expectation values of the linear terms in ε of the interaction (52):

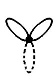
$$
-\langle\mathcal{A}_{\rm int}^{\text{lin in }\varepsilon}\rangle = \int_0^\beta d\tau\,\left\langle\varepsilon q^2(\tau)\dot q^2(\tau) - \delta(0)\varepsilon q^2(\tau)\right\rangle. \tag{53}
$$

Thus there exists only three diagrams, two originating from the kinetic term and one from the Jacobian action:

ε  $+\,2\,\varepsilon$  $-\,\varepsilon\,\delta(0)$  . (54)

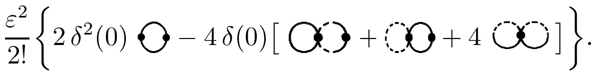
To order ε^2, we need to calculate only connected diagrams contained in the term $\langle\mathcal{A}_{\rm int}^2\rangle/2$ in (48), all disconnected ones being obtainable from the cumulant relation $\langle\mathcal{A}_{\rm int}^2\rangle = \langle\mathcal{A}_{\rm int}^2\rangle_c + \langle\mathcal{A}_{\rm int}\rangle^2$. We distinguish several contributions.

First, there are two local three-loop diagrams and one two-loop local diagram coming from the kinetic term and the Jacobian of the interaction (52), respectively:

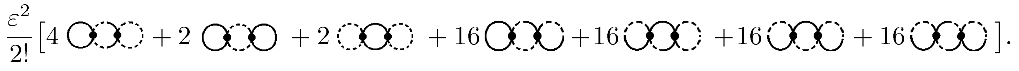
$\left(-\frac{3}{2}\varepsilon^2\right)\Big[\ 3$  $+\ 12$  $-\ \delta(0)$  $\Big]$. (55)

We call a diagram local if it involves only equal-time Wick contractions.

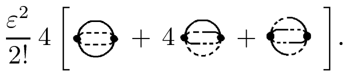
The Jacobian part of the action (52) contributes further the nonlocal diagrams:


(56)

The remaining diagrams come from the kinetic term of the interaction (52) only. They are either of the three-bubble type, or of the watermelon type, each with all possible combinations of the three line types (5): The sum of all three-bubbles diagrams is


(57)

The watermelon-type diagrams contribute


(58)

Path Integral in Curved Space. – Before we start evaluating the above Feynman diagrams, we observe that the same diagrams appear if we define path integral in a higher-dimensional target space q^i. The generalization of the formal expression (45) is obvious: we replace $g(q)$ by $g_{ij}(q)$ in the kinetic term, and by $g(q) \to \det(g_{ij}(q))$ in the measure, where $g_{ij}(q)$ is the metric induced by the coordinate transformation from cartesian to curvilinear coordinates. In a further step, we shall also consider $g_{ij}(q)$ more generally as a metric in a curved space, which can be reached from a flat space only by a nonholonomic coordinate transformation [19]. It was shown in the textbook [1] that under nonholonomic coordinate transformations, the measure of a time-sliced path integral transforms from the flat-space form $\prod_n dx_n$ to $\prod_n dq\sqrt{g_n}\exp(\Delta t R_n/6)$, which has the consequence that the amplitude satisfies a Schrödinger equation with the pure Laplace-Beltrami operator in the kinetic Hamiltonian, containing no extra R-term. Here we shall see that a similar thing must happen for perturbatively defined path integrals, where the nonholonomic transformation must carry the flat-space measure

$$\mathcal{D}x \to \mathcal{D}q\,\sqrt{g}\,\exp\left(\int_0^\beta d\tau\, R/8\right). \tag{59}$$

The proof of this rather technical issue is relegated to a separate paper.

For n-dimensional manifolds with a general metric $g_{ij}(q)$ we make use of the coordinate invariance to be proved by the vanishing of the expansion (54)–(58). This will allow us to bring the metric to the most convenient coordinates which are the Riemannian normal coordinates. Assuming n-dimensional manifold to be a homogeneous space, as in a standard nonlinear σ-model, we expand the metric and its determinant in the normal coordinates as follows

$$g_{ij}(q) \;=\; \delta_{ij} + \varepsilon\,\frac{1}{3}\,R_{ik_1jk_2}\,q^{k_1}q^{k_2} + \varepsilon^2\,\frac{2}{45}R_{k_1jk_2}{}^{l}\,R_{k_3ik_4l}\,q^{k_1}q^{k_2}q^{k_3}q^{k_4} + \dots, \tag{60}$$

$$g(q) \;=\; 1 - \varepsilon\,\frac{1}{3}\,R_{ij}\,q^iq^j + \varepsilon^2\,\frac{1}{18}\left(R_{ij}\,R_{kl} + \frac{1}{5}\,R_{inj}{}^{m}\,R_{kml}{}^{n}\right)q^iq^jq^kq^l + \dots\,. \tag{61}$$

In our conventions, the Riemann and Ricci tensors are $R_{ijk}{}^{l} = \partial_i\Gamma_{jk}{}^{l} - \dots$, $R_{jk} = R_{ijk}{}^{i}$, and the curvature $R = R_i{}^{i}$ has the positive sign for a sphere. The expansions (60) and (61) have obviously a similar power structure in q^i as the previous expansion (51).

In normal coordinates, the interaction (47) becomes, up to order ε^2:

$$\mathcal{A}_{\rm int}[q] = \int_0^\beta d\tau \Big\{ \Big[\varepsilon\frac{1}{6}R_{ikjl}\, q^k q^l + \varepsilon^2\frac{1}{45}R_{mjn}{}^l\, R_{risl}\, q^m q^n q^r q^s\Big]\dot q^i \dot q^j + \varepsilon\frac{1}{6}\delta(0)R_{ij}\, q^i q^j + \varepsilon^2\frac{1}{180}\delta(0)R_{imj}{}^n\, R_{knl}{}^m\, q^i q^j q^k q^l\Big\}, \qquad (62)$$

with the same powers of q^i as in Eq. (52). The interactions (52) and (62) yield the same diagrams in the perturbation expansions in powers of ε. In one dimension and with the trivial vertices in the interaction (52), the sum of all diagrams will be shown to vanish in the case of a flat space. In a curved space with the more complicated vertices proportional to R_{ijkl} and R_{ij}, the same Feynman integrals will yield a nontrivial short-time amplitude. The explicit R_{ijkl}-dependence coming from the interaction vertices is easily identified in the diagrams: all bubbles in (56)–(57) yield results proportional to R_{ij}^2, while the watermelon-like diagrams (58) carry a factor R_{ijkl}^2. In our previous work [2–4], all integrals were calculated in d dimension, taking the limit $d \to 1$ at the end. In this way we confirmed that the sum of all Feynman diagrams contributing to each order in ε vanishes. It is easy to verify that the same results are found using the procedure developed above.

From Coordinate Independence to DeWitt-Seeley Expansion. – With the same procedure we now calculate the first two terms in the short-time expansion of the time-evolution amplitude. The results will be compared with the similar expansion obtained from the operator expression for the amplitude $e^{\beta D^2\beta/2}$ with the Laplace-Beltrami operator $D^2 = g^{-1/2}\partial_i g^{1/2} g^{ij}(q)\partial_j$ first derived by DeWitt [20] (see also [21]):

$$(q,\beta \mid q',0) = (q \mid e^{\beta D^2/2} \mid q') = \frac{1}{\sqrt{2\pi\beta}^n}\, e^{-g_{ij}\Delta q^i \Delta q^j/2\beta} \sum_{k=0}^{\infty} \beta^k a_k(q,q'), \qquad (63)$$

with the expansion coefficients being for a homogeneous space

$$\begin{aligned} a_0(q,q') &\equiv 1 + \frac{1}{12}R_{ij}\Delta q^i \Delta q^j + \left(\frac{1}{360}R^{i\ j}_{k\ l}R_{imjn} + \frac{1}{288}R_{kl}R_{mn}\right)\Delta q^k \Delta q^l \Delta q^m \Delta q^n + \dots, \\ a_1(q,q') &\equiv \frac{1}{12}R + \left(\frac{1}{144}R\ R_{ij} + \frac{1}{360}R^{kl}R_{kilj} + \frac{1}{360}R^{klm}{}_i R_{klmj} - \frac{1}{180}R^n_i R_{nj}\right)\Delta q^i \Delta q^j + \dots, \\ a_2(q,q') &\equiv \frac{1}{288}R^2 + \frac{1}{720}R^{ijkl}R_{ijkl} - \frac{1}{720}R^{ij}R_{ij} + \dots, \end{aligned} \qquad (64)$$

where $\Delta q^i \equiv (q-q')^i$. For $\Delta q^i = 0$ this simplifies to

$$(0,\beta \mid 0,0) = \frac{1}{\sqrt{2\pi\beta}^n}\left\{1 + \frac{\beta}{12}R(q) + \frac{\beta^2}{72}\left[\frac{1}{4}R^2 + \frac{1}{10}\left(R^{ijkl}R_{ijkl} - R^{ij}R_{ij}\right)\right]\right\}. \qquad (65)$$

The derivation is sketched in Appendix B.

For coordinate independence, the sum of the first-order diagrams (54) has to vanish. Analytically, this amounts to the equation

$$\varepsilon\left\{\int_0^\beta d\tau\, \left[\Delta(\tau,\tau)\,{}^{\bullet}\Delta^{\bullet}(\tau,\tau) + 2\,{}^{\bullet}\Delta^2(\tau,\tau) - \delta(0)\Delta(\tau,\tau)\right]\right\} = 0. \qquad (66)$$

In its d-dimensional extension, correlation function ${}^{\bullet}\Delta^{\bullet}(\tau,\tau)$ at equal times is the limit $d \to 1$ of the contracted correlation function ${}_\mu\Delta_\mu(x,x)$ which satisfies the d-dimensional field equation (31). Thus we can use Eq. (10) to replace ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ by $\delta(0)-1/\beta$. This removes the infinite factor $\delta(0)$ in Eq. (66) coming from the measure. For the remainder we calculate directly

$$\int_0^\beta d\tau \left[-\frac{1}{\beta}\Delta(\tau,\tau)+2\,{}^{\cdot}\Delta^{2}(\tau,\tau)\right]=0. \tag{67}$$

This result is obtained without subtleties, since by Eqs. (8) and (9)

$$\Delta(\tau,\tau)=\tau-\frac{\tau^2}{\beta},\qquad {}^{\cdot}\Delta^{2}(\tau,\tau)=\frac{1}{4}-\frac{\Delta(\tau,\tau)}{\beta}, \tag{68}$$

whose integrals yield

$$\frac{1}{2\beta}\int_0^\beta d\tau\Delta(\tau,\tau)=\int_0^\beta d\tau\,{}^{\cdot}\Delta^{2}(\tau,\tau)=\frac{\beta}{12}. \tag{69}$$

The same first-order diagrams (54) appear in curved space, albeit in different combinations:

$$\frac{1}{6}R\int_0^\beta d\tau\left[\Delta(\tau,\tau){}^{\cdot}\Delta^{\cdot}(\tau,\tau)-{}^{\cdot}\Delta^{2}(\tau,\tau)-\delta(0)\Delta(\tau,\tau)\right], \tag{70}$$

which is evaluated, using the integrals (69), to

$$-\frac{1}{6}R\int_0^\beta d\tau\left[\frac{1}{\beta}\Delta(\tau,\tau)+{}^{\cdot}\Delta^{2}(\tau,\tau)\right]=-\frac{\beta}{24}R. \tag{71}$$

This has to be supplemented by a similar contribution coming from the nonholonomically transformed measure (59). Both terms together yield the first-order DeWitt-Seeley expansion

$$(0\,\beta|0\,0)\equiv\langle e^{\beta D^2/2}\rangle=\frac{1}{\sqrt{2\pi\beta}^{n}}\left(1+\frac{\beta}{12}R\right), \tag{72}$$

in agreement with (65).

We now turn to the evaluation of the second-order diagrams. The sum of the local diagrams (55) is given by

$$\sum(55)=-\frac{3}{2}\varepsilon^2\int_0^\beta d\tau\left[3\Delta^2(\tau,\tau){}^{\cdot}\Delta^{\cdot}(\tau,\tau)+12\Delta(\tau,\tau){}^{\cdot}\Delta^{2}(\tau,\tau)-\delta(0)\Delta^2(\tau,\tau)\right]. \tag{73}$$

Replacing ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ in Eq. (73) again by $\delta(0)-1/\beta$, and taking into account the equality

$$\int_0^\beta d\tau\,\Delta(\tau,\tau)\left[\frac{1}{\beta}\Delta(\tau,\tau)-4{}^{\cdot}\Delta^{2}(\tau,\tau)\right]=0 \tag{74}$$

following from Eq. (68), we find only the divergent term

$$\sum(55)=\varepsilon^2\left[-3\delta(0)\int_0^\beta d\tau\Delta^2(\tau,\tau)\right]=\varepsilon^2\left[-\frac{\beta^3}{10}\,\delta(0)\right]. \tag{75}$$

The sum of all bubbles diagrams (56)–(57) resembles a Russian doll, where the partial sums of different diagrams are embedded into each other. Therefore, we begin the calculation with the sum (56) whose analytic form is

$$\sum(56)=\frac{\varepsilon^2}{2}\int_0^\beta\int_0^\beta d\tau\,d\tau'\left\{2\delta^2(0)\Delta^2(\tau,\tau')\right.$$
$$\left.-4\,\delta(0)\left[\Delta(\tau,\tau){}^{\cdot}\Delta^{2}(\tau,\tau')+4{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau'){}^{\cdot}\Delta(\tau,\tau')+\Delta^2(\tau,\tau'){}^{\cdot}\Delta^{\cdot}(\tau,\tau)\right]\right\}. \tag{76}$$

Inserting Eq. (10) into the last equal-time term, we obtain

$$\begin{aligned}\sum(56) &= \frac{\varepsilon^2}{2}\int_0^\beta\!\int_0^\beta d\tau\, d\tau' \left\{-2\delta^2(0)\Delta^2(\tau,\tau')\right.\\ &\left.-4\delta(0)\left[\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{2}(\tau,\tau') + 4\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta(\tau,\tau') - \Delta^2(\tau,\tau')/\beta\right]\right\}. \qquad (77)\end{aligned}$$

As we shall see below, the explicit evaluation of the integrals in this sum is not necessary. Just for completeness, we give the result:

$$\begin{aligned}\sum(56) &= \frac{\varepsilon^2}{2}\left\{-2\delta^2(0)\frac{\beta^4}{90} - 4\delta(0)\left[\frac{\beta^3}{45} + 4\frac{\beta^3}{180} - \frac{1}{\beta}\cdot\frac{\beta^4}{90}\right]\right\}\\ &= \varepsilon^2\left\{-\frac{\beta^4}{90}\delta^2(0) - \frac{\beta^3}{15}\delta(0)\right\}. \qquad (78)\end{aligned}$$

We now turn to the three-bubbles diagrams (57). Among these, there exist only three involving the correlation function ${}_\mu\Delta_\nu(x,x') \to {}^{\cdot}\Delta^{\cdot}(\tau,\tau')$ for which Eq. (10) is not applicable: the second, fourth, and sixth diagram. The other three-bubble diagrams in (57) containing the generalization ${}_\mu\Delta_\mu(x,x)$ of the equal-time propagator ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ can be calculated using Eq. (10).

Consider first a partial sum consisting of the first, third, and fifth three-bubble diagrams in the sum (57). This has the analytic form

$$\begin{aligned}\sum_{1,3,5}(57) &= \frac{\varepsilon^2}{2}\int_0^\beta\!\int_0^\beta d\tau\, d\tau' \left\{4\,\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{2}(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau')\right.\\ &+\left. 2\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau)\Delta^2(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau') + 16\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau')\right\}. \qquad (79)\end{aligned}$$

Replacing ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ and ${}^{\cdot}\Delta^{\cdot}(\tau',\tau')$ by $\delta(0) - 1/\beta$ we see that Eq. (79) contains, with opposite sign, precisely the previous sum (77) of all one-and two-bubble diagrams. The remainder reads

$$\begin{aligned}\sum_{1,3,5}(57) + \sum(56) &= \frac{\varepsilon^2}{2}\int_0^\beta\!\int_0^\beta d\tau\, d\tau' \left\{-\frac{4}{\beta}\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{2}(\tau,\tau')\right.\\ &\left.+\frac{2}{\beta^2}\Delta^2(\tau,\tau') - \frac{16}{\beta}\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta(\tau,\tau')\right\}. \qquad (80)\end{aligned}$$

and can be evaluated directly to

$$\sum_{1,3,5}(57) + \sum(56) = \frac{\varepsilon^2}{2}\left(-\frac{4}{\beta}\frac{\beta^2}{45} + \frac{2}{\beta^2}\frac{\beta^4}{90} - \frac{16}{\beta}\frac{\beta^3}{180}\right) = \frac{\varepsilon^2}{2}\left(-\frac{7}{45}\beta^2\right). \qquad (81)$$

By the same direct calculation, the Feynman integral in the seventh three-bubble diagram in (57) yields

$$I_7 = \int_0^\beta\!\int_0^\beta d\tau\, d\tau'\; {}^{\cdot}\Delta(\tau,\tau)\,{}^{\cdot}\Delta(\tau,\tau')\,\Delta^{\cdot}(\tau,\tau')\,\Delta^{\cdot}(\tau',\tau') = -\frac{\beta^2}{720}. \qquad (82)$$

The explicit results (81) and (82) are again not needed, since the last term in Eq. (80) is equal, with opposite sign, to the partial sum of the sixth and seventh three-bubble diagrams in Eq. (57). To see this, consider the Feynman integral associated with the sixth three-bubble diagram in Eq. (57):

$$I_6 \;=\; \int_0^\beta\!\int_0^\beta d\tau\, d\tau'\; {}^{\bullet}\Delta\,(\tau,\tau)\Delta(\tau,\tau')\,{}^{\bullet}\Delta^{\bullet}(\tau,\tau')\,\Delta^{\bullet}(\tau',\tau'), \tag{83}$$

whose d-dimensional extension is

$$I_6^d \;=\; \int_0^\beta\!\int_0^\beta d^dx\, d^dx'\; {}_\mu\Delta(x,x)\Delta(x,x')_\mu\Delta_\nu(x,x')\Delta_\nu(x',x'). \tag{84}$$

Adding this to the seventh Feynman integral (82) and performing a partial integration, we find in one dimension

$$\begin{aligned}\sum_{6,7}(57) = \frac{\varepsilon^2}{2}\,16\,(I_6+I_7) \;&=\; \frac{\varepsilon^2}{2}\int_0^\beta\!\int_0^\beta d\tau\, d\tau'\,\frac{16}{\beta}\,{}^{\bullet}\Delta\,(\tau,\tau)\,{}^{\bullet}\Delta\,(\tau,\tau')\Delta(\tau,\tau')\\ &=\; \frac{\varepsilon^2}{2}\left(\frac{4}{45}\beta^2\right),\end{aligned} \tag{85}$$

where we have used $d\left[{}^{\bullet}\Delta\,(\tau,\tau)\right]/d\tau = -1/\beta$ obtained by differentiating (68). Comparing (85) with (80), we find the sum of all bubbles diagrams, except for the second and fourth three-bubble diagrams in Eq. (57), to be given by

$$\sum_{2,4}{}'(57) + \sum(56) = \frac{\varepsilon^2}{2}\left(-\frac{\beta^2}{15}\right). \tag{86}$$

The prime on the sum denotes the exclusion of the diagrams indicated by subscripts. The correlation function ${}^{\bullet}\Delta^{\bullet}(\tau,\tau')$ in the two remaining diagrams of Eq. (57), whose d-dimensional extension is ${}_\mu\Delta_\nu(x,x')$, cannot be replaced via Eq. (10), and the expression can only be simplified by applying partial integration to the fourth diagram in Eq. (57), yielding

$$\begin{aligned} I_4 \;&=\; \int_0^\beta\!\int_0^\beta d\tau\, d\tau'\;\Delta(\tau,\tau)\,{}^{\bullet}\Delta\,(\tau,\tau')\,{}^{\bullet}\Delta^{\bullet}(\tau,\tau')\,\Delta^{\bullet}(\tau',\tau')\\ &\to\; \int_0^\beta\!\int_0^\beta d^dx\, d^dx'\;\Delta(x,x)\,{}_\mu\Delta(x,x')_\mu\Delta_\nu(x,x')\Delta_\nu(x',x')\\ &=\; \frac{1}{2}\int_0^\beta\!\int_0^\beta d^dx\, d^dx'\;\Delta(x,x)\Delta_\nu(x',x')\partial'_\nu\left[{}_\mu\Delta(x,x')\right]^2\\ &\to\; \frac{1}{2}\int_0^\beta\!\int_0^\beta d\tau\, d\tau'\;\Delta(\tau,\tau)\,{}^{\bullet}\Delta\,(\tau',\tau')\frac{d}{d\tau'}\left[{}^{\bullet}\Delta^{\,2}(\tau,\tau')\right]\\ &=\; \frac{1}{2\beta}\int_0^\beta\!\int_0^\beta d\tau\, d\tau'\Delta(\tau,\tau)\,{}^{\bullet}\Delta^{\,2}(\tau,\tau') = \frac{\beta^2}{90}. \end{aligned} \tag{87}$$

The second diagram in the sum (57) diverges linearly. As before, we add and subtract the divergence

$$\begin{aligned} I_2 \;&=\; \int_0^\beta\!\int_0^\beta d\tau\, d\tau'\;\Delta(\tau,\tau)\,{}^{\bullet}\Delta^{\bullet 2}(\tau,\tau')\Delta(\tau',\tau')\\ &=\; \int_0^\beta\!\int_0^\beta d\tau\, d\tau'\;\Delta(\tau,\tau)\left[{}^{\bullet}\Delta^{\bullet 2}(\tau,\tau') - \delta^2(\tau-\tau')\right]\Delta(\tau',\tau')\\ &+\; \int_0^\beta\!\int_0^\beta d\tau\, d\tau'\;\Delta^2(\tau,\tau)\delta^2(\tau-\tau'). \end{aligned} \tag{88}$$

In the first, finite term we go to d dimensions and replace $\delta(\tau-\tau') \to \delta(x-x') = -\Delta_{\nu\nu}(x,x')$ using the field equation (31). After this, we apply partial integration and find

$$\begin{aligned} I_2^R \;\to\; & \int_0^\beta \int_0^\beta d^d x \, d^d x' \, \Delta(x,x) \left[{}_\mu\Delta_\nu^2(x,x') - \Delta_{\lambda\lambda}^2(x,x') \right] \Delta(x',x') \\ =\; & \int_0^\beta \int_0^\beta d^d x \, d^d x' \, \{ -\partial_\mu \left[\Delta(x,x)\right] \Delta_\nu(x,x') \, {}_\mu\Delta_\nu(x,x') \Delta(x',x') \\ & \qquad\qquad + \Delta(x,x)\Delta_\nu(x,x')\Delta_{\lambda\lambda}(x,x')\partial'_\nu \left[\Delta(x',x')\right] \} \\ \to\; & \int_0^\beta \int_0^\beta d\tau \, d\tau' \, 2 \, \{ -{}^{\cdot}\Delta(\tau,\tau) \, \Delta^{\cdot}(\tau,\tau') \, {}^{\cdot}\Delta^{\cdot}(\tau,\tau') \Delta(\tau',\tau') + \\ & \qquad\qquad \Delta(\tau,\tau) \, \Delta^{\cdot}(\tau,\tau') \, {}^{\cdot}\Delta(\tau',\tau') \, \Delta^{\cdot\cdot}(\tau,\tau') \} \,. \end{aligned} \tag{89}$$

In going to the last line we have used $d[\Delta(\tau,\tau)]/d\tau = 2\, {}^{\cdot}\Delta(\tau,\tau)$ following from (68). By interchanging the order of integration $\tau \leftrightarrow \tau'$, the first term in Eq. (89) reduced to the integral (87). In the last term we replace $\Delta^{\cdot\cdot}(\tau,\tau')$ using the field equation (6) and the trivial equation

$$\int d\tau \, \epsilon(\tau)\, \delta(\tau) = 0. \tag{90}$$

Thus we obtain

$$I_2 = I_2^R + I_2^{\rm div} \tag{91}$$

with

$$I_2^R = 2\left(-\frac{\beta^2}{90} - \frac{\beta^2}{120}\right) = \frac{1}{2}\left(-\frac{7\beta^2}{90}\right), \tag{92}$$

$$I_2^{\rm div} = \int_0^\beta \int_0^\beta d\tau d\tau' \Delta^2(\tau,\tau)\delta^2(\tau-\tau'). \tag{93}$$

Using Eqs. (87) and (91) yields the sum of the second and fourth three-bubble diagrams in Eq. (57):

$$\sum_{2,4}(57) = \frac{\varepsilon^2}{2}\left(2I_2 + 16 I_4\right) = \varepsilon^2 \left\{ \int_0^\beta \int_0^\beta d\tau d\tau' \Delta^2(\tau,\tau)\delta^2(\tau-\tau') + \frac{\beta^2}{20} \right\}. \tag{94}$$

Finally, inserting this into Eq. (86), we have the sum of all bubbles diagrams

$$\sum(57) + \sum(56) = \varepsilon^2 \left\{ \int_0^\beta \int_0^\beta d\tau d\tau' \Delta^2(\tau,\tau)\delta^2(\tau-\tau') + \frac{\beta^2}{60} \right\}. \tag{95}$$

Note that the finite part of this is independent of ambiguous integrals of type (26).

The contributions of the watermelon diagrams (58) correspond to the Feynman integrals

$$\begin{aligned} \sum(58) = 2\varepsilon^2 & \int_0^\beta \int_0^\beta d\tau d\tau' \left[\Delta^2(\tau,\tau') \, {}^{\cdot}\Delta^{\cdot 2}(\tau,\tau') \right. \\ & \left. + 4\, \Delta(\tau,\tau') \, {}^{\cdot}\Delta(\tau,\tau') \, \Delta^{\cdot}(\tau,\tau') \, {}^{\cdot}\Delta^{\cdot}(\tau,\tau') + {}^{\cdot}\Delta^{2}(\tau,\tau') \, \Delta^{\cdot 2}(\tau,\tau') \right]. \end{aligned} \tag{96}$$

The third integral is unique and can be calculated directly:

$$I_{10} = \int_0^\beta d\tau \int_0^\beta d\tau' \, {}^{\cdot}\Delta^{2}(\tau,\tau') \, \Delta^{\cdot 2}(\tau,\tau') = \varepsilon^2 \frac{\beta^2}{90}. \tag{97}$$

The second integral reads in d dimensions

$$I_9 \;=\; \int\int d^dx d^dx' \Delta(x,x')_\mu \Delta(x,x') \Delta_\nu(x,x')\,_\mu\Delta_\nu(x,x'). \tag{98}$$

This is integrated partially to yields, in one dimension,

$$I_9 = -\frac{1}{2} I_{10} - \frac{1}{2}\int\int d\tau\, d\tau' \Delta(\tau,\tau')\, {}^{\cdot}\!\Delta^{\cdot\,2}(\tau,\tau')\, {}^{\cdot\cdot}\!\Delta\,(\tau,\tau'). \tag{99}$$

The integral on the right-hand side is the one-dimensional version of

$$I_{9'} \;=\; \int_0^\beta \int_0^\beta d^dx d^dx' \Delta(x,x') \Delta^2_\nu(x,x')\;_{\mu\mu}\Delta(x,x'). \tag{100}$$

Using the field equation (31), going back to one dimension, and inserting $\Delta(\tau,\tau')$, $\Delta^{\cdot}(\tau,\tau')$, and ${}^{\cdot\cdot}\!\Delta\,(\tau,\tau')$ from (8), (9), and (6), we perform all unique integrals and obtain

$$I_{9'} = -\beta^2 \left\{ \frac{1}{24} \int d\tau\, \epsilon^2(\tau)\, \delta(\tau) + \frac{1}{120} \right\}. \tag{101}$$

Inserting this and (97) into Eq. (99) gives, finally,

$$I_9 = \left\{ \frac{1}{48} \int d\tau\, \epsilon^2(\tau)\, \delta(\tau) - \frac{1}{720} \right\} \beta^2. \tag{102}$$

We now evaluate the first integral in Eq. (96). Adding and subtracting the linear divergence yields

$$\begin{aligned} I_8 &= \int_0^\beta \int_0^\beta d\tau\, d\tau'\, \Delta^2(\tau,\tau')\, {}^{\cdot}\!\Delta^{\cdot\,2}(\tau,\tau') \\ &= \int_0^\beta \int_0^\beta d\tau d\tau' \Delta^2(\tau,\tau') \left[{}^{\cdot}\!\Delta^{\cdot\,2}(\tau,\tau') - \delta^2(\tau-\tau') \right] + \varepsilon^2 \int_0^\beta \int_0^\beta d\tau d\tau' \Delta^2(\tau,\tau) \delta^2(\tau-\tau'). \end{aligned} \tag{103}$$

The finite part of the integral (103) has the d-dimensional extension

$$I_8^R \;=\; \int\int d^dx\, d^dx' \Delta^2(x,x') \left[{}_\mu\Delta^2_\nu(x,x') - \Delta^2_{\lambda\lambda}(x,x') \right] \tag{104}$$

which after partial integration and going back to one dimension reduces to a combination of integrals Eqs. (102) and (101):

$$I_8^R = -2I_9 + 2I_{9'} = -\left\{ \frac{1}{8} \int d\tau\, \epsilon^2(\tau) \delta(\tau) + \frac{1}{72} \right\} \beta^2. \tag{105}$$

The divergent part of I_8 coincides with $I_2^{\rm div}$ in Eq. (93):

$$I_8^{\rm div} = \int_0^\beta \int_0^\beta d\tau d\tau' \Delta^2(\tau,\tau) \delta^2(\tau-\tau') = I_2^{\rm div}. \tag{106}$$

Inserting this together with (97) and (102) into Eq. (96), we obtain the sum of watermelon diagrams

$$\begin{aligned} \sum(58) \;&=\; 2\varepsilon^2 (I_8 + 4I_9 + I_{10}) \\ &=\; \varepsilon^2 \left\{ 2 \int_0^\beta \int_0^\beta d\tau\, d\tau' \Delta^2(\tau,\tau) \delta^2(\tau-\tau') - \frac{\beta^2}{12} \int_0^\beta d\tau\, \epsilon^2(\tau) \delta(\tau) - \frac{\beta^2}{60} \right\}. \end{aligned} \tag{107}$$

For a flat space in curvilinear coordinates, the sum of the first-order diagrams vanish. To second order, the requirement of coordinate independence implies the vanishing the sum of all connected diagrams (55)–(58). Setting the sum of Eqs. (75), (95), and (107) to zero leads directly to the integration rule (26) and, in addition, to the rule

$$\int_0^\beta \int_0^\beta d\tau\, d\tau'\, \Delta^2(\tau,\tau)\delta^2(\tau-\tau') = \delta(0) \int d\tau\, \Delta^2(\tau,\tau), \tag{108}$$

which we postulated before in Eq. (20) to cancel the $\delta(0)$s coming from the measure at the one-loop level.

The procedure can easily be continued to higher-loop diagrams to define integrals over higher singular products of ϵ- and δ-functions. In this way we obtain the confirmation of the rule (20). We have seen that at the one-loop level, the cancellation of $\delta(0)$s requires

$$\int d\tau\, \Delta(\tau,\tau)\delta(0) = \delta(0) \int d\tau\, \Delta(\tau,\tau). \tag{109}$$

The second-order equation (108) contains the second power of $\Delta(\tau,\tau)$. To n-order we find the equation

$$\int d\tau_1 \dots d\tau_n \Delta(\tau_1,\tau_2)\delta(\tau_1,\tau_2)\cdots\Delta(\tau_n,\tau_1)\delta(\tau_n,\tau_1) = \delta(0)\int d\tau\, \Delta^n(\tau,\tau). \tag{110}$$

which reduces to

$$\int\int d\tau_1 d\tau_n\, \Delta^n(\tau_1,\tau_1)\delta^2(\tau_1-\tau_n) = \delta(0)\int d\tau\, \Delta^n(\tau,\tau), \tag{111}$$

and this is satisfied given the rule (20). See Appendix C for a general derivation of these rules.

Let us now see what the above integrals imply for the perturbation expansion of the short-time amplitude in curved space in Riemann normal coordinates. Taking into account the nonzeroth contribution (72) of the first-order diagrams reproduces immediately the first term in the second-order operator expansion (64):

$$\frac{1}{2}\langle \mathcal{A}_{\rm int}\rangle^2 = \frac{1}{2}\left(\varepsilon\,\frac{R}{12}\,\beta\right)^2 = \varepsilon^2\frac{R^2}{288}\,\beta^2\,. \tag{112}$$

The sum of the local diagrams (55) involves both tensors R^2_{ij} and R^2_{ijkl}. To order ε^2, we find

$$\sum(55) = -\varepsilon^2\frac{\beta^3}{30}\left(\frac{1}{36}R^2_{ij}+\frac{1}{24}R^2_{ijkl}\right)\delta(0) + \varepsilon^2\frac{\beta^2}{24}\left(\frac{1}{45}R^2_{ij}+\frac{1}{30}R^2_{ijkl}\right). \tag{113}$$

The contribution of all bubbles diagrams (56) and (57) contains only R^2_{ij}:

$$\sum(56)+\sum(57) = \varepsilon^2\frac{\beta^3}{1080}\,R^2_{ij}\,\delta(0) - \varepsilon^2\frac{\beta^2}{432}\,R^2_{ij}. \tag{114}$$

This compensates exactly the $\delta(0)$-term proportional to R^2_{ij} in Eq. (113) and yields correctly the third second-order term $-R^2_{ij}/720$ in the operator expansion (64).

Before turning to the contribution of the second-order watermelon diagrams (58) which contain initially ambiguous Feynman integrals we make an important observation. Comparison with Eq. (64) shows that Eq. (113) contains already the correct part of the second-order

DeWitt-Seeley coefficient $R_{ijkl}^2/720$. Therefore, the only role of contributions of the watermelon diagrams (58) which are proportional to R_{ijkl}^2 must be to cancel a corresponding divergent part of the sum (113). In fact, the sum of the second-order watermelon diagrams (58) reads now,

$$\sum(58) = \frac{\varepsilon^2}{24} R_{ijkl}^2 \left(I_8 - 2I_9 + I_{10}\right\}, \tag{115}$$

where the integrals I_8, I_9, and I_{10} are given before in Eqs. (106), (105), (102), and (97). Substituting these into Eq. (115) and using the rules (20) and (26), we obtain

$$\sum(58) = \frac{\varepsilon^2}{24} R_{ijkl}^2 \int_0^\beta \int_0^\beta d\tau\, d\tau' \Delta^2(\tau,\tau) \delta^2(\tau-\tau') = \varepsilon^2 \frac{\beta^3}{720} R_{ijkl}^2\, \delta(0), \tag{116}$$

thus compensating the $\delta(0)$-term proportional to R_{ijkl}^2 in Eq. (113) and no finite contribution.

For one-component target space as well as for n-component curved space in normal coordinates, our calculation procedure using only the essence of the d-dimensional extension together with the rules (20) and (26) yields unique results which guarantee the coordinate independence of path integrals and agrees with the DeWitt-Seeley expansion of the short-time amplitude. The need for this agreement fixes the initially ambiguous integrals I_8 and I_9 to satisfy the equations

$$I_8^R + 4I_9 + I_{10} = -\frac{\beta^2}{120}, \tag{117}$$

$$I_8^R - 2I_9 + I_{10} = 0, \tag{118}$$

as we can see from Eqs. (107) and (115). Since the integral $I_{10} = \beta^2/90$ is unique, we must have $I_9 = -\beta^2/720$ and $I_8^R = -\beta^2/72$, and this is what our integration rules indeed gave us.

The main role of the dimensional extension in this context is to forbid the application of Eq. (10) to correlation functions $\dot{\Delta}\dot{}(\tau,\tau')$. This would have immediately fixed the finite part of the integral I_8 to the wrong value $I_8^R = -\beta^2/18$, leaving only the integral I_9 which would define the integral over distributions (26). In this way, however, we could only satisfy one of the equations (117) and (118), the other would always be violated. Thus, any regularization different from ours will ruin immediately coordinate independence.

It must be noted that if we were to use arbitrary rather than Riemann normal coordinates, one can fix ambiguous integrals already at the two-loop level, and obtains the conditions (13). Thus, although the calculation in normal coordinates are simpler and can be carried more easily to higher orders, the perturbation in arbitrary coordinates help to fix more ambiguous integrals.

Let us finally compare our procedure with the previous discussion of the same problem by F. Bastianelli, P. van Nieuwenhuizen, and others in Refs. [7–15]. Those authors suggested for almost ten years two regularization schemes for perturbative calculation on a finite-time interval: mode regularization (MR) [9–11] and time discretization (TS) [11–13]. They gave a detailed comparison of both schemes up to three loops in Ref. [14]. Their main goal was to calculate of trace anomalies of quantum field theory by means of path integrals [9, 13, 15]. From the present point of view of extended distribution theory, mode regularization (MR) amounts to setting

$$\int d\tau\, \epsilon^2(\tau)\, \delta(\tau) \equiv \frac{1}{3}. \tag{119}$$

With this rule, the ambiguous integrals I_8 and I_9 yield $I_8^R = -\beta^2/18$, $I_9 = \beta^2/180$. However, these values do not allow for coordinate independence, nor do they lead to the correct short-time DeWitt-Seeley expansion of the amplitudes. This is what forced the authors to add an unpleasant noncovariant "correction term" $\mathcal{A}^{\rm fudge} = -\int d\tau \Gamma^i_{jk}\Gamma^l_{mn} g_{il} g^{im} g^{kn}/24$ to the classical action, in violation of Feynman's construction rules for path integrals. In doing this they followed earlier work by Salomonson in Ref. [8].

Their time discretization scheme (TS), on the other hand, amounts to setting

$$\int d\tau \epsilon^2(\tau)\,\delta(\tau) = 0. \tag{120}$$

They applied this to purely one-dimensional calculations which, as we have shown in this paper, leads to the contradictory results depending on where partial integration or field equations are used. While I_8 is again $I_8^R = -\beta^2/18$, the result for $I_9 = 7\beta^2/360$ is not unique. To obtain coordinate independence as well as the correct DeWitt-Seeley expansion, they had now to add another noncovariant "correction term" $\mathcal{A}^{\rm fudge} = \int d\tau \Gamma^i_{jk}\Gamma^j_{il} g^{kl}/8$, thereby following the original work of Gervais and Jevicki in Ref. [7].

In recent papers [16–18], the authors of Refs. [10] and [14] have begun following our method of dimensional regularization [2,3], adapting it to a finite time interval in [17]. They now obtain, of course, correct coordinate-independent results without noncovariant additional terms in the action. They do not, however, exhibit the precise location of ambiguities as we did here, and most importantly, they do not derive from their results rules for integrating products of ϵ- and δ-functions, which are central to the present paper. In particular, they do not realize that dimensional regularization amounts to the integration rule Eq. (120).

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Appendix A: Integrals I_{14} and I_{15}^R from two-loop expansion in arbitrary coordinates. – To order ε, the metric and its determinant have the expansions:

$$\begin{aligned} g_{ij}(q) &= \delta_{ij} + \sqrt{\varepsilon}(\partial_k g_{ij})q^k + \varepsilon\frac{1}{2}(\partial_l\partial_k g_{ij})q^l q^k, \\ \log g(q) &= \sqrt{\varepsilon} g^{ij}(\partial_k g_{ij})q^k + \varepsilon\frac{1}{2}\, g^{ij}\,[(\partial_l\partial_k g_{ij}) - g^{mn}(\partial_l g_{im})(\partial_k g_{jn})]q^l q^k. \end{aligned} \tag{121}$$

The interaction (47) becomes

$$\begin{aligned} \mathcal{A}_{\rm int}[q] = \int_0^\beta d\tau \Big\{ &\Big[\frac{1}{2}\sqrt{\varepsilon}\,(\partial_k g_{ij})q^k + \frac{1}{4}\varepsilon\,(\partial_l\partial_k g_{ij})q^l q^k\Big]\dot q^i \dot q^j \\ &-\frac{1}{2}\sqrt{\varepsilon}\,\delta(0) g^{ij}(\partial_k g_{ij})q^k - \frac{1}{4}\varepsilon\,\delta(0)\,g^{ij}\Big[(\partial_l\partial_k g_{ij}) - g^{mn}\,(\partial_l g_{im})(\partial_k g_{jn})\Big]q^l q^k\Big\}. \end{aligned} \tag{122}$$

To the first-order in ε, the perturbation expansion (48) with the interaction (122) consist of two sets of diagrams proportional to $\Gamma_{ij,\,k}$ and $\Gamma^2_{ij,\,k}$, respectively. First, there are the same local diagrams as in Eq. (54): the first two local diagrams, coming from the kinetic term in Eq. (122), carry a factor $\Gamma_{ij,\,k}$, while the last local diagram, coming from the measure in Eq. (122), involves both factors $\Gamma_{ij,\,k}$ and $\Gamma^2_{ij,\,k}$. Omitting the $\Gamma^2_{ij,\,k}$-part of the last diagram, the terms linear in the Christoffel symbol $\Gamma_{ij,\,k}$ coming from the sum of local diagrams in (54) reads

$$\sum(54) = -\frac{\varepsilon}{4}(\partial_l\partial_k g_{ij})\int_0^\beta d\tau\Big\{g^{ij}g^{kl}\,{}^{\bullet}\Delta^{\bullet}(\tau,\tau)\Delta(\tau,\tau) + 2g^{ik}g^{jl}\ {}^{\bullet}\Delta^{2}(\tau,\tau) - \delta(0)g^{ij}g^{kl}\Delta(\tau,\tau)\Big\}$$
$$= \beta\frac{\varepsilon}{24}(\partial_l\partial_k g_{ij})(g^{ij}g^{kl} - g^{ik}g^{jl}) = \beta\frac{\varepsilon}{24}g^{ij}g^{kl}(\partial_l\Gamma_{ik,\,j} - \partial_i\Gamma_{lk,\,j}). \tag{123}$$

In addition, the interaction (122) generates also nonlocal first-order diagrams proportional to $\Gamma^2_{ij,\,k}$. Together with nonlinear in Christoffel symbol part of the last local diagram in Eq. (54), they are represented as follows

$\frac{\varepsilon}{2}g^{lk}\,\Gamma_{li}{}^{i}\,\Gamma_{kj}{}^{j}\,[$  $-\,2\delta(0)$  $+\ \delta^2(0)$ $]$

$+\,\varepsilon\Gamma_{li}{}^{i}\,(g^{lk}\,\Gamma_{kj}{}^{j} + g^{jk}\,\Gamma_{jk}{}^{l})\,[$  $-\ \delta(0)$  $]$

$+\,\frac{\varepsilon}{2}(g^{il}\,g^{kn}\,\Gamma_{il}{}^{j}\,\Gamma_{kn,\,j} + g^{ij}\,\Gamma_{ik}{}^{k}\,\Gamma_{jl}{}^{l} + 2g^{ij}\,\Gamma_{ij}{}^{k}\,\Gamma_{kl}{}^{l})$ 

$+\,\frac{\varepsilon}{2}(g^{ik}\,g^{jl}\,\Gamma_{il}{}^{n}\,\Gamma_{kj,\,n} + 3g^{ik}\,\Gamma_{il}{}^{n}\,\Gamma_{nk}{}^{l})$ 

$+\,\frac{\varepsilon}{2}g^{lk}\,(\Gamma_{lj}{}^{i}\,\Gamma_{ik}{}^{j} + g^{in}\,\Gamma_{nk}{}^{j}\,\Gamma_{il,\,j})\,[$  $-\ \delta(0)$  $]$ (124)

The Feynman integrals associated with the diagrams in the first and second lines of Eq. (124) read

$$I_{11} = \int\!\!\int d\tau\, d\tau'\ \big\{{}^{\bullet}\Delta^{\bullet}(\tau,\tau)\Delta(\tau,\tau')\,{}^{\bullet}\Delta^{\bullet}(\tau',\tau') - 2\delta(0)\,{}^{\bullet}\Delta^{\bullet}(\tau,\tau)\,\Delta(\tau,\tau') + \delta^2(0)\,\Delta(\tau,\tau')\big\} \tag{125}$$

and

$$I_{12} = \int\!\!\int d\tau\, d\tau'\ \{{}^{\bullet}\Delta\,(\tau,\tau)\,{}^{\bullet}\Delta\,(\tau,\tau')\,{}^{\bullet}\Delta^{\bullet}(\tau',\tau') - \delta(0)\,{}^{\bullet}\Delta\,(\tau,\tau)\,{}^{\bullet}\Delta\,(\tau,\tau')\}\,, \tag{126}$$

respectively. Replacing in Eqs. (125) and (126) ${}^{\bullet}\Delta^{\bullet}(\tau,\tau)$ and ${}^{\bullet}\Delta^{\bullet}(\tau',\tau')$ by $\delta(0) - 1/\beta$ leads to cancellation of the infinite factors $\delta(0)$ and $\delta^2(0)$ coming from the measure, such that we are left with

$$I_{11} = \frac{1}{\beta^2}\int_0^\beta d\tau\int_0^\beta d\tau'\Delta(\tau,\tau') = \frac{\beta}{12} \tag{127}$$

and

$$I_{12} = -\frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau'\ {}^{\bullet}\Delta\,(\tau,\tau)\,{}^{\bullet}\Delta\,(\tau,\tau') = -\frac{\beta}{12}. \tag{128}$$

The Feynman integral of the diagram in the third line of Eq. (124) has d-dimensional extension

$$I_{13} = \int\!\!\int d\tau\, d\tau'\ {}^{\bullet}\Delta\,(\tau,\tau)\,\Delta^{\bullet}(\tau',\tau')\,{}^{\bullet}\Delta^{\bullet}(\tau,\tau') \to \int\!\!\int d^dx\, d^dx'\,{}_{\mu}\Delta(x,x)\Delta_{\nu}(x',x')_{\mu}\Delta_{\nu}(x,x'). \tag{129}$$

Integrating this partially yields

$$I_{13} = \frac{1}{\beta}\int\!\!\int d\tau\, d\tau'\ \Delta^{\bullet}(\tau,\tau')\,{}^{\bullet}\Delta\,(\tau',\tau') = \frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau'\ {}^{\bullet}\Delta\,(\tau,\tau)\,{}^{\bullet}\Delta\,(\tau,\tau') = \frac{\beta}{12}, \tag{130}$$

where we have interchanged the order of integration $\tau \leftrightarrow \tau'$ in the second line of Eq. (130) and used $d[{}^{\cdot}\Delta(\tau,\tau)]/d\tau = -1/\beta$. Multiplying the integrals (127), (128), and (130) by corresponding vertices in Eq. (124) and adding them together, we obtain

$$\sum_{1,2,3}(124) = \frac{\varepsilon\beta}{24}\, g^{ij} g^{kl}\, \Gamma_{ij}{}^{n}\, \Gamma_{kl,\,n}. \tag{131}$$

The contributions of the last three diagrams in the fourth and the fifth line of Eq. (124) correspond to the ambiguous integrals (11) and (12), respectively. Moreover, the difference of two diagrams in the last line of Eq. (124) contains only the finite part of the integral (12), since its divergent part (21) is canceled by the contribution of the local diagram with the factor $\delta(0)$. Multiplying these integrals by corresponding vertices in Eq. (124) yields the sum of diagrams in the fourth and the fifth line of Eq. (124) as follows

$$\sum_{4,5}(124) = \frac{\varepsilon}{2}\left\{ g^{ik}\, g^{jl}\, \Gamma_{il}{}^{n}\, \Gamma_{kj,\,n}\ \left(I_{14} + I_{15}^{R}\right) + g^{lk}\, \Gamma_{lj}{}^{i}\, \Gamma_{ik}{}^{j}\ \left(3I_{14} + I_{15}^{R}\right) \right\}. \tag{132}$$

On the other hand, to guarantee the coordinate independence of path integrals, this sum must be

$$\sum_{4,5}(124) = -\frac{\varepsilon\beta}{24}\, g^{ij} g^{kl}\, \Gamma_{ik}{}^{n}\, \Gamma_{jl,\,n}. \tag{133}$$

Adding this to (131), we find the sum of all diagrams in (124) as follows

$$\sum(124) = \frac{\varepsilon\beta}{24}\, g^{ij} g^{kl}\, \left(\Gamma_{ij}{}^{n}\Gamma_{kl,\,n} - \Gamma_{ik}{}^{n}\Gamma_{jl,\,n}\right). \tag{134}$$

Together with the sum over all diagrams in (54) calculated in (123) this yields, finally, the sum of all first-order diagrams

$$\sum(54) + \sum(124) = \frac{\varepsilon\beta}{24}\, g^{ij} g^{kl}\, R_{likj} = -\frac{\varepsilon\beta}{24}\, R. \tag{135}$$

The result is perfectly covariant and agrees, of course, with Eq. (71) derived in normal coordinate. Comparing now Eq. (132) with (133), we find

$$\begin{aligned} I_{14} + I_{15}^{R} &= -\frac{\beta}{12}\,, \\ 3I_{14} + I_{15}^{R} &= 0\,. \end{aligned} \tag{136}$$

Thus, coordinate independence specifies the initially ambiguous integrals (11) and (12) to have indeed the values (13).

Appendix B: Operator derivation of short-time DeWitt-Seeley expansion. – Here we give a short derivation of the DeWitt-Seeley expansion (63). In a neighborhood of some arbitrary point q_0^i we expand the Laplace-Beltrami operator in normal coordinate system (61) as

$$D^2 = \partial^2 - \frac{1}{3} R_{ik_1jk_2}(q_0)(q-q_0)^{k_1}(q-q_0)^{k_2}\partial_i\partial_j - \frac{2}{3} R_{ij}(q_0)(q-q_0)^{i}\partial_j. \tag{137}$$

To find the coefficients $a_k(q,q')$ in Eq. (63), we resort to perturbation theory. The time displacement operator $H = -D^2/2$ in the exponent of Eq. (63) is separated into a free part

H_0 and an interaction part $H_{\rm int}$ as follows

$$H_0 = -\frac{1}{2}\partial^2, \tag{138}$$

$$H_{\rm int} = \frac{1}{6}R_{ik_1jk_2}(q-q_0)^{k_1}(q-q_0)^{k_2}\partial_i\partial_j + \frac{1}{3}R_{ij}(q-q_0)^i\partial_j. \tag{139}$$

The transition amplitude (63) satisfies the integral equation

$$\begin{aligned}(q,\beta\mid q',0) &= \langle q\mid e^{-\beta(H_0+H_{\rm int})}\mid q'\rangle = \langle q\mid e^{-\beta H_0}\left[1-\int_0^\beta d\sigma e^{\sigma H_0}H_{\rm int}e^{-\sigma H}\right]\mid q'\rangle \\ &= (q,\beta\mid q',0)_0 - \int_0^\beta d\sigma\int d^n\bar{q}\,(q,\beta-\sigma\mid\bar{q},0)_0\,H_{\rm int}(\bar{q})\,(\bar{q},\sigma\mid q,0),\end{aligned} \tag{140}$$

where

$$(q,\beta\mid q',0)_0 = \langle q\mid e^{-\beta H_0}\mid q'\rangle = \frac{1}{\sqrt{2\pi\beta}^n}\,e^{-(\Delta q)^2/2\beta}. \tag{141}$$

To first order in $H_{\rm int}$ we obtain

$$(q,\beta\mid q',0) = (q,\beta\mid q',0)_0 - \int_0^\beta d\sigma\int d^n\bar{q}\;(q,\beta-\sigma\mid\bar{q},0)_0\,H_{\rm int}(\bar{q})\,(\bar{q},\sigma\mid q,0)_0. \tag{142}$$

Inserting (139) and choosing $q_0 = q'$, we find

$$\begin{aligned}(q,\beta\mid q',0) &= (q,\beta\mid q',0)_0\left\{1+\int_0^\beta d\sigma\int\frac{d^n(\Delta\bar{q})}{\sqrt{2\pi a}^n}\,e^{-[\Delta\bar{q}-(\sigma/\beta)\,\Delta q]^2/2a}\right. \\ &\times\left.\left[-\frac{1}{6}R_{ik_1jk_2}\Delta\bar{q}^{k_1}\Delta\bar{q}^{k_2}\left(-\frac{\delta^{ij}}{\sigma}+\frac{\Delta\bar{q}^i\Delta\bar{q}^j}{\sigma^2}\right)+\frac{1}{3}R_{ij}\frac{\Delta\bar{q}^i\Delta\bar{q}^j}{\sigma}\right]\right\},\end{aligned} \tag{143}$$

where we have replaced the integrating variable $\bar{q}$ by $\Delta\bar{q} = \bar{q}-q'$ and used the notation $a = (\beta-\sigma)\sigma/\beta$. There is initially also a term of fourth order in $\Delta\bar{q}$ which vanishes, however, because of the antisymmetry of R_{ikjl} in ik and jl. The remaining Gaussian integrals are performed after shifting $\Delta\bar{q}\to\Delta\bar{q}+\sigma\,\Delta q/\beta$, and we obtain

$$\begin{aligned}(q,\beta\mid q',0) &= (q,\beta\mid q',0)_0\Big\{1+\frac{1}{6}\int_0^\beta d\sigma\Big[\frac{\sigma}{\beta^2}R_{ij}(q')\Delta q^i\Delta q^j+\frac{a}{\sigma}R(q')\Big]\Big\} \\ &= (q,\beta\mid q',0)_0\Big[1+\frac{1}{12}R_{ij}(q')\Delta q^i\Delta q^j+\frac{\beta}{12}R(q')\Big].\end{aligned} \tag{144}$$

Note that all geometrical quantities are evaluated at the initial point q'. They can be re-expressed in power series around the final position q using the fact that in normal coordinates

$$g_{ij}(q') = g_{ij}(q)+\frac{1}{3}R_{ik_1jk_2}(q)\Delta q^{k_1}\Delta q^{k_2}+\dots\ , \tag{145}$$

$$g_{ij}(q')\Delta q^i\Delta q^j = g_{ij}(q)\Delta q^i\Delta q^j, \tag{146}$$

the latter equation being true to all orders in Δq due to the antisymmetry of the tensors R_{ijkl} in all terms of the expansion (145), which is just another form of writing the expansion (60).

Going back to the general coordinates, we obtain all coefficients of the expansion (63) linear in the curvature tensor

$$(q,\beta \mid q',0) \simeq \frac{1}{\sqrt{2\pi\beta}^{n}}\, e^{-g_{ij}(q)\Delta q^i \Delta q^j/2\beta}\Big[1+\frac{1}{12}R_{ij}(q)\Delta q^i\Delta q^i+\frac{\beta}{12}R(q)\Big]. \tag{147}$$

The higher terms in (63) can be derived similarly, although with much more effort.

A simple cross check of the expansion (63) to high orders is possible if we restrict the space to a sphere of radius r in D dimensions. Then

$$R_{ijkl} = -\frac{1}{r^2}\,(g_{ik}\,g_{jl} - g_{il}\,g_{jk})\,, \quad i,j=1,2,\dots,n=D-1, \tag{148}$$

where $n=2$ is dimension of a sphere, and $D=3$ is dimension of a flat embedding space, respectively. Contractions yield Ricci tensor and scalar curvature

$$R_{ij} \;=\; R_{kij}{}^{k} = \frac{D-2}{r^2}\,g_{ij}, \qquad R = R_i{}^{i} = \frac{(D-1)(D-2)}{r^2} \tag{149}$$

and further:

$$R^2_{ijkl} = \frac{2(D-1)(D-2)}{r^4}, \qquad R^2_{ij} = \frac{(D-1)(D-2)^2}{r^4}. \tag{150}$$

Inserting these into (64), we obtain the DeWitt-Seeley short-time expansion of the amplitude from $q=0$ to $q=0$ up to prder β^2:

$$(0,\beta \mid 0,0) = \frac{1}{\sqrt{2\pi\beta}^{D-1}}\left[1+(D-1)(D-2)\frac{\beta}{12r^2}+(D-1)(D-2)(5D^2-17D+18)\frac{\beta^2}{1440r^4}\right]. \tag{151}$$

On the other hand, we may follow Ref. [22], and calculate explicitly the partition function for this system

$$Z(\beta) = \sum_{l=0}^{\infty} d_l\, \exp[-l(l+D-2)\beta/2r^2]\,, \tag{152}$$

where $-l(l+D-2)$ are the eigenvalues of the Laplace-Beltrami operator on a sphere and $d_l=(2l+D-2)(l+D-3)!/l!(D-2)!$ their degeneracies. Since the space is homogeneous, the amplitude $(0,\beta \mid 0,0)$ is obtained from this by dividing out the constant surface of a sphere:

$$(0,\beta \mid 0,0) = \frac{\Gamma(D/2)}{2\pi^{D/2}r^{D-1}}Z(\beta). \tag{153}$$

For any given D, the sum in (152) easily be expanded in powers of β. As an example, take $D=3$ where

$$Z(\beta) = \sum_{l=0}^{\infty}(2l+1)\,\exp[-l(l+1)\beta/2r^2]\,. \tag{154}$$

In the small β-limit, the sum (154) is evaluated as follows

$$Z(\beta) = \int_0^{\infty} d\,[l(l+1)]\,\exp[-l(l+1)\beta/2r^2] + \sum_{l=0}^{\infty}(2l+1)\,\big[1-l(l+1)\beta/2r^2+\dots\big]\,. \tag{155}$$

The integral is immediately done and yields

$$\int_0^\infty dz\, \exp(-z\beta/2r^2) = \frac{2r^2}{\beta}\,. \tag{156}$$

The sums are divergent but can be evaluated by analytic continuation from negative powers of l to positive ones with the help of Riemann zeta functions $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$, which vanishes for all even negative arguments. Thus we find

$$\sum_{l=0}^{\infty}(2l+1) \;=\; 1+\sum_{l=1}^{\infty}(2l+1) = 1+2\zeta(-1)-\frac{1}{2} = \frac{1}{3}\,, \tag{157}$$

$$-\frac{\beta}{2r^2}\sum_{l=0}^{\infty}(2l+1)l(l+1) \;=\; -\frac{\beta}{2r^2}\sum_{l=1}^{\infty}(2l^3+l) = -\frac{\beta}{2r^2}\left[2\zeta(-3)+\zeta(-1)\right] = \frac{\beta}{30r^2}\,. \tag{158}$$

Substituting these into (155), we find

$$Z(\beta) \;=\; \frac{2r^2}{\beta}\left(1+\frac{\beta}{6r^2}+\frac{\beta^2}{60r^4}+\dots\right). \tag{159}$$

Dividing out the constant surface of a sphere $4\pi r^2$ as required by Eq. (153), we obtain indeed the expansion (151) for $D = 3$.

Appendix C: Cancellation of all powers of $\delta(0)$. – There is a simple way of proving the cancellation of all UV-divergences $\delta(0)$. Consider a free particle whose mass depends on the time with an action

$$\mathcal{A}_{\rm tot}[q] = \int_0^\beta d\tau\left[\frac{1}{2}Z(\tau)\dot{q}^2(\tau)-\frac{1}{2}\delta(0)\log Z(\tau)\right], \tag{160}$$

where $Z(\tau)$ is some function of τ but independent now of the path $q(\tau)$. The last term is the simplest nontrivial form of the Jacobian action in (45). Since it is independent of q, it is conveniently taken out of the path integral as a factor

$$J = e^{(1/2)\delta(0)\int_0^\beta d\tau \log Z(\tau)}. \tag{161}$$

We split the action into a sum of a free and an interacting part

$$\mathcal{A}_0 = \int_0^\beta d\tau\,\frac{1}{2}\dot{q}^2(\tau), \qquad \mathcal{A}_{\rm int} = \int_0^\beta d\tau\,\frac{1}{2}\left[Z(\tau)-1\right]\dot{q}^2(\tau), \tag{162}$$

and calculate the transition amplitude (48) as a sum of all connected diagrams in the cumulant expansion

$$\begin{aligned}\langle 0,\beta|0,0\rangle \;&=\; J\int\mathcal{D}q(\tau)e^{-\mathcal{A}_0[q]-\mathcal{A}_{\rm int}[q]} = J\int\mathcal{D}q(\tau)e^{-\mathcal{A}_0[q]}\left(1-\mathcal{A}_{\rm int}+\frac{1}{2}\mathcal{A}_{\rm int}^2-\dots\right)\\ &=\; (2\pi\beta)^{-1/2}J\left[1-\langle\mathcal{A}_{\rm int}\rangle+\frac{1}{2}\langle\mathcal{A}_{\rm int}^2\rangle-\dots\right]\\ &=\; (2\pi\beta)^{-1/2}J\,e^{-\langle\mathcal{A}_{\rm int}\rangle_c+\frac{1}{2}\langle\mathcal{A}_{\rm int}^2\rangle_c-\dots}.\end{aligned} \tag{163}$$

We now show that the infinite series the of $\delta(0)$-powers appearing in a Taylor expansion of the exponential (161) is precisely compensated by the sum of all terms in the perturbation expansion (163). Being interested only in these singular terms, we may extend the τ-interval to the entire time axis. Then Eq. (10) yields the propagator ${}^{\cdot}\Delta^{\cdot}(\tau,\tau')=\delta(\tau-\tau')$, and we find the first-order expansion term

$$\langle \mathcal{A}_{\rm int}\rangle_c=\int d\tau\,\frac{1}{2}[Z(\tau)-1]\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau)=-\frac{1}{2}\delta(0)\int d\tau\,[1-Z(\tau)]. \tag{164}$$

To second order, divergent integrals appear involving products of distributions, thus requiring an intermediate extension to d dimensions as follows

$$\begin{aligned}\langle \mathcal{A}^2_{\rm int}\rangle_c &= \int\!\!\int d\tau_1\,d\tau_2\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,2\,{}^{\cdot}\Delta^{\cdot}(\tau_1,\tau_2)\,{}^{\cdot}\Delta^{\cdot}(\tau_2,\tau_1)\\ &\to \int\!\!\int d^dx_1\,d^dx_2\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,2\,{}_{\mu}\Delta_{\nu}(x_1,x_2)\,{}_{\nu}\Delta_{\mu}(x_2,x_1)\\ &= \int\!\!\int d^dx_1\,d^dx_2\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,2\,\Delta_{\mu\mu}(x_2,x_1)\,\Delta_{\nu\nu}(x_1,x_2)\,, \end{aligned} \tag{165}$$

the last line following from partial integrations. For brevity, we have abbreviated $[1-Z(\tau_i)]$ by $(1-Z)_i$. Using the field equation (31) and going back to one dimension yields

$$\langle \mathcal{A}^2_{\rm int}\rangle_c=\frac{1}{2}\int\!\!\int d\tau_1\,d\tau_2\,(1-Z)_1\,(1-Z)_2\,\delta^2(\tau_1,\tau_2). \tag{166}$$

To third order we calculate

$$\begin{aligned}\langle \mathcal{A}^3_{\rm int}\rangle_c &= \int\!\!\int\!\!\int d\tau_1\,d\tau_2\,d\tau_3\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,\frac{1}{2}(Z-1)_3\,8\,{}^{\cdot}\Delta^{\cdot}(\tau_1,\tau_2)\,{}^{\cdot}\Delta^{\cdot}(\tau_2,\tau_3)\,{}^{\cdot}\Delta^{\cdot}(\tau_3,\tau_1)\\ &\to \int\!\!\int\!\!\int d^dx_1\,d^dx_2\,d^dx_3\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,\frac{1}{2}(Z-1)_3\,8\,{}_{\mu}\Delta_{\nu}(x_1,x_2)\,{}_{\nu}\Delta_{\sigma}(x_2,x_3)\,{}_{\sigma}\Delta_{\mu}(x_3,x_1)\\ &= -\int\!\!\int\!\!\int d^dx_1\,d^dx_2\,d^dx_3\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,\frac{1}{2}(Z-1)_3\,8\,\Delta_{\mu\mu}(x_3,x_1)\,\Delta_{\nu\nu}(x_1,x_2)\,\Delta_{\sigma\sigma}(x_2,x_3). \end{aligned} \tag{167}$$

Applying again the field equation (31) and going back to one dimension, this reduces to

$$\langle \mathcal{A}^3_{\rm int}\rangle_c=-\int\!\!\int\!\!\int d\tau_1\,d\tau_2\,d\tau_3\,(1-Z)_1\,(1-Z)_2\,(1-Z)_3\delta(\tau_1,\tau_2)\,\delta(\tau_2,\tau_3)\,\delta(\tau_3,\tau_1). \tag{168}$$

Continuing to n-order and substituting Eqs. (164), (166), (168), etc. into (163), we obtain in the exponent of Eq. (163) as sum

$$-\langle \mathcal{A}_{\rm int}\rangle_c+\frac{1}{2}\langle \mathcal{A}^2_{\rm int}\rangle_c-\frac{1}{3!}\langle \mathcal{A}^3_{\rm int}\rangle_c+\,\dots\,=\frac{1}{2}\sum_1^{\infty}\frac{c_n}{n}, \tag{169}$$

with

$$c_n=\int d\tau_1\dots d\tau_n\,C(\tau_1,\tau_2)\,C(\tau_2,\tau_3)\,\dots\,C(\tau_n,\tau_1) \tag{170}$$

where

$$C(\tau,\tau')=[1-Z(\tau)]\,\delta(\tau,\tau'). \tag{171}$$

Substituting this into Eq. (170) and using the rule (20) yields

$$c_n=\int\!\!\int d\tau_1 d\tau_n\,[1-Z(\tau_1)]^n\,\delta^2(\tau_1-\tau_n)=\delta(0)\int d\tau\,[1-Z(\tau)]^n. \tag{172}$$

Inserting these numbers into the expansion (169), we obtain

$$-\langle \mathcal{A}_{\rm int}\rangle_c + \frac{1}{2}\langle \mathcal{A}_{\rm int}^2\rangle_c - \frac{1}{3!}\langle \mathcal{A}_{\rm int}^3\rangle_c + \dots = \frac{1}{2}\,\delta(0)\int d\tau \sum_1^\infty \frac{[1-Z(\tau)]^n}{n} = -\frac{1}{2}\,\delta(0)\int d\tau \,\log Z(\tau), \qquad (173)$$

which compensates precisely the Jacobian factor J in (163).

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See Eq. (27).