Option Pricing from Path Integral for Non-Gaussian Fluctuations. Natural Martingale and Application to Truncated Lèvy Distributions*

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Within a path integral formalism for non-Gaussian price fluctuations we set up a simple stochastic calculus and *derive* a natural martingale for option pricing from the wealth balance of options, stocks, and bonds. The resulting formula is evaluated for truncated Lèvy distributions.

I. INTRODUCTION

As pointed out in Pareto's 19th century work [1] and reemphasized by Mandelbrot in the 1960s, the logarithms of assets prices in financial markets do not fluctuate with Gaussian distributions, but possess much larger tails which may be approximated by various other distributions such as truncated Lévy distributions [2–5], Meixner distributions [6,7], generalized hyperbolic distributions or simplified versions thereof [8–32]. This has the unpleasant consequence that the associated stochastic differential equations cannot be treated with the popular Ito calculus. The mathematicians call such markets *incomplete*, implying that there are different choices of martingale distributions with which one can calculate option prices. Many of these have been discussed in the literature, and mathematicians have invented various sophisticated criteria under which one would be preferable over the others for calculating financial risks. Davis, for instance, has introduced a so-called *utility function* [33] which is supposed to select optimal martingales for different purposes.

In this paper we want to point out that in a path integral formulation of the problem, a straightforward extension of the old chain of arguments which led Black-Scholes to their famous formula produces a specific simple martingale which is different from presently popular versions based on Esscher transforms. Before we come to this we briefly review the derivation of a stochastic calculus for non-Gaussian processes [34] from path integrals which extends Ito's calculus in the simplest way.

II. GAUSSIAN APPROXIMATION TO FLUCTUATION PROPERTIES OF STOCK PRICES

Let $S(t)$ denote the price of some stock. On the average, stocks grow exponentially, and as a first approximation, one may consider a stock price as an exponential $S(t) = e^{x(t)}$ of a fluctuating variable $x(t)$ which obeys a stochastic differential equation

$$\dot{x}(t) = r_x + \eta(t), \tag{1}$$

where $\eta(t)$ is a white noise of unit strength with the correlation functions

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \sigma^2 \delta(t - t'). \tag{2}$$

The stock price $S(t)$ itself satisfies the stochastic differential equation for exponential growth

$$\frac{\dot{S}(t)}{S(t)} = r_S + \eta(t), \tag{3}$$

where The standard deviation σ is a measure for the volatility of the stock price, which defined by the expectation value

$$\left\langle \left[\frac{\dot{S}(t)}{S(t)} \right]^2 \right\rangle = \sigma^2. \tag{4}$$

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The rate constants in Eqs. (1) and (3) differ from one another due the stochastic nature of $x(t)$ and $S(t)$. According to Ito's rule, we may expand

$$\begin{aligned}\dot{x}(t) &= \frac{dx}{dS}\dot{S}(t) + \frac{1}{2}\frac{d^2x}{dS^2}\dot{S}^2(t)\,dt + \dots \\ &= \frac{\dot{S}(t)}{S(t)} - \frac{1}{2}\left[\frac{\dot{S}(t)}{S(t)}\right]^2 dt + \dots,\end{aligned} \tag{5}$$

and replace the last term by its expectation value (4):

$$\begin{aligned}\dot{x}(t) &= \frac{\dot{S}(t)}{S(t)} - \frac{1}{2}\left\langle\left[\frac{\dot{S}(t)}{S(t)}\right]^2\right\rangle dt \\ &= \frac{\dot{S}(t)}{S(t)} - \frac{1}{2}\sigma^2 + \dots\ .\end{aligned} \tag{6}$$

Inserting Eq. (3), we obtain the well-known Ito relation

$$r_x = r_S - \frac{1}{2}\sigma^2. \tag{7}$$

In praxis, this relation implies that if we fit a straight line of slope r_{s_X} through a plot of the logarithms of stock prices, the forward extrapolation of the average stock price itself is given by

$$\langle S(t)\rangle = S(0)\,e^{r_S t} = S(0)\,e^{(r_x+\sigma^2/2)t}. \tag{8}$$

III. NON-GAUSSIAN DISTRIBUTIONS

The description of the fluctuations of the logarithms of the stock prices around the linear trend by a Gaussian distribution is only a rough approximation to the real stock prices. Given a certain time scale, for instance days, they follow some distribution $\tilde{D}(x)$ which is the Fourier transform of some function $D(p)$, which we shall write as an exponential $e^{-H(p)}$:

$$\tilde{D}(x) = \int \frac{dp}{2\pi} e^{ipx} D(p) = \int \frac{dp}{2\pi} e^{ipx - H(p)}. \tag{9}$$

In general, $H(p)$ will have a power series expansion

$$H(p) = ic_1 p + \frac{1}{2}c_2\,p^2 - i\frac{1}{3!}c_3 p^3 - \frac{1}{4!}c_4\,p^4 + i\frac{1}{5!}c_5 p^5 + \dots, \tag{10}$$

The coefficients c_n in the expansion (10) are the *cumulants* of the distribution $\tilde{D}(x)$, from which they can be obtained by the connected expectation values

$$c_1 = \langle x\rangle_c \equiv \langle x\rangle \equiv \int dx\,x\,\tilde{D}(x), \quad c_2 = \langle x^2\rangle_c \equiv \langle x^2\rangle - \langle x\rangle^2 \equiv \int dx\,(x - \langle x\rangle)^2 \tilde{D}(x) \equiv \sigma^2, \dots\ . \tag{11}$$

By analogy with mechanics, we call $H(p)$ the *Hamiltonian* of the fluctuations. A Gaussian Hamiltonian $H(p) = \sigma^2 p^2/2$ coincides with the mechanical energy of a free particle of mass $1/\sigma^2$. Non-Gaussian Hamiltonians are standard in elementary particle physics. A relativistic particle, for example, has a Hamiltonian $H(p) = \sqrt{p^2 + m^2}$, and there is no problem in defining and solving the associated path integral [35].

We also introduce a modified Hamiltonian $\bar{H}(p)$, whose Fourier transform has zero average. Its expansion (10) has the linear term $ic_1 p$ subtracted. We further define a modified Hamiltonian $H_r(p)$ by adding to $\bar{H}(p)$ a linear term irp, i.e.,

$$\bar{H}(p) \equiv H(p) - H'(0)p, \quad H_r(p) \equiv \bar{H}(p) + irp = H(p) - H'(0)p + irp. \tag{12}$$

A. Path Integral for Non-Gaussian Fluctuations

It is easy to calculate the properties of a process (1) whose fluctuations are distributed according to a general non-Gaussian distribution. If we assume the rate r_x to coincide with the linear coefficient c_1 in the expansion (10), such that we can abbreviate $H_{r_x}(p)$ by $H(p)$, the stochastic differential equation reads

$$\dot{x}(t) = \eta(t), \tag{13}$$

and the probability distribution of the endpoints of paths starting at a certain initial point is given by the path integral

$$P(x_b t_b | x_a t_a) = \int \mathcal{D}\eta \int \mathcal{D}x \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \delta[\dot{x} - \eta], \tag{14}$$

with the initial condition $x(t_a) = x_a$. The final point is, of course, $x_b = x(t_b)$. The function $\tilde{H}(\eta)$ is defined by the negative logarithm of the non-Gaussian distributions $\tilde{D}(x)$, such that

$$e^{-\tilde{H}(x)} \equiv \tilde{D}(x). \tag{15}$$

The path integral is defined á la Feynman [35] by slicing the time axis at times $t_n = \epsilon n$ with $n = 0, \ldots, N$, and integrating over all $x(t_n)$. At the end, the limit $N \to \infty$ is taken. In this way, we select from the space of all fluctuating paths a well-behaved set of measure zero, which is in most cases sufficient to obtain the correct limit $N \to \infty$, in particular, it will be sufficient for the typical non-Gaussian distributions encountered in stock markets (see Fig. 1). This is completely analogous to Riemann's procedure of defining ordinary integrals, which are approximated by sums over values of a function on a set of measure zero.

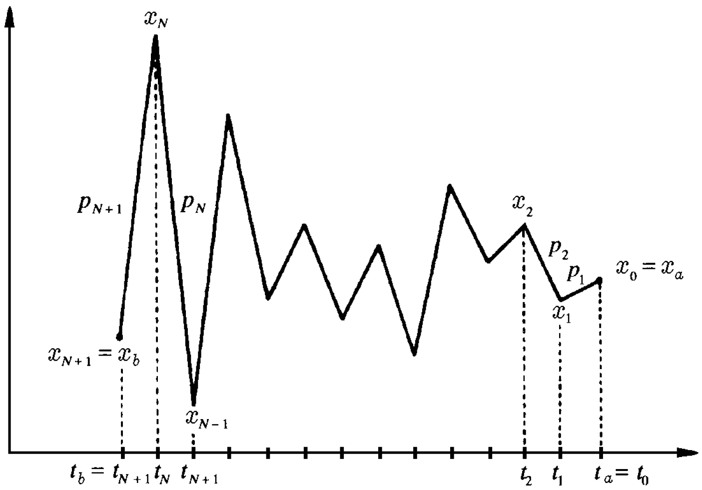

FIG. 1. Time-sliced approximation to paths with non-Gaussian fluctuations.

With the help of the path integral (14), the correlation functions of the noise in the path integral (14) can be found by straightforward functional differentiation. For this purpose, we express the noise distribution $P[\eta] \equiv \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]$ as a Fourier path integral

$$P[\eta] = \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} \exp\left\{\int_{t_a}^{t_b} dt\, [ip(t)\eta(t) - H(p(t))]\right\}, \tag{16}$$

and note that the correlation functions can be obtained from the functional derivatives

$$\langle \eta(t_1)\cdots\eta(t_n)\rangle = (-i)^n \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} \left[\frac{\delta}{\delta p(t_1)} \cdots \frac{\delta}{\delta p(t_n)} e^{i\int_{t_a}^{t_b} dt\, p(t)\eta(t)} \right] e^{-\int_{t_a}^{t_b} dt\, H(p(t))}. \tag{17}$$

After n partial integrations, this becomes

$$\begin{aligned} \langle \eta(t_1)\cdots\eta(t_n)\rangle &= i^n \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} e^{i\int_{t_a}^{t_b} dt\, p(t)\eta(t)} \frac{\delta}{\delta p(t_1)} \cdots \frac{\delta}{\delta p(t_n)} e^{-\int_{t_a}^{t_b} dt\, H(p(t))} \\ &= i^n \left[\frac{\delta}{\delta p(t_1)} \cdots \frac{\delta}{\delta p(t_n)} e^{-\int_{t_a}^{t_b} dt\, H(p(t))} \right]_{p(t)\equiv 0}. \end{aligned} \tag{18}$$

By expanding the exponential $e^{-\int_{t_a}^{t_b} dt\, H(p(t))}$ in a power series using (10), it is straightforward to calculate

$$\langle \eta(t_1)\rangle \equiv Z^{-1} \int \mathcal{D}\eta\, \eta(t_1) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t)) \right] = c_1, \tag{19}$$

$$\begin{aligned} \langle \eta(t_1)\eta(t_2)\rangle &\equiv Z^{-1} \int \mathcal{D}\eta\, \eta(t_1)\eta(t_2) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t)) \right] \\ &= c_2\delta(t_1-t_2) + c_1^2, \end{aligned} \tag{20}$$

$$\begin{aligned} \langle \eta(t_1)\eta(t_2)\eta(t_3)\rangle &\equiv Z^{-1} \int \mathcal{D}\eta\, \eta(t_1)\eta(t_2)\eta(t_3) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t)) \right] \\ &= c_3\delta(t_1-t_2)\delta(t_1-t_3) \\ &+ c_2c_1\left[\delta(t_1-t_2) + \delta(t_2-t_3) + \delta(t_1-t_3)\right] + c_1^3 + c_1^3, \end{aligned} \tag{21}$$

$$\begin{aligned} \langle \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4)\rangle &\equiv Z^{-1} \int \mathcal{D}\eta\, \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t)) \right] \\ &= c_4\delta(t_1-t_2)\delta(t_1-t_3)\delta(t_1-t_4) \\ &+ c_3c_1\left[\delta(t_1-t_2)\delta(t_1-t_3) + 3\,\text{cyclic perms}\right] \\ &+ c_2^2\left[\delta(t_1-t_2)\delta(t_3-t_4) + \delta(t_1-t_3)\delta(t_2-t_4) + \delta(t_1-t_4)\delta(t_2-t_3)\right] \\ &+ c_2c_1^2\left[\delta(t_1-t_2) + 5\,\text{pair terms}\right] + c_1^4, \end{aligned} \tag{22}$$

where Z is the normalization integral

$$Z \equiv \int \mathcal{D}\eta\, \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t)) \right]. \tag{23}$$

The higher correlation functions are obvious generalizations of these equations. The different contributions on the right-hand side of Eqs. (19)–(22) are distinguishable by their connectedness structure.

An important property of the probability (14) is that it satisfies the semigroup property of path integrals

$$P(x_ct_c|x_at_a) = \int dx_b\, P(x_ct_c|x_bt_b)P(x_bt_b|x_at_a). \tag{24}$$

The the experimental asset distributions do satisfy this property approximately. For truncated Lévy distributions this is shown in Fig. 2.

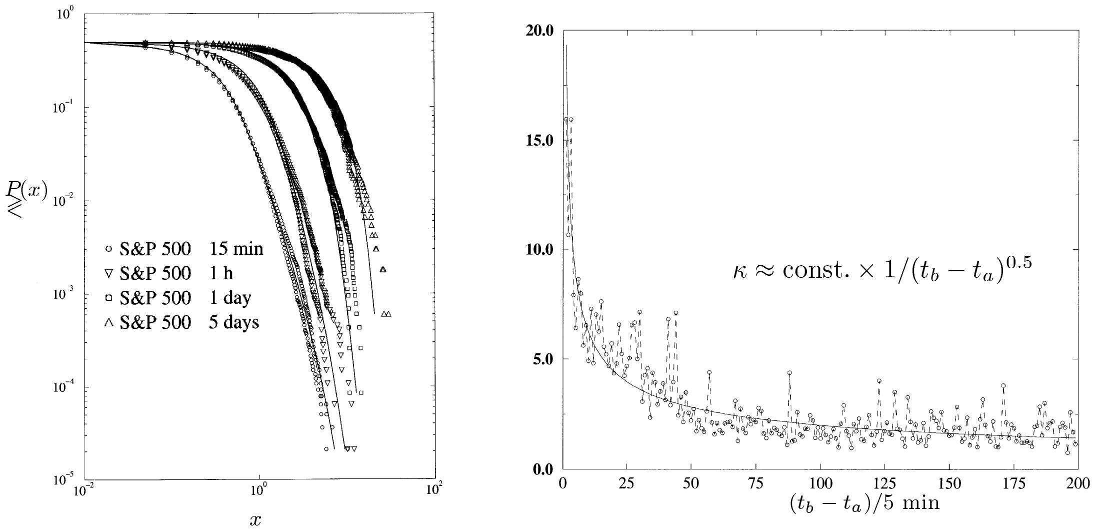

FIG. 2. Left-hand side: Cumulative distributions obtained by repeated convolutions of the 15-min distribution (from Ref. [5]). Apart from the tails, the semigroup property (24) is reasonably well satisfied. Right-hand side: Falloff of kurtosis is slower than $1/(t_b - t_a)$ expected from convolution property (see Ref. [36]).

Apart from the far ends of the tails, the semigroup property (24) is reasonably well satisfied.

B. Lévy-Khintchine Formula

If is sometimes useful to represent the Hamiltonian in the form of a Fourier integral

$$H(p) = -\int dx\, e^{ipx} F(x). \tag{25}$$

Due to the the special significance of the linear term in $H(p)$ governing the drift, this is usually subtracted out of the integral by rewriting (25) as

$$H_r(p) = irp + \int dx \left(e^{ipx} - 1 - ipx\right) F(x). \tag{26}$$

The first subtraction ensures the property $H_r(0) = 0$ which guarantees the unit normalization of the distribution. This subtracted representation is known as the *Lévy-Khintchine formula*, and the function $F(x)$ is the so-called of the distribution. Some people also subtract out the quadratic term and write

$$H_r(p) = irp + \frac{\sigma^2}{2}p^2 + \int dx \left(e^{ipx} - 1\right) \bar{F}(x). \tag{27}$$

and use a weight function $\bar{F}(x)$ which has no first and second moment (i.e., $\int F(x)x = 0$, $\int F(x)x^2 = 0$. to avoid redundancy in the representation.

C. Fokker-Planck-Type Equation

The δ-functional may be represented by a Fourier integral leading to

$$P(x_b t_b | x_a t_a) = \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} \exp\left\{\int_{t_a}^{t_b} dt \left[ip(t)\dot{x}(t) - ip(t)\eta(t) - \tilde{H}(\eta(t))\right]\right\}. \tag{28}$$

Integrating out the noise variable $\eta(t)$ at each time t we obtain

$$P(x_b t_b|x_a t_a) = \int \frac{\mathcal{D}p}{2\pi} \exp\left\{\int_{t_a}^{t_b} dt\, [ip(t)\dot{x}(t) - H(p(t))]\right\}. \tag{29}$$

Integrating over all $x(t)$ with fixed end points enforces a constant momentum along the path, and we remain with a single integral

$$P(x_b t_b|x_a t_a) = \int \frac{dp}{2\pi} \exp\left[-(t_b - t_a)H(p) + ip(x_b - x_a)\right]. \tag{30}$$

From this representation it is easy to verify that this probability satisfies a Fokker-Planck-type equation

$$\partial_t P(x_b t_b|x_a t_a) = -H(-i\partial_x) P(x_b t_b|x_a t_a). \tag{31}$$

The general solution $\psi(x,t)$ of this differential equation with the initial condition $\psi(x,0)$ is given by the path integral generalizing (14)

$$\psi(x,t) = \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \psi\left(x - \int_{t_a}^{t} dt' \eta(t')\right). \tag{32}$$

To verify that this satisfies indeed the Fokker-Planck-type equation (31) we consider $\psi(x,t)$ at a slightly later time $t+\epsilon$ and expand

$$\begin{aligned}
\psi(x,t+\epsilon) &= \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \psi\left(x - \int_{t_a}^{t} dt'\eta(t') - \int_{t}^{t+\epsilon} dt'\eta(t')\right). \\
&= \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \left\{\psi\left(x - \int_{t_a}^{t} dt'\eta(t')\right)\right. \\
&- \quad \psi'\left(x - \int_{t_a}^{t} dt'\eta(t')\right) \int_{t}^{t+\epsilon} dt'\eta(t') \\
&+ \frac{1}{2}\psi''\left(x - \int_{t_a}^{t} dt'\eta(t')\right) \int_{t}^{t+\epsilon} dt_1 dt_2\, \eta(t_1)\eta(t_2) \\
&- \frac{1}{3!}\psi'''\left(x - \int_{t_a}^{t} dt'\eta(t')\right) \int_{t}^{t+\epsilon} dt_1 dt_2 dt_3\, \eta(t_1)\eta(t_2)\eta(t_3) \\
&+ \left.\frac{1}{4!}\psi^{(4)}\left(x - \int_{t_a}^{t} dt'\eta(t')\right) \int_{t}^{t+\epsilon} dt_1 dt_2 dt_3 dt_4\, \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) + \dots \right\}.
\end{aligned} \tag{33}$$

Using the correlation functions (19)–(22) we obtain

$$\begin{aligned}
\psi(x,t+\epsilon) &= \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \\
&\times \left[-\epsilon c_1 \partial_x + \left(\epsilon c_2 + \epsilon^2 c_1\right)\frac{1}{2}\partial_x^2 - \left(\epsilon c_3 + 3\epsilon^2 c_2 c_1\right)\frac{1}{3!}\partial_x^3\right. \\
&+ \left.(\epsilon c_4 + \epsilon^2 4 c_3 c_1 + \epsilon^2 3 c_2^2 + \epsilon^3 c_2 c_1^2 + \epsilon^4 c_1^2)\frac{1}{4!}\partial_x^4 + \dots\right] \psi\left(x - \int_{t_a}^{t} dt'\eta(t')\right).
\end{aligned} \tag{34}$$

In the limit $\epsilon \to 0$, only the linear terms in ϵ contribute, which all descend from the connected parts of the correlation functions of $\eta(t)$. The differential operators in the brackets can now be pulled out of the integral and we find the differential equation

$$\partial_t \psi(x,t) = \left[-c_1\partial_x + c_2\frac{1}{2}\partial_x^2 - c_3\frac{1}{3!}\partial_x^3 + c_4\frac{1}{4!}\partial_x^4 + \dots\right] \psi(x,t). \tag{35}$$

We now replace $c_1 \to r_x$ and express using (10) the differential operators in brackets as Hamiltonian operator $-H_{r_x}(-i\partial_x)$. This leads to the Schrödinger-like equation

$$\partial_t \psi(x,t) = -H_{r_x}(-i\partial_x)\, \psi(x,t). \tag{36}$$

Due to the many derivatives in $H(i\partial_x)$, this equation is in general non-local. This can be made explicit with the help of the Lévy-Khintchine weight function $F(x)$ in the Fourier representation (25). In this case, the right-hand side

$$-H(i\partial_x)\psi(x,t) = \int dx'\, e^{-x'\partial_x}\, F(x')\psi(x,t) = \int dx'\, F(x')\psi(x-x',t). \tag{37}$$

and the Fokker-Planck-like equation (36) takes the form of an integral equation. Some people like to use the subtracted form (27) of the Lévy-Khintchine and arrive at the integro-differential equation

$$\partial_t\psi(x,t) = \left[-c_1\partial_x - \frac{c_2}{2}\partial_x^2\right]\psi(x,t) + \int dx'\,\bar{F}(x')\psi(x-x',t). \tag{38}$$

The integral term can then be treated as a perturbation to an ordinary Fokker-Planck equation.

By a similar procedure as in the derivation of Eq. (35) it is possible to derive a generalization of Ito's rule (6) to functions of noise variable with non-Gaussian distributions. As in (5) we expand $f(x(t+\epsilon))$:

$$\begin{aligned} f(x(t+\epsilon)) = f(x(t)) &+ f'(x(t))\int_t^{t+\epsilon} dt'\,\dot{x}(t') \\ &+ \frac{1}{2}f''(x(t))\int_t^{t+\epsilon} dt_1 \int_t^{t+\epsilon} dt_2\,\dot{x}(t_1)\dot{x}(t_2) \\ &+ \frac{1}{3!}f^{(3)}(x(t))\int_t^{t+\epsilon} dt_1 \int_t^{t+\epsilon} dt_2 \int_t^{t+\epsilon} t_3\,\dot{x}(t_1)\dot{x}(t_2)\dot{x}(t_3) + \dots\ , \end{aligned} \tag{39}$$

where $\dot{x}(t) = \eta(t)$ is the stochastic differential equation with a nonzero expectation value $\langle\eta(t)\rangle = c_1$. In contrast to (5) which had to be carried out only up to second order in $\dot{x}$, we must now keep *all* orders in the noise variable. Evaluating the noise averages of the multiple integrals on the right-hand side using the correlation functions (19)–(22), we find the time derivative of the expectation value of an arbitrary function of the fluctuating variable $x(t)$

$$\begin{aligned} \langle f(x(t+\epsilon))\rangle = &\langle f(x(t))\rangle + \langle f'(x(t))\rangle\epsilon c_1 + \frac{1}{2}\langle f''(x(t))\rangle(\epsilon c_2 + \epsilon^2 c_1^2) \\ &+ \frac{1}{3!}\langle f^{(3)}(x(t))\rangle(\epsilon c_3 + \epsilon^2 c_2 c_1 + \epsilon^3 c_1^3) + \dots \\ = &\,\epsilon\left[-c_1\partial_x + c_2\frac{1}{2}\partial_x^2 - c_3\frac{1}{3!}\partial_x^3 + \dots\right]\langle f(x(t))\rangle + \mathcal{O}(\epsilon^2). \end{aligned} \tag{40}$$

After the replacement $c_1 \to r_x$ the function $f(x(t))$ obeys therefore the following equation:

$$\langle\dot{f}(x(t))\rangle = -H_{r_x}(i\partial_x)\langle f(x(t))\rangle. \tag{41}$$

Taking out the lowest-derivative term this takes a form

$$\langle\dot{f}(x(t))\rangle = \langle\partial_x f(x(t))\rangle\langle\dot{x}(t)\rangle - \bar{H}_{r_x}(i\partial_x)\langle f(x(t))\rangle. \tag{42}$$

In postpoint time slicing, this may be viewed as the expectation value of the stochastic differential equation

$$\dot{f}(x(t)) = \partial_x f(x(t))\dot{x}(t) - \bar{H}_{r_x}(i\partial_x)f(x(t)). \tag{43}$$

This is the direct generalization of Ito's rule (6).

For an exponential function $f(x) = e^{Px}$, this becomes

$$\frac{d}{dt}e^{Px(t)} = \left[\dot{x}(t) - H_{r_x}(iP)\right]e^{Px(t)}. \tag{44}$$

As a consequence of this equation for $P = 1$, the rate r_S with which a stock price $S(t) = e^{x(t)}$ grows according to formula (3) is now related to r_x by

$$r_S = r_x - \bar{H}(i) = r_x - [H(i) - iH'(0)] = -H_{r_x}(i), \tag{45}$$

which replaces the simple Ito relation $r_S = r_x + \sigma^2/2$ in Eq. (7). Recall the definition of $\bar{H}(p)$ in Eq. (12). The corresponding generalization of the left-hand part of Eq. (1) reads

$$\frac{\dot{S}}{S} = \dot{x}(t) - \bar{H}(i) = \dot{x}(t) - [H(i) - iH'(0)] = \dot{x}(t) - r_x - H_{r_x}(i). \tag{46}$$

The forward price of a stock must therefore be calculated using the generalization of formula (8):

$$\langle S(t)\rangle = S(0)e^{r_S t} = S(0)\langle e^{r_x t + \int_0^t dt'\,\eta(t')}\rangle = S(0)e^{-H_{r_x}(i)t} = S(0)e^{\{r_x - [H(i) - iH'(0)]t\}}. \tag{47}$$

Note that may derive the differential equation of an arbitrary function $f(x(t))$ in Eq. (41) from a simple mnemonic rule, expanding sloppily

$$\begin{aligned} f(x(t+dt)) = f(x(t) + \dot{x}dt) = f(x(t)) + f'(x(t))\dot{x}(t)dt + \frac{1}{2}f''(x(t))\dot{x}^2(t)dt^2 \\ + \frac{1}{3!}f^{(3)}(x(t))\dot{x}^3(t)dt^3 + \dots\ , \end{aligned} \tag{48}$$

and replacing (6),

$$\langle \dot{x}(t)\rangle dt \to c_1 dt, \quad \langle \dot{x}^2(t)\rangle dt^2 \to c_2 dt, \quad \langle \dot{x}^3(t)\rangle dt^2 \to c_3 dt, \dots . \tag{49}$$

IV. MARTINGALES

In financial mathematics, an often-encountered concept is that of a *martingale* [37]. The name stems from a casino strategy in which a gambler doubles his stake each time a bet is lost. A stochastic variable is called a martingale, if its expectation value is time-independent. The noise variable $\eta(t)$ with vanishing average is a trivial martingale.

A. Gaussian Martingale

For a harmonic noise variable, the exponential $e^{\int_0^t dt'\eta(t') - \sigma^2 t/2}$ is a nontrivial martingale, due to Eq. (8). For the same reason, a stock price $S(t) = e^{x(t)}$ with $x(t)$ obeying the stochastic differential equation (1) can be made a martingale by a time-dependent multiplicative factor $e^{-r_S t}e^{\dot{x}} = e^{-r_x t}e^{\dot{x} - \sigma^2 t/2}$. An explicit distribution which makes $S(t) = e^{x(t)}$ a martingale is

$$P^M(x_b t_b|x_a t_a) \equiv \frac{e^{-r_S t}}{\sqrt{2\pi\sigma^2(t_b - t_a)}} \exp\left\{-\frac{[x_b - x_a - r_x(t_b - t_a)]^2}{2\sigma^2(t_b - t_a)}\right\}. \tag{50}$$

I can easily be verified by direct integration that the expectation value of $S(t) = e^{x(t)}$

$$\langle S(t_b)\rangle = \langle e^{x(t_b)}\rangle = \int dx_b\, e^{x_b}\, P^M(x_b t_b|x_a t_a) \tag{51}$$

is independent of the time t_b.

At this place we can make an important observation: There exists an entire family of distributions for which $S(t) = e^{x(t)}$ is a martingale, namely

$$P^{M\,r}(x_b t_b|x_a t_a) \equiv \frac{e^{-rt}}{\sqrt{2\pi\sigma^2(t_b - t_a)}} \exp\left\{-\frac{[x_b - x_a - r_x(t_b - t_a)]^2}{2\sigma^2(t_b - t_a)}\right\}. \tag{52}$$

for any r and $r_x = r - \sigma^2/2$. Such distributions which differ only in the drift r are called *equivalent*. The prefactor e^{-rt} is referred to as a *discount factor* with the rate r.

B. Non-Gaussian Martingales

For $S(t) = e^{x(t)}$ with an arbitrary non-Gaussian noise $\eta(t)$, there are many ways of constructing martingales. the relation (45) allows us to construct immediately the simplest martingale

1. Natural Martingale

Relation (45) allows us to write down immediately the simplest martingale. It is given by an obvious generalization of the Gaussian expression $e^{\int_0^t dt'\eta(t')-\sigma^2 t/2}$ which is the exponential $e^{\int_0^t dt'\eta(t')+\bar{H}(i)t}$, whose expectation value is time-independent due to Eq. (45) for the stock price $S(t) = e^{x(t)}$, we obtain therefore the simplest possible martingale

$$e^{-r_S t} S(t) = e^{-r_S t} e^{r_x t + \int_0^t dt' \eta(t')} \tag{53}$$

if r_S and r_x are related by $r_S = r_x - \bar{H}(i) = -H_{r_x S}(i)$.

It is easy to write down a distribution function which makes $S(t) = e^{x(t)}$ itself aa martingale:

$$P^M(x_b t_b | x_a t_a) = e^{-r_S t} \int \mathcal{D}\eta \int \mathcal{D}x \exp\left\{ - \int_{t_a}^{t_b} dt\, \tilde{H}_{r_x}(\eta(t)) \right\} \delta[\dot{x} - \eta]. \tag{54}$$

There exists also here an entire family of equivalent distribution functions for which the integrals of the type (51) over e^{x_b} are independent of t_b. This is done as follows. One obvious family of this type is a straightforward generalization of the Gaussian family (52) which we shall refer to as *natural martingales*:

$$P^{M\,r}(x_b t_b | x_a t_a) = e^{-rt} \int \mathcal{D}\eta \int \mathcal{D}x \exp\left\{ - \int_{t_a}^{t_b} dt\, \tilde{H}_{r_x}(\eta(t)) \right\} \delta[\dot{x} - \eta], \tag{55}$$

with arbitrary $r = r_x - \bar{H}(i) = -H_{r_x}(i)$. Indeed, multiplying this with e^{x_b} and integrating over x_b gives rise to a δ-function $\delta(p - i)$ and produces the same result e^{x_a} for all times t_b. As explained in Section 3.3, the path integral leads to the Fourier integral [compare (30)]

$$P^{M\,r}(x_b t_b | x_a t_a) = e^{-rt} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left[ip(x_b - x_a) - (t_b - t_a) H_{r_x}(p) \right]. \tag{56}$$

2. Esscher Martingale

In the literature on mathematical finance, much attention is paid to another family of equivalent martingale measures. It has been used a long time ago to estimate risks of actuaries [38] and introduced more recently into the theory of option prices [39,40] where it is now of wide use [41]– [46]. This family is constructed as follows. Let $\tilde{D}(x)$ be an arbitrary distribution function with a Fourier transform

$$\tilde{D}(x) = \int_{\infty}^{\infty} \frac{dp}{2\pi}\, e^{-H(p)}\, e^{ipx}, \tag{57}$$

and $H(0) = 0$, to guarantee a unit normalization $\int dx \tilde{D}(x) = 1$. We now introduce an *Esscher-transformed* distribution function. It is obtained by slightly tilting the initial distribution $\tilde{D}(x)$ by multiplication with an asymmetric exponential factor $e^{\theta x}$:

$$D^\theta(x) \equiv e^{H(i\theta)}\, e^{\theta x} \tilde{D}(x). \tag{58}$$

The constant prefactor $e^{H(i\theta)}$ is necessary to conserve the total probability. This distribution can be written as a Fourier transform

$$D^\theta(x) = \int_{\infty}^{\infty} \frac{dp}{2\pi}\, e^{-H^\theta(p)}\, e^{ipx}. \tag{59}$$

with the Esscher-transformed Hamiltonian

$$H^{\theta}(p) \equiv H(p+i\theta) - H(i\theta). \tag{60}$$

Since $H^{\theta}(0) = 0$, and the transformed distribution is properly normalized: $\int dx D^{\theta}(x) = 1$. We now define the Esscher-transformed expectation value

$$\langle F(x)\rangle^{\theta} \equiv \int dx D^{\theta}(x) F(x). \tag{61}$$

It is related to the original expectation value by

$$\langle F(x)\rangle^{\theta} \equiv e^{H(i\theta)} \left\langle e^{\theta x} F(x)\right\rangle \tag{62}$$

For the specific function $F(x) = e^{x}$, Eq. (62) becomes

$$\langle e^{x}\rangle^{\theta} = e^{-H^{\theta}(i)} \equiv e^{H^{\theta}(i\theta)} \left\langle e^{(\theta+1)x}\right\rangle = e^{H^{\theta}(i\theta)-H^{\theta}(i\theta+i)}. \tag{63}$$

Applying the transformation (59) to each time slice in the general path integral (14), we obtain the *Esscher-transformed* path integral

$$P^{\theta}(x_b t_b|x_a t_a) = e^{-r_S t} e^{H_{r_x}(i\theta)t} \int \mathcal{D}\eta \int \mathcal{D}x \exp\left\{\int_{t_a}^{t_b} dt \left[\theta\eta(t) - \tilde{H}_{r_x}(\eta(t))\right]\right\} \delta[\dot{x}-\eta], \tag{64}$$

This leads to the Fourier integral [compare (56)]

$$P^{\theta}(x_b t_b|x_a t_a) = e^{-r_S t} e^{H_{r_x}(i\theta)t} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left[ip(x_b - x_a) - (t_b - t_a) H_{r_x}(p+i\theta)\right]. \tag{65}$$

Let us denote the expectation values calculated with this probability by $\langle\ \dots\ \rangle^{\theta}$. Then we find for $S(t) = e^{x(t)}$ the time dependence

$$\langle S(t)\rangle^{\theta} = e^{-H^{\theta}_{r_x}(i)t}. \tag{66}$$

This equation shows, that the exponential of a stochastic variable $x(t)$ can be made a martingale with respect to any Esscher-transformed distribution if we remove the exponential growth factor $\exp(r^{\theta}t)$ with

$$r^{\theta} \equiv -H^{\theta}_{r_x}(i) = -H_{r_x}(i+i\theta) + H_{r_x}(i\theta). \tag{67}$$

Thus, a family of equivalent martingale distributions for the stock price $S(t) = e^{x(t)}$ is

$$P^{M\,\theta}(x_b t_b|x_a t_a) \equiv e^{-r^{\theta}t} P^{\theta}(x_b t_b|x_a t_a) \tag{68}$$

for any choice of the parameter θ.

For a harmonic distribution function (50), the Esscher martingales and the previous ones are equivalent. Indeed, starting from (50) in which $r_S = r_x + \sigma^2/2$, the Esscher transform leads us after a quadratic completion to the family of natural martingales (50) with the rate parameter $r = r_x + \theta\sigma^2$.

3. Other Non-Gaussian Martingales

Many other non-gaussian martingales have been discussed in the literature. Mathematicians have invented various sophisticated criteria under which one would be preferable over the others for calculating financial risks. Davis has introduced a so-called *utility function* [33] which is supposed to select optimal martingales for different purposes.

For the upcoming development of a theory of option pricing, only the initial *natural martingale* will be relevant.

V. OPTION PRICING

The most important use of path integrals in financial markets is made in the determination of a fair price of financial derivatives, in particular options.[1] Options are an ancient financial tool. They are used for speculative purposes or for hedging major market transactions against unexpected changes in the market environment. These can sometimes produce dramatic price explosions or erosions, and options are supposed to prevent the destruction of huge amounts of capital. Ancient Romans, Grecians, and Phoenicians traded options against outgoing cargos from their local seaports. In financial markets, options are contracted between two parties in which one party has the right but not the obligation to do something, usually to buy or sell some underlying asset. Having rights without obligations has a value, so option holders must pay a price for acquiring them. The price depends on the value of the associated asset, which is why they are also called *derivative assets* or briefly *derivatives*. *Call options* are contracts giving the option holder the right to buy something, while *put options* entitle the holder to sell something. The price of an option is called *premium*. Usually, options are associated with stock, bonds, or commodities like oil, metals or other raw materials. In the sequel we shall consider call options on stocks, to be specific.

Modern option pricing techniques have their roots in early work by Charles Castelli who published in 1877 a book entitled *The Theory of Options in Stocks and Shares*. This book presented an introduction to the hedging and speculation aspects of options. l. Twenty three years later, Louis Bachelier offered the earliest known analytical valuation for options in his dissertation at the Sorbonne [47]. Remarkably, he discovered the treatment of stochastic phenomena five years before Einstein's related but much more famous work on Brownian motion [48], and twenty three years before Wiener's mathematical development [49]. The stochastic differential equations considered by him still had an important defect of allowing for negative security prices, and for option prices exceeding the price of the underlying asset. Bachelier's work was continued by Paul Samuelson, who wrote in 1955 an unpublished paper entitled *Brownian Motion in the Stock Market*. During that same year, Richard Kruizenga, one of Samuelson's students, cited Bachelier's work in his dissertation *Put and Call Options: A Theoretical and Market Analysis*. In 1962, a dissertation by A. James Boness with the title *A Theory and Measurement of Stock Option Value* developed a more satisfactory pricing model which was further improved by Fischer Black and Myron Scholes In 1973 they published their famous *Black and Scholes Model* [50] which, together with the improvements introduced by R. Merton, earned the Nobel prize in 1997.[2]

A. Black-Scholes Option Pricing Model

In the early seventies, Fisher Black was working on a valuation model for stock warrants and observed that his formulas resembled very much the well-known equations for heat transfer. Soon after this, Myron Scholes joined Black and together they discovered the approximate option pricing model which is still of wide use.

The Black and Scholes Model is based on the following assumptions:

1. The return is normally distributed. We remarked before that there are considerable deviations from a normal distribution which call for improvement of the model to be developed below.

2. Markets are efficient. This assumption implies that the market operates continuously with share prices following a continuous stochastic process without memory. It also implies that different markets have the same asset prices.

   This is not quite true. Different markets do in general have slightly different prices. Their differences are kept small by the existence of arbitrage dealers. There also exist correlations over a short time scale which make it possible, in principle, to profit without risk from the so-called statistical arbitrage. This possibility is, however, strongly limited by transaction fees.

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[1] This description of options can be found on the internet under
http://bradley.bradley.edu/~arr/bsm/model.html
[2] For F. Black the prize came too late—he had died two years earlier.

3. No commissions are charged.

   This assumption is not fulfilled. Usually market participants have to pay a commission to buy or sell assets. Even floor traders pay some fee, although this is usually very small. The fees payed by individual investors is more substantial and can distort the output of the model.

4. Interest rates remain constant and known.

   The Black and Scholes model assumes the existence of a risk-free rate to represent this constant and known rate. In reality there is no such thing as the risk-free rate. As an approximation, one uses the discount rate on U.S. Government Treasury Bills with 30 days left until maturity. During periods of rapidly changing interest rates, these 30 day rates are often subject to change, thereby violating one of the assumptions of the model.

5. The stock pays no dividends during the option's life.dividends

   Most companies pay dividends to their share holders, so this is a limitation to the model since higher dividend lead to lower call premiums. There is, however, a simple possibility of adjusting the model to the real situation by subtracting the discounted value of a future dividend from the stock price.

6. European exercise terms are used. European exercise terms imply the exercise of an option only on the expiration date. This is in contrast to the American exercise terms which allow for this at any time during the life of the option. This greater flexibility makes an American option more valuable than the European one.

   The difference is, however, not dramatic in praxis because very few calls are ever exercised before the last few days of their life, since an early exercise means giving away the remaining time value on the call. Different exercise times towards the end of the life of a call are irrelevant since the remaining time value is very small and the intrinsic value has a small time dependence, barring a dramatic event right before expiration date.

Since 1973, the original Black and Scholes Option Pricing Model has been improved and extended considerably. In the same year, Robert Merton [51] included the effect of dividends. Three years later, Jonathan Ingerson relaxed the assumption of no taxes or transaction costs, and Merton removed the restriction of constant interest rates. At present we are in a position of being able to determine a large variety of different values of options.

The relevance of path integrals to this field was recognized first in 1988 by a theoretical physicist J.W. Dash, who wrote two unpublished papers on the subject entitled *Path Integrals and Options I* and *II* [52]. Since then many theoretical physicists have entered the field, and papers on this subject have begun appearing on the Los Alamos server [5,53,54].

B. Evolution Equations of Portfolios with Options

The option price $O(t)$ has a larger fluctuations than the associated stock price. It usually varies with an efficiency factor $\partial O(S(t),t)/\partial S(t)$. For this reason it is possible, in the ideal case of Gaussian price fluctuations, to guarantee a steady growth of a portfolio by mixing $N_S(t)$ stocks with $N_O(t)$ options and a certain amount of short-term bonds (usually those with 30 days to maturity) whose number is denoted by $N_B(t)$. The composition $[N_S(t), N_O(t), N_B(t)]$ is referred to as the *strategy* of the portfolio manager. The total wealth has the value

$$W(t) = N_S(t)S(t) + N_O(t)O(S,t) + N_B(t)B(t). \tag{69}$$

The goal is to make $W(t)$ grow with a smooth exponential curve *without fluctuations*

$$\dot{W}(t) \approx r_W W(t). \tag{70}$$

As we shall see immediately, this is possible provided the short-term bonds grow without any fluctuations.

$$\dot{B}(t) \approx r_B B(t). \tag{71}$$

The rate r_B is referred to as *riskfree interest rate* encountered property exists in true markets only if here are no events which change the value of short-term bonds excessively.

The existence of arbitrage dealers will ensure that the growth rate r_W is equal to that of the short-term bonds

$$r_W \approx r_B. \tag{72}$$

Otherwise the dealers would change from one investment to the other.

In the decomposition (69), the desired growth (70) reads

$$\begin{aligned} N_S(t)\dot{S}(t) + N_O(t)\dot{O}(S,t) + N_B(t)\dot{B}(t) + \dot{N}_S(t)S(t) + \dot{N}_O(t)O(S,t) + \dot{N}_B(t)B(t) \\ = r_W\left[N_S(t)S(t) + N_O(t)O(S,t) + N_B(t)B(t)\right]. \end{aligned} \tag{73}$$

Due to (71) and (72), the terms containing $N_B(t)$ without a dot drop out. Moreover, if no extra money is inserted into or taken from the system, i.e., if stocks, options, and bonds are only traded against each other, this does not change the total wealth, assuming the absence of commissions. This so-called *self-financing strategy* is expressed in the equation

$$\dot{N}_S(t)S(t) + \dot{N}_O(t)O(S,t) + \dot{N}_B(t)B(t) = 0. \tag{74}$$

Thus the growth equation (70) translates into

$$\dot{W}(t) = N_S\dot{S} + N_O\dot{O} + N_B\dot{B} = r_W\left(N_S S + N_O O + N_B\dot{B}\right). \tag{75}$$

Due to the equality of the rates $r_W = r_B$ and Eq. (71), the entire contribution of $B(t)$ cancels, and we obtain

$$N_S\dot{S} + N_O\dot{O} = r_W\left(N_S S + N_O O\right) \tag{76}$$

The important observation is now that there exists an optimal ratio between the number of stocks N_S and the number of options N_O, which is inversely equal to the efficiency factor

$$\frac{N_S(t)}{N_O(t)} = -\frac{\partial O(S(t),t)}{\partial S(t)}. \tag{77}$$

Then Eq. (76) becomes

$$N_S\dot{S} + N_O\dot{O} = r_W\left(-\frac{\partial O}{\partial x} + O\right)N_O. \tag{78}$$

The two terms on the left-hand side are treated as follows: First we use the relation (77) to rewrite

$$N_S\dot{S} = -\frac{\partial O(S,t)}{\partial S}\dot{S} = -\frac{\partial O(S,t)}{\partial x}\frac{\dot{S}}{S}, \tag{79}$$

and further, with the help of Eq. (46), as

$$N_S\dot{S} = -\frac{\partial O(S,t)}{\partial x}\left[\dot{x} - \bar{H}(i)\right]. \tag{80}$$

In the second term on the left-hand side of (78), we expand the total time dependence of the option price in a Taylor series

$$\begin{aligned} \frac{dO}{dt} &= \frac{1}{dt}\Big[O(x(t)+\dot{x}(t)\,dt, t+dt) - O(x(t),t)\Big] \\ &= \frac{\partial O}{\partial t} + \frac{\partial O}{\partial x}\dot{x} + \frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2\,dt + \frac{1}{3!}\frac{\partial^3 O}{\partial x^3}\dot{x}^3\,dt^2 + \ldots\,. \end{aligned} \tag{81}$$

We we have gone over to the logarithmic stock price variable $x(t)$ rather than $S(t)$ itself. Some of the derivatives on the right-hand side are denoted by special symbols in financial mathematics: the quantities $\vartheta \equiv \partial O/\partial t$, $\Delta \equiv \partial O/\partial S = \partial O/\partial x S$, and $\Gamma \equiv \partial^2 O/\partial S^2 = (\partial^2 O/\partial x^2 - \partial O/\partial x)/S^2$ are called the "Theta", "Delta", and "Gamma" of the option. Another derivative with a standard name is the "Vega" $V \equiv \partial O/\partial\sigma$.

The expansion (81) is carried to arbitrary powers of $\dot{x}$ as in (48). It is, of course, only an abbreviated notation for the proper expansion in powers of a stochastic variable to be performed as in Eq. (39). Inserting (81) and (80) on the left-hand side of Eq. (78), this becomes

$$\begin{aligned} N_S\dot{S}+N_O\dot{O} &= -\,N_O\frac{\partial O}{\partial x}\left[\dot{x}-\bar{H}(i)\right] \\ &\quad + N_O\left(\frac{\partial O}{\partial t}+\frac{\partial O}{\partial x}\,\dot{x}+\frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2dt+\frac{1}{3!}\frac{\partial O}{\partial x}\dot{x}^3dt+\dots\right) \\ &= N_O\left[\bar{H}(i)\frac{\partial O}{\partial x}+\frac{\partial O}{\partial t}+\frac{1}{2}\frac{\partial O}{\partial x}\dot{x}^2dt+\frac{1}{3!}\frac{\partial O}{\partial x}\dot{x}^3dt+\dots\right]. \end{aligned} \tag{82}$$

Remarkably, the fluctuating variable $\dot{x}$ drops out in this equation, which also becomes *independent* of the growth rate r_S of the stock price. This is the reason why the total wealth $W(t)$ will increase without fluctuations. This happens since we may treat the Taylor series

$$\frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2\,dt+\frac{1}{6}\frac{\partial^3 O}{\partial x^3}\dot{x}^3\,dt^2+\dots \tag{83}$$

in the same way as the expansion (48), using the rules (49), such that (83) becomes

$$\frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2\,dt+\frac{1}{6}\frac{\partial^3 O}{\partial x^3}\dot{x}^3\,dt^2+\dots=-\bar{H}(i\partial_x)O. \tag{84}$$

In this way we find for the option price $O(x,t)$ the Fokker-Planck-like differential equation

$$\frac{\partial O}{\partial t}=r_WO-r_{xW}\frac{\partial O}{\partial x}+\bar{H}(i\partial_x)O, \tag{85}$$

where we have defined, by analogy with (45), an auxiliary rate parameter

$$r_{xW}\equiv r_W+\bar{H}(i). \tag{86}$$

Note, however, that in contrast to the relation between r_x and r_S defined for a fluctuating stock price $S(t)$ and its logarithm $x(t)$, the parameter r_{xW} *does not* have the physical interpretation of governing the logarithm of $W(t)$ since the absence of fluctuations in the wealth $W(t)$ makes $\log W(t)$ grow linearly with the *same* rate r_W that governs the exponential growth e^{r_Wt} of $W(t)$ itself.

If we rename t as t_a, The general solution of the differential equation (85), which at some time $t=t_b$ starts out like $\delta(x-x_b)$, has the Fourier representation

$$P(x_bt_b|x_at_a)=e^{-r_W(t_b-t_a)}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x_b-x_a)}\exp\left\{-\left[\bar{H}(p)+ir_{xW}p\right](t_b-t_a)\right\}, \tag{87}$$

if the *initial* variables x_a and t_a are identified with x and t, respectively. A convergent integral exists only for $t_b>t_a$.

Comparing this expression with Eq. (56) we recognize it as a member of the family of equivalent martingale measures $P^{M\,r}(x_bt_b|x_at_a)$, in which the discount factor r coincides with the risk-free interest rate r_W.

C. Option Pricing for Gaussian Fluctuations

For Gaussian fluctuations where $H(p)=\sigma^2p^2/2$, the integral in (87) can easily be performed and yields

$$P(x_bt_b|x_at_a)=\Theta(t_b-t_a)\frac{e^{-r_W(t_b-t_a)}}{\sqrt{2\pi\sigma^2(t_b-t_a)}}\exp\left\{-\frac{[x_b-x_a-r_{xW}(t_b-t_a)]^2}{2\sigma^2(t_b-t_a)}\right\}. \tag{88}$$

This probability distribution is obviously the solution of the path integral

$$P(x_bt_b|x_at_a) = \Theta(t_b - t_a)e^{-r_W(t_b-t_a)} \int \mathcal{D}x \ \exp\left\{-\frac{1}{2\sigma^2}\int_{t_a}^{t_b} [\dot{x} - r_{xW}]^2\right\}. \tag{89}$$

Recalling the discussion in Section IV, the distribution function (88) is recognized as a member of the equivalent family of martingale distributions (52) for the stock price $S(t) = e^{x(t)}$. It is the particular distribution in which the discount factor contains the risk-free interest rate r_W, i.e., (88) is equal to the martingale distribution $P^{M\,r_W}(x_bt_b|x_at_a)$. This distribution is referred to as the *risk-neutral* equivalent martingale distribution.

An option is written for a certain *strike price* E of the stock. The value of the option at its expiration date t^b is given by the difference between the stock price on expiration date and the strike price:

$$O(x_b, t_b) = \Theta(S_b - E)(S_b - E) = \Theta(x_b - x_E)(e^{x_b} - e^{x_E}), \tag{90}$$

where

$$x_E \equiv \log E. \tag{91}$$

The Heaviside function accounts for the fact that only for $S_b > E$ it is worthwhile to execute the option.

From (90) we calculate the option price at an arbitrary earlier time using the time evolution amplitude (88)

$$O(x_a, t_a) = \int_{-\infty}^{\infty} dx_b \, O(x_b, t_b) \, P^{M\,r_W}(x_bt_b|x_at_a). \tag{92}$$

Inserting (90) we obtain the sum of two terms

$$O(x_a, t_a) = O_S(x_a, t_a) - O_E(x_a, t_a), \tag{93}$$

where

$$O_S(x_a, t_a) = \frac{e^{-r_W(t_b-t_a)}}{\sqrt{2\pi\sigma^2(t_b - t_a)}} \int_{x_E}^{\infty} dx_b \ \exp\left\{x_b - \frac{[x_b - x_a - r_{xW}(t_b - t_a)]^2}{2\sigma^2(t_b - t_a)}\right\}. \tag{94}$$

and

$$O_E(x_a, t_a) = Ee^{-r_W(t_b-t_a)} \frac{1}{\sqrt{2\pi\sigma^2(t_b - t_a)}} \int_{x_E}^{\infty} dx_b \ \exp\left\{-\frac{[x_b - x_a - r_{xW}(t_b - t_a)]^2}{2\sigma^2(t_b - t_a)}\right\}. \tag{95}$$

In the second integral we set

$$x_- \equiv x_a + r_{xW}(t_b - t_a) = x_a + \left(r_W - \frac{1}{2}\sigma^2\right)(t_b - t_a), \tag{96}$$

and obtain

$$O_E(x_a, t_a) = E\frac{e^{-r_W(t_b-t_a)}}{\sqrt{2\pi\sigma^2(t_b - t_a)}} \int_{x_E - x_-}^{\infty} dx_b \ \exp\left\{-\frac{x_b^2}{2\sigma^2(t_b - t_a)}\right\}. \tag{97}$$

After rescaling the integration variable $x_b \to -\xi\sigma\sqrt{t_b - t_a}$, this can be rewritten as

$$O_E(x_a, t_a) = e^{-r_W(t_b-t_a)} E \, N(y_-), \tag{98}$$

where $N(y)$ is the Gaussian distribution function

$$N(y) \equiv \int_{-\infty}^{y} \frac{d\xi}{\sqrt{2\pi}} e^{-\xi^2/2}. \tag{99}$$

evaluated at

$$
\begin{aligned}
y_- \equiv \frac{x_- - x_E}{\sqrt{\sigma^2(t_a-t_b)}} &= \frac{\log[S(t_a)/E] + r_{xW}(t_b-t_a)}{\sqrt{\sigma^2(t_a-t_b)}} \\
&= \frac{\log[S(t_a)/E] + \left(r_W - \frac{1}{2}\sigma^2\right)(t_b-t_a)}{\sqrt{\sigma^2(t_a-t_b)}}.
\end{aligned} \tag{100}
$$

The integral in the first contribution (94) to the option price is found after completing the exponent in the integrand quadratically as follows:

$$
\begin{aligned}
x_b &- \frac{[x_b - x_a - r_{xW}(t_b-t_a)]^2}{2\sigma^2(t_b-t_a)} \\
&= -\frac{[x_b - x_a - (r_{xW}+\sigma^2)(t_b-t_a)]^2 - 2r_W\sigma^2(t_b-t_a) - 2x_a\sigma^2(t_b-t_a)}{2\sigma^2(t_b-t_a)}.
\end{aligned} \tag{101}
$$

Introducing now

$$
x_+ \equiv x_a + \left(r_{xW} + \sigma^2\right)(t_b-t_a) = x_a + \left(r_W + \frac{1}{2}\sigma^2\right)(t_b-t_a), \tag{102}
$$

and rescaling x_b as before, we obtain

$$
O_S(x_a,t_a) = S(t_a)N(y_+), \tag{103}
$$

with

$$
\begin{aligned}
y_+ \equiv \frac{x_+ - x_E}{\sqrt{\sigma^2(t_a-t_b)}} &= \frac{\log[S(t_a)/E] + \left(r_{xW} + \sigma^2\right)(t_b-t_a)}{\sqrt{\sigma^2(t_a-t_b)}} \\
&= \frac{\log[S(t_a)/E] + \left(r_W + \frac{1}{2}\sigma^2\right)(t_b-t_a)}{\sqrt{\sigma^2(t_a-t_b)}}.
\end{aligned} \tag{104}
$$

The combined result

$$
O(x_a,t_a) = S(t_a)N(y_+) - e^{-r_W(t_b-t_a)}E\,N(y_-) \tag{105}
$$

is the celebrated *Black-Scholes formula* of option pricing.

In Fig. 3 we illustrate how the dependence of the call price on the stock price varies with different times to expiration $t_b - t_a$ and with different volatilities σ.

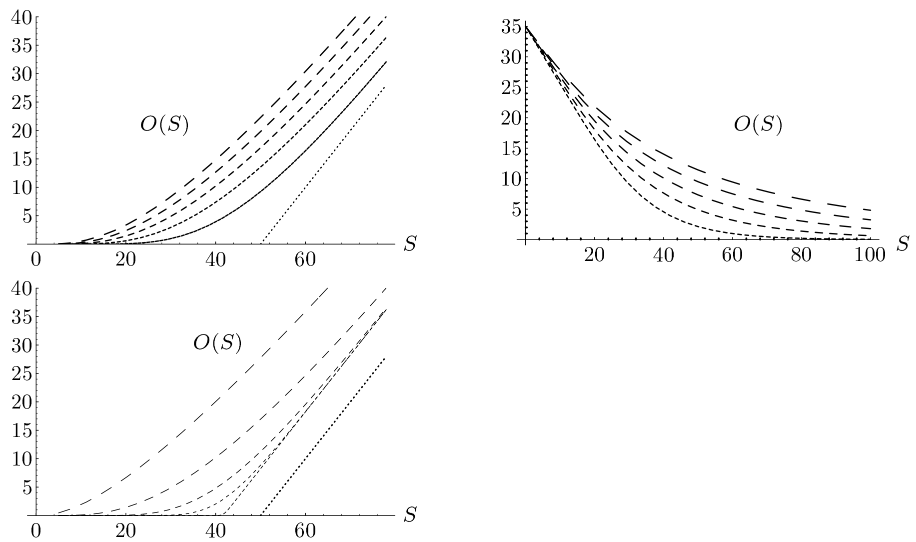

FIG. 3. Left: Dependence of call price O on the stock price S for different times before expiration date (increasing dash length: $1,2,3,4,5$ months). from left to right: $1,2,3,5,6$ months). The parameters are $E = 50\,$US\$, $\sigma = 40\%$, $r_W = 6\%$ per month. Right: Dependence on the strike price E for fixed stock price $35\,$US\$ and the same times to expiration (increasing with dash length). Bottom: Dependence on the volatilities (from left to right: $80\%, 60\%, 20\%, 10\%, 1\%$) at a fixed time $t_b - t_a = 3$ months before expiration.

D. Option Pricing for Non-Gaussian Fluctuations

For non-Gaussian fluctuations, the option price must be calculated numerically from Eqs. (92) and (90). Inserting the Fourier representation (87) and using the Hamiltonian

$$H_{r_{xW}}(p) \equiv \bar{H}(p) + ir_{xW}p \tag{106}$$

defined as in (12), this becomes

$$\begin{aligned} O(x_a,t_a) &= \int_{x_E}^{\infty} dx_b\,(e^{x_b} - e^{x_E})P(x_b t_b|x_a t_a) \\ &= e^{-r_W(t_b-t_a)}\int_{x_E}^{\infty} dx_b\,(e^{x_b} - e^{x_E})\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x_b-x_a)-H_{r_{xW}}(p)(t_b-t_a)}. \end{aligned} \tag{107}$$

The integrand can be rearranged as follows:

$$O(x_a,t_a) = e^{-r_W(t_b-t_a)}\int_{x_E}^{\infty} dx_b\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[e^{x_a}e^{(ip+1)(x_b-x_a)} - e^{x_E}e^{ip(x_b-x_a)}\right]e^{-H_{r_{xW}}(p)(t_b-t_a)}, \tag{108}$$

Two integrations are required. This would make a numerical calculation quite time consuming, Fortunately, one integration can be done analytically. For this purpose we write the integral in the form

$$O(x_a,t_a) = e^{-r_W(t_b-t_a)}\int_{x_E}^{\infty} dx_b\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x_b-x_a)}f(x_a,x_E;p), \tag{109}$$

with

$$f(p) \equiv e^{x_a}e^{-H_{r_{xW}}(p+i)(t_b-t_a)} - e^{x_E}e^{-H_{r_{xW}}(p)(t_b-t_a)}. \tag{110}$$

We have suppressed the arguments $x_a, x_E, t_b - t_a$ in $f(p)$, for brevity. The integral over x_b in (109) runs over the Fourier transform

$$\tilde{f}(x_b - x_a) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x_b-x_a)}f(p), \tag{111}$$

of the function $f(p)$. It is then convenient to express the integral $\int_{x_E}^{\infty} dx_b$ in terms of the Heaviside function $\Theta(x_b - x_E)$ as $\int_{-\infty}^{\infty} dx_b\,\Theta(x_b - x_E)$ and use the Fourier representation

$$\Theta(x_b - x_E) = \int\frac{dq}{2\pi}\frac{i}{q+i\eta}e^{-iq(x_b-x_E)}. \tag{112}$$

of the Heaviside function to write

$$\int_{x_E}^{\infty} dx_b\,\tilde{f}(x_b - x_a) = \int_{-\infty}^{\infty} dx_b\int\frac{dq}{2\pi}\frac{i}{q+i\eta}e^{-iq(x_b-x_E)}\tilde{f}(x_b - x_a). \tag{113}$$

Inserting here the Fourier representation (111), we can perform the integral over x_b and obtain the momentum space representation of the option price

$$O(x_a,t_a) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x_E-x_a)}\frac{i}{p+i\eta}f(p). \tag{114}$$

For numerical integrations, the singularity at $p = 0$ is inconvenient. We therefore employ the well-known decomposition

$$\frac{i}{p+i\eta} = \frac{\mathcal{P}}{p} + \pi\delta(p), \tag{115}$$

to write

$$O(x_a, t_a) = \frac{1}{2} f(0) + i \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{ip(x_E - x_a)} f(p) - f(0)}{p}. \tag{116}$$

We have used the fact that the principal value of the integral over $1/p$ vanishes to subtract the constant $f(0)$ from $e^{ip(x_E-x_a)}f(p)$. After this the integrand is regular, does not need any more the principal-value specification, and allows for a numerical integration.

For x_a very much different from x_E, we may approximate

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{ip(x_E - x_a)} f(p) - f(0)}{p} \approx \frac{1}{2} \epsilon(x_a - x_E) f(0), \tag{117}$$

where $\epsilon(x) \equiv 1 + 2\Theta(x)$ is the step function, and obtain

$$O(x_a, t_a) \approx \frac{1}{2} \left[1 + \Theta(x_a - x_E) \right] f(0). \tag{118}$$

Using (86) we have $e^{-H_{r_x W}(i)} = e^{r_W(t_b - t_a)}$, and since $e^{-H_{r_x W}(0)} = 1$ we see that $O(x_a, t_a)$ goes to zero for $x_a \to -\infty$ and has the large-x_a behavior

$$O(x_a, t_a) \approx \left(e^{x_a} - e^{x_E} e^{-r_W(t_b - t_a)} \right) = S(t_a) - e^{-r_W(t_b - t_a)} E. \tag{119}$$

This is the same behavior as in the Black-Scholes formula (105).

In Fig. 4 we display the difference between the option prices emerging from our formula (116) with a truncated Lèvy distribution of kurtosis $\kappa = 4$, and the Black-Scholes formula (105) for the same data as in the upper left of Fig. 3.

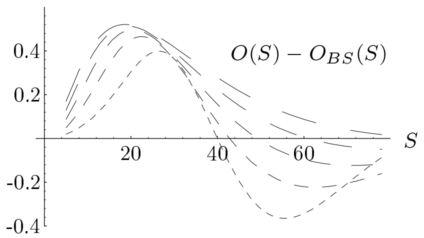

FIG. 4. Difference of the call price O obtained from truncated Lèvy distribution with kurtosis $\kappa = 4$ and the Black-Scholes price as a function of the stock price S for different times before expiration date (increasing dash length: $1, 2, 3, 4, 5$ months). The parameters are $E = 50\,$US\$, $\sigma = 40\%$, $r_W = 6\%$ per month.

VI. CONCLUSION

The stochastic calculus and the option pricing formulas developed in this paper will be useful for estimating financial risks of a variety of investments. In particular, it will help developing a more realistic theory of fair option prices. More details can be found in the textbook Ref. [35].

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