

Gauge-Invariant Critical Exponents for the Ginzburg-Landau Model

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The critical behavior of the Ginzburg-Landau model is described in a manifestly gauge-invariant manner. The gauge-invariant correlation-function exponent is computed to first order in the $4 - d$ and $1/n$ expansion, and found to agree with the ordinary exponent obtained in the covariant gauge, with the parameter $\alpha = 1 - d$ in the gauge-fixing term $(\partial_\mu A_\mu)^2/2\alpha$.

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Despite being one of the most studied field-theoretic models in theoretical physics, the critical behavior of the Ginzburg-Landau model is still poorly understood due to nontrivial gauge properties. The model is defined by the Hamiltonian

$$\mathcal{H} = |(\partial_\mu + ieA_\mu)\phi|^2 + m^2|\phi|^2 + \lambda|\phi|^4 + \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2, \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with $\mu, \nu = 1, \dots, d$, e and m are electric charge and mass, and λ parametrizes the self-interaction. The last (covariant) term with parameter α fixes the gauge. The phase transition occurs where m^2 changes sign. The complex field ϕ has a nonzero expectation value in the ordered phase.

Apart from the standard field-theoretic interpretation, the Ginzburg-Landau model can be equivalently understood as describing a random tangle of intertwined electric current loops of arbitrary length and shape [1,2]. In the normal state, only a few current loops are present due to a finite line tension θ . At the critical temperature T_c , the tension vanishes and the current loops become infinitely long. An important characteristic of these geometrical objects is their fractal dimension D , which at T_c is related to Fisher's critical exponent η , determining the anomalous dimension of the order field (see below). In the absence of gauge fields, this exponent manifests itself in the power behavior of the correlation function

$$G(x - x') \equiv \langle \phi(x)\phi^\dagger(x') \rangle \quad (2)$$

at the critical point as being $G(x) \sim 1/x^{d-2+\eta_\phi}$. The free theory has $G(x) \sim 1/x^{d-2}$, corresponding to $\eta_\phi = 0$. A nonzero value of the critical exponent η_ϕ implies that the dimension of ϕ deviates from the canonical, or engineering dimension $(d - 2)/2$.

In this note, we wish to clarify the properties of this important exponent, which has been controversially discussed in the past, and very recently also in the context of quantum electrodynamics (QED)—the fermionic counterpart of the Ginzburg-Landau model (see Refs. [3–6],

and references therein). The poor understanding of this exponent is because in a gauge theory, the correlation function (2) depends on the gauge parameter α in Eq. (1). In ordinary local quantum field theories without gauge fields, one can prove that it must be greater or equal to zero. In contrast, renormalization group studies have always produced negative (albeit gauge dependent) values, starting with the historic paper by Halperin, Lubensky, and Ma [7]. In gauge theories, the proof of a non-negative η is not applicable due to the nonlocal nature of the gauge-invariant correlation function [8].

We first consider the model close to the upper critical dimension in $d = 4 - \epsilon$ dimensions, and extend the Hamiltonian (1) for later discussions to contain $n/2$ complex fields with an $O(n/2)$ -symmetric self-interaction.

A one-loop perturbative treatment of the Hamiltonian (1) yields the first term in the ϵ expansion of the critical exponents. For η_ϕ , the well-known result is [7,9]:

$$\eta_\phi = \hat{e}_*^2 \frac{\alpha - 3}{8\pi^2} = \frac{6}{n}(\alpha - 3)\epsilon, \quad (3)$$

where $\hat{e}_*^2 = 48\pi^2\epsilon/n$ is the value of the charge at the fixed point. For an infrared-stable fixed point to exist, n must satisfy $n > 12(15 + 4\sqrt{15}) \approx 365.9$ to this order. These results are for the massless model. To avoid infrared divergences, Feynman diagrams are evaluated at finite external momentum κ . Being the only scale available, κ is used to remove the dimension from dimensionful parameters, for example, $\hat{e}^2 = e^2\kappa^{d-4}$. Most calculations reported in the literature are performed in the Landau gauge ($\alpha = 0$) for which [7] $\eta_\phi \rightarrow -18\epsilon/n$.

Since the correlation function (2) is not gauge invariant, it is not a physical quantity. It is therefore not surprising to find that the critical exponent η_ϕ depends on the gauge parameter α . As long as the gauge is fixed by the last term in Eq. (1), the correlation function is nevertheless well defined and the critical exponents satisfy the usual scaling laws: $\beta_\phi = \frac{1}{2}\nu(d - 2 + \eta_\phi)$, $d\nu = \beta_\phi(\delta_\phi + 1)$, with η_ϕ given in Eq. (3), as, following Ref. [10], can be verified explicitly to first order in ϵ . Since the correlation length exponent ν is gauge independent, β_ϕ and δ_ϕ depend on α .

A physical correlation function must be gauge invariant, i.e., invariant under the combined transformations $\phi(x) \rightarrow \exp[ie\Lambda(x)]\phi(x)$, $A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Lambda(x)$. Such a correlation function is obtained by including a path-dependent Schwinger phase factor in Eq. (2), forming

$$\mathbf{G}(x - x') \equiv \langle \phi(x)\phi^\dagger(x')e^{-ie \int_x^{x'} d\bar{x}_\mu A_\mu(\bar{x})} \rangle, \quad (4)$$

where the line integral extends from x to x' , and the average, denoted by angle brackets, is taken with respect to the Hamiltonian (1). At the critical point, we now expect a power behavior $G(x) \sim 1/x^{d-2+\eta_{\text{GI}}}$, where in contrast to η_ϕ , the exponent η_{GI} has no α dependence. The exponential in Eq. (4) can alternatively be written in terms of an external electric current line $J_\mu(z)$ as $e^{-i \int d^d z J_\mu(z) A_\mu(z)}$, where $J_\mu(z) = e \int_x^{x'} d\bar{x}_\mu \delta(z - \bar{x})$ is a delta function along the path from x to x' which satisfies the current conservation law $\partial_\mu J_\mu(z) = e\delta(z - x) - e\delta(z - x')$, with a current source at x and a sink at x' . In Refs. [11,12], the gauge-invariant correlation function (4), with its external current line, was studied in $d = 3$ and found to behave differently in the normal and superconductive state. In the normal state, where the line tension θ of the current line was shown to be finite, this correlation function decreases exponentially for large separation, $\mathbf{G}(r) \sim e^{-\theta r}$, with $r_\mu = x'_\mu - x_\mu$ being the distance vector. In the superconductive state, on the other hand, the line tension vanishes and the correlation function was found to behave instead as $\mathbf{G}(r) \sim \exp(e^2 \lambda_L^2 / 4\pi r)$, with λ_L being the London penetration depth. Rather than tending to zero, the correlation function now reaches a finite value for large separation. The finite expectation value at infinite separation signals that the current lines have lost their line tension and have become infinitely long. In the correlation function (4) it manifests itself in an independence on the path over which the line integral is taken. Only the end points of the line connecting x and x' are physical. The exponent in the correlation function contains a Coulomblike interaction between these end points. (Note that the combination $e\lambda_L$ is independent of the electric charge.)

To compute the exponent η_{GI} to first order in the ϵ expansion, the gauge-invariant correlation function (4) is expanded to order e^2 . Then, using Wick's theorem, three contributions are obtained besides the lowest order

$$\mathbf{G}(r) = G + T_0 + T_1 + T_2, \quad (5)$$

containing, respectively, no, one, and two Schwinger phase factors. The first contribution T_0 is given by

$$e^2 \int d^d z d^d z' [G(x - z) \overleftrightarrow{\partial}_{z_\mu} G(z' - x') \overleftrightarrow{\partial}_{z'_\nu} G(z' - z)] \\ \times D_{\mu\nu}(z - z'),$$

where the right minus left derivatives $\overleftrightarrow{\partial}_{z_\mu} \equiv \partial_{z_\mu} - \overleftarrow{\partial}_{z_\mu}$

operate only within the square brackets, and $D_{\mu\nu}$ is the correlation function of the gauge-field A_μ in Eq. (1), with the Fourier components

$$D_{\mu\nu}(q) = \frac{1}{q^2} \left[\delta_{\mu\nu} - (1 - \alpha) \frac{q_\mu q_\nu}{q^2} \right]. \quad (6)$$

In momentum space this yields

$$T_0 = e^2 \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{e^{ik \cdot r}}{k^4} \frac{(q + \frac{3}{2}k)_\mu (q + \frac{3}{2}k)_\nu}{(q + k/2)^2} \\ \times D_{\mu\nu}(q - k/2).$$

Since the fixed-point value of e^2 is of order ϵ [see below Eq. (3)], the integrals can either be evaluated directly in $d = 4$ or using dimensional regularization and taking the limit $d \rightarrow 4$ at the end. Either way gives

$$T_0(r) = \hat{e}^2 \frac{\alpha - 3}{8\pi^2} \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot r}}{k^2} \ln \frac{k}{\kappa} \\ = -\hat{e}^2 \frac{\alpha - 3}{8\pi^2} G(r) \ln \kappa r$$

with logarithmic accuracy. Adding the free scalar correlation function $G(r)$, we obtain

$$G(r) + T_0(r) = G(r) \left[1 - \hat{e}^2 \frac{\alpha - 3}{8\pi^2} \ln \kappa r \right] \\ \approx G(r) r^{-\hat{e}^2(\alpha-3)/8\pi^2}, \quad (7)$$

which reproduces the old result (3).

The last term in Eq. (5),

$$T_2(r) = -\frac{1}{2} G(r) \int d^d z d^d z' J_\mu(z) D_{\mu\nu}(z - z') J_\nu(z'), \quad (8)$$

factorizes from the start in a scalar and gauge part, with the second factor—which plays a central role in the study of the gauge-invariant order parameter of the Ginzburg-Landau model [11,12]—denoting the Biot-Savart interaction between two segments of the external current line. Since the integrals in Eq. (8) are line integrals, their value depends on the path chosen. Because of the finite string tension, the external current line connects the two end points x and x' on the shortest path. The relevant integration paths are therefore straight lines, and

$$T_2(r) = -\frac{e^2}{2} G(r) \int_0^1 du du' r_\mu D_{\mu\nu}[(u' - u)r] r_\nu, \quad (9)$$

after the reparametrization $\bar{x}_\mu = x_\mu + ur_\mu$, $\bar{x}'_\mu = x'_\mu + u'r_\mu$, with a fixed distance vector $r_\mu = x'_\mu - x_\mu$ and $0 \leq u, u' \leq 1$. The integrals are easily evaluated following Ref. [3], with the result

$$-\frac{e^2}{(3-d)(4-d)} \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} \left[1 - \frac{1}{2}(1 - \alpha)(3 - d) \right] \\ \times G(r) r^{4-d}, \quad (10)$$

which for d near 4 yields

$$T_2(r) = \hat{e}^2 \frac{3 - \alpha}{8\pi^2} G(r) \left[\ln(\kappa r) + \frac{1}{\epsilon} \right]. \quad (11)$$

Because of the appearance of the logarithm multiplying $G(r)$, this gives the contribution

$$\eta_2 = \hat{e}_*^2 \frac{\alpha - 3}{8\pi^2} \quad (12)$$

to the Fisher exponent. When both contributions obtained so far are subtracted, one obtains a result (which happens to be zero) independent of α . As first noted in the context of QED [4], this is because the combination $\eta_\phi - \eta_2$ characterizes the correlation function

$$\langle \phi(x) \phi^\dagger(x') \rangle \left\langle \exp \left[i e \int_x^{x'} d\bar{x}_\mu A_\mu(\bar{x}) \right] \right\rangle^{-1}, \quad (13)$$

which, although not fully gauge invariant, is independent of the gauge-fixing parameter α within the class of covariant gauges.

Next, the third, or mixed term in Eq. (5), given by

$$e \int d^d z d^d z' [G(x - z) \vec{\partial}_{z_\mu} G(z - x')] J_\nu(z') D_{\mu\nu}(z - z'), \quad (14)$$

is evaluated. We expect an α -dependent contribution that precisely cancels the dependence on the gauge parameter found in Eqs. (3) and (12). To extract the term of the form $G(r) \ln(r)$, we use the approximation, cf. Ref. [4],

$$T_1(r) \approx e G(r) \int d^d z d^d z' [\partial_{z_\mu} G(z - x') - \partial_{z_\mu} G(x - z)] \times J_\nu(z') D_{\mu\nu}(z - z'), \quad (15)$$

valid with logarithmic accuracy. Both terms in the square brackets give the same contribution. Partially integrating this expression and using the identity $\partial_\mu(x_\mu x_\nu/x^4) = [(3 - d)/2] \partial_\nu x^{-2}$ in $d = 4$, we obtain

$$T_1(r) = -e^2 \frac{\alpha}{2\pi^2} G(r) \int d^d z G(z - x') \frac{1}{(z - x)^2}, \quad (16)$$

giving $(\hat{e}^2 \alpha / 4\pi^2) G(r) \ln(\kappa r)$ and thus a contribution to η

$$\eta_1 = -\hat{e}_*^2 \frac{\alpha}{4\pi^2}. \quad (17)$$

As expected, this contribution precisely cancels the α dependence in Eqs. (3) and (12). More specifically, we obtain for the gauge-invariant correlation function

$$\eta_{\text{GI}} = \eta_\phi + \eta_1 + \eta_2 = -\hat{e}_*^2 \frac{3}{4\pi^2} = -\frac{36}{n} \epsilon. \quad (18)$$

This value for η_{GI} is twice that for η_ϕ obtained in the Landau gauge ($\alpha = 0$). Both results coincide, however, when $\alpha = -3$.

In the current loop description, the critical exponent η_{GI} determines the fractal dimension D of the current

lines via [13,14] $D = 2 - \eta_{\text{GI}}$. With $\eta_{\text{GI}} < 0$, the fractal dimension is larger than that of Brownian random walks for which $D = 2$, implying that the current lines are self-seeking, which makes them more crumpled than Brownian random walks. Although higher-order corrections may well change the sign of η_{GI} , nothing in the context of the Ginzburg-Landau model forbids negative values [8,11], provided $\eta_{\text{GI}} > 2 - d$, or $D < d$.

Instead of an ϵ expansion, we may compute the gauge-invariant critical exponent η_{GI} nonperturbatively in the limit of a large number n of field components. Then η can be expanded in powers of $1/n$ for all $2 < d < 4$.

The leading contribution in $1/n$ generated by fluctuations in the gauge field is obtained by dressing its correlation function with arbitrary many bubble insertions, and summing the entire set of Feynman diagrams [15]. The resulting series is a simple geometrical one, which leads to the following change in the denominator of the prefactor in the correlation function (6):

$$q^2 \rightarrow q^2 + e^2 \frac{n}{2} \frac{c(d)}{(d-1)} q^{d-2}, \quad (19)$$

where the second term dominates the first one for small q in $2 < d < 4$. In Eq. (19), $c(d)$ stands for the 1-loop integral

$$c(d) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k+p)^2} \Big|_{p^2=1} = \frac{\Gamma(2 - d/2) \Gamma^2(d/2 - 1)}{(4\pi)^{d/2} \Gamma(d-2)}, \quad (20)$$

where analytic regularization is used to handle the ultraviolet divergences. To leading order in $1/n$ the value of η_ϕ for $2 < d < 4$ reads [16,17]

$$\eta_\phi = \frac{24 - d - (d-1)[4(d-1) - d\alpha]}{n (4\pi)^{d/2} c(d) \Gamma(d/2 + 1)}, \quad (21)$$

which depends on the gauge parameter α . For $d = 4 - \epsilon$, this result reduces to Eq. (3) obtained to first order in the ϵ expansion. The gauge dependence of η_ϕ is not always obvious from the results quoted in the literature as often a specific gauge is chosen from the start, for example, the Landau gauge [7], where $\eta_\phi \rightarrow -40/\pi^2 n$ for $d = 3$.

The term (8) with the modified gauge-field correlation function can be evaluated as before. To extract the dependence on $\ln(r)$ it proves useful to replace q^{d-2} with $q^{d-2+\delta}$ in Eq. (19), so that the gauge-field correlation function in the large- n limit becomes

$$D_{\mu\nu}(q) = \frac{2}{ne^2} \frac{d-1}{c(d)} \frac{1}{q^{d-2+\delta}} \left[\delta_{\mu\nu} - (1-\alpha) \frac{q_\mu q_\nu}{q^2} \right], \quad (22)$$

and to let $\delta \rightarrow 0$ at the end. This leads to

$$T_2(r) = \frac{1}{n} \frac{8(d-1)}{(4\pi)^{d/2} c(d) \Gamma(d/2-1)} \left(1 + \frac{1-\alpha}{d-2}\right) G(r) \frac{r^\delta}{\delta}, \quad (23)$$

with $r^\delta/\delta \rightarrow 1/\delta + \ln(r)$, and

$$\eta_2 = -\frac{4(d-1)(d-1-\alpha)}{n(4\pi)^{d/2} c(d) \Gamma(d/2)}. \quad (24)$$

As a check note that, when subtracted from η_ϕ given in Eq. (21), this yields an α -independent result characterizing the correlation function (13),

$$\eta_\phi - \eta_2 = \frac{1}{n} \frac{16}{(4\pi)^{d/2} c(d) \Gamma(d/2-2)}. \quad (25)$$

This expression is negative for all $2 < d < 4$. Specifically, $\eta_\phi - \eta_2 = -4\epsilon^2/n + \mathcal{O}(\epsilon^3)$ for $d = 4 - \epsilon$.

To calculate the mixed term (14), the gauge-field correlation is needed in coordinate space. Fourier transforming Eq. (22), we arrive at

$$D_{\mu\nu}(x) = \frac{2}{ne^2} \frac{d-1}{c(d)} \frac{4}{(4\pi)^{d/2} \Gamma(d/2)} \frac{1}{x^{2-\delta}} \times \left[\frac{1}{2} (d-3+\alpha) \delta_{\mu\nu} + (1-\alpha) \frac{x_\mu x_\nu}{x^2} \right]. \quad (26)$$

Proceeding as before, we obtain

$$T_1(r) = -\frac{\alpha}{n} \frac{16(d-1)}{(4\pi)^{d/2} c(d) \Gamma(d/2-1)} G(r) \times \int d^d z G(z-x') \frac{1}{(z-x)^{2-\delta}} = -\eta_1 G(r) \ln(r), \quad (27)$$

with

$$\eta_1 = -\frac{\alpha}{n} \frac{8(d-1)}{(4\pi)^{d/2} c(d) \Gamma(d/2)}, \quad (28)$$

which, being proportional to α , should cancel the α dependence in η_ϕ and η_2 . And indeed, the sum

$$\eta_{\text{GI}} = \eta_\phi + \eta_1 + \eta_2 = -\frac{4}{n} \frac{(d^2 + 2d - 6) \Gamma(d-2)}{\Gamma(2-d/2) \Gamma^2(d/2-1) \Gamma(d/2)} \quad (29)$$

is independent of the gauge parameter α . Remarkably, for the d -dependent gauge choice $\alpha = 1 - d$ (of which our $\alpha = -3$ found in the ϵ expansion is a special case) the η_ϕ of Eq. (21) coincides with η_{GI} to this order in $1/n$. With this gauge choice, the trace of the gauge-field correlation function vanishes, $D_{\mu\mu}(q) = 0$. Since this observation does not depend on the matter part of the theory, we expect it to hold also in QED. And indeed, the values [4,5] for the two η exponents obtained in first order in $1/n$ in $d = 3$ and also in the ϵ expansion coincide when $\alpha = 1 - d$. Although higher-order corrections might change this simple relation between the two exponents,

we speculate that in first order the gauge choice $\alpha = 1 - d$ provides a shortcut for obtaining the gauge invariant result of other quantities such as the effective potential and mass renormalization.

The expression (29) is negative for all $2 < d < 4$, with $\eta_{\text{GI}} = -72/\pi^2 n$ for $d = 3$ and $-36\epsilon/n + \mathcal{O}(\epsilon^2)$ for $d = 4 - \epsilon$. The latter result is in accord with Eq. (18) obtained in perturbation theory. We repeat that in the context of the Ginzburg-Landau model, negative values are allowed, provided $\eta_{\text{GI}} > 2 - d$, so that the fractal dimension $D = 2 - \eta_{\text{GI}}$ of the current lines is smaller than the dimension of the embedding space. A negative value merely indicates that the current lines are self-seeking.

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