Perturbatively Defined Effective Classical Potential in Curved Space

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Abstract

The partition function of a quantum statistical system in flat space can always be written as an integral over a classical Boltzmann factor $\exp[-\beta V^{\mathrm{eff\,cl}}(\mathbf{x}_0)]$, where $V^{\mathrm{eff\,cl}}(\mathbf{x}_0)$ is the so-called *effective classical potential* containing the effects of all quantum fluctuations. The variable of integration is the temporal path average $\mathbf{x}_0 \equiv \beta^{-1}\int_0^\beta d\tau\, \mathbf{x}(\tau)$. We show how to generalize this concept to paths $q^\mu(\tau)$ in curved space with metric $g_{\mu\nu}(q)$, and calculate perturbatively the high-temperature expansion of $V^{\mathrm{eff\,cl}}(q_0)$. The requirement of independence under coordinate transformations $q^\mu(\tau) \to q'^\mu(\tau)$ introduces subtleties in the definition and treatment of the path average q_0^μ, and covariance is achieved only with the help of a suitable Faddeev-Popov procedure.

1 Introduction

Path integrals for particles in curved space are defined unambiguously as nonholonomic images of flat-space path integrals (the procedure following the so-called *nonholonomic mapping principle*) [1]. The resulting time evolution amplitudes satisfy automatically the correct Schrödinger equation *without* an extra R-term [2], and they are invariant under arbitrary coordinate transformations. For perturbatively defined path integrals, a similar implementation of the nonholonomic mapping principle in curved space has so far not been found. There the absence of an extra R-term must be inferred from the above time-sliced theory. Even coordinate invariance was a problem for a long time, resolved only recently by our treatment via good-old dimensional regularization [3]. Moreover, we were able to find well-defined calculation rules for dealing with products of distributions, which enable us now to perform perturbation expansions *without* a tedious extension of the dimension of the time axis. Due to the local nature of these rules, they can be applied to Feynman integrals with infinite as well as finite propagation times [4, 5].

One important aspect of perturbatively defined path integrals with a finite propagation time has, however, remained puzzling. The results have so far been found correctly only if calculations are done with propagators satisfying Dirichlet boundary conditions. Attempts with periodic paths have led to noncovariant results [6].

Some years ago it has been pointed out by Feynman and one of the authors (H.K.) that in flat space, the temporal average $\mathbf{x}_0 \equiv \bar{\mathbf{x}}(\tau) = \beta^{-1} \int_0^\beta d\tau\, \mathbf{x}(\tau)$ of periodic paths plays a special role in isolating the classical fluctuations in a path integral over periodic paths [7].

An ordinary integral over $\mathbf{x}_0$ which has the form of a classical partition function can produce the full quantum statistical result, if it is performed over a Boltzmann factor containing the so-called *effective classical potential* $V^{\mathrm{eff\ cl}}(\mathbf{x}_0)$. If a similar quantity is calculated in curved space keeping the temporal average $q_0 \equiv \bar{q}(\tau) \equiv \beta^{-1} \int_0^\beta d\tau q(\tau)$ fixed, the two-loop perturbative result for $V^{\mathrm{eff\ cl}}(q_0)$ turned out to deviate from the covariant one by a noncovariant total derivative [6], in contrast to the covariant result obtained with Dirichlet boundary conditions. For this reason, perturbatively defined path integrals with periodic boundary conditions in curved space have been of limited use in the presently popular first-quantized worldline approach to quantum field theory (also called the string-inspired approach reviewed in Ref. [8]). In particular, is has so far been impossible to calculate with periodic boundary conditions interesting quantities such as curved-space effective actions, gravitational anomalies, and index densities [9], all results having been reproduced with Dirichlet boundary conditions [8].

The purpose of this paper is to improve the situation by developing a manifestly covariant integration procedure for periodic paths. It is an adaption of similar procedures used before in the effective action formalism of two-dimensional sigma-models [10, 11]. Covariance is achieved by expanding the fluctuations in the neighborhood of any given point in powers of geodesic coordinates, and by a covariant definition of a path average different from the naive temporal average. As a result, we shall find the same locally covariant perturbation expansion of the effective classical potential as in earlier work with Dirichlet boundary conditions [4].

An important role in the development is played by the Faddeev-Popov method, which produces a Jacobian and an associated new effective interaction necessary to guarantee covariance.

2 Partition function

Consider a quantum particle moving in a compact Riemannian space with metric $g_{\mu\nu}(q)$ and coordinates $q^\mu(\tau),\ \mu = 1, \ldots, D$. The partition function can be written as an integral over the *partition function density* $z(q)$:

$$Z \;=\; \int d^D q \sqrt{g(q)}\, z(q), \tag{1}$$

where $g = \det g_{\mu\nu}$. The partition function density is equal to the diagonal time evolution amplitude $\langle q^\mu \beta | q^\mu 0 \rangle$, and has the path integral representation

$$z(q) = \langle q^\mu \beta | q^\mu 0 \rangle \;=\; \int_{q^\mu(0)=q^\mu}^{q^\mu(\beta)=q^\mu} \mathcal{D}^D q(\tau) \sqrt{g(q(\tau))} e^{\mathcal{A}_{\mathrm{e}}[q]}, \tag{2}$$

with the euclidean action

$$\mathcal{A}_{\rm e}[q] = \int_0^\beta d\tau \frac{1}{2} g_{\mu\nu}(q)\dot{q}^\mu(\tau)\dot{q}^\nu(\tau). \tag{3}$$

The invariant measure represents formally the product

$$\mathcal{D}^D q\sqrt{g(q)} \equiv \prod_{\mu,\tau}\left[dq^\mu(\tau)\sqrt{g(q(\tau))}\right]. \tag{4}$$

In our notation, a single symbol $\sqrt{g(q(\tau))}$ in the measure on the left-hand side symbolizes a factor $\sqrt{g(q(\tau))}$ for *each* time point.

For small inverse temperature β, the path integral (2) can be calculated perturbatively using Green function with Dirichlet boundary conditions, leading to a manifestly covariant high-temperature expansion, whose initial terms are [4, 5, 1]

$$\langle q^\mu \beta | q^\mu 0\rangle = \frac{1}{\sqrt{2\pi\beta}^D}\left[1 - \frac{1}{24}R(q)\beta + \dots\right]. \tag{5}$$

This differs from the well-known quantum-mechanical DeWitt-Seeley expansion of the exponential of the Laplace-Beltrami operator $\Delta = \sqrt{g}^{-1}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu$,

$$\langle q^\mu | e^{\beta\Delta/2} | q^\mu\rangle = \frac{1}{\sqrt{2\pi\beta}^D}\left[1 + \frac{1}{12}R(q)\beta + \dots\right] \tag{6}$$

by a term $R(q)\beta/8$ in the brackets. Since the correct path integral defined by the nonholonomic mapping principle in Ref. [1] agrees with (6), the invariant volume element (4) of path integration must be corrected by a factor $\exp[\int_0^\beta d\tau R(q)/8]$.

For consistency of the perturbative approach, we must of course be able to calculate the same partition function (1) with the same result (5) by performing a functional integral over all periodic paths

$$Z^{\rm P} \;=\; \oint \mathcal{D}^D q\sqrt{g(q)}e^{-\mathcal{A}_{\rm e}[q]}, \tag{7}$$

where the symbol $\oint$ indicates the periodicity of the paths. A method for doing this will now be developed.

3 Covariant fluctuation expansion

For small β, the path integral (7) is dominated by the constant paths $q^\mu(\tau) = q_0^\mu$ which are solutions of the classical equations of motion whose classical action vanishes. To derive the high-temperature expansion of the path integral (7), we parametrize the small fluctuations around q_0^μ covariantly by geodesic coordinates $\xi^\mu(\tau)$. This is done with a *nonlinear* decomposition

$$q^\mu(\tau) = q_0^\mu + \eta^\mu(q_0, \xi), \tag{8}$$

where $\eta^\mu(q_0,\xi)=0$ for $\xi^\mu=0$. The geodesic coordinates $\xi^\mu(\tau)$ are the tangent vectors at q_0^μ to the geodesic connecting the points q_0^μ and $q_0^\mu+\eta^\mu$. The functions $\eta^\mu(q_0,\xi)$ have the expansion

$$\eta^\mu(q_0,\xi)=\xi^\mu-\frac{1}{2}\Gamma_{(\sigma\tau)}{}^\mu(q_0)\xi^\sigma\xi^\tau-\frac{1}{6}\Gamma_{(\sigma\tau\kappa)}{}^\mu(q_0)\xi^\sigma\xi^\tau\xi^\kappa-\dots, \tag{9}$$

where $\Gamma_{\sigma\tau}{}^\mu(q_0)$ with two lower indices is the usual Christoffel symbol, while the other coefficients $\Gamma_{\sigma\tau\dots\kappa}{}^\mu(q_0)$ are its successive covariant derivatives with respect to lower indices only,

$$\Gamma_{\sigma\tau\kappa}{}^\mu(q_0)=\nabla_\kappa\Gamma_{\sigma\tau}{}^\mu=\partial_\kappa\Gamma_{\sigma\tau}{}^\mu-2\Gamma_{\kappa\sigma}{}^\nu\Gamma_{\nu\tau}{}^\mu,\dots \tag{10}$$

The parentheses around the subscripts in (9) indicate symmetrization with respect to all possible cyclic permutations.

If the initial coordinates q^μ are themselves geodesic at q_0^μ, all coefficients $\Gamma_{(\sigma\tau\dots\kappa)}{}^\mu(q_0)$ in Eq. (9) are zero, so that $\eta^\mu(\tau)=\xi^\mu(\tau)$, and the decomposition (8) is linear. In this case, the derivatives of the Christoffel symbols can be expressed directly in terms of the curvature tensor:

$$\partial_\kappa\Gamma_{\tau\sigma}{}^\mu(q_0)=-\frac{1}{3}\left[R_{\tau\kappa\sigma}{}^\mu(q_0)+R_{\sigma\kappa\tau}{}^\mu(q_0)\right],\qquad\text{for normal coordinates.} \tag{11}$$

In arbitrary coordinates, however, $\eta^\mu(\tau)$ does not transform like a vector under coordinate transformations, and we must use the nonlinear decomposition (8).

We now transform the path integral (7) to the new coordinates $\xi^\mu(\tau)$ using Eqs. (8)–(10). The perturbation expansion for the transformed path integral over $\xi^\mu(\tau)$ is constructed for any chosen q_0^μ by expanding the action (3) and the measure (4) in powers of small linear fluctuations $\xi^\mu(\tau)$. The expansion starts out like

$$\begin{aligned}\mathcal{A}_{\rm e}[q] &= \int_0^\beta d\tau\frac{1}{2}\left[g_{\mu\nu}(q_0)+\partial_\sigma g_{\mu\nu}(q_0)\eta^\sigma+\frac{1}{2}\partial_\sigma\partial_\tau g_{\mu\nu}(q_0)\eta^\sigma\eta^\tau+\dots\right]\dot\eta^\mu\dot\eta^\nu\\ &= \int_0^\beta d\tau\frac{1}{2}\left[g_{\mu\nu}(q_0)+\frac{1}{3}R_{\mu\lambda_1\nu\lambda_2}(q_0)\xi^{\lambda_1}\xi^{\lambda_2}+\dots\right]\dot\xi^\mu\dot\xi^\nu.\end{aligned} \tag{12}$$

The leading small-β behavior of the path integral (7) is given by the quadratic term in $\xi^\mu(\tau)$, which we write after a partial integration as

$$\mathcal{A}_{\rm e}^{(0)}[q_0,\xi]=\int_0^\beta d\tau\frac{1}{2}\xi^\mu(\tau)[-g_{\mu\nu}(q_0)d_\tau^2]\xi^\nu(\tau). \tag{13}$$

The next term in β is caused by the interaction of fourth order in the fluctuations:

$$\mathcal{A}_{\rm e}^{\rm int,4}[q_0,\xi]=\int_0^\beta d\tau\frac{1}{6}R_{\mu\lambda_1\nu\lambda_2}(q_0)\xi^{\lambda_1}(\tau)\xi^{\lambda_2}(\tau)\dot\xi^\mu(\tau)\dot\xi^\nu(\tau). \tag{14}$$

A further contribution comes from the invariant measure (4). This is transformed to the coordinates $\eta^\mu(\tau)$ as

$$\mathcal{D}^Dq\sqrt{g(q)}=\prod_{\mu,\tau}\left[dq^\mu(\tau)\sqrt{g(q(\tau))}\right]=\sqrt{g(q_0)}^N\prod_{\mu,\tau}\left[d\eta^\mu(\tau)\frac{\sqrt{g(q_0+\eta(\tau))}}{\sqrt{g(q_0)}}\right], \tag{15}$$

and further to the geodesic coordinates $\xi^\mu(\tau)$ as

$$\mathcal{D}^D q\sqrt{g(q)} = \sqrt{g(q_0)}^N J(q_0,\xi)\prod_{\mu,\tau}\left[d\xi^\mu(\tau)\frac{\sqrt{g(q_0+\eta(q_0,\xi))}}{\sqrt{g(q_0)}}\right], \tag{16}$$

where $J(q_0,\xi)$ is the Jacobian of the transformation (9):

$$J(q_0,\xi) = \exp\left\{\int_0^\beta d\tau\,\delta(\tau,\tau)\,\mathrm{tr}\log\left(\frac{\partial\eta^\mu}{\partial\xi^\nu}\right)\right\}. \tag{17}$$

The trace of the logarithm in the exponent has the small-ξ^μ expansion

$$\mathrm{tr}\log\left(\frac{\partial\eta^\mu}{\partial\xi^\nu}\right) = -\Gamma_{\mu\sigma}{}^\mu\xi^\sigma + \frac{1}{3}\left(\frac{1}{2}\Gamma_{\nu\tau}{}^\mu\Gamma_{\mu\sigma}{}^\nu + \Gamma_{\tau\sigma}{}^\nu\Gamma_{\nu\mu}{}^\mu - \partial_\sigma\Gamma_{\mu\tau}{}^\mu - \frac{1}{2}\partial_\mu\Gamma_{\sigma\tau}{}^\mu\right)\xi^\sigma\xi^\tau + \dots. \tag{18}$$

The exponent contains also an infinite quantity

$$N = \int_0^\beta d\tau\,\delta(\tau,\tau) = \beta\,\delta(0), \tag{19}$$

which formally represents the total number of points on the time axis and counts simultaneously the number of eigenvalues of the operator $-g_{\mu\nu}(q_0)d_\tau^2$ in the space of periodic functions $\xi^\mu(\tau)$. By rewriting also the product on the right-hand side of Eq. (16) as an exponential

$$\prod_\tau \frac{\sqrt{g(q_0+\eta(q_0,\xi))}}{\sqrt{g(q_0)}} = \exp\left\{\int_0^\beta d\tau\,\delta(\tau,\tau)\frac{1}{2}\log\frac{g(q_0+\eta(q_0,\xi))}{g(q_0)}\right\}, \tag{20}$$

and expanding

$$\frac{1}{2}\log\frac{g(q_0+\eta)}{g(q_0)} = \Gamma_{\mu\sigma}{}^\mu\eta^\sigma + \frac{1}{2}\partial_\sigma\Gamma_{\tau\mu}{}^\mu\eta^\sigma\eta^\tau + \dots\ , \tag{21}$$

and this further into

$$\frac{1}{2}\log\frac{g(q_0+\eta(q_0,\xi))}{g(q_0)} = \Gamma_{\mu\sigma}{}^\mu\xi^\sigma + \frac{1}{2}\left(\partial_\sigma\Gamma_{\tau\mu}{}^\mu - \Gamma_{\nu\mu}{}^\mu\Gamma_{\sigma\tau}{}^\nu\right)\xi^\sigma\xi^\tau + \dots, \tag{22}$$

we may combine the expansions (20), (22) with (17), (18), and obtain

$$\prod_{\mu,\tau}\left[dq^\mu(\tau)\sqrt{g(q(\tau))}\right] = \sqrt{g(q_0)}^N\prod_{\mu,\tau}\left[d\xi^\mu(\tau)\right]\exp\left\{-\mathcal{A}_\mathrm{e}^\mathrm{meas}[q_0,\xi]\right\}, \tag{23}$$

where $\mathcal{A}_\mathrm{e}^\mathrm{meas}[q_0,\xi]$ plays the role of an interaction coming from the invariant measure. Its expansion starts out like

$$\mathcal{A}_\mathrm{e}^\mathrm{meas}[q_0,\xi] = \int_0^\beta d\tau\,\delta(\tau,\tau)\frac{1}{6}R_{\mu\nu}(q_0)\xi^\mu(\tau)\xi^\nu(\tau) + \dots\ . \tag{24}$$

Collecting all terms, we obtain the desired expansion of the partition function (7) in terms of the coordinates $\xi^\mu(\tau)$ around the origin

$$Z^{\rm P} = \oint \mathcal{D}^D\xi(\tau)\sqrt{g(q_0)}\, e^{-\mathcal{A}_{\rm e}^{(0)}[q_0,\xi]-\mathcal{A}_{\rm e}^{\rm int}[q_0,\xi]}, \tag{25}$$

with the total interaction

$$\mathcal{A}_{\rm e}^{\rm int}[q_0,\xi] = \mathcal{A}_{\rm e}^{\rm int,4}[q_0,\xi] + \mathcal{A}_{\rm e}^{\rm meas}[q_0,\xi], \tag{26}$$

and the measure written down in the notation (4).

The path integral (25) cannot immediately be calculated perturbatively in the standard way, since the quadratic form of the free action (13) is degenerate. The spectrum of the operator $-d_\tau^2$ in the space of periodic functions $\xi^\mu(\tau)$ has a zero mode. The zero mode is associated with the fluctuations of the temporal average of $\xi^\mu(\tau)$:

$$\xi_0^\mu = \bar{\xi}^\mu \equiv \beta^{-1}\int_0^\beta d\tau\, \xi^\mu(\tau). \tag{27}$$

Small fluctuations of ξ_0^μ have the effect of moving the path as a whole infinitesimally through the manifold. The same movement can be achieved by changing q_0^μ infinitesimally. Thus we can replace the integral over the path average ξ_0^μ by an integral over q_0^μ, provided that we properly account for the change of measure arising from such a variable transformation.

Anticipating such a change, the path average (27) can be set equal to zero eliminating the zero mode in the fluctuation spectrum. The basic free correlation function $\langle\xi^\mu(\tau)\xi^\nu(\tau')\rangle$ can then easily be found from its spectral representation. We solve the trivial eigenvalue problem of the operator $-d_\tau^2$ in the quadratic action (13):

$$-d_\tau^2\, u_m(\tau) = \lambda_m u_m(\tau), \tag{28}$$

and impose periodic boundary conditions $u_m(0) = u_m(\beta)$. The obvious eigenfunctions are $u_m(\tau) = e^{-i\omega_m\tau}$, where $\omega_m = 2\pi m/\beta$ are the Matsubara frequencies with $m = 0, \pm1, \pm2, \dots$. The eigenvalues are $\lambda_m = \omega_m^2$. The eigenfunctions are orthonormal,

$$\frac{1}{\beta}\int_0^\beta d\tau\, u_m^*(\tau)u_{m'}(\tau) = \delta_{m,m'}. \tag{29}$$

and satisfy the completeness relation

$$\frac{1}{\beta}\sum_m u_m^*(\tau)u_m(\tau') = \delta(\tau-\tau'). \tag{30}$$

Fixing $\xi_0^\mu = 0$ removes the troublesome eigenmode $\lambda_0 = 0$ from the spectral representation. This leads to the correlation function

$$\langle\xi^\mu(\tau)\xi^\nu(\tau')\rangle^{q_0} = g^{\mu\nu}(q_0)[-d_\tau^2]^{-1}(\tau-\tau') = g^{\mu\nu}(q_0)\Delta'(\tau,\tau'), \tag{31}$$

where $\Delta'(\tau,\tau')$ is the Green function of the operator $-d_\tau^2$ without the zero mode:

$$\Delta'(\tau,\tau') = \frac{1}{\beta}\sum_{m\neq0}\frac{u_m^*(\tau)u_m(\tau')}{\lambda_m} = \frac{1}{\beta}\sum_{m\neq0}\frac{e^{-i\omega_m(\tau-\tau')}}{\omega_m^2}. \tag{32}$$

Performing the sum with the help of the formula

$$\sum_{m=1}^{\infty} \frac{\cos m\tau}{m^2} = \frac{1}{6}\pi^2 - \frac{1}{2}\pi|\tau| + \frac{1}{4}\tau^2, \quad \tau \in [0, \beta), \tag{33}$$

yields for τ, τ' in the presently relevant interval $[0, \beta)$ the translationally invariant expression:

$$\Delta'(\tau, \tau') = \Delta'(\tau - \tau') \equiv \frac{|\tau - \tau'|^2}{2\beta} - \frac{|\tau - \tau'|}{2} + \frac{\beta}{12}\,. \tag{34}$$

which satisfies the inhomogeneous differential equation

$$-d_\tau^2\,\Delta'(\tau - \tau') = \delta'(\tau - \tau')\,, \tag{35}$$

where the right-hand side contains an extra term on the right-hand side due to the missing zero eigenmode in the spectral representation:

$$\delta'(\tau - \tau') \equiv \frac{1}{\beta}\sum_{m\neq 0} e^{-i\omega_m(\tau-\tau')} = \delta(\tau - \tau') - \frac{1}{\beta}\,. \tag{36}$$

Both $\delta'(\tau - \tau')$ and $\Delta'(\tau - \tau')$ are periodic in the interval $\tau - \tau' \in [0, \hbar\beta)$.

4 Arbitrariness of q_0^μ

We now take advantage of an important property of the perturbation expansion of the partition function (25) around $q^\mu(\tau) = q_0^\mu$: the *independence* of the choice of q_0^μ. The separation (8) into a constant q_0^μ and a time-dependent $\xi^\mu(\tau)$ paths must lead to the same result for any nearby constant $q_0'^\mu$ on the manifold. The result must therefore be invariant under an arbitrary infinitesimal displacement

$$q_0^\mu \to q_{0\varepsilon}^\mu = q_0^\mu + \varepsilon^\mu, \quad |\varepsilon| \ll 1. \tag{37}$$

In the path integral, this will be compensated by some translation of fluctuation coordinates $\xi^\mu(\tau)$, which will have the general nonlinear form

$$\xi^\mu \to \xi_\varepsilon^\mu = \xi^\mu - \varepsilon^\nu Q_\nu{}^\mu(q_0, \xi). \tag{38}$$

The transformation matrix $Q_\nu{}^\mu(q_0, \xi)$ satisfies the obvious initial condition $Q_\nu{}^\mu(q_0, 0) = \delta_\nu{}^\mu$. The path $q^\mu(\tau) = q^\mu(q_0, \xi(\tau))$ must remain invariant under simultaneous transformations (37) and (38), which implies that

$$\delta q^\mu \equiv q_\varepsilon^\mu - q^\mu = \varepsilon^\nu D_\nu q^\mu(q_0, \xi) = 0\,, \tag{39}$$

where D_μ is the infinitesimal transition operator

$$D_\mu = \frac{\partial}{\partial q_0^\mu} - Q_\mu{}^\nu(q_0, \xi)\frac{\partial}{\partial \xi^\nu}\,. \tag{40}$$

Geometrically, the matrix $Q_\nu{}^\mu(q_0,\xi)$ plays the role of a locally flat nonlinear connection [11]. It can be calculated as follows. We express the vector $q^\mu(q_0,\xi)$ in terms of the geodesic coordinates ξ^μ using Eqs. (8), (9), and (10), and substitute this into Eq. (39). The coefficients of ε^ν yield the equations

$$\delta_\nu{}^\mu + \frac{\partial\eta^\mu(q_0,\xi)}{\partial q_0^\nu} - Q_\nu{}^\kappa(q_0,\xi)\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\kappa} = 0, \tag{41}$$

where by Eq. (9):

$$\frac{\partial\eta^\mu(q_0,\xi)}{\partial q_0^\nu} = -\frac{1}{2}\partial_\nu\Gamma_{(\sigma\tau)}{}^\mu(q_0)\xi^\sigma\xi^\tau - \dots\ , \tag{42}$$

and

$$\begin{aligned}\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\nu} &= \delta_\nu{}^\mu - \Gamma_{(\nu\sigma)}{}^\mu(q_0)\xi^\sigma - \frac{1}{2}\Gamma_{(\nu\sigma\tau)}{}^\mu(q_0)\xi^\sigma\xi^\tau - \dots\\ &= \delta_\nu{}^\mu - \Gamma_{\nu\sigma}{}^\mu\xi^\sigma - \frac{1}{3}\left(\partial_\sigma\Gamma_{\nu\tau}{}^\mu + \frac{1}{2}\partial_\nu\Gamma_{\sigma\tau}{}^\mu - 2\Gamma_{\tau\nu}{}^\kappa\Gamma_{\kappa\sigma}{}^\mu - \Gamma_{\tau\sigma}{}^\kappa\Gamma_{\kappa\nu}{}^\mu\right)\xi^\sigma\xi^\tau - \dots\ .\end{aligned} \tag{43}$$

To find $Q_\nu{}^\mu(q_0,\xi)$, we invert the expansion (43) to

$$\begin{aligned}\left[\left(\frac{\partial\eta(q_0,\xi)}{\partial\xi}\right)^{-1}\right]_\nu{}^\mu &= \delta_\nu{}^\mu + \Gamma_{\nu\sigma}{}^\mu\xi^\sigma + \frac{1}{3}\left(\partial_\sigma\Gamma_{\nu\tau}{}^\mu + \frac{1}{2}\partial_\nu\Gamma_{\sigma\tau}{}^\mu + \Gamma_{\tau\nu}{}^\kappa\Gamma_{\kappa\sigma}{}^\mu - \Gamma_{\tau\sigma}{}^\kappa\Gamma_{\kappa\nu}{}^\mu\right)\xi^\sigma\xi^\tau + \dots \\ &= \left(\frac{\partial\xi^\mu(q_0,\eta)}{\partial\eta^\nu}\right)_{\eta=\eta(q_0,\xi)},\end{aligned} \tag{44}$$

where the last equality indicating that the result (44) can also be obtained from the inverted expansion (9):

$$\xi^\mu(q_0,\eta) = \eta^\mu + \frac{1}{2}\tilde\Gamma_{(\sigma\tau)}{}^\mu(q_0)\eta^\sigma\eta^\tau + \frac{1}{6}\tilde\Gamma_{(\sigma\tau\kappa)}{}^\mu(q_0)\eta^\sigma\eta^\tau\eta^\kappa + \dots, \tag{45}$$

with

$$\begin{aligned}\tilde\Gamma_{\sigma\tau}{}^\mu(q_0) &= \Gamma_{\sigma\tau}{}^\mu,\\ \tilde\Gamma_{\sigma\tau\kappa}{}^\mu(q_0) &= \Gamma_{\sigma\tau\kappa}{}^\mu + 3\Gamma_{\kappa\sigma}{}^\nu\Gamma_{\nu\tau}{}^\mu = \partial_\kappa\Gamma_{\sigma\tau}{}^\mu + \Gamma_{\kappa\sigma}{}^\nu\Gamma_{\nu\tau}{}^\mu,\\ &\vdots\ \ .\end{aligned} \tag{46}$$

Indeed, differentiating (45) with respect to η^ν, and reexpressing the result in terms of ξ^μ via Eq. (9), we find once more (44).

Multiplying both sides of Eq. (41) by (44), we express the nonlinear connection $Q_\nu{}^\mu(q_0,\xi)$ by means of geodesic coordinates $\xi^\mu(\tau)$ as

$$Q_\nu{}^\mu(q_0,\xi) = \delta_\nu{}^\mu + \Gamma_{\nu\sigma}{}^\mu(q_0)\xi^\sigma + \frac{1}{3}R_{\sigma\nu\tau}{}^\mu(q_0)\xi^\sigma\xi^\tau + \dots\ . \tag{47}$$

The effect of simultaneous transformations (37), (38) upon the fluctuation function $\eta^\mu = \eta^\mu(q_0, \xi)$ in Eq. (9) is

$$\eta^\mu \to \eta'^\mu = \eta^\mu - \varepsilon^\nu \bar{Q}_\nu{}^\mu(q_0, \eta), \quad \bar{Q}_\nu{}^\mu(q_0, 0) = \delta_\nu{}^\mu, \tag{48}$$

where the matrix $\bar{Q}_\nu{}^\mu(q_0, \eta)$ is related to $Q_\nu{}^\mu(q_0, \xi)$ as follows

$$\bar{Q}_\nu{}^\mu(q_0, \eta) = \left[Q_\nu{}^\kappa(q_0, \xi) \frac{\partial \eta^\mu(q_0, \xi)}{\partial \xi^\kappa} - \frac{\partial \eta^\mu(q_0, \xi)}{\partial q_0^\nu} \right]_{\xi = \xi(q_0, \eta)}. \tag{49}$$

Applying Eq. (41) to the right-hand side of Eq. (49) yields $\bar{Q}_\nu{}^\mu(q_0, \eta) = \delta_\nu{}^\mu$, as it should to compensate the translation (37).

The above independence of q_0^μ will be essential for constructing the correct perturbation expansion for the path integral (25). For some special cases of the Riemannian manifold, such as a surface of sphere in $D+1$ dimensions which forms a homogeneous space $O(D)/O(D-1)$, all points are equivalent, and the local independence becomes global. This will be discussed further in Section 8.

5 Zero-Mode Properties

We are now prepared to eliminate the zero mode by the condition of vanishing average $\bar{\xi}^\mu = 0$. As mentioned before, the vanishing fluctuation $\xi^\mu(\tau) = 0$ is obviously a classical saddle-point for the path integral (25). In addition, because of the symmetry (38) there exist other equivalent extrema $\xi_\varepsilon^\mu(\tau) = -\varepsilon^\mu = \text{const}$. The D components of ε^μ correspond to D zero modes which we shall eliminate in favor of a change of q_0^μ. The proper way of doing this is provided by the Faddeev-Popov procedure. We insert into the path integral (25) the trivial unit integral, rewritten with the help of (37):

$$1 = \int d^D q_0 \, \delta^{(D)}(q_{0\varepsilon} - q_0) = \int d^D q_0 \, \delta^{(D)}(\varepsilon), \tag{50}$$

and decompose the measure of path integration over all periodic paths $\xi^\mu(\tau)$ into a product of an ordinary integral over the temporal average $\xi_0^\mu = \bar{\xi}^\mu$, and a remainder containing only nonzero Fourier components [12]:

$$\oint \mathcal{D}^D \xi(\tau) = \int \frac{d^D \xi_0}{\sqrt{2\pi\beta}^D} \oint \mathcal{D}'^D \xi(\tau) \, . \tag{51}$$

According to Eq. (38), the path average $\bar{\xi}^\mu$ is translated under ε^μ as follows

$$\bar{\xi}^\mu \to \bar{\xi}_\varepsilon^\mu = \bar{\xi}^\mu - \varepsilon^\nu \frac{1}{\beta} \int_0^\beta d\tau \, Q_\nu{}^\mu(q_0, \xi(\tau)). \tag{52}$$

Thus we can replace

$$\int \frac{d^D \xi_0}{\sqrt{2\pi\beta}^D} \to \int \frac{d^D \varepsilon}{\sqrt{2\pi\beta}^D} \, \det \left[\frac{1}{\beta} \int_0^\beta d\tau \, Q_\nu{}^\mu(q_0, \xi(\tau)) \right]. \tag{53}$$

Performing this replacement in (51) and performing the integral over ε^μ in the inserted unity (50), we obtain the measure of path integration in terms of q_0^μ and geodesic coordinates of zero temporal average

$$\begin{aligned}\oint \mathcal{D}^D\xi(\tau) &= \int d^D q_0 \oint \frac{d^D\xi_0}{\sqrt{2\pi\beta}^D}\delta^{(D)}(\varepsilon) \oint \mathcal{D}'^D\xi(\tau) \\ &= \int \frac{d^D q_0}{\sqrt{2\pi\beta}^D} \oint \mathcal{D}'^D\xi(\tau)\ \det\left[\frac{1}{\beta}\int_0^\beta d\tau\, Q_\nu{}^\mu(q_0,\xi(\tau))\right]. \end{aligned} \tag{54}$$

The factor on the right-hand side is the Faddeev-Popov determinant $\Delta[q_0,\xi]$ for the change from ξ_0^μ to q_0^μ. We shall write it as an exponential:

$$\Delta[q_0,\xi] = \det\left[\frac{1}{\beta}\int_0^\beta d\tau\, Q_\nu{}^\mu(q_0,\xi)\right] = e^{-\mathcal{A}_{\rm e}^{\rm FP}[q_0,\xi]}\,, \tag{55}$$

where $\mathcal{A}_{\rm e}^{\rm FP}[q_0,\xi]$ is an auxiliary action accounting for the Faddeev-Popov determinant

$$\mathcal{A}_{\rm e}^{\rm FP}[q_0,\xi] \equiv -{\rm tr}\log\left[\frac{1}{\beta}\int_0^\beta {\rm d}\tau\, {\rm Q}_\nu{}^\mu({\rm q}_0,\xi)\right]\,, \tag{56}$$

which must be included into the interaction (26). Inserting (47) into Eq. (56), we find explictly

$$\begin{aligned}\mathcal{A}_{\rm e}^{\rm FP}[q_0,\xi] &= -{\rm tr}\log\left[\delta_\nu{}^\mu + (3\beta)^{-1}\int_0^\beta d\tau\, R_{\sigma\nu\tau}{}^\mu(q_0)\xi^\sigma(\tau)\xi^\tau(\tau)+\dots\right] \\ &= \frac{1}{3\beta}\int_0^\beta d\tau\, R_{\mu\nu}(q_0)\xi^\mu\xi^\nu + \dots\ . \end{aligned} \tag{57}$$

The contribution of this action will crucial for obtaining the correct perturbation expansion of the path integral (25).

With the new interaction

$$\mathcal{A}_{\rm e,new}^{\rm int}[q_0,\xi] = \mathcal{A}_{\rm e}^{\rm int}[q_0,\xi] + \mathcal{A}_{\rm e}^{\rm FP}[q_0,\xi] \tag{58}$$

the partition function (25) can be written as a classical partition function

$$Z^{\rm P} = \int \frac{d^D q_0}{\sqrt{2\pi\beta}^D}\sqrt{g(q_0)}\, e^{-\beta V^{\rm eff\,cl}(q_0)}\,, \tag{59}$$

where $V^{\rm eff\,cl}(q_0)$ is the curved-space version of the effective classical partition function of Ref. [7]. The effective classical Boltzmann factor

$$B(q_0) \equiv e^{-\beta V^{\rm eff\,cl}(q_0)} \tag{60}$$

is given by the path integral

$$B(q_0) = \oint \mathcal{D}'^D\xi(\tau)\sqrt{g(q_0)}e^{-\mathcal{A}_{\rm e}^{(0)}[q_0,\xi]-\mathcal{A}_{\rm e,new}^{\rm int}[q_0,\xi]}. \tag{61}$$

Since the zero mode is absent in the fluctuations on the right-hand side, the perturbation expansion is now straightforward. We expand the path integral (61) in powers of the interaction (58) around the free Boltzmann factor

$$B_0(q_0) = \oint \mathcal{D}'^D\xi(\tau)\sqrt{g(q_0)}e^{-\int_0^\beta d\tau\,\frac{1}{2}g_{\mu\nu}(q_0)\dot\xi^\mu\dot\xi^\nu} \tag{62}$$

as follows:

$$B(q_0) = B_0(q_0)\left[1 - \langle \mathcal{A}^{\rm int}_{\rm e,new}[q_0,\xi]\rangle^{q_0} + \frac{1}{2}\langle \mathcal{A}^{\rm int}_{\rm e,new}[q_0,\xi]^2\rangle^{q_0} - \dots\right], \tag{63}$$

where the q_0-dependent correlation functions are defined by the Gaussian path integrals

$$\langle \dots\rangle^{q_0} = B^{-1}(q_0)\oint \mathcal{D}'^D\xi(\tau)\,[\dots]^{q_0}\,e^{-\mathcal{A}^{(0)}_{\rm e}[q_0,\xi]}. \tag{64}$$

By taking the logarithm of (62), we obtain directly a cumulant expansion for the effective classical potential $V^{\rm eff\,cl}(q_0)$.

For a proper normalization of the Gaussian path integral (62) we diagonalize the free action in the exponent by changing the components of the fluctuations

$$\xi^\mu(\tau) \to \xi^a(\tau) = e^a{}_\mu(q_0)\xi^\mu(\tau),\ \ a = 1,\dots,D. \tag{65}$$

The basis vectors $e^a{}_\mu(q_0)$ are "square-roots" of the metric at q_0^μ:

$$g_{\mu\nu}(q_0) = e^a{}_\mu(q_0)e^a{}_\nu(q_0), \qquad \sqrt{g(q_0)} = \det e^a{}_\mu(q_0) = [\det e_a{}^\mu(q_0)]^{-1}, \tag{66}$$

satisfying the orthogonality relation $e^a{}_\mu(q_0)e^{b\mu}(q_0) = \delta^{ab}$. Substituting $\xi^\mu(\tau) = e_a{}^\mu(q_0)\xi^a(\tau)$ into Eq. (62) and taking into account that

$$\oint \mathcal{D}'^D\xi^\mu(\tau)\sqrt{g(q_0)} = \oint \mathcal{D}'^D\xi^a(\tau), \tag{67}$$

we find

$$B_0(q_0) = \oint \mathcal{D}'^D\xi^a(\tau)e^{-\int_0^\beta d\tau\,\frac{1}{2}(\dot\xi^a)^2}. \tag{68}$$

We have kept the superscript in the measure of integration to clarify which components of ξ are being considered. If we expand the fluctuations $\xi^a(\tau)$ into the eigenfunctions $e^{-i\omega_m\tau}$ of the operator $-d_\tau^2$ for periodic boundary conditions $\xi^a(0) = \xi^a(\beta)$,

$$\xi^a(\tau) = \sum_m \xi^a_m u_m(\tau) = \xi^a_0 + \sum_{m\neq 0}\xi^a_m u_m(\tau),\ \ \xi^a_{-m} = \xi^{a*}_m,\ \ m > 0, \tag{69}$$

and substitute this into the path integral (68), the exponent becomes

$$-\frac{1}{2}\int_0^\beta d\tau\ [\xi^a(\tau)]^2 = -\frac{\beta}{2}\sum_{m\neq 0}\omega_m^2\,\xi_{-m}\xi_m = -\beta\sum_{m>0}\omega_m^2\,\xi^{\alpha *}{}_m\xi^\alpha_m\,. \tag{70}$$

The measure has the Fourier decomposition [12]

$$\oint \mathcal{D}'^D \xi^a(\tau) = \prod_{a,m>0} \int \frac{d\xi_m^{\rm re} d\xi_m^{\rm im}}{N_m}\,, \tag{71}$$

where the normalization factor N_m regularize the divergent product of eigenvalues $\lambda_m = \omega_m^2$. The proper values are $N_m = \pi/\beta\omega_m^2$, and we find after performing the Gaussian integrals in (68) the correct result for a free-particle Boltzmann factor

$$B_0(q_0) = 1, \tag{72}$$

corresponding to a vanishing effective classical potential in Eq. (60). As a consequence, the partition function (59) becomes

$$Z^{\rm P} = \int \frac{d^D q_0}{\sqrt{2\pi\beta}^D} \sqrt{g(q_0)}\; B(q_0), \tag{73}$$

with the perturbation expansion

$$B(q_0) = 1 - \langle \mathcal{A}_{\rm e,new}^{\rm int}[q_0,\xi]\rangle^{q_0} + \frac{1}{2}\langle \mathcal{A}_{\rm e,new}^{\rm int}[q_0,\xi]^2\rangle^{q_0} - \dots\ . \tag{74}$$

The expectation values on the right-hand side are to be calculated with the help of Wick contractions involving the basic correlation functions of $\xi^a(\tau)$ associated with the unperturbed action in (68):

$$\langle \xi^a(\tau)\xi^b(\tau')\rangle^{q_0} = \delta^{ab}\Delta'(\tau,\tau')\,, \tag{75}$$

which are of course consistent with (31) via Eqs. (65) and (66).

6 Covariant perturbation expansion

We now perform all possible Wick contractions of the fluctuations $\xi^\mu(\tau)$ in the expectation values (74) using the correlation function (31). We restrict our attention to the lowest-order terms only, since all problems of previous treatments arise already there. Making use of Eqs. (34) and (35), we find for the interaction (26):

$$\begin{aligned}\langle \mathcal{A}_{\rm e}^{\rm int}[q_0,\xi]\rangle^{q_0} &= \int_0^\beta d\tau\, \frac{1}{6}\Big[R_{\mu\lambda_1\nu\lambda_2}(q_0)\langle \xi^{\lambda_1}\xi^{\lambda_2}\dot\xi^\mu\dot\xi^\nu\rangle^{q_0} + \delta(\tau,\tau)R_{\mu\nu}(q_0)\langle\xi^\mu\xi^\nu\rangle^{q_0}\Big]\\ &= \frac{1}{72}R(q_0)\beta,\end{aligned} \tag{76}$$

and for (57):

$$\langle \mathcal{A}_{\rm e}^{\rm FP}[q_0,\xi]\rangle^{q_0} = \int_0^\beta d\tau\, \frac{1}{3\beta}R_{\mu\nu}(q_0)\langle\xi^\mu\xi^\nu\rangle^{q_0} = \frac{1}{36}R(q_0)\beta. \tag{77}$$

The sum of the two contributions yields the manifestly covariant high-temperature expansion up to two loops:

$$B(q_0) = 1 - \langle \mathcal{A}^{\rm int}_{\rm e,new}[q_0,\xi]\rangle^{q_0} + \ldots = 1 - \frac{1}{24}R(q_0)\beta + \ldots \tag{78}$$

in agreement with the partition function density (5) calculated from Dirichlet boundary conditions. The associated partition function (73) coincides with the partition function (1). Note the crucial role of the action (57) coming from the Faddeev-Popov determinant in obtaining the correct two-loop coefficient in Eq. (78) and the normalization in Eq. (73).

The intermediate transformation to the geodesic coordinates $\xi^\mu(\tau)$ has made our calculations rather lengthy if the action is given in arbitrary coordinates, but it guarantees complete independence of the coordinates in the result (78). The entire derivation simplify, of course, drastically if we choose from the outset geodesic coordinates to parametrize the curved space.

7 Covariant result from noncovariant expansion

Having found the proper way of calculating the Boltzmann factor $B(q_0)$ we can easily set up a procedure for calculating the same covariant result without the use of the geodesic fluctuations $\xi^\mu(\tau)$. Thus we would like to evaluate the path integral (73) by a direct expansion of the action in powers of the noncovariant fluctuations $\eta^\mu(\tau)$ in Eq. (8). In order to make q_0^μ equal to the path average, $\bar{q}(\tau)$, we now require $\eta^\mu(\tau)$ to have a vanishing temporal average $\eta_0^\mu = \bar{\eta}^\mu = 0$.

The expansions of the action (3) and the measure (4) in powers of $\eta^\mu(\tau)$ were already given in Eqs. (12), (15), (20) and (21). The free action reads now,

$$\mathcal{A}^{(0)}_{\rm e}[q_0,\eta] = \int_0^\beta d\tau \frac{1}{2}\eta^\mu(\tau)[-g_{\mu\nu}(q_0)d_\tau^2]\eta^\nu(\tau)\,, \tag{79}$$

and the small-β behavior of the path integral (73) is governed by the interaction

$$\begin{aligned}\mathcal{A}^{\rm int}_{\rm e}[q_0,\eta] &= \mathcal{A}^{\rm int,4}_{\rm e}[q_0,\eta] + \mathcal{A}^{\rm meas}_{\rm e}[q_0,\eta] \\ &= \frac{1}{2}\int_0^\beta d\tau \left\{\left[\partial_\sigma g_{\mu\nu}(q_0)\eta^\sigma + \frac{1}{2}\partial_\sigma\partial_\tau g_{\mu\nu}(q_0)\eta^\sigma\eta^\tau\right]\dot{\eta}^\mu\dot{\eta}^\nu - \delta(\tau,\tau)\partial_\sigma\Gamma_{\tau\mu}{}^\mu(q_0)\eta^\sigma\eta^\tau\right\}.\end{aligned} \tag{80}$$

The measure of functional integration over η-fluctuations without zero mode $\eta_0^\mu = \bar{\eta}^\mu$ can be deduced from the proper measure of nonzero ξ-fluctuations:

$$\oint \mathcal{D}'^D\xi(\tau)J(q_0,\xi)\Delta[q_0,\xi] \equiv \oint \mathcal{D}^D\xi(\tau)J(q_0,\xi)\delta^{(D)}(\xi_0)\Delta[q_0,\xi]. \tag{81}$$

This is transformed to coordinates $\eta^\mu(\tau)$ via Eqs. (45) and (46) yielding

$$\oint \mathcal{D}'^D\xi(\tau)J(q_0,\xi)\Delta[q_0,\xi] = \oint \mathcal{D}'^D\eta(\tau)\bar{\Delta}[q_0,\eta]\,, \tag{82}$$

where $\bar{\Delta}[q_0,\eta]$ is obtained from the Faddeev-Popov determinant $\Delta[q_0,\xi]$ in Eq. (55) by expressed the coordinates $\eta^\mu(\tau)$ in terms of $\xi^\mu(\tau)$ and multiplying the result with a Jacobian accounting for the change of the δ-function of ξ_0 to a δ-function of η_0 via the transformation Eq. (45):

$$\bar{\Delta}[q_0,\eta] \quad = \quad \Delta[q_0,\xi(q_0,\eta)] \times \det\left(\frac{\partial\bar{\eta}^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)} . \tag{83}$$

The last determinant has the exponential form

$$\det\left(\frac{\partial\bar{\eta}^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)} = \exp\left\{\mathrm{tr}\log\left[\frac{1}{\beta}\int_0^\beta d\tau\,\left(\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)}\right]\right\} , \tag{84}$$

where the matrix in the exponent has small-η expansion

$$\begin{aligned}\left(\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)} &= \delta_\nu{}^\mu - \Gamma_{\nu\sigma}{}^\mu\eta^\sigma \\ &- \frac{1}{3}\Big(\partial_\sigma\Gamma_{\nu\tau}{}^\mu + \frac{1}{2}\partial_\nu\Gamma_{\sigma\tau}{}^\mu - 2\Gamma_{\tau\nu}{}^\kappa\Gamma_{\kappa\sigma}{}^\mu + \frac{1}{2}\Gamma_{\tau\sigma}{}^\kappa\Gamma_{\kappa\nu}{}^\mu\Big)\eta^\sigma\eta^\tau + \dots .\end{aligned} \tag{85}$$

The factor (83) leads to a new contribution to the interaction (80), if we rewrite it as

$$\bar{\Delta}[q_0,\eta] = e^{-\bar{\mathcal{A}}_{\rm e}^{\rm FP}[q_0,\eta]} . \tag{86}$$

Combining Eqs. (55) and (84), we find a new Faddeev-Popov type action for η^μ-fluctuations at vanishing η_0^μ:

$$\begin{aligned}\bar{\mathcal{A}}_{\rm e}^{\rm FP}[q_0,\eta] &= \mathcal{A}_{\rm e}^{\rm FP}[q_0,\xi(q_0,\eta)] - \mathrm{tr}\log\left[\frac{1}{\beta}\int_0^\beta d\tau\,\left(\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)}\right] \\ &= \frac{1}{2\beta}\int_0^\beta d\tau\; T_{\sigma\tau}(q_0)\eta^\sigma\eta^\tau + \dots ,\end{aligned} \tag{87}$$

where

$$T_{\sigma\tau}(q_0) = (\partial_\mu\Gamma_{\sigma\tau}{}^\mu - 2\Gamma_{\sigma\kappa}{}^\mu\Gamma_{\mu\tau}{}^\kappa + \Gamma_{\kappa\mu}{}^\mu\Gamma_{\sigma\tau}{}^\kappa) . \tag{88}$$

The unperturbed correlation functions associated with the action (79) are:

$$\langle\eta^\mu(\tau)\eta^\nu(\tau')\rangle^{q_0} = g^{\mu\nu}(q_0)\Delta'(\tau,\tau') \tag{89}$$

and the free Boltzmann factor is the same as in Eq. (72). The perturbation expansion of the interacting Boltzmann factor is to be calculated from an expansion like (74):

$$B(q_0) = 1 - \langle\mathcal{A}_{\rm e,new}^{\rm int}[q_0,\eta]\rangle^{q_0} + \frac{1}{2}\langle\big(\mathcal{A}_{\rm e,new}^{\rm int}[q_0,\eta]\big)^2\rangle^{q_0} - \dots . \tag{90}$$

where the interaction is now

$$\mathcal{A}_{\rm e,new}^{\rm int}[q_0,\eta] = \mathcal{A}_{\rm e}^{\rm int}[q_0,\eta] + \bar{\mathcal{A}}_{\rm e}^{\rm FP}[q_0,\eta] . \tag{91}$$

The Wick contractions of $\eta^\mu(\tau)$ are more numerous and complicated that those of the manifestly covariant coordinates $\xi^\mu(\tau)$. In particular, the divergences containing powers of $\delta(\tau,\tau)=\delta(0)$ no longer cancel order by order, but different orders conspire to remove them in the final result. Consider for example the first term in (90):

$$
\begin{aligned}
-\langle \mathcal{A}_{\rm e}^{\rm int}[q_0,\eta]\rangle^{q_0} &= \frac{\beta}{24} g^{\sigma\tau}\left(\partial_\sigma \Gamma_{\tau\mu}{}^{\mu} + g^{\mu\nu} g_{\delta\kappa}\Gamma_{\tau\mu}{}^{\delta}\Gamma_{\sigma\nu}{}^{\kappa} + \Gamma_{\tau\nu}{}^{\mu}\Gamma_{\sigma\mu}{}^{\nu}\right)\\
&- \frac{\beta}{24} g^{\sigma\tau}\left(\partial_\mu \Gamma_{\sigma\tau}{}^{\mu} - 2\Gamma_{\sigma\nu}{}^{\mu}\Gamma_{\mu\tau}{}^{\nu} + \Gamma_{\mu\kappa}{}^{\mu}\Gamma_{\sigma\tau}{}^{\kappa}\right)\\
&- \frac{\beta^2}{24}\delta(0) g^{\sigma\tau}\left(g^{\mu\nu} g_{\delta\kappa}\Gamma_{\tau\mu}{}^{\delta}\Gamma_{\sigma\nu}{}^{\kappa} + \Gamma_{\tau\mu}{}^{\nu}\Gamma_{\sigma\nu}{}^{\mu}\right) , \qquad (92)
\end{aligned}
$$

where the term in the second line is the contribution from the Faddeev-Popov action (87). The last term contains the divergent quantity $\delta(0)$. This is canceled by the same expression in the second-order contribution to (90):

$$
\begin{aligned}
\frac{1}{2}\left(\mathcal{A}_{\rm e}^{\rm int}[q_0,\eta]\right)^2\rangle^{q_0} = &- \frac{\beta}{24} g^{\sigma\tau}\left(g^{\mu\nu} g_{\delta\kappa}\Gamma_{\tau\mu}{}^{\delta}\Gamma_{\sigma\nu}{}^{\kappa} + 2\Gamma_{\tau\nu}{}^{\mu}\Gamma_{\sigma\mu}{}^{\nu}\right)\\
&+ \frac{\beta^2}{24}\delta(0) g^{\sigma\tau}\left(g^{\mu\nu} g_{\delta\kappa}\Gamma_{\tau\mu}{}^{\delta}\Gamma_{\sigma\nu}{}^{\kappa} + \Gamma_{\tau\mu}{}^{\nu}\Gamma_{\sigma\nu}{}^{\mu}\right) . \qquad (93)
\end{aligned}
$$

In calculating this, there is an additional complication caused by the appearance of initially undetermined integrals over products of distributions of the type $\int d\tau\ \epsilon^2(\tau,\tau)\delta(\tau,\tau)$. Such integrals are determined uniquely by the new calculus of distributions in one dimension developed in Ref. [4] from the coordinate invariance of path integrals.

The sum of Eqs. (92) and (93) is of course finite leading to the same covariant perturbation expansion as before in Eq. (78). Neglecting the contribution of the action (87) as done by other authors in Ref. [6] will produce in Eq. (90) an additional noncovariant term $g^{\sigma\tau}T_{\sigma\tau}(q_0)/24$. This may be rewritten as a covariant divergence of a nonvectorial quantity

$$
g^{\sigma\tau}T_{\sigma\tau} = \nabla_\mu V^\mu, \quad V^\mu(q_0) = g^{\sigma\tau}(q_0)\Gamma_{\sigma\tau}{}^{\mu}(q_0). \qquad (94)
$$

As such it does not contribute to the integral over q_0^μ in Eq. (73), but it is nevertheless a wrong noncovariant result for the Boltzmann factor (78).

The appearance of a noncovariant term in a treatment where q_0^μ is the path average of $q^\mu(\tau)$ is not surprising. If the time dependence of a path shows an acceleration, the average of a path is not an invariant concept even for an infinitesimal time. One may covariantly impose the condition of a vanishing temporal average only upon fluctuation coordinates which have no acceleration. This is the case of geodesic coordinates $\xi^a(\tau)$ since their equation of motion at q_0^μ is $\ddot{\xi}^a(\tau)=0$.

8 Quantum particle on unit sphere

A special treatment exits for particle in homogeneous spaces. As an example, consider a quantum particle moving on a unit sphere in a flat $D+1$-dimensional space. The partition function is defined by Eq. (7) with the euclidean action (3) and the invariant measure (4), where the metric and its determinant are

$$g_{\mu\nu}(q) = \delta_{\mu\nu} + \frac{q_\mu q_\nu}{1-q^2}, \qquad g(q) = \frac{1}{1-q^2}. \tag{95}$$

It is, of course, possible to calculate the Boltzmann factor $B(q_0)$ with the procedure of Section 5. Instead of doing this, we shall, however, exploit the homogeneity of the sphere. The invariance under reparametrizations of general Riemannian space becomes here an isometry of the metric (95). Consequently, the Boltzmann factor $B(q_0)$ in Eq. (73) becomes *completely independent* of the choice of q_0^μ, and the integral over q_0^μ in (59) yields simply the total surface of the sphere times the Boltzmann factor $B(q_0)$.

The homogeneity of the space allows us to treat paths $q^\mu(\tau)$ themselves as small quantum fluctuations around the origin $q_0^\mu = 0$, which extremizes the path integral (7). The possibility of this expansion is due to the fact that at $q^\mu(\tau) = 0$, the movement is free of acceleration, this being similar to the situation in geodesic coordinates.

As before we now take account of the fact that there are other equivalent saddle-points due to isometries of the metric (95) on the sphere (see, e.g., [13]). The infinitesimal translations of a small vector q^μ:

$$q_\varepsilon^\mu = q^\mu + \varepsilon^\mu\sqrt{1-q^2}, \quad \varepsilon^\mu = \text{const}, \ \mu = 1,\dots,D \tag{96}$$

move the origin $q_0^\mu = 0$ into ε^μ. Due to rotational symmetry, these fluctuations have a vanishing action. They may be eliminated from the path integral (7) by including a factor $\delta^{(D)}(\bar{q})$ to enforce a vanishing path average. The associated Faddeev-Popov determinant $\Delta(q)$ is determined by the integral

$$\Delta(q)\int d^D\varepsilon\,\delta^{(D)}\left(\bar{q}_\varepsilon\right) = \Delta(q)\int d^D\varepsilon\,\delta^{(D)}\left(\varepsilon^\mu\frac{1}{\beta}\int_0^\beta d\tau\,\sqrt{1-q^2}\right) = 1\,. \tag{97}$$

The result has the exponential form

$$\Delta(q) = \left(\frac{1}{\beta}\int_0^\beta d\tau\,\sqrt{1-q^2}\right)^D = e^{-\mathcal{A}_{\rm e}^{\rm FP}[q]}\,, \tag{98}$$

where $\mathcal{A}_{\rm e}^{\rm FP}[q]$ must be added to the action (3):

$$\mathcal{A}_{\rm e}^{\rm FP}[q] = -D\log\left(\frac{1}{\beta}\int_0^\beta d\tau\,\sqrt{1-q^2}\right). \tag{99}$$

The Boltzmann factor $B(q_0) \equiv B$ is then given by the path integral without zero modes

$$B = \oint\prod_{\mu,\tau}\left[dq^\mu(\tau)\sqrt{g(q(\tau))}\right]\delta^{(D)}(\bar{q})\Delta(q)e^{-\mathcal{A}_{\rm e}[q]} = \oint\mathcal{D}'^D q(\tau)\sqrt{g(q(\tau))}\Delta(q)e^{-\mathcal{A}_{\rm e}[q]}\,, \tag{100}$$

where the measure $\mathcal{D}'^D q$ is defined as in Eq. (71). This can also be written as

$$B = \oint\mathcal{D}'^D q(\tau)e^{-\mathcal{A}_{\rm e}[q]-\mathcal{A}_{\rm e}^J[q]-\mathcal{A}_{\rm e}^{\rm FP}[q]}\,, \tag{101}$$

where $\mathcal{A}_{\rm e}^J[q]$ is a contribution to the action (3) coming from the product

$$\prod_{\tau}\sqrt{g(q(\tau))}\equiv e^{-\mathcal{A}_{\rm e}^J[q]}. \tag{102}$$

By inserting (95), this becomes

$$\mathcal{A}_{\rm e}^J[q]=-\int_0^\beta d\tau\,\frac{1}{2}\delta(\tau,\tau)\log g(q)=\int_0^\beta d\tau\,\frac{1}{2}\delta(\tau,\tau)\log(1-q^2)\,. \tag{103}$$

The total partition function is, of course, obtained from B by multiplication it with the surface of the unit sphere in $D+1$ dimensions $2\pi^{(D+1)/2}/\Gamma(D+1)/2$.

To calculate B from (101), we now expand $\mathcal{A}_{\rm e}[q]$, $\mathcal{A}_{\rm e}^J[q]$ and $\mathcal{A}_{\rm e}^{\rm FP}[q]$ in powers of $q^\mu(\tau)$. The metric $g_{\mu\nu}(q)$ and its determinant $g(q)$ in Eq. (95) have the expansions

$$g_{\mu\nu}(q)=\delta_{\mu\nu}+q_\mu q_\nu+\dots\ ,\qquad g(q)=1+q^2+\dots\ , \tag{104}$$

and the unperturbed action reads

$$\mathcal{A}_{\rm e}^{(0)}[q]=\int_0^\beta d\tau\,\frac{1}{2}\dot{q}^2(\tau). \tag{105}$$

In the absence of the zero eigenmodes due to the δ-function over $\bar{q}$ in Eq. (100), we find as in Eq. (72) the free Boltzmann factor

$$B_0=1. \tag{106}$$

The free correlation function looks similar to (75):

$$\langle q^\mu(\tau)q^\nu(\tau')\rangle=\delta^{\mu\nu}\Delta'(\tau,\tau'). \tag{107}$$

The interactions coming from the higher expansions terms in Eq. (104) begin with

$$\mathcal{A}_{\rm e}^{\rm int}[q]=\mathcal{A}_{\rm e}^{{\rm int},4}[q]+\mathcal{A}_{\rm e}^J[q]=\int_0^\beta d\tau\,\frac{1}{2}\left[(q\dot{q})^2-\delta(\tau,\tau)q^2\right]. \tag{108}$$

To the same order, the Faddeev-Popov interaction (99) contributes

$$\mathcal{A}_{\rm e}^{\rm FP}[q]=\frac{D}{2\beta}\int_0^\beta d\tau\,q^2. \tag{109}$$

This has an important effect upon the two-loop perturbation expansion of the Boltzmann factor

$$B(q_0)=1-\langle\mathcal{A}_{\rm e}^{\rm int}[q]\rangle^{q_0}-\langle\mathcal{A}_{\rm e}^{\rm FP}[q]\rangle^{q_0}+\dots\ \equiv B. \tag{110}$$

Performing the Wick contractions with the help of Eq. (107), we find from Eqs. (108), (109):

$$\langle\mathcal{A}_{\rm e}^{\rm int}[q]\rangle^{q_0}=-\frac{D}{24}\beta \tag{111}$$

and

$$\langle \mathcal{A}_{\rm e}^{\rm FP}[q] \rangle^{q_0} = \frac{D^2}{24}\beta. \tag{112}$$

Their combination in Eq. (110) yields the high-temperature expansion

$$B = 1 - \frac{D(D-1)}{24}\beta + \dots \ . \tag{113}$$

This is in perfect agreement with Eqs. (5) and (78), since the scalar curvature for a unit sphere in $D+1$ dimensions is $R = D(D-1)$. It is remarkable how the contribution (112) of the Faddeev-Popov determinant has made the noncovariant result (111) covariant.

References

[1] H. Kleinert, *Path Integrals in Quantum Mechanics,Statistics, and Polymer Physics*, (World Scientific, Singapore, Second Edition, 1995). (`http://www.physik.fu-berlin.de/~kleinert/kleiner_reb3/3rded.html`).

[2] Such an R-term may be outruled by the following argument: If it exists, it should have the same universal coefficient $c\hbar^2$ in all Schrödinger equations (proposed candidates are 1/12, 1/8, 1/24), otherwise particles would fall with different speed in a gravitational field. Due to the quantum-mechanical equivalence of coordinate and momentum representations, the same term should also be present in Schrödinger equations written in curved momentum space. For the hydrogen atom, such an equation lives on the surface of a sphere in four dimensions. See
H. Kleinert, *Group Dynamics of the Hydrogen Atom,* Lectures in Theoretical Physics, edited by W.E. Brittin and A.O. Barut, Gordon and Breach, N.Y. 1968, pp. 427-482 (`http://www.physik.fu-berlin.de/~kleinert/4`).
In that equation, however, an extra R-term would change the Rydberg spectrum to $E_n = -1/2(n^2 + 3/2c)$, $n = 1, 2, 3, \dots$. Such a distortion of the Rydberg spectrum would clearly have been detected for all proposed c-values. For details see
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