Dependence of variational perturbation expansions on strong-coupling behavior: Inapplicability of δ expansion to field theory

B. Hamprecht* and H. Kleinert†
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany
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We show that in applications of variational theory to quantum field theory it is essential to account for the correct Wegner exponent ω governing the approach to the strong coupling, or scaling, limit. Otherwise the procedure does not converge at all or to the wrong limit. This casts doubt on all papers applying the so-called δ expansion to quantum field theory.

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I. INTRODUCTION

Variational perturbation theory is a powerful tool for extracting nonperturbative strong-coupling results from weak-coupling expansions. It was initially invented in quantum mechanics as a reexpansion of the perturbation series of the action [1]

$$\mathcal{A}=\int_{t_a}^{t_b}dt\left[\frac{M}{2}\dot{x}^2-\frac{\omega^2}{2}x^2-V^{\rm int}(x)\right], \tag{1.1}$$

which arises from splitting the potential into a quadratic part $V_\Omega^{(0)}\equiv\Omega^2x^2/2$, with an arbitrary trial frequency Ω, and an interacting part

$$V_\Omega^{\rm int}\equiv\delta\left[\frac{(\omega^2-\Omega^2)}{2}x^2+V^{\rm int}(x)\right]. \tag{1.2}$$

The perturbation expansion is then performed in powers of δ, setting $\delta=1$ at the end, and optimizing the result in Ω guided by the *principle of minimal sensitivity* [2]. The history and convergence properties are discussed in the textbook [3]. Because of the prefactor δ in Eq. (1.2), the procedure is often called δ expansion [1]. For the anharmonic oscillator, convergence was proved to be exponentially fast for finite [4] as well as for infinite coupling strength [3,5,6].

In recent years the method has been extended in a simple but essential way to allow for the resummation of divergent perturbation expansions in quantum field theories [7,8]. The most important new feature of this *field-theoretic variational perturbation theory* is that it accounts for the anomalous power approach to the strong-coupling limit which the δ expansion cannot do. This approach is governed by an irrational critical exponent ω as was first shown by Wegner [9] in the context of critical phenomena. In contrast to the δ expansion, the field-theoretic variational perturbation expansions *cannot* be derived from the action by adding and subtracting a harmonic term as in Eq. (1.2). The new theory has led to the most accurate determination so far of critical exponents via quantum field theory, as amply demonstrated in the textbook [10]. In particular, the theory has perfectly explained the experimentally best known critical exponent α of the specific heat of the λ transition measured in a satellite orbiting around the earth [11].

In spite of the existence of this reliable quantum-field-theoretic variational perturbation theory, the literature keeps offering applications of the above quantum-mechanical δ expansion to quantum field theory, for instance in recent papers by Braaten and Radescu (BR) [12,13] and de Souza Cruz *et al.* and Kneur *et al.* [14] (see also [15]).

It is the purpose of this paper to show what goes wrong with such unjustified applications, and how the proper quantum field-theoretic variational perturbation theory corrects the mistakes.

II. REVIEW OF THE METHOD

Suppose the function $f(g)$ is given by a divergent series expansion around the point $g=0$:

$$f_L(g)=\sum_{l=0}^{L}a_lg^l, \tag{2.1}$$

typically with factorial growth of the coefficients a_l. Suppose, furthermore, that the expected leading behavior of $f(g)$ for large g has the general power structure:

$$f_M(g)=g^\alpha\sum_{m=0}^{M}b_mg^{-\omega m}, \tag{2.2}$$

where ω is the Wegner exponent of approach to the strong-coupling limit. In quantum mechanics, this exponent is easily found from the naive scaling properties of the action. In quantum field theory, however, it is an initially unknown number which has to be determined from the above weak-coupling expansion by a procedure to be called the *dynamical determination* of ω.

Assuming for a moment that this has been done, the Lth order approximation to the leading coefficient b_0 is given by [16]

$$b_0^{(L)}(z)=z^{-\alpha}\sum_{l=0}^{L}a_lz^l\binom{L-l+(l-\alpha)/\omega}{L-l}, \tag{2.3}$$

*Email address: bodo.hamprecht@physik.fu-berlin.de
†Email address: hagen.kleinert@physik.fu-berlin.de

where the $z \equiv g/\Omega^{2/\omega}$ is the variational parameter to be optimized for minimal sensitivity on z. A short reminder of the derivation of this formula is given in Appendix A. An application to a simple known function is shown in Appendix B. For a successful application to the quantum-mechanical anharmonic oscillators, the reader is referred to the textbook [3]. The exponent ω is equal to 2/3 for an x^4-anharmonic oscillator, and the exponentially fast convergence has an error decreasing like $e^{-\text{const}\times L^{1-\omega}}$. For the oscillator, the number ω is found directly from the dimensional analysis in Appendix A. As mentioned above, such an analysis will not be applicable in quantum field theory, where ω is anomalous and must be determined dynamically.

Most often we want to calculate a quantity $f(g)$ which goes to a constant in the strong-coupling limit $f(g) \to f^*$. This is the case for all critical exponents. Then we must set $\alpha = 0$ in Eqs. (2.2) and (2.3), which implies that for infinite g:

$$\beta(g) = \frac{d \log f(g)}{d \log g}\bigg|_{g\to\infty} = 0. \tag{2.4}$$

If $\beta(g)$ is reexpressed as a function of f, this implies $\beta(f^*) = 0$, the standard requirement for the existence of a critical point in quantum field theory if $f(g) = g_R(g)$ is the renormalized coupling strength as a function of the bare coupling strength g.

The dynamical determination of ω proceeds now by treating not only $f(g)$ but also the beta function (2.4) according to the rules of variational perturbation theory [7], and determining ω to make $\beta^* = \beta(\infty)$ vanish, which is done by optimizing the following equation of z:

$$\beta(\infty, z) = \sum_{l=0}^{L} \beta_l z^l \binom{L - l + l/\omega}{L - l} = 0, \tag{2.5}$$

where β_l are the coefficients of the expansion of Eq. (2.4) in powers of g. Minimal sensitivity is reached for a vanishing derivative with respect to z:

$$\frac{\partial}{\partial z}\beta(\infty, z) = \sum_{l=1}^{L} \beta_l l z^{l-1} \binom{L - l + l/\omega}{L - l} = 0, \tag{2.6}$$

so that z and ω are to be found as simultaneous solutions of Eqs. (2.5) and (2.6).

III. ANOMALOUS DIMENSIONS

As mentioned above, a number of authors have applied the δ expansion to field theories. Most recently, this was done for the purpose of calculating the shift of the critical temperature in a Bose-Einstein condensate caused by a small interaction [13,14]. Since the perturbation expansion for this quantity is a function of g/μ, where μ is the chemical potential which goes to zero at the critical point, we are faced with a typical strong-coupling problem of critical phenomena. In order to justify the application of the δ expansion to this problem, BR [12] studied the convergence properties of the method by applying it to a certain amplitude $\Delta(g)$ of an $O(N)$-symmetric ϕ^4-field theory in the limit of large N, where the model is exactly solvable.

Their procedure can be criticized in two ways. First, the amplitude $\Delta(g)$ they considered is not a good candidate for a resummation by a δ expansion since it does not possess the characteristic strong-coupling power structure (2.2) of quantum mechanics and field theory, which the final resummed expression will always have. The power structure is disturbed by additional logarithmic terms. Second, the δ expansion is equivalent to choosing, on dimensional grounds, the exponent $\omega = 2$ in Eq. (2.2), which is far from the approximate optimal value 0.842 to be derived below. Thus the δ expansion is inapplicable, and this explains the problems into which BR run in their resummation attempt. Most importantly, they do not find a well-shaped plateau of the variational expressions $\Delta^{(L)}(g,z)$ as a function of z which would be necessary for invoking the principle of minimal sensitivity. Instead, they observe that the zeros of the first derivatives $\partial_z \Delta^{(L)}(g,z)$ run away far into the complex plain. Choosing the complex solutions to determine their final resummed value they miss the correct result by 3% up to the 35th order.

One may improve the situation by trying out various different ω values and choosing the best of them, yielding an acceptable plateau in $\Delta(g,z)$. This happens for $\omega \approx 0.842$. The numerical analysis indicates that then convergence may be achieved.

Let us explain these points in more detail. BR consider the weak-coupling series with the reexpansion parameter δ:

$$\Delta(g) = -\sum_{l=1}^{\infty} \left(-\frac{\delta g}{\sqrt{1-\delta}} \right)^l a_l, \tag{3.1}$$

where

$$a_l \equiv \int_0^\infty K(x) f^l(x) dx,$$

with

$$K(x) \equiv \frac{4x^2}{\pi(1+x^2)^2}, \quad f(x) \equiv \frac{2}{x}\arctan\frac{x}{2}. \tag{3.2}$$

The geometric series in Eq. (3.1) can be summed exactly, and the result may formally be reexpanded into a strong-coupling series in $h \equiv \sqrt{1-\delta}/(\delta g)$:

$$\Delta(g) = \int_0^\infty K(x) \frac{\delta g f(x)}{\sqrt{1-\delta} + \delta g f(x)} dx$$

$$= \sum_{m=0}^{\infty} b_m (-h)^m,$$

where

$$b_m = \int_0^\infty K(x) f^{-m}(x) dx. \tag{3.3}$$

The strong-coupling limit is found for $h \to 0$ where $\Delta \to b_0 = \int_0^\infty dx K(x) = 1$. The approach to this limit is, however, *not* given by a strong-coupling expansion of the form (3.3). This would only happen if all the integrals b_m were to exist

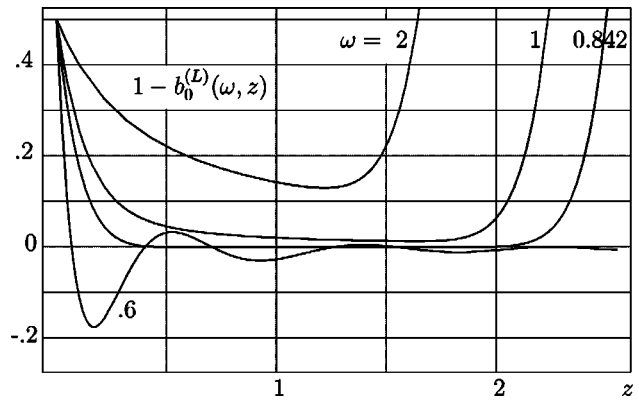

FIG. 1. Plot of $1-b_0^{(L)}(\omega,z)$ versus z for $L=10$ and $\omega=0.6,0.842,1,2$. The curve with $\omega=0.6$ shows oscillations. They decrease with increasing ω and the curve becomes flat at about $\omega=0.842$. Further increase of ω tilts the plateau and shows no regime of minimal sensitivity. At the same time, the minimum of the curve rises rapidly above the correct value of $1-b_0=0$, as can be seen from the upper two curves for $\omega=1$ and $\omega=2$, respectively.

which, unfortunately, is not the case since all integrals for b_m with $m>0$ diverge at the upper limit, where

$$f(x)=\frac{2}{x}\arctan\frac{x}{2}\sim\frac{\pi}{x}. \tag{3.4}$$

The exact behavior of Δ in the strong-coupling limit $h\to 0$ is found by studying the effect of the asymptotic π/x contribution of $f(x)$ to the integral in Eq. (3.3). For $f(x)=\pi/x$ we obtain

$$\int_0^\infty K(x)\frac{1}{1+h/f(x)}dx = \frac{\pi^4+2\pi^2h-\pi^2h^2+2h^3+4\pi^2h\log h/\pi}{(\pi^2+h^2)^2}. \tag{3.5}$$

The logarithm of h shows a mismatch with Eq. (2.2) and requires extra care if the expansion (3.1) is to become a candidate for variational perturbation theory.

We now explain the second criticism. Suppose we ignore the just-demonstrated fundamental obstacle and follow the rules of the δ expansion, to find the Lth order approximant to $\Delta(\infty)$ by expanding (3.1) in powers of δ up to order δ^L, setting $\delta=1$, and replacing $z\to g$. Then we obtain the Lth variational expression for b_0:

$$b_0^{(L)}(\omega,z)=\sum_{l=1}^{L} a_l z^l \binom{L-l+l/\omega}{L-l}, \tag{3.6}$$

with $\omega=2$, to be optimized in z. This ω value would only be adequate if the approach to the strong-coupling limit behaved like $A+B/h^2+\cdots$, rather than Eq. (3.5). This is the reason why BR find no real regime of minimal sensitivity on z. Had there been no logarithmic term in Eq. (3.5), $\omega=1$ would have been the correct choice.

As mentioned above it is possible to improve the situation by allowing for an effective anomalous dimension in the resummation of Eq. (3.5). Thus we determine ω dynamically from Eq. (2.4), and we find that for $\omega\approx 0.842$ there is definitely a flat regime of minimal sensitivity as can be seen in Fig. 1 by comparing plots of $1-b_0^{(L)}(\omega,z)$ versus z for several different ω values. The optimal ω is quite far from the naive value $\omega=2$, and also way off from $\omega=1$.

This value can also be estimated by inspecting plots of $1-b_0^{(L)}(\omega,z)$ versus z for several different ω values, selecting the one producing minimal sensitivity. It produces good results also in higher orders, as is seen in Fig. 2. The approximations appear to converge rapidly.

As a final step, we are pushing the order up to $L=100$, focussing on $\omega=1$ and $\omega=0.842\ldots$. We determine the optimum of the variational parameter z from the plateau, making use of the vanishing of the second derivative of $b_0^{(L)}(\omega,z)$ with respect to z. All real zeros are exploited to win an approximation for the strong-coupling coefficient

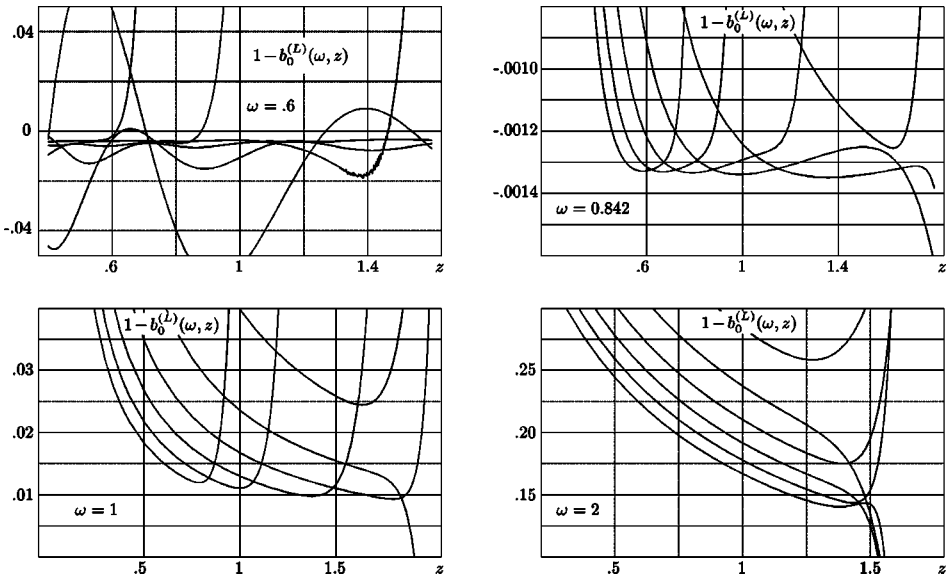

FIG. 2. The functions $1-b_0^{(L)}(\omega,z)$ are shown for various orders $L=10,17,24,31,38,45$ in each plot. The critical exponent is chosen to be $\omega=0.6,0.842,1,2$, respectively. The optimal $\omega=0.842$ ensures that the plateau is well pronounced and for increasing orders L stays near zero, which is the correct value. The other choices for ω do not have this property. In particular, for $\omega=2$ of the δ expansion of BR there is no convergence.

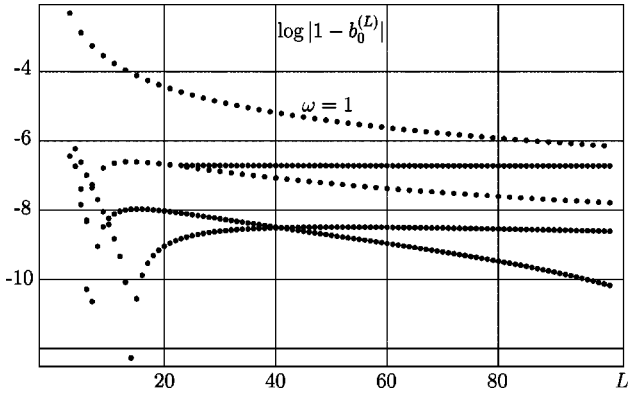

FIG. 3. The absolute logarithmic error of the leading strong-coupling coefficient $\log|1-b_0^{(L)}|$ is plotted as a function of the order L for $\omega=1$ and $\omega=0.842$. The choice $\omega=1$ would be the correct dimension, if the logarithm was absent from 3.5, whereas $\omega=0.842$ has been determined dynamically as an effective anomalous Wegner exponent to achieve a plateau of minimal sensitivity. The real zeros of the second derivative of $b_0^{(L)}(\omega,z)$ with respect to z have been used throughout to fix z. For $\omega=1$ there is only one zero for every L, leading to the highest curve. For $\omega=0.842$, there are four zeros contributing at each L, assembling themselves into four families for sufficiently large L. One of them seems to have an error, exponentially decreasing with the order L. All others, including the one for $\omega=1$, level off and either converge slowly to some value, which may not be the correct one, or converging rapidly to a wrong value. It cannot be concluded whether the exponential behavior of the ''good'' family carries through to much higher orders, since the numerical data have always a limited accuracy. Here we have employed an accuracy of at least 200 digits. It is remarkable that for closely lying neighboring values such as $\omega=0.841$ or $\omega=0.843$, the exponentially decreasing families disappear. Apparently, smaller and smaller adjustments of $\omega=0.842$ in higher decimals are necessary to ensure exponential convergence to arbitrarily high orders.

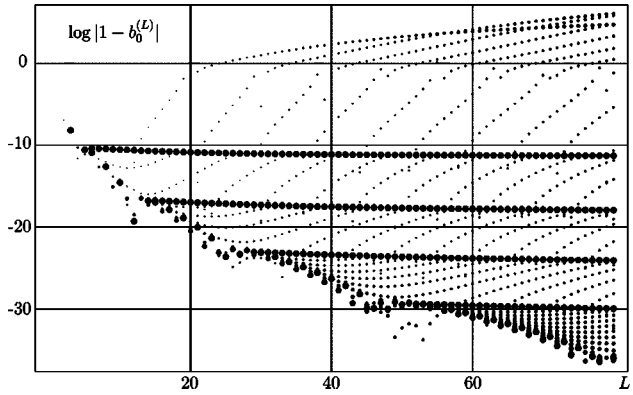

FIG. 4. Logarithmic error of the leading strong-coupling coefficient $b_0^{(L)}$ of the ground state energy of the anharmonic oscillator with x^4 potential. The errors are plotted over the order L of the variational perturbation expansion. At each order, all zeros of the first derivative have been exploited. Only the real parts of the complex roots have been used to evaluate $b_0^{(L)}$. The fat points show the results from real zeros, the smaller points those from complex zeros, with the size decreasing with distance from the real axis.

$b_0^{(L)}$. Figure 3 shows the logarithm of its absolute error $\log|1-b_0^{(L)}|$, plotted over the order L. Obviously the dynamically determined $\omega=0.842$ gives a better result than $\omega=1$. Figure 3 does not give sufficient evidence for the latter to reach the correct result, but if so, it will certainly not do so exponentially fast. It is interesting to see that the exponentially convergent result given by one of the roots for $\omega=0.842$ seems to depend very sensitively on the correct choice of the Wegner dimension. We have found that small changes in ω make the result oscillate around the correct value. Adjusting ω as the order L increases, we may push the transition from exponential to oscillatory behavior to higher and higher orders.

This implies that the failure of the strong-coupling behavior (3.5) to have a pure power structure (2.2) requires a departure from naive dimensions in favor of dynamically determined ones. To corroborate this issue we change the function $f(x)$ in Eq. (3.2) slightly into $f(x)\to\widetilde{f}(x)=f(x)+1$, which makes the integrals for $\widetilde{b}_m$ in Eq. (3.3) convergent. The exact limiting value 1 of $\widetilde{\Delta}$ remains unchanged, but $\overline{b_0^{(L)}}$ acquires now the correct strong-coupling power structure (2.2). The reader may easily verify that the application of variational theory with a dynamical determination of ω yields the correct strong-coupling limit 1 with the exponentially fast convergence of the successive approximations for $L\to\infty$ like $\overline{b_0^{(L)}}\approx 1-\exp(-1.909-1.168\,L)$.

In the next section we are going to point out that an escape to complex zeros which BR propose to remedy the problems of the δ expansion is really of no help.

IV. RULING OUT COMPLEX ZEROS

It has been claimed [17] and repeatedly quoted [18] that the study of the anharmonic oscillator in quantum mechanics suggests the use of complex extrema to optimize the δ expansion. In particular, the use of so-called *families* of optimal candidates for the variational parameter z has been suggested. We are now going to show that by following these suggestions one obtains bad resummation results for the anharmonic oscillator [19]. Thus we expect such procedures to lead to even worse results in field-theoretic applications.

In quantum mechanical applications there are no anomalous dimensions in the strong-coupling behavior of the energy eigenvalues. The growth parameters α and ω can be directly read off from the Schrödinger equation; they are $\alpha=1/3$ and $\omega=2/3$ for the anharmonic oscillator (see Appendix A). The variational perturbation theory is applicable for all coupling strengths g as long as $b_0^{(L)}(z)$ becomes stationary for a certain value of z. For higher orders L it must exhibit a well-developed plateau. Within the range of the plateau, various derivatives of $b_0^{(L)}(z)$ with respect to z will vanish. In addition there will be complex zeros with small imaginary parts clustering around the plateau. They are, however, of limited use for designing an automatized computer program for localizing the position of the plateau. The study of several examples shows that plotting $b_0^{(L)}(z)$ for various values of α and ω and judging visually the plateau is by far the safest method, showing immediately which values

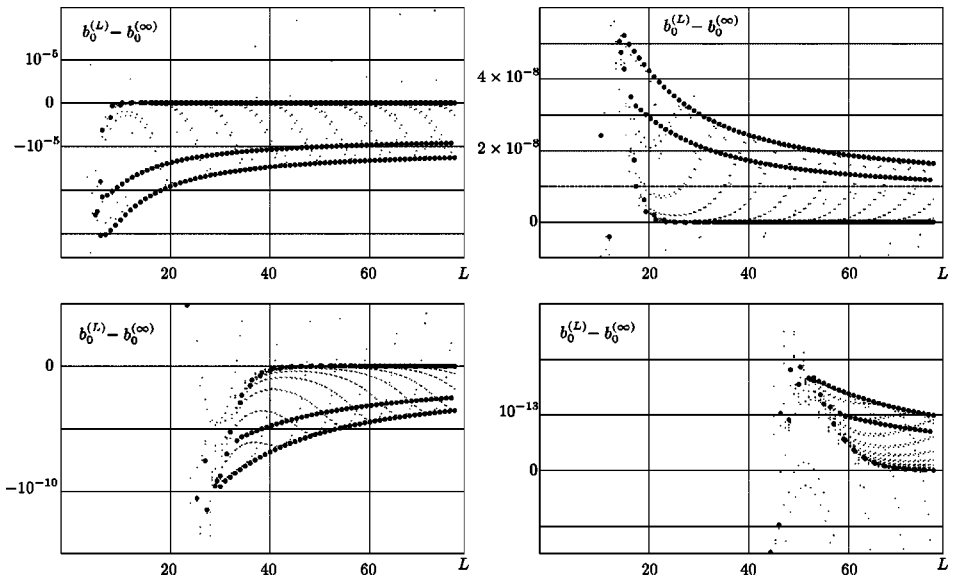

FIG. 5. Deviation of the coefficient $b_0^{(L)}$ from the exact value is shown as a function of perturbative order L on a linear scale. As before, fat dots represent real zeros. In addition to Fig. 4, the results obtained from zeros of the second derivative of $b_0^{(L)}$ are included. They give rise to their own families with smaller errors by about 30%. At $N=6$, the upper left plot shows the start of two families belonging to the first and second derivative of $b_0^{(L)}$, respectively. The deviations of both families are negative. On the upper right-hand figure, an enlargement visualizes the next two families starting at $N=15$. Their deviations are positive. The bottom row shows two more enlargements of families starting at $N=30$ and $N=53$, respectively. The deviations alternate again in sign.

of α and ω lead to a well-shaped plateau.

Let us review briefly the properties of the results obtained from real and complex zeros of $\partial_z b_0^{(L)}(z)$ for the anharmonic oscillator. In Fig. 4 the logarithmic error of $b_0^{(L)}$ is plotted versus the order L. At each order, all zeros of the first derivative are exploited. To test the rule suggested in [17], only the real parts of the complex roots have been used to evaluate $b_0^{(L)}$. The fat points represent the results of real zeros; the thin points stem from the real parts of complex zeros. It is readily seen that the real zeros give the better result. Only by chance may a complex zero yield a smaller error. Unfortunately, there is no rule to detect these accidental events. Most complex zeros produce large errors.

We observe the existence of families described in detail in the textbook [3] and rediscovered in Ref. [17]. These families start at about $N=6,15,30,53$, respectively. But each family fails to converge to the correct result. Only a sequence of selected members in each family leads to an exponential convergence. Consecutive families alternate around the correct result, as can be seen more clearly in a plot of the deviations of $b_0^{(L)}$ from their $L\to\infty$ limit in Fig. 5, where values derived from the zeros of the second derivative of $b_0^{(L)}$ have been included. These give rise to accompanying families of similar behavior, deviating with the same sign pattern from the exact result, but lying closer to the correct result by about 30%.

V. TEMPERATURE SHIFT FOR $N=2$ REVISITED

Much attention has been paid to a field theoretic model with $O(2)$ symmetry [13,14,20] to calculate in a realistic context the coefficient c_1, which enters into the temperature shift of the Bose-Einstein condensation parametrized as

$$\frac{\Delta T_c}{T_c^{(0)}} = c_1 a n^{1/3}. \tag{5.1}$$

Presently, five coefficients of the relevant perturbation expansion are known for the weak-coupling expansion [13,14,20]

$$F(x) = \sum_{n=-1}^{3} a_n x^n, \tag{5.2}$$

whose asymptotic value for $x\to\infty$ coincides with c_1: $c_1 = F^* \equiv \lim_{x\to\infty} F(x)$. The known coefficients are $a_{-1} = -13.9707$, $a_0 = 0$, $a_1 = -0.446572$, $a_2 = 0.264412$, $a_3 = -0.199$.

We would like to offer an alternative resummation result for this series to that in Ref. [20]. It is based on considering the function $xF(x)$ containing no negative powers of x. The desired number c_1 is the leading coefficient b_0 of the strong-coupling expansion

$$F(x) = x\left(c_1 + \sum_{n=1} b_n x^{-\omega n} \right). \tag{5.3}$$

The result for c_1 should be unaffected by this modification of the function, and given by the optimized Lth-order approximations

$$c_1^{(L)}(z,\omega) = \sum_{l=0}^{L} a_l z^{l-1} \binom{L-l+(l-1)/\omega}{L-l}. \tag{5.4}$$

For the available orders $L \leqslant 4$, this set of functions is now inspected for plateaus. For $L<3$ there is none. For $L=3$ and $L=4$, a plateau can be identified unambiguously as the only horizontal turning point solving simultaneously $\partial_z c_1^{(L)}(z,\omega)=0$ and $\partial_z^2 c_1^{(L)}(z,\omega)=0$. The results are

$$L=3,\ \ z^{(3)}=1.089,\ \ \omega^{(3)}=1.071,\ \ c_1^{(3)}=0.940, \quad (5.5)$$

$$L=4,\quad z^{(4)}=2.057,\quad \omega^{(4)}=0.571,\quad c_1^{(4)}=1.282. \quad (5.6)$$

Given only two approximations for c_1 it is unrealistic to attempt an extrapolation to $L\to\infty$, as done with another selection rule of optima in Ref. [20], but it is interesting to note that the value of the coefficient for the temperature shift $c_1^{(4)}=1.282$ is in excellent agreement with the latest Monte Carlo result of $c_1\approx 1.30$ [21].

VI. RENORMALIZATION GROUP AND VARIATIONAL PERTURBATION

The most convincing evidence for the power of the field-theoretic variational perturbation theory with anomalous dimensions comes from applications to critical exponents in $4-\epsilon$ dimensions [8,10]. The results obtained turn out to be immediately resummed expressions of the ϵ expansions, which can be recovered as a Taylor series. The renormalization group function $\beta(g)$ is obtained from the weak-coupling expansion of the renormalized coupling constant g in terms of the bare coupling constant g_B [10,22]:

$$\beta(g,\epsilon)=-\epsilon g\frac{d\log g(g_B,\epsilon)}{d\log g_B}=-\epsilon g\left[\frac{d\log g_B(g,\epsilon)}{d\log g}\right]^{-1}. \quad (6.1)$$

Due to renormalizability, $\beta(g)$ necessarily has the form

$$\beta(g,\epsilon)=-\epsilon g+\beta_0(g). \quad (6.2)$$

Perturbation theory with minimal subtractions yields the weak-coupling expansion

$$g=g_B+\sum_{k=1}^{\infty} f_k(g_B)\epsilon^{-k}, \quad (6.3)$$

where $f_k(g_B)$ possesses an expansion in powers of g_B, starting with g_B^{k+1}. By suitably normalizing g and g_B, the leading coefficient of f_1 can be made equal to minus one: $f_1(g_B)=-g_B^2+O(g_B)^3$. The function $\beta_0(g)$ can be expressed in terms of the residue $f_1(g_B)$ of the ϵ pole in Eq. (6.3) alone

$$\beta_0(g)=f_1(g)-gf_1'(g). \quad (6.4)$$

Recall the standard proof for this based on combining Eqs. (6.1) and (6.2) to

$$\beta_0(g)=\epsilon g-\epsilon g_B\partial_{g_B}g(g_B,\epsilon), \quad (6.5)$$

which becomes, after inserting Eq. (6.3):

$$\beta_0\left[g_B+\sum_{k=1}^{\infty} f_k(g_B)\epsilon^{-k}\right]=\epsilon\sum_{k=1}^{\infty}[f_k(g_B)-g_Bf_k'(g_B)]\epsilon^{-k}. \quad (6.6)$$

The limit $\epsilon\to\infty$ leads directly to the property (6.4).

Another well-known fact is that all the functions $f_k(g_B)$ for $k>1$ can be expressed in terms of the residues $f_1(g_B)$ only [10]. Indeed, taking the derivatives of $\beta_0(g)$ in Eq. (6.5) with respect to g_B and ϵ:

$$\beta_0'(g)\partial_{g_B}g=-\epsilon g_B\partial_{g_B}^2 g, \quad (6.7)$$

$$\beta_0'(g)\partial_\epsilon g=g+\epsilon\partial_\epsilon g-g_B\partial_{g_B}g-\epsilon g_B\partial_{g_B}\partial_\epsilon g, \quad (6.8)$$

eliminating $\beta_0'(g)$ between these two equations, and inserting the expansion (6.3), we obtain order by order in $1/\epsilon$ a recursive set of differential equations for the functions $f_k(g_B)$ with $k>1$, which are power series in g_B. If we now expand

$$f_1(g_B)=-g_B^2+\sum_{j=3}^{\infty}\gamma_j g_B^j,\quad f_k(g_B)=\sum_{j=k+1}^{\infty}\gamma_{k,j}g_B^j, \quad (6.9)$$

a solution is readily found, beginning with

$$\gamma_{k,k+1}=(-1)^k,\quad \gamma_{2,4}=-\frac{8}{3}\gamma_3,\quad \gamma_{2,5}=\frac{3}{2}\gamma_3^2-\frac{7}{2}\gamma_4, \quad (6.10)$$

$$\gamma_{3,5}=\frac{29}{6}\gamma_3,\quad \gamma_{2,6}=\frac{18}{5}\gamma_3\gamma_4-\frac{22}{5}\gamma_5,$$

$$\gamma_{3,6}=-\frac{32}{5}\gamma_3^2+\frac{39}{5}\gamma_4, \quad (6.11)$$

$$\gamma_{4,6}=-\frac{37}{5}\gamma_3,\quad \gamma_{2,7}=2\gamma_4^2+\frac{13}{3}\gamma_3\gamma_5-\frac{16}{3}\gamma_6,$$

$$\gamma_{3,7}=\frac{5}{2}\gamma_3^3-\frac{551}{30}\gamma_3\gamma_4+\frac{59}{5}\gamma_5, \quad (6.12)$$

$$\gamma_{4,7}=\frac{751}{45}\gamma_3^2-\frac{141}{10}\gamma_4,\quad \gamma_{5,7}=\frac{103}{10}\gamma_3. \quad (6.13)$$

In the renormalization group approach, a fixed point $g^*\neq 0$ is determined by the zero of the β function: $\beta(g^*)=0$. The Wegner exponent ω governing the approach to scaling is given by the slope at the fixed point: $\omega=\beta'(g^*)$. The two quantities have ϵ expansions

$$g^*=\sum_{j=1}^{\infty}\alpha_j\epsilon^j,\quad \omega=\sum_{j=1}^{\infty}\omega_j\epsilon^j. \quad (6.14)$$

The coefficients α_j and ω_j are determined from the residues γ_j as

$$\alpha_1=1,\quad \alpha_2=2\gamma_3,\quad \alpha_3=8\gamma_3^2+3\gamma_4, \quad (6.15)$$

$$\alpha_4 = 40\gamma_3^3 + 30\gamma_3\gamma_4 + 4\gamma_5,$$

$$\alpha_5 = 224\gamma_3^4 + 252\gamma_3^2\gamma_4 + 27\gamma_4^2 + 48\gamma_3\gamma_5 + 5\gamma_6, \quad (6.16)$$

and

$$\omega_1 = 1, \quad \omega_2 = -2\gamma_3, \quad \omega_3 = -8\gamma_3^2 - 6\gamma_4,$$

$$\omega_4 = -40\gamma_3^3 - 48\gamma_3\gamma_4 - 12\gamma_5. \quad (6.17)$$

We can now convince ourselves that precisely the same results can be derived from variational perturbation theory applied to the weak-coupling expansion (6.3) (and as shown in [23] from the expansion of any other critical exponent). We determine ω dynamically solving Eq. (2.5), while assuming for ω an unknown ϵ expansion of the form (6.14). The variational parameter z is again adjusted to make (2.5) stationary. Then, since for $\epsilon \to 0$ the weak-coupling coefficients of $g(g_B)$ in the expansion (6.3) behave like $\sim \epsilon^{1-l}$, z has to scale with ϵ, so that we may put $z = \zeta_1\epsilon + \zeta_2\epsilon^2 + \zeta_3\epsilon^3 + O(\epsilon^4)$, and solve Eqs. (2.5) and (2.6) for each perturbative order L, order by order in ϵ. This leads to a rapidly increasing number of nonlinear and not even independent equations for the unknown ζ_l and ω_l, some depending also on the order L.

Despite these possible complications, the solutions turn out to be well structured and easily obtained. At each L to lowest order in ϵ, the term independent of ϵ in Eq. (2.5) and the coefficient of ϵ^{-1} in Eq. (2.6) demand that $\zeta_1 = 1$. In addition, they require $\gamma_{k,k+1} = (-1)^k$ for some k, in agreement with Eqs. (6.10). Such conditions imposed on $\gamma_{k,l}$ can, of course, not depend on the order L, but must be enforced in general. Raising the order of ϵ in Eqs. (2.5) and (2.6), and imposing $\zeta_1 = 1$ as well as the conditions already established for the $\gamma_{k,l}$, all dependences on the ω_k and ζ_k disappear, and we are left with conditions on $\gamma_{k,l}$ alone, which reproduce exactly the relations (6.10) through (6.13). This shows that the variational perturbation method is completely compatible with the well-known ϵ expansions, if the input divergent series has a structure satisfying the renormalization group equation (6.2).

After having reproduced $\gamma_{k,l}$, there are further equations to be solved. Going to the next higher order in ϵ, either for Eq. (2.5) or for (2.6), gives a relation involving exactly one of the expansion coefficients of ϵ/ω, which are simply related to the coefficients ω_l of ω. In this way, the renormalization group results of Eq. (6.17) are exactly reproduced. These solutions are stable in the sense that with increasing order L, the expansion coefficients ω_l for $l < L$ remain unchanged. This proves that the variational method produces the same ϵ expansions of all critical exponents as renormalization group theory. At the same time this implies that the standard δ expansion which does not allow for the anomalous dimension ω is bound to fail.

It is noteworthy that several other conditions are automatically satisfied up to some order ϵ^L, ϵ^{L-1}, or ϵ^{L-2}, respectively. Among them is the variationally transcribed second logarithmic derivative of the weak-coupling series and the derivative thereof:

$$\sum_{l=0}^{L} h_l z^l \binom{L - l + l/\omega}{L - l} = -1 - \omega, \quad (6.18)$$

$$\sum_{l=1}^{L} h_l l z^{l-1} \binom{L - l + l/\omega}{L - l} = 0, \quad (6.19)$$

where the h_l are the expansion coefficients of

$$\frac{g_B g''(g_B)}{g'(g_B)}. \quad (6.20)$$

Of some computational benefit is the observation that with the same accuracy in ϵ the first and second derivatives of the variational series (2.3) themselves vanish (here for $\alpha = 0$). This means that the function has a flat plateau. For a typical field-theoretic application with only a few known perturbation coefficients, the plateau is easily found by inspection. Therefore, if the model possesses a well-behaved β function satisfying Eq. (6.2), we expect a reliable result for the anomalous dimension ω if it is chosen such as to produce an acceptable plateau. The ordinate of the plateau is the most promising variational perturbative value for the quantity analyzed to the respective order.

VII. CONCLUSION

Summarizing this paper we have learned that the so-called δ expansion is inapplicable to quantum field theory, since it does not account for the Wegner exponent ω of approach to the strong-coupling limit. Only the field-theoretic variational perturbation theory yields the correct results by incorporating ω in an essential way.

APPENDIX A

Here we review briefly how the strong-coupling parameters α and ω in Eq. (2.2) and the variational equation (2.3) for the leading strong-coupling coefficient are found for the anharmonic oscillator with the Schrödinger equation in natural units

$$-\frac{1}{2}\Psi'' + \frac{x^2}{2}\Psi + g x^{2\kappa}\Psi = E\Psi. \quad (A1)$$

We rescale the space coordinate x so that the potential becomes

$$V(x) = \frac{1}{2} g^{-2/(\kappa+1)} x^2 + x^{2\kappa}. \quad (A2)$$

Any eigenvalue has the obvious strong-coupling expansion

$$E = g^{1/(\kappa+1)} \sum_{l=0}^{\infty} b_l (g^{-2/(\kappa+1)})^l, \quad (A3)$$

where b_l are the strong-coupling coefficients. The aim is to determine them from the known weak-coupling coefficients a_n of the divergent perturbation expansion:

$$E=\sum_{l=0}^{\infty} a_l g^l. \tag{A4}$$

The solution of this problem comes from physical intuition, suggesting that the perturbation expansion should be performed around an effective harmonic potential $\Omega^2 x^2/2$, whose frequency is different from the bare value 1/2 in Eq. (A1), depending on g and the order L of truncation of Eq. (A3). Thereafter only the difference between the anharmonic part and the effective harmonic part is to be treated by perturbation methods. The trial frequency Ω of the effective potential can be fixed later by the consideration that the resulting quantity of interest should be as independent as possible of Ω, according to the principle of minimal sensitivity. With the harmonic trial potential $V_\Omega^{(0)}=\Omega^2 x^2/2$, the interaction potential (1.2) reads $V_\Omega^{\rm int}=\delta[gx^{2\kappa}-(\Omega^2-1)x^2/2]$. The parameter δ organizes the reexpansion and is set equal to 1 at the end. The expansion proceeds from the rescaled Schrödinger equation (A1):

$$-\frac{1}{2}\Psi''+\frac{x^2}{2}\Psi+\frac{\delta g x^{2\kappa}\Psi}{\beta^{N+1}}=\frac{E}{\beta}\Psi, \tag{A5}$$

where $\beta=\sqrt{\Omega^2-\delta(\Omega^2-1)}$. To order L, the energy has the reexpansion

$$E^{(L)}(\Omega,g)=\beta\sum_{l=0}^{L} a_l^{(i)}\left(\frac{\delta g}{\beta^{\kappa+1}}\right)^l, \tag{A6}$$

with the well-known weak-coupling expansion coefficients as defined in Eq. (A4). The strong-coupling behavior (A3) suggests changing the variational parameter from Ω to $z:=g/\Omega^{\kappa+1}$. In the limit $g\to\infty$ we obtain the reexpansion which must be optimized in z:

$$E^{(L)}(z)=g^\alpha\sum_{l=0}^{L} a_l^{(i)} z^{l-\alpha}\binom{L-l+(l-\alpha)/\omega}{L-l}, \tag{A7}$$

where $\omega=2/(\kappa+1)$ and $\alpha=1/(\kappa+1)$. For the leading coefficient of the strong coupling expansion of the ground state energy, Eq. (A7) leads directly to the variational equation (2.3).

APPENDIX B

In order to gain further insight into the working of the variational resummation procedure, we apply it to the simple test function

$$f(x)=(1+x)^\alpha=x^\alpha\left(1+\frac{1}{x}\right)^\alpha \tag{B1}$$

with weak coupling coefficients $a_n=\binom{\alpha}{n}$ and a leading strong-coupling behavior $f\sim x^\alpha(1+\alpha/x+\cdots)$, so that $b_0=1$. Inserting this information into Eq. (2.3), we obtain the variational leading coefficient to Lth order:

$$b_0^{(L)}(z)=\sum_{l=0}^{L}\binom{\alpha}{l}\binom{L-\alpha}{L-l}z^{l-\alpha}, \tag{B2}$$

which is easily transformed into the expression

$$b_0^{(L)}(z)=\binom{\alpha}{L}\sum_{l=0}^{L}\binom{L}{l}\frac{L-\alpha}{l-\alpha}(-1)^{L+l}z^{l-\alpha}. \tag{B3}$$

Determining the variational parameter z according to the principle of minimal sensitivity requires a well developed plateau of $b_0^{(L)}$ as a function of z. For the simple test function, the derivative $\partial_z b_0^{(L)}(z)$ can be obtained in the closed form:

$$\frac{d}{dz}b_0^{(L)}(z)=(-1)^{L+1}\frac{L-\alpha}{z^{\alpha+1}}\binom{\alpha}{L}\sum_{l=0}^{L}(-z)^l\binom{L}{l} \tag{B4}$$

$$=(-1)^{L+1}(L-\alpha)\binom{\alpha}{L}\frac{(1-z)^L}{z^{\alpha+1}}. \tag{B5}$$

This exhibits a flat plateau around $z=1$ if the order L is much larger than α. An equally flat plateau is found for $b_0^{(L)}(z)$. The value of the leading strong coupling coefficient $b_0^{(L)}$ at the plateau is

$$b_0^{(L)}(1)=\binom{\alpha}{L}\sum_{l=0}^{L}\binom{L}{l}\frac{L-\alpha}{l-\alpha}(-1)^{L+l}=1, \tag{B6}$$

in perfect agreement with the exact result, thus confirming the applicability of the resummation scheme for this class of problems.

[1] R. Seznec and J. Zinn-Justin, J. Math. Phys. **20**, 1398 (1979); T. Barnes and G.I. Ghandour, Phys. Rev. D **22**, 924 (1980); B.S. Shaverdyan and A.G. Usherveridze, Phys. Lett. **123B**, 316 (1983); K. Yamazaki, J. Phys. A **17**, 345 (1984); H. Mitter and K. Yamazaki, *ibid.* **17**, 1215 (1984); P.M. Stevenson, Phys. Rev. D **30**, 1712 (1984); **32**, 1389 (1985); P.M. Stevenson and R. Tarrach, Phys. Lett. B **176**, 436 (1986); A. Okopinska, Phys. Rev. D **35**, 1835 (1987); **36**, 2415 (1987); W. Namgung, P.M. Stevenson, and J.F. Reed, Z. Phys. C **45**, 47 (1989); U. Ritschel, Phys. Lett. B **227**, 44 (1989); Z. Phys. C **51**, 469 (1991); M.H. Thoma, *ibid.* **44**, 343 (1991); I. Stancu and P.M. Stevenson, Phys. Rev. D **42**, 2710 (1990); R. Tarrach, Phys. Lett. B **262**, 294 (1991); H. Haugerud and F. Raunda, Phys. Rev. D **43**, 2736 (1991); A.N. Sissakian, I.L. Solovtsov, and O.Y. Shevchenko, Phys. Lett. B **313**, 367 (1993).

[2] P.M. Stevenson, Phys. Rev. D **30**, 1712 (1984); **32**, 1389 (1985); P.M. Stevenson and R. Tarrach, Phys. Lett. B **176**, 436 (1986).

[3] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, Second extended edition (World Scientific, Singapore, 1995), pp. 1–850 (http://www.physik.fu-berlin.de/~kleinert/b3).

[4] I.R.C. Buckley, A. Duncan, and H.F. Jones, Phys. Rev. D **47**, 2554 (1993); C.M. Bender, A. Duncan, and H.F. Jones, *ibid.* **49**, 4219 (1994).

[5] H. Kleinert and W. Janke, Phys. Lett. A **206**, 283 (1995).
[6] R. Guida, K. Konishi, and H. Suzuki, Ann. Phys. (N.Y.) **249**, 109 (1996).
[7] H. Kleinert, Phys. Rev. D **57**, 2264 (1998); addendum: Phys. Rev. D **58**, 107702 (1998); **60**, 085001 (1999).
[8] H. Kleinert, Phys. Lett. B **434**, 74 (1998).
[9] F.J. Wegner, Phys. Rev. B **5**, 4529 (1972); **6**, 1891 (1972).
[10] H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of* Φ^4*-Theories* (World Scientific, Singapore, 2001).
[11] J.A. Lipa, D.R. Swanson, J. Nissen, Z.K. Geng, P.R. Williamson, D.A. Stricker, T.C.P. Chui, U.E. Israelson, and M. Larson, Phys. Rev. Lett. **84**, 4894 (2000); H. Kleinert, Phys. Rev. D **60**, 085001 (1999); Phys. Lett. A **277**, 205 (2000).
[12] E. Braaten and E. Radescu, Phys. Rev. Lett. **89**, 271602 (2002).
[13] E. Braaten and E. Radescu, cond-mat/0206186v1.
[14] F.F. de Souza Cruz, M.B. Pinto, and R.O. Ramos, Phys. Rev. A **65**, 053613 (2002); Phys. Rev. B **64**, 014515 (2001); J.-L. Kneur, M.B. Pinto, and R.O. Ramos, Phys. Rev. Lett. **89**, 210403 (2002).
[15] A similar program had been proposed a long time ago for fermion systems by P. Nozières, *Theory of Interacting Fermi Systems* (Benjamin, New York, 1964); and recently been applied by A. Neumayr and W. Metzner, ''Renormalized perturbation theory for Fermi systems: Fermi surface deformation and superconductivity in the two-dimensional Hubbard model,'' cond-mat/0208431. There are also rigorous convergence proofs in this context by J. Feldman, J. Magnen, V. Rivasseau, and E. Trubowitz, in *The State of Matter*, edited by M. Aizenmann and H. Araki, Advanced Series in Mathematical Physics Vol. 20 (World Scientific, Singapore, 1994); J. Feldman, H. Knörrer, M. Salmhofer, and E. Trubowitz, J. Stat. Phys. **94**, 113 (1999). These authors do not realize, however, that the limit will in general produce the *wrong* result in the most interesting limit of strong coupling (which governs the critical behavior), as shown with the help of simple models in the present paper.
[16] See Eq. (5.251) in H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, third ed. (World Scientific, Singapore, 2003), pp. 1–1463.
[17] B. Bellet, P. Garcia, and A. Neveu, Int. J. Mod. Phys. A **11**, 5587 (1997). The family structure of optimal variational parameters emphasized in this paper was discussed in great detail earlier in Chap. 5 of the textbook [3], but with the correct application rules.
[18] J.-L. Kneur and D. Reynaud, Phys. Rev. D **66**, 085020 (2002), and references [12–14].
[19] In a forthcoming letter, in Phys. Lett. A (to be published), we were able to show that complex zeros can be used effectively to sum divergent series in there nonborel regimes. B. Hamprecht and H. Kleinert, hep-th/0302124; hep-th/0303163.
[20] H. Kleinert, ''Five-Loop Critical Temperature Shift in Weakly Interacting Homogeneous Bose-Einstein Condensate,'' cond-mat/0210162, and references quoted in this paper.
[21] P. Grueter, D. Ceperley, and F. Laloe, Phys. Rev. Lett. **79**, 3549 (1997).
[22] D.J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978).
[23] H. Kleinert, Phys. Lett. B **463**, 69 (1999).