Path-integral approach to 't Hooft's derivation of quantum physics from classical physics

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(Received 21 September 2004; published 27 May 2005)

We present a path-integral formulation of 't Hooft's derivation of quantum physics from classical physics. The crucial ingredient of this formulation is Gozzi *et al.*'s supersymmetric path integral of classical mechanics. We quantize explicitly two simple classical systems: the planar mathematical pendulum and the Rössler dynamical system.

DOI: 10.1103/PhysRevA.71.052507 PACS number(s): 31.15.Kb, 03.65.−w, 45.20.Jj, 11.30.Pb

I. INTRODUCTION

In recent decades, various classical, i.e., deterministic, approaches to quantum theory have been proposed. Examples are Bohmian mechanics [1], and the stochastic quantization procedures of Nelson [2], Guerra and Ruggiero [3], and Parisi and Wu [4,5]. Such approaches are finding increasing interest in the physics community. This might be partially ascribed to the fact that such alternative formulations help in explaining some quantum phenomena that cannot be easily explained with the usual formalisms. Examples are multiple tunneling [6], critical phenomena at zero temperature [7], mesoscopic physics and quantum Brownian oscillators [8], and quantum-field-theoretical regularization procedures which manifestly preserve all symmetries of the bare theory such as gauge symmetry, chiral symmetry, and supersymmetry [9]. They allow one to quantize gauge fields, both Abelian and non-Abelian, without gauge fixing and the ensuing cumbersome Faddeev-Popov ghosts [10], etc.

The primary objective of a reformulation of quantum theory in the language of classical, i.e., deterministic, theory is basically twofold. On the formal side, it is hoped that this will help in attacking quantum-mechanical problems from a different direction using hopefully more efficient mathematical techniques than the conventional ones. Such techniques may be based on stochastic calculus, supersymmetry, or various new numerical approaches (see, e.g., Refs. [5,11] and citations therein). On the conceptual side, deterministic scenarios are hoped to shed new light on some old problems of quantum mechanics, such as the origin of the superposition rule for amplitudes and the theory of quantum measurement. It may lead to new ways of quantizing chaotic dynamical systems, and ultimately a long-awaited consistent theory of quantum gravity. There is, however, a price to be paid for this; such theories must have a built-in nonlocality to escape problems with Bell's inequalities. Nonlocality may be incorporated in numerous ways—the Bohm-Hiley quantum potential [1,12], Nelson's osmotic potential [2], or Parisi and Wu's *fifth-time* parameter [4,5].

Another deterministic access to quantum-mechanical systems was recently proposed by 't Hooft [13,14] with subsequent applications in Refs. [15–21]. It is motivated by black-hole thermodynamics (and particularly by the so-called *holographic principle* [22,23]), and hinges on the concept of *information loss*. This and certain accompanying nontrivial geometric phases are able to explain the observed nonlocality in quantum mechanics. The original formulation has appeared in two versions: one involving a discrete time axis [16], the second continuous times [14]. The goal of this paper is to discuss further and gain more understanding of the latter model. The reader interested in the discrete-time model may find some practical applications in Refs. [24,25]. It is not our purpose to dwell on the conceptual foundations of 't Hooft's proposal. Our aim is to set up a possible useful alternative formulation of 't Hooft's model and quantization scheme that is based on path integrals [11]. It makes use of Gozzi *et al.*'s path-integral formulation of classical mechanics [26,27] which appears to be a natural mathematical framework for such a discussion. The condition of the information loss, which is basically a first-class subsidiary constraint, can then be incorporated into path integrals by standard techniques. Although 't Hooft's procedure differs in its basic rationale from stochastic quantization approaches, we show that they share a common key feature, which is a hidden BRST invariance, related to the so-called Nicolai map [28]. To be specific, we shall apply our formulation to two classical systems: a planar mathematical pendulum and the simplest deterministic chaotic system—the Rössler attractor. Suitable choices of the "loss of information" condition then allow us to identify the emergent quantum systems with a free particle, a quantum harmonic oscillator, and a free particle weakly coupled to Duffing's oscillator.

Our paper is organized as follows In Sec. II we quantize 't Hooft's Hamiltonian system by expressing it in terms of a path integral which is singular due to the presence of second-class primary constraints. The singularity is removed with the help of the Faddeev-Senjanovic prescription [29,30]. It is then shown that the fluctuating system produces a classical partition function. In Sec. III we briefly review Gozzi *et al.*'s path-integral formulation of classical mechanics in configu-

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ration space. The corresponding phase-space formulation is more involved and will not be considered here. By imposing the condition of a vanishing ghost sector, which is characteristic for the underlying deterministic system, we find that the most general Hamiltonian system compatible with such a condition is the one proposed by 't Hooft. In Sec. IV we introduce 't Hooft's constraint which expresses the property of information loss. This condition not only explicitly breaks the BRST symmetry but, when coupled with the Dirac-Bergmann algorithm, it also allows us to recast the classical generating functional into a form representing a proper quantum-mechanical partition function. Section V is devoted to application of our formalism to practical examples. We conclude with Sec. VI. For the reader's convenience the paper is supplemented with four appendixes which clarify some finer mathematical points needed in the paper.

II. QUANTIZATION OF 't HOOFT'S MODEL

Consider the class of systems described by Hamiltonians of the form

$$H = \sum_{a=1}^{N} p_a f_a(\mathbf{q}). \tag{1}$$

Such systems emerge in diverse physical situations, for example, Fermi fields, chiral oscillators [20], and noncommutative magnetohydrodynamics [31]. The relevant example in the present context is the use of Eq. (1) by 't Hooft to formulate his *deterministic* proposal [13].

An immediate problem with the above Hamiltonian is its unboundedness from below. This is due to the absence of a leading kinetic term quadratic in the momenta $p_a^2/2M$, and we shall dwell more on this point in Sec. IV. The equations of motion following from Eq. (1) are

$$\dot{q}_a = f_a(\mathbf{q}), \quad \dot{p}_a = -p_a \frac{\partial f_a(\mathbf{q})}{\partial q_a}. \tag{2}$$

Note that the equation for q_a is autonomous, i.e., it is decoupled from the conjugate momenta p_a. The absence of a quadratic term makes it impossible to find a Lagrangian via a Legendre transformation. This is because the system is singular—its Hess matrix $H^{ab} \equiv \partial^2 H/\partial p_a \partial p_b$ vanishes.

A Lagrangian yielding the equations of motion (2) can nevertheless be found, but at the expense of doubling the configuration space by introducing additional auxiliary variables $\overline{q}_a (a=1,\ldots,N)$. This *extended* Lagrangian has the form

$$\overline{L} \equiv \sum_{a=1}^{N} [\dot{\overline{q}}_a \dot{q}_a - \overline{q}_a f_a(\mathbf{q})] \tag{3}$$

and it allows us to define canonically conjugate momenta in the usual way: $p_a \equiv \partial \overline{L}/\partial \dot{q}_a$, $\overline{p}_a \equiv \partial \overline{L}/\partial \dot{\overline{q}}_a$. A Legendre transformation produces the Hamiltonian

$$\overline{H}(p_a, q_a, \overline{p}_a, \overline{q}_a) = \sum_{a=1}^{N} p_a \dot{q}_a + \overline{p}_a \dot{\overline{q}}_a - L = \sum_{a=1}^{N} \overline{q}_a f_a(\mathbf{q}). \tag{4}$$

The rank of the Hess matrix is zero, which gives rise to $2N$ primary constraints, which can be chosen as

$$\phi_1^a = p_a - \overline{q}_a \approx 0, \quad \phi_2^a = \overline{p}_a \approx 0. \tag{5}$$

The use of the symbol $\approx$ instead of $=$ is due to Dirac [32] and it has a special meaning: two quantities related by this symbol are equal after all constraints have been enforced. The system has no secondary constraints (see Appendix A). The matrix formed by the Poisson brackets of the primary constraints,

$$\{\phi_1^a(t), \phi_2^b(t)\} = -\delta_{ab}, \tag{6}$$

has a nonzero determinant, implying that all constraints are of the second class. Note that on the constraint manifold the *canonical* Hamiltonian (4) coincides with 't Hooft's Hamiltonian (1).

To quantize 't Hooft's system we utilize the general Faddeev-Senjanovic path integral formula [29,30] for time evolution amplitudes[1]

$$\langle \mathbf{q}_2, t_2 | \mathbf{q}_1, t_1 \rangle = \mathcal{N} \int \mathcal{D}\mathbf{p}\, \mathcal{D}\mathbf{q} \sqrt{|\det\|\{\phi_i, \phi_j\}\||} \prod_i \delta[\phi_i] \times \exp\left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} dt [\mathbf{p}\dot{\mathbf{q}} - \overline{H}(\mathbf{q}, \mathbf{p})] \right\}. \tag{7}$$

Using the shorthand notation $\phi_i = \{\phi_1^1, \phi_2^1, \phi_1^2, \phi_2^2, \ldots, \phi_1^N, \phi_2^N\}$ $(i=1,\ldots,2N)$, Eq. (7) implies in our case that

$$\begin{aligned} \langle \mathbf{q}_2, t_2 | \mathbf{q}_1, t_1 \rangle &= \mathcal{N} \int \mathcal{D}\mathbf{p}\, \mathcal{D}\mathbf{q}\, \mathcal{D}\overline{\mathbf{p}}\, \mathcal{D}\overline{\mathbf{q}}\, \delta[\mathbf{p} - \overline{\mathbf{q}}]\delta[\overline{\mathbf{p}}] \\ &\quad \times \exp\left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} dt [\mathbf{p}\dot{\mathbf{q}} + \overline{\mathbf{p}}\dot{\overline{\mathbf{q}}} - \overline{H}(\mathbf{q}, \overline{\mathbf{q}}, \mathbf{p}, \overline{\mathbf{p}})] \right\} \\ &= \mathcal{N} \int_{\mathbf{q}(t_1)=\mathbf{q}_1}^{\mathbf{q}(t_2)=\mathbf{q}_2} \mathcal{D}\mathbf{q}\, \mathcal{D}\overline{\mathbf{q}} \exp\left[\frac{i}{\hbar} \int_{t_1}^{t_2} \overline{L}(\mathbf{q}, \overline{\mathbf{q}}, \dot{\mathbf{q}}, \dot{\overline{\mathbf{q}}}) dt \right] \\ &= \mathcal{N} \int_{\mathbf{q}(t_1)=\mathbf{q}_1}^{\mathbf{q}(t_2)=\mathbf{q}_2} \mathcal{D}\mathbf{q} \prod_a \delta[\dot{q}_a - f_a(\mathbf{q})], \end{aligned} \tag{8}$$

where $\delta[\mathbf{f}] \equiv \Pi_t \delta(\mathbf{f}(t))$ is the functional version of Dirac's δ function. This result shows that after quantization the system described by the Hamiltonian (1) retains its deterministic character. The paths are squeezed onto the classical trajectories determined by the differential equations $\dot{q}_a = f_a(\mathbf{q})$. The time evolution amplitude (8) contains a sum over only the classical trajectories—there are no quantum fluctuations driving the system away from the classical paths, which is precisely what we expect from a deterministic dynamics.

The amplitude (8) can be brought to a more intuitive form by utilizing the identity

$$\delta[\mathbf{f}(\mathbf{q}) - \dot{\mathbf{q}}] = \delta[\mathbf{q} - \mathbf{q}_{\text{cl}}](\det M)^{-1}, \tag{9}$$

where M is a functional matrix formed by the second derivatives of the action $\overline{\mathcal{A}}[\mathbf{q}, \overline{\mathbf{q}}] \equiv \int dt\, \overline{L}(\mathbf{q}, \overline{\mathbf{q}}, \dot{\mathbf{q}}, \dot{\overline{\mathbf{q}}})$:

[1] Other path-integral s of systems with second-class constraints such as that of Fradkin and Fradkina [33] would lead to the same result (8).

$$M_{ab}(t,t') = \left. \frac{\delta^2 \overline{\mathcal{A}}}{\delta q_a(t)\, \delta \overline{q_b}(t')} \right|_{\mathbf{q}=\mathbf{q}_{\rm cl}}. \quad (10)$$

The Morse index theorem then ensures that for sufficiently short time intervals $t_2 - t_1$ (before the system reaches its first focal point), the classical solution with the initial condition $\mathbf{q}(t_1) = \mathbf{q}_1$ is unique. Note, however, that because of the first-order character of the equations of motion we are dealing with a Cauchy problem, which may happen to possess no classical trajectory satisfying the two Dirichlet boundary conditions $\mathbf{q}(t_1) = \mathbf{q}_1$, $\mathbf{q}(t_2) = \mathbf{q}_2$. If a trajectory exists, Eq. (8) can be brought to the form

$$\langle \mathbf{q}_2, t_2 | \mathbf{q}_1, t_1 \rangle = \overline{\mathcal{N}} \int_{\mathbf{q}(t_1)=\mathbf{q}_1}^{\mathbf{q}(t_2)=\mathbf{q}_2} \mathcal{D}\mathbf{q}\; \delta[\mathbf{q} - \mathbf{q}_{\rm cl}], \quad (11)$$

where $\overline{\mathcal{N}} \equiv \mathcal{N}/(\det M)$. We close this section by observing that $\det M$ can be recast into more expedient form. To do this we formally write

$$\begin{aligned}
\det M &= \det \left\| \left(\partial_t \delta_a^b + \frac{\partial f_a(\mathbf{q}(t))}{\partial q_b(t)} \right) \delta(t-t') \right\| \\
&= \exp\left[\mathrm{Tr}\ln \left\| \left(\partial_t \delta_a^b + \frac{\partial f_a(\mathbf{q}(t))}{\partial q_b(t)} \right) \delta(t-t') \right\| \right] \\
&= \exp\left[\mathrm{Tr}\ln \partial_t \left\| \delta_a^b \delta(t-t') + G(t-t') \frac{\partial f_a(\mathbf{q}(t'))}{\partial q_b(t')} \right\| \right] \\
&= \exp[\mathrm{Tr}(\ln \partial_t)] \exp\left[\mathrm{Tr}\ln \left\| \delta_a^b \delta(t-t') \right.\right. \\
&\quad \left.\left. + G(t-t') \frac{\partial f_a(\mathbf{q}(t'))}{\partial q_b(t')} \right\| \right].
\end{aligned} \quad (12)$$

Here $G(t-t')$ is the Green's function satisfying the equation

$$\partial_t G(t-t') = \delta(t-t').$$

Choosing $G(t-t') = \theta(t-t')$, and noting that the first factor in Eq. (12) is an irrelevant constant that can be assimilated into $\mathcal{N}$ we have

$$\begin{aligned}
\det M &= \exp\left[\mathrm{Tr}\ln \left\| \delta_a^b \delta(t-t') + G(t-t') \frac{\partial f_a(\mathbf{q}(t'))}{\partial q_b(t')} \right\| \right] \\
&= \exp\left[\mathrm{Tr} \left\| \theta(t-t') \frac{\partial f_a(\mathbf{q}(t))}{\partial q_b(t)} \right\| \right] \\
&= \exp\left[\frac{1}{2} \int_{t_1}^{t_2} dt\, \nabla_{\mathbf{q}} \mathbf{f}(\mathbf{q}) \right].
\end{aligned} \quad (13)$$

In deriving Eq. (13) we have used the fact that due to the product of the θ function in the expansion of the logarithm, all terms vanish but the first one. In evaluating the generalized function $\theta(x)$ at the origin we have used the only consistent midpoint rule [11]: $\theta(0) = 1/2$. Using the identity

$$\begin{aligned}
&\left. \exp\left[\frac{1}{2} \int_{t_1}^{t_2} dt\, \nabla_{\mathbf{q}} \mathbf{f}(\mathbf{q}) \right] \right|_{\mathbf{q}=\mathbf{q}_{\rm cl}} \\
&\quad = \int \mathcal{D}\overline{\mathbf{q}}\; \delta[\overline{\mathbf{q}} - \overline{\mathbf{q}}_{\rm cl}] \exp\left[-\frac{1}{2} \int_{t_1}^{t_2} dt\, \nabla_{\overline{\mathbf{q}}} \dot{\overline{\mathbf{q}}} \right],
\end{aligned} \quad (14)$$

we can finally write the amplitude of transition in the suggestive form

$$\begin{aligned}
\langle \mathbf{q}_2, t_2 | \mathbf{q}_1, t_1 \rangle &= \mathcal{N} \int_{\mathbf{q}(t_1)=\mathbf{q}_1}^{\mathbf{q}(t_2)=\mathbf{q}_2} \mathcal{D}\mathbf{q}\, \mathcal{D}\overline{\mathbf{q}}\, \delta[\mathbf{q} - \mathbf{q}_{\rm cl}] \delta[\overline{\mathbf{q}} - \overline{\mathbf{q}}_{\rm cl}] \\
&\quad \times \exp\left[-\frac{1}{2} \int_{t_1}^{t_2} dt\, \nabla_{\overline{\mathbf{q}}} \dot{\overline{\mathbf{q}}} \right] = \mathcal{N} \int_{\mathbf{q}(t_1)=\mathbf{q}_1}^{\mathbf{q}(t_2)=\mathbf{q}_2} \mathcal{D}\mathbf{q}\, \mathcal{D}\overline{\mathbf{q}} \\
&\quad \times \delta[\mathbf{q} - \mathbf{q}_{\rm cl}] \delta[\overline{\mathbf{q}} - \overline{\mathbf{q}}_{\rm cl}] \sqrt{\frac{\det K(t_2)}{\det K(t_1)}}.
\end{aligned} \quad (15)$$

Here $K(t)$ is the fundamental matrix of the solutions of the system

$$\dot{\overline{q}}_a = -\overline{q}_b \frac{\partial f_b(\mathbf{q})}{\partial q_a}. \quad (16)$$

$\det K(t)$ is then the corresponding Wronskian. Note that in the particular case when $\nabla_{\mathbf{q}} \mathbf{f}(\mathbf{q}) \equiv 0$, i.e., when the phase flow preserves the volume of any domain in the *configuration* space, the exponential in Eq. (15) can be dropped.[2] Because the exponent depends only on the end points of the $\overline{\mathbf{q}}$ variable it can be removed by performing the trace over $\overline{\mathbf{q}}$. As a result we can cast the quantum-mechanical partition function (or generating functional) Z into the form

$$\begin{aligned}
Z &= \mathcal{N} \int \mathcal{D}\mathbf{q}\, \mathcal{D}\overline{\mathbf{q}}\, \delta[\mathbf{q} - \mathbf{q}_{\rm cl}] \delta[\overline{\mathbf{q}} - \overline{\mathbf{q}}_{\rm cl}] \\
&\quad \times \exp\left[\int_{t_1}^{t_2} [\mathbf{J}(t)\mathbf{q}(t) + \overline{\mathbf{J}}(t)\overline{\mathbf{q}}(t)] dt \right] \\
&= \mathcal{N} \int \mathcal{D}q_a \delta[q_a - (q_a)_{\rm cl}] \exp\left[\int_{t_1}^{t_2} dt\, J_a(t) q_a(t) \right].
\end{aligned} \quad (17)$$

Here the doubled vector notation $q_a = \{\mathbf{q}, \overline{\mathbf{q}}\}$ and $J_a \equiv \{\mathbf{J}, \overline{\mathbf{J}}\}$ was used.

III. PATH-INTEGRAL FORMULATION OF CLASSICAL MECHANICS: CONFIGURATION-SPACE APPROACH

Expressions (11) and (17) formally coincide with the path-integral formulation of classical mechanics in configuration space proposed by Gozzi [26] and further developed by Gozzi, Reuter, and Thacker [27] (see also Ref. [21] for recent applications). Let us briefly review aspects of this which will be needed here. Consider the path-integral of the generating functional of a quantum-mechanical system with action $\mathcal{A}[\mathbf{q}]$:

[2]This corresponds to the situation when there are no attractors in the configuration space $\Gamma_{\mathbf{q}}$.

$$Z_{\mathrm{QM}} = \mathcal{N}\int \mathcal{D}\mathbf{q}\, e^{-i\mathcal{A}[\mathbf{q}]/\hbar}\exp\left[\int \mathbf{J}(t)\mathbf{q}(t)dt\right]. \tag{18}$$

We assume in this context that there are no constraints that would make the measure more complicated as in Eq. (7). Gozzi *et al.* proposed to describe classical mechanics by a generating functional of the form (18) with an obviously modified integration measure which gives equal weight to all classical trajectories and zero weight to all others,

$$Z_{\mathrm{CM}} = \tilde{\mathcal{N}}\int \mathcal{D}\mathbf{q}\; \delta[\mathbf{q}-\mathbf{q}_{\mathrm{cl}}]\exp\left[\int \mathbf{J}(t)\mathbf{q}(t)dt\right]. \tag{19}$$

Although the form of the partition function (19) is not derived but *postulated*, we show in Appendix B that it can be heuristically understood either as the "classical" limit of the stochastic-quantization partition function, or as a result of the classical limit of the closed-time-path integral for transition probability of systems coupled to a heat bath. This, in turn, indicates that it would be formally more correct to associate Eq. (19) with the *probability* of transition or (via the stochastic-quantization passage) with the *Euclidean* amplitude of transition [34]. Albeit Eq. (19) cannot be generally obtained from Eq. (18) by a semiclassical limit as in the WKB method (which can be recognized by the absence of a phase factor $\exp[i/\hbar\mathcal{A}(q_{\mathrm{cl}})]$ in Eq. (19)) it may happen that even ordinary amplitudes of transition possess this form. This is the case, for instance, when the number of degrees of freedom is doubled or when one deals with closed-time-path formulation of thermal quantum theory. Yet, whatever is the origin or motivation for Eq. (19), it will be its formal structure and mathematical implications that will interest us here most.

To proceed we note that an alternative way of writing Eq. (19) is

$$Z_{\mathrm{CM}} = \tilde{\mathcal{N}}\int \mathcal{D}\mathbf{q}\; \delta\left[\frac{\delta\mathcal{A}}{\delta\mathbf{q}}\right]\det\left|\frac{\delta^2\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')}\right| \times\exp\left[\int \mathbf{J}(t)\mathbf{q}(t)dt\right]. \tag{20}$$

By representing the δ functional in the usual way as a functional Fourier integral,

$$\delta\left[\frac{\delta\mathcal{A}}{\delta\mathbf{q}}\right] = \int \mathcal{D}\lambda\, \exp\left(i\int_{t_1}^{t_2} dt\, \lambda(t)\frac{\delta\mathcal{A}}{\delta\mathbf{q}(t)}\right), \tag{21}$$

and the functional determinant as a functional integral over two real time-dependent Grassmannian *ghost variables* $c_a(t)$ and $\bar{c}_a(t)$,

$$\det\left|\frac{\delta^2\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')}\right| = \int \mathcal{D}\mathbf{c}\,\mathcal{D}\bar{\mathbf{c}}\exp\left[\int_{t_1}^{t_2} dt\int_{t_1}^{t_2} dt'\,\bar{c}_a(t) \times\frac{\delta^2\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')}c_b(t')\right], \tag{22}$$

we obtain

$$Z_{\mathrm{CM}} = \int \mathcal{D}\mathbf{q}\,\mathcal{D}\lambda\,\mathcal{D}\mathbf{c}\,\mathcal{D}\bar{\mathbf{c}}\exp\left[i\mathcal{S} + \int_{t_1}^{t_2} dt\, \mathbf{J}(t)\mathbf{q}(t)\right], \tag{23}$$

with the new action

$$\mathcal{S}[\mathbf{q},\bar{\mathbf{c}},\mathbf{c},\lambda] \equiv \int_{t_1}^{t_2} dt\, \lambda(t)\frac{\delta\mathcal{A}}{\delta\mathbf{q}(t)} - i\int_{t_1}^{t_2} dt\int_{t_1}^{t_2} dt'\,\bar{c}_a(t)\frac{\delta^2\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')}c_b(t'). \tag{24}$$

Since Z_{CM} together with the action (24) formally result from the classical limit of the stochastic-quantization partition function, it comes as no surprise that $\mathcal{S}$ exhibits BRST (and anti-BRST) supersymmetry. It is simple to check that $\mathcal{S}$ does not change under the supersymmetry transformations

$$\delta_{\mathrm{BRST}}\mathbf{q} = \bar{\boldsymbol{\varepsilon}}\mathbf{c}, \quad \delta_{\mathrm{BRST}}\mathbf{c} = 0, \quad \delta_{\mathrm{BRST}}\bar{\mathbf{c}} = -i\bar{\varepsilon}\lambda, \quad \delta_{\mathrm{BRST}}\lambda = 0, \tag{25}$$

where $\bar{\varepsilon}$ is a Grassmann-valued parameter [the corresponding anti-BRST transformations are related with Eq. (25) by charge conjugation]. Indeed, the variations of the two terms in Eq. (24) read

$$\delta_{\mathrm{BRST}}\left[\int_{t_1}^{t_2} dt\, \lambda(t)\frac{\delta\mathcal{A}}{\delta\mathbf{q}(t)}\right] = \bar{\varepsilon}\int_{t_1}^{t_2} dt\int_{t_1}^{t_2} dt'\,\lambda_a(t) \times\frac{\delta^2\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')}c_b(t'), \tag{26}$$

$$\begin{aligned}\delta_{\mathrm{BRST}}&\left[\int_{t_1}^{t_2} dt\int_{t_1}^{t_2} dt'\,\bar{c}_a(t)\frac{\delta^2\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')}c_b(t')\right]\\ &= -i\bar{\varepsilon}\int_{t_1}^{t_2} dt\int_{t_1}^{t_2} dt'\,\lambda_a(t)\frac{\delta^2\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')}c_b(t')\\ &+ \int_{t_1}^{t_2} dt\int_{t_1}^{t_2} dt'\int_{t_1}^{t_2} dt''\,\bar{c}_a(t)\\ &\times\frac{\delta^3\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')\,\delta q_c(t'')}\bar{\varepsilon}c_c(t'')c_b(t').\end{aligned} \tag{27}$$

The second term on the right-hand side RHS of Eq. (27) vanishes because the functional derivative of $\mathcal{A}$ is symmetric in $c\leftrightarrow b$ whereas the term $c_c c_b$ is anti-symmetric. Inserting Eqs. (26) and (27) into the action we clearly find $\delta_{\mathrm{BRST}}\mathcal{S} = 0$. As noted in [27], the ghost fields $\bar{\mathbf{c}}$ and $\mathbf{c}$ are mandatory at the classical level as their role is to cut off the fluctuations *perpendicular* to the classical trajectories. On the formal side, $\bar{\mathbf{c}}$ and $\mathbf{c}$ may be identified with Jacobi fields [27,35]. The corresponding BRST charges are related to Poincaré-Cartan integral invariants [36].

By analogy with the stochastic quantization the path integral (23) can, of course, be rewritten in a compact form with the help of a superfield [26,34]

$$\Phi_a(t,\theta,\bar{\theta}) = q_a(t) + i\theta c_a(t) - i\bar{\theta}\bar{c}_a(t) + i\bar{\theta}\theta\lambda_a(t), \qquad (28)$$

in which θ and $\bar{\theta}$ are anticommuting coordinates extending the configuration space of q_a variable to a superspace. The latter is nothing but the degenerate case of supersymmetric field theory in $d=1$ in the superspace formalism of Salam and Strathdee [37]. In terms of superspace variables we see that

$$\begin{aligned}\int d\bar{\theta}\,d\theta\,\mathcal{A}[\mathbf{\Phi}] &= \int dt\,d\bar{\theta}\,d\theta\,L[\mathbf{q}(t) + i\theta\mathbf{c}(t) - i\bar{\theta}\bar{\mathbf{c}}(t) + i\bar{\theta}\theta\lambda(t)] \\ &= \int d\bar{\theta}\,d\theta\,\mathcal{A}[\mathbf{q}] + \int dt\,d\bar{\theta}\,d\theta \times[i\theta\mathbf{c}(t) - i\bar{\theta}\bar{\mathbf{c}}(t) + i\bar{\theta}\theta\lambda]\frac{\delta\mathcal{A}}{\delta\mathbf{q}(t)} \\ &\quad + \int dt\,dt'\,d\bar{\theta}\,d\theta\,\theta c_a(t)\frac{\delta^2\mathcal{A}}{\delta q_a(t)\,\delta q_b(t')}\bar{\theta}\bar{c}(t').\end{aligned} \qquad (29)$$

Using the standard integration rules for Grassmann variables, this becomes equal to $-i\mathcal{S}$. Together with the identity $\mathcal{D}\mathbf{\Phi} = \mathcal{D}\mathbf{q}\,\mathcal{D}\mathbf{c}\,\mathcal{D}\bar{\mathbf{c}}\,\mathcal{D}\lambda$ we may therefore express the classical partition functions (19) and (20) as a supersymmetric path integral with fully fluctuating paths in superspace

$$\begin{aligned}Z_{\mathrm{CM}} = \int \mathcal{D}\mathbf{\Phi}\,\exp\bigg\{ &-\int d\theta\,d\bar{\theta}\,\mathcal{A}[\mathbf{\Phi}](\theta,\bar{\theta}) \\ &+ \int dt\,d\theta\,d\bar{\theta}\,\mathbf{\Gamma}(t,\theta,\bar{\theta})\mathbf{\Phi}(t,\theta,\bar{\theta})\bigg\}.\end{aligned} \qquad (30)$$

Here we have defined the supercurrent $\mathbf{\Gamma}(t,\theta,\bar{\theta}) = \bar{\theta}\theta\mathbf{J}(t)$.

It is interesting to find the most general form of an action $\mathcal{A}$ for which the classical path integral (30) coincides with the quantum-mechanical path integral of the system, or, in other words, for which a theory would possess at the same time deterministic and quantal character. As already mentioned, the Grassmannnian ghost variables are responsible for the deterministic nature of the partition function. It is obvious that if the ghost sector could somehow be factored out we would extend the path integration to all fluctuating paths in $\mathbf{q}$ space. By formally writing

$$\frac{\delta^2\mathcal{A}}{\delta q_k(t)\,\delta q_l(t')} = \mathcal{F}_{kl}\left(t,t',q_m,\frac{\delta\mathcal{A}}{\delta q_n}\right), \quad k,l,m,n = 1,\ldots,N, \qquad (31)$$

we see that the factorization will occur if and only if the (distribution valued) functional $\mathcal{F}_{kl}(\cdots)$ is q_m independent when evaluated on shell, i.e., $\mathcal{F}_{kl}(t,t',q_m,0) = F_{kl}(t,t')$. This is a simple consequence of Eq. (20) where the determinant is factorizable if and only if it is $\mathbf{q}$ independent at $\delta\mathcal{A}/\delta\mathbf{q} = 0$.

In order to provide a correct Feynman weight to every path we must, in addition, identify

$$\mathcal{A}[\mathbf{q}] = \int_{t_1}^{t_2} dt\,\lambda_m\frac{\delta\mathcal{A}[\mathbf{q}]}{\delta q_m}, \qquad (32)$$

as can be seen from Eq. (24) after factoring out the second term. Assuming that $L = L(q_l,\dot{q}_l)$ (i.e., a scleronomic system) and that the Hessian is regular, the condition (32) shows that $\lambda_k = \lambda_k(q_l,\dot{q}_k)$. In addition, it is obvious on dimensional grounds that $[\lambda_l] = [q_l]$. This, in turn, implies that $\lambda_k = \alpha_{kl}q_l$, where α_{lk} is some real (t-independent) matrix. To determine the latter we functionally expand $\mathcal{A}$ in Eq. (32) around q_k and compare both sides. The resulting integrability condition reads

$$(\delta_{ji} - \alpha_{ji})\frac{\delta\mathcal{A}}{\delta q_j(t)}\delta(t-t') = \alpha_{lj}q_j(t)\frac{\delta^2\mathcal{A}}{\delta q_l(t)\,\delta q_i(t')}, \qquad (33)$$

which is evidently compatible with the condition (31). When α_{ij} is diagonalizable we can pass to a polar basis and write Eq. (32) in more manageable form, namely,

$$\mathcal{A}[\mathbf{q}] = \int_{t_1}^{t_2} dt\sum_i \alpha_i q_i(t)\frac{\delta\mathcal{A}[\mathbf{q}]}{\delta q_i(t)}. \qquad (34)$$

For simplicity, we do not use new symbols for transformed $\mathbf{q}$'s.

To proceed we assume that the kinetic energy is quadratic in $\mathbf{q}$ and $\dot{\mathbf{q}}$. Then Eq. (34) implies that L_{kin} must be linear in $\dot{\mathbf{q}}$. As such, one can always write (modulo the total derivative)

$$L_{\mathrm{kin}} = \sum_{i,j} B_{ij}q_i(t)\dot{q}_j(t), \qquad (35)$$

with B being an upper triangular matrix. Comparing L_{kin} on both sides of Eq. (34) we arrive at the equation

$$(\alpha_m - 1)B_{im} = B_{mi}\alpha_m \Rightarrow (B - B^{\mathsf{T}})\alpha = B, \qquad (36)$$

with no Einstein's summation convention applied here. Because B is upper triangular, the first part of Eq. (36) implies that the only eigenvalues of α_{ij} are 1 and 0. Thus, α can be reduced to the block form

$$\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (37)$$

where 1 is an $r\times r$ ($r \leqslant N$) unit matrix. Using the equation $(B - B^{\mathsf{T}})\alpha = B$ we see that B has the block structure

$$B = \begin{bmatrix} 0 & B_2 \\ 0 & 0 \end{bmatrix} \qquad (38)$$

where B_2 is an $(N-r)\times r$ matrix. To determine r we use the fact that α is idempotent, i.e. $\alpha^2 = \alpha$. Multiplying $(B - B^{\mathsf{T}})\alpha = B$ by α we find

$$B\alpha = B, \quad B^{\mathsf{T}}\alpha = 0. \qquad (39)$$

From $B\alpha = B$ it follows that $\mathrm{rank}(B) = \mathrm{rank}(\alpha) = r$, whereas $B^{\mathsf{T}}(1-\alpha) = B^{\mathsf{T}}$ implies that $\mathrm{rank}(B^{\mathsf{T}}) = \mathrm{rank}(1-\alpha)$. Utilizing the identity $\mathrm{rank}(B) = \mathrm{rank}(B^{\mathsf{T}})$ we derive $r = \mathrm{rank}(\alpha) = \mathrm{rank}(1-\alpha) = (N-r)$, and thus $r = N/2$. Thus the condition (34) can be satisfied only for an even number N of degrees of freedom. An immediate further consequence of Eq. (38) is that we can rewrite Eq. (35) as

$$L_{\rm kin} = \sum_{i,j=1}^{N/2} B_{i,(N/2+j)}\dot{q}_i q_{N/2+j}. \qquad (40)$$

Denoting $\alpha_{N/2+i}$, $q_{N/2+i}$ and $\lambda_{N/2+i}$ $(i=1,\ldots,N/2)$ as $\overline{\alpha_i}$, $\overline{q_i}$, and $\overline{\lambda}_i$, respectively (hence, $\boldsymbol{\lambda}=\mathbf{0}$ and $\overline{\boldsymbol{\lambda}}=\overline{\mathbf{q}}$), then Eq.(34) reads

$$\overline{\mathcal{A}}[\mathbf{q},\overline{\mathbf{q}}] = \int_{t_1}^{t_2} dt\, \overline{\mathbf{q}}(t)\frac{\delta\overline{\mathcal{A}}[\mathbf{q},\overline{\mathbf{q}}]}{\delta\overline{\mathbf{q}}(t)}. \qquad (41)$$

Here $\overline{\mathcal{A}}[\mathbf{q},\overline{\mathbf{q}}]=\mathcal{A}[q_1,\ldots,q_{2N}]$.

The result (41) can be obtained also in a different way. Indeed, in Appendix C we show that Eq. (34) is a so-called Euler-like functional

$$\mathcal{A}[\mathbf{q}] = \int_{t_1}^{t_2} dt\, r(t)L\left(r^{-\alpha_1}(t)q_1(t),\ldots,r^{-\alpha_N}(t)q_N(t),\frac{d[r^{-\alpha_1}(t)q_1(t)]}{dt},\ldots,\frac{d[r^{-\alpha_N}(t)q_N(t)]}{dt}\right), \qquad (42)$$

with $r(t)$ being an arbitrary function of q_k whose variations vanish at the ends $\delta r(t_i)=\delta r(t_f)=0$ if all δq_k's have this property. In particular, we may chose r to be any finite power q_k^{1/α_k} (for $k=1,\ldots,N$), in which case

$$\mathcal{A}[\mathbf{q}] = \int_{t_1}^{t_2} dt\, q_k^{1/\alpha_k}L\left(\frac{q_1}{q_k^{\alpha_1/\alpha_k}},\ldots,\overset{\overset{k}{\downarrow}}{1},\ldots,\frac{q_N}{q_k^{\alpha_N/\alpha_k}},\frac{d\left(q_1/q_k^{\alpha_1/\alpha_k}\right)}{dt},\ldots,\overset{\overset{k}{\downarrow}}{0},\ldots,\frac{d\left(q_N/q_k^{\alpha_N/\alpha_k}\right)}{dt}\right). \qquad (43)$$

Assuming, as before, that the kinetic term in L is quadratic in $\mathbf{q}$ and $\dot{\mathbf{q}}$, we arrive at α as in (37), and the action (43) reduces again to (41).

One can incorporate the constraints on α_i (or λ_i) by inserting a corresponding δ functional into the path integral (23). This leads to the most general generating functional with the above-stated property:

$$\begin{aligned} Z_{\rm CM} &= \int \mathcal{D}\mathbf{q}\,\mathcal{D}\overline{\mathbf{q}}\,\mathcal{D}\lambda\,\mathcal{D}\overline{\lambda}\,\delta[\lambda]\,\delta[\overline{\boldsymbol{\lambda}}-\overline{\mathbf{q}}] \\ &\quad\times\exp\left[i\int_{t_1}^{t_2} dt\,\lambda\frac{\delta\overline{\mathcal{A}}[\mathbf{q},\overline{\mathbf{q}}]}{\delta\mathbf{q}} + i\int_{t_1}^{t_2} dt\,\overline{\lambda}\frac{\delta\overline{\mathcal{A}}[\mathbf{q},\overline{\mathbf{q}}]}{\delta\overline{\mathbf{q}}} \right.\\ &\quad\left. + \int_{t_1}^{t_2} dt\sum_{k=1}^{N} J_k q_k \right] \\ &= \int \mathcal{D}\mathbf{q}\,\mathcal{D}\overline{\mathbf{q}}\exp\left[i\int_{t_1}^{t_2} dt\,\overline{\mathbf{q}}\frac{\delta\overline{\mathcal{A}}[\mathbf{q},\overline{\mathbf{q}}]}{\delta\overline{\mathbf{q}}} + \int_{t_1}^{t_2} dt\sum_{k=1}^{N} J_k q_k \right] \\ &= \int \mathcal{D}\mathbf{q}\,\mathcal{D}\overline{\mathbf{q}}\exp\left[i\int_{t_1}^{t_2} dt\,\overline{L} + \int dt\sum_{k=1}^{N} J_k q_k \right]. \end{aligned} \qquad (44)$$

An irrelevant normalization factor has been dropped. The Lagrangian $\overline{L}$ coincides precisely with the Lagrangian (3), and describes therefore 't Hooft's deterministic system. Hence within the above assumptions there are no other systems with the peculiar property that their full quantum properties are classical. Among other things, the latter also indicates that the Koopman–von Neumann operatorial formulation of classical mechanics [38] when applied to 't Hooft systems must agree with its canonically quantized counterpart.

IV. 't HOOFT'S INFORMATION LOSS AS A FIRST-CLASS PRIMARY CONSTRAINT

As observed in Sec. II, the Hamiltonian (1) is not bounded from below, and this is true for any function f_i. Thus, no deterministic system with dynamical equations $\dot{q}_i=f_i(\mathbf{q})$ can describe a physically acceptable *quantum world*. Its Hamiltonian would not be stable and we could build a perpetual motion machine. To deal with this problem we will employ 't Hooft's procedure [13]. We assume that the system (1) has n conserved, irreducible charges C_i, i.e.,

$$\{C_i,H\} = 0, \quad i = 1,\ldots,n. \qquad (45)$$

In order to enforce a lower bound upon H, 't Hooft split the Hamiltonian as $H=H_+-H_-$ with both H_+ and H_- having lower bounds. Then he imposed the condition that H_- should be zero on the physically accessible part of phase space, i.e.,

$$H_- \approx 0. \qquad (46)$$

This will make the actual dynamics governed by the reduced Hamiltonian H_+, which is bounded from below, by definition.

To ensure that the above splitting is conserved in time one must require that $\{H_-,H\}=\{H_+,H\}=0$. The latter is equivalent to the statement that $\{H_+,H_-\}=0$. Since the charges C_i in Eq. (45) form an irreducible set, the Hamiltonians H_+ and H_- must be functions of the charges and H: $H_+=F_+(C_k,H)$ and $H_-=F_-(C_k,H)$. There is a certain amount of flexibility in finding F_- and F_+, but for convenience's sake we confine ourselves to the following choice:

$$H_+ = \frac{\left[H + \sum_i a_i(t)C_i\right]^2}{4\sum_i a_i(t)C_i}, \quad H_- = \frac{\left[H - \sum_i a_i(t)C_i\right]^2}{4\sum_i a_i(t)C_i}, \tag{47}$$

where $a_i(t)$ are independent of $\mathbf{q}$ and $\mathbf{p}$ and will be specified later. The lower bound is then achieved by choosing $\Sigma_i a_i(t)C_i$ to be positive definite. In the following it will also be important to select the combination of C_i's in such a way that it depends solely on $\mathbf{q}$ [this condition may not necessarily be achievable for general $f_a(\mathbf{q})$]. Thus by imposing $H_- \approx 0$ we obtain the weak reduced Hamiltonian $H \approx H_+ \approx \Sigma_i a_i(t)C_i$.

The constraint (46) [or (47)] can be motivated by dissipation or information loss [14,15,19]. In Appendix D we show that the *explicit* constraint (46) does not generate any new (i.e., secondary) constraints when added to the existing constraints (5). In addition, this new set of constraints corresponds to $2N$ second-class constraints and *one* first-class constraint (see also Appendix D). It is well known in the theory of constrained systems that the existence of first-class constraints signals the presence of a gauge freedom in Hamiltonian theory. This is so because the Lagrange multipliers affiliated with first-class constraints cannot be fixed from dynamical equations alone [32]. The time evolution of observable (physical) quantities, however, cannot be affected by the arbitrariness in Lagrange multipliers. To remove this superfluous freedom that is left in the formalism we must pick up a gauge, i.e., impose a set of conditions that will eliminate the above redundancy from the description. It is easy to see that the number of independent gauge conditions must match the number of first-class constraints. Indeed, the requirement on a physical quantity (say f) to have a unique time evolution on the constraint submanifold $\mathcal{M}$, i.e.,

$$\dot{f} \approx \{f, \overline{H}\} + \sum_{i=1}^{m} v_i\{f, \varphi_i\} + \sum_{k=1}^{m'} u_k\{f, \phi_k\}, \tag{48}$$

implies that

$$\{f, \varphi_i\} \approx 0. \tag{49}$$

The constraints φ_i and ϕ_k represent first- and second-class constraints, respectively. First-class constraints have, by definition, weakly vanishing Poisson brackets with all other constraints; any other constraint that is not first class is second class. While the Lagrange multipliers u_k can be uniquely fixed from the dynamics by consistency conditions (see Appendixes A and D) this cannot be done for the v_i's. In this way (49) represents an obligatory condition for a quantity f to be observable. Equation (49) can be considered as a set of m first-order differential equations on the constrained surface with the relation $\{\varphi_i, \varphi_j\} \approx 0$ serving as the integrability conditions [32,39].Thus, f is uniquely defined by its values on the the submanifold of the initial conditons for (49). As a result, the above initial value surface describes the true degrees of freedom. By denoting the dimension of the constraint manifold as D we see that the dimension of the submanifold of initial conditions must be $D-m$. We can take this submanifold to be a surface Γ^* specified by the equations

$$\varphi_i = 0, \quad i = 1, \ldots, m,$$

$$\phi_k = 0, \quad k = 1, \ldots, m',$$

$$\chi_l = 0, \quad l = 1, \ldots, m. \tag{50}$$

The m subsidiary conditions χ_l are the sought gauge constraints. The functions χ_l must clearly satisfy the condition

$$\det\|\{\chi_l, \varphi_i\}\| \neq 0, \tag{51}$$

as only in such a case can we determine specific values for the multipliers v_i from the dynamical equation for χ_l (this is because the time derivative of any constraint, and hence also χ_l, must be zero). Therefore only when the condition (51) is satisfied do the constraints (50) indeed describe the surface of the initial conditions.

The preceding discussion implies that in our case the surface Γ^* is defined by

$$\varphi(\mathbf{q}, \overline{\mathbf{q}}, \mathbf{p}, \overline{\mathbf{p}}) = 0, \quad \chi(\mathbf{q}, \overline{\mathbf{q}}, \mathbf{p}, \overline{\mathbf{p}}) = 0, \tag{52}$$

$$\phi_i(\mathbf{q}, \overline{\mathbf{q}}, \mathbf{p}, \overline{\mathbf{p}}) = 0, \quad i = 1, \ldots, 2N. \tag{53}$$

The explicit form of φ is found in Appendix D where we show that $\varphi \approx H - \Sigma a_i C_i$. Apart from condition (51) we shall further restrict our choice of χ to functions satisfying the simultaneous equations

$$\{\chi, \phi_i\} = 0, \quad i = 1, \ldots, 2N. \tag{54}$$

Such a choice is always possible (at least in a weak sense) [30] and it will prove crucial in the following.

In order to proceed further we begin by reexamining Eq. (44). The latter basically states that

$$Z_{\text{CM}} = \int \mathcal{D}\mathbf{q}\ \delta[\mathbf{q} - \mathbf{q}_{cl}]\exp\left[\int_{t_1}^{t_2} dt\ \mathbf{q}(t)\mathbf{J}(t)\right]. \tag{55}$$

We may now formally invert the steps leading to Eq. (8), i.e., we introduce auxiliary momentum integrations and go over to the canonical of (55). Correspondingly Eq. (55) can be recast into

$$Z_{\text{CM}} = \int \mathcal{D}\mathbf{p}\ \mathcal{D}\mathbf{q}\ \mathcal{D}\overline{\mathbf{p}}\ \mathcal{D}\overline{\mathbf{q}}\sqrt{|\det\|\{\phi_i, \phi_j\}\||}\prod_{i=1}^{2N} \delta[\phi_i]$$
$$\times \exp\left[i\int_{t_1}^{t_2} dt[\mathbf{p}\dot{\mathbf{q}} + \overline{\mathbf{p}}\dot{\overline{\mathbf{q}}} - H] + \int_{t_1}^{t_2} dt[\mathbf{q}\mathbf{J} + \overline{\mathbf{q}}\overline{\mathbf{J}}]\right].$$

Due to δ functions in the integration we could substitute 't Hooft's Hamiltonian H for the canonical Hamiltonian $\overline{H}$. It should be stressed that despite its formal appearance and the phase-space disguise, the latter is still the classical partition function of Gozzi *et al.*

To include the constraints (52) into Eq. (44) we must be a bit cautious. A naive intuition would dictate that the functional δ functions $\delta[\chi]$ and $\delta[\varphi]$ should be inserted into the path-integral measure for Z_{CM}. This would be, however, too simplistic as a mere inclusion of δ functions into Z_{CM} would not guarantee that the physical content of the theory that resides in the generating functional Z_{CM} is independent of the choice χ. Indeed, utilizing the fact that the generators of gauge transformations are the first-class constraints [39] we can write that

$$\delta\chi = \varepsilon\{\chi,\varphi\} + C\varphi \approx \varepsilon\{\chi,\varphi\}. \tag{56}$$

Here ε is an infinitesimal quantity. The corresponding gauge generator $\varepsilon\varphi$ generates the infinistesimal canonical transformations

$$\mathbf{q}\to\mathbf{q}+\delta\mathbf{q},\quad \mathbf{p}\to\mathbf{p}+\delta\mathbf{p},\quad \delta\mathbf{q}=\{\varepsilon\varphi,\mathbf{q}\},\quad \mathbf{p}=\{\varepsilon\varphi,\mathbf{p}\},$$
$$\overline{\mathbf{q}}\to\overline{\mathbf{q}}+\delta\overline{\mathbf{q}},\quad \overline{\mathbf{p}}\to\overline{\mathbf{p}}+\delta\overline{\mathbf{p}},\quad \delta\overline{\mathbf{q}}=\{\varepsilon\varphi,\overline{\mathbf{q}}\},\quad \overline{\mathbf{p}}=\{\varepsilon\varphi,\overline{\mathbf{p}}\}. \tag{57}$$

It follows immediately that the corresponding generating function is

$$G(\mathbf{q},\overline{\mathbf{q}},\mathbf{P},\overline{\mathbf{P}}) = \mathbf{qP} + \overline{\mathbf{q}}\overline{\mathbf{P}} + \varepsilon\varphi + o(\varepsilon^2). \tag{58}$$

The canonical transformations (57) result in changing φ and ϕ_i by

$$\delta\varphi = A\varphi, \tag{59}$$

$$\delta\phi_i = \varepsilon\{\phi_i,\varphi\} = B_i\varphi + D_{ij}\phi_j. \tag{60}$$

Here A, B_i, C and D_{ij} are some phase-space functions of order ε. Note that in our case the gauge algebra is Abelian. As a consequence of Eqs. (59) and (60) we find

$$\delta[\varphi] \to |1+\mathrm{Tr}(A)|^{-1}\delta[\varphi], \tag{61}$$

$$\prod_i \delta[\phi_i] \to |1+\mathrm{Tr}(D)|^{-1}\prod_i \delta[\phi_i], \tag{62}$$

$$\sqrt{|\det\|\{\phi_i,\phi_j\}\||} \to |1+\mathrm{Tr}(D)|\sqrt{|\det\|\{\phi_i,\phi_j\}\||} \tag{63}$$

[here $\mathrm{Tr}(A)=\Sigma_t A(t)$, etc.]. In (63) we have used the fact that in the path-integral measure are present $\delta[\varphi]$ and $\delta[\phi_i]$, and so we have dropped on the RHS's of (61)–(63) the vanishing terms. The infinitesimal gauge transformations described hitherto clearly show that Z_{CM} is dependent on the choice of χ [the term with $|1+\mathrm{Tr}(A)|$ does not get canceled]. To ensure the gauge invariance we need to factor out the "orbit volume" from the definition of Z_{CM}. This will be achieved by a procedure that is akin to the Faddeev–Popov–De Witt trick. We define the functional

$$(\Delta_\chi)^{-1} = \int \mathcal{D}g\ \delta[\chi^g], \tag{64}$$

with χ^g representing the gauge transformed χ. The superscript g in Eq. (64) denotes an element of the Abelian gauge group generated by φ. We point out that the functional (64) is manifestly gauge invariant since

$$(\Delta_{\chi^{g'}})^{-1} = \int \mathcal{D}g\ \delta[\chi^{g'g}] = \int \mathcal{D}(g'g)\delta[\chi^{g'g}] = (\Delta_\chi)^{-1}. \tag{65}$$

The second identity holds because of the invariance of the group measure under composition, i.e., $\mathcal{D}g=\mathcal{D}(g'g)$. Equations (64) and (65) allow as to write "1" as

$$1 = \Delta_\chi\delta[\chi]\int \mathcal{D}g. \tag{66}$$

To find an explicit form of $\Delta[\chi]$ we can apply the infinitesimal gauge transformation (56). Then

$$\begin{aligned}\chi^g &= \chi + \varepsilon\{\chi,\varphi\} + C\varphi\\ &\Rightarrow (\Delta_\chi)^{-1} = \int \mathcal{D}\varepsilon\ \delta[\chi+\varepsilon\{\chi,\varphi\}+C\varphi]\\ &\Rightarrow (\Delta_\chi)^{-1}|_{\Gamma^*} = |\det\|\{\chi,\varphi\}\||^{-1},\end{aligned} \tag{67}$$

with the obvious notation $\det\|\{\chi(t),\varphi(t')\}\| = \Pi_t\{\chi(t),\varphi(t)\}$. Upon insertion of Eq. (66) into Z_{CM} we obtain

$$Z_{\mathrm{CM}} = \int \mathcal{D}\mathbf{p}\,\mathcal{D}\mathbf{q}\,\mathcal{D}\overline{\mathbf{p}}\,\mathcal{D}\overline{\mathbf{q}}\,|\det\|\{\chi,\varphi\}\||\sqrt{|\det\|\{\phi_i,\phi_j\}\||}\,\delta[\chi]\delta[\varphi]\prod_{i=1}^{2N}\delta[\phi_i]\exp\left[i\int_{t_1}^{t_2}dt[\mathbf{p}\dot{\mathbf{q}}+\overline{\mathbf{p}}\dot{\overline{\mathbf{q}}}-\overline{H}]+\int_{t_1}^{t_2}dt[\mathbf{qJ}+\overline{\mathbf{q}}\overline{\mathbf{J}}]\right], \tag{68}$$

where[3] the group volume $G_V=\int\mathcal{D}g$ has been factored out as desired. The partition function (68) is now clearly (locally) independent of the choice of the gauge constraints χ. This is because under the transformation (59) we have

$$\det\|\{\chi,\varphi\}\| \to [1+\mathrm{Tr}(A)]\det\|\{\chi+\delta\chi,\varphi\}\|, \tag{69}$$

and hence the partition function Z_{CM} as obtained by Eq. (68) takes the same form as the untransformed one, but with χ replaced by $\chi+\delta\chi$. Because we deal with canonical transformations it is implicit in our derivation that the action in the new variables is identical, to within a boundary term, with the original action. In path integrals this might be invalidated

[3] If $\mathcal{F}$ is any phase-space function then $[\delta_\varepsilon,\delta_\eta]\mathcal{F}=\delta_\varepsilon\delta_\eta\mathcal{F}-\delta_\eta\delta_\varepsilon\mathcal{F}=\varepsilon\eta\{\mathcal{F},\{\varphi,\varphi\}\}=0$.

by the path roughness and related ordering problems.[4] For simplicity's sake we shall further assume that the latter are absent or harmless. This happens, for instance, when canonical transformations are linear. In such cases an infinitesimal change in χ does not alter the physical content of the theory present in Z_{CM}. This conclusion may generally not be true globally throughout phase space. Global gauge invariance, however, is mandatory in our case since we need a global equivalence between the partition functions Z_{CM} and Z_{QM} and not mere perturbative correspondence. Thus the potentiality of Gribov's copies must be checked in every individual problem separately.

In passing we may notice that if we arrange the constraints in one set $\{\eta_a\}=\{\chi,\varphi,\phi_i\}$ we can write Eq. (68) as

$$Z_{\text{CM}} = \int \mathcal{D}\mathbf{p}\,\mathcal{D}\mathbf{q}\,\mathcal{D}\bar{\mathbf{p}}\,\mathcal{D}\bar{\mathbf{q}}\sqrt{|\det\|\{\eta_a,\eta_b\}\||}\prod_{a=1}^{2N+2}\delta[\eta_a] \times \exp\left[i\int_{t_1}^{t_2}dt[\bar{\mathbf{p}}\dot{\mathbf{q}}+\bar{\mathbf{p}}\dot{\bar{\mathbf{q}}}-H]+\int_{t_1}^{t_2}dt[\mathbf{qJ}+\bar{\mathbf{q}}\bar{\mathbf{J}}]\right]. \quad (70)$$

By comparison with Eq. (7) we retrieve a well known result [39,41], namely, that the set $\{\eta_a\}$ of $2N+2$ constraints can be viewed as a set of second-class constraints. Thus, by fixing a gauge we have effectively converted the original system of $2N$ second-class and *one* first-class constraints into $2N+2$ second-class constraints.

In view of Eqs. (6) and (54), we can perform a canonical transformation in the full phase space in such a way that the new variables are $P_1=\chi, Q_{1+i}=\phi_{2i}, P_{1+i}=\phi_{2i-1}; i=1,\ldots,N$. After a trivial integration over P_a and Q_{1+i} we find that

$$Z_{\text{CM}} = \int \mathcal{D}\bar{\mathbf{P}}\,\mathcal{D}\bar{\mathbf{Q}}\,\mathcal{D}Q_1\left(\delta[\varphi]\left|\det\left\|\frac{\delta\varphi}{\delta Q_1}\right\|\right|\right) \times\exp\left[i\int_{t_1}^{t_2}dt[\bar{\mathbf{P}}\dot{\bar{\mathbf{Q}}}-K]+\int_{t_1}^{t_2}dt\,\bar{\mathbf{Q}}\mathbf{j}\right], \quad (71)$$

where $\bar{P}_a$ and $\bar{Q}_a$ are the remaining canonical variables spanning the $(2N-2)$-dimensional phase space. To within a time derivative term the new Hamiltonian is done by the prescription $K(\bar{\mathbf{P}},\bar{\mathbf{Q}},Q_1)=H(\bar{\mathbf{P}},\bar{\mathbf{Q}},P_1=0,Q_1,Q_{1+i}=0,P_{1+i}=0)$. The sources $\mathbf{j}$ are correspondingly transformed sources $\mathbf{J}$ and $\bar{\mathbf{J}}$. Utilizing the identity

$$\delta[\varphi]\left|\det\left\|\frac{\delta\varphi}{\delta Q_1}\right\|\right| = \delta[Q_1-Q_1^*(\bar{\mathbf{P}},\bar{\mathbf{Q}})], \quad (72)$$

we can finally write

$$Z_{\text{CM}} = \int \mathcal{D}\bar{\mathbf{P}}\,\mathcal{D}\bar{\mathbf{Q}}\exp\left[i\int_{t_1}^{t_2}dt[\bar{\mathbf{P}}\dot{\bar{\mathbf{Q}}}-K^*]+\int_{t_1}^{t_2}dt\,\bar{\mathbf{Q}}\mathbf{j}\right]. \quad (73)$$

Here $K^*(\bar{\mathbf{P}},\bar{\mathbf{Q}})=K(\bar{\mathbf{P}},\bar{\mathbf{Q}},Q_1=Q_1^*(\bar{\mathbf{P}},\bar{\mathbf{Q}}))$. In view of Eq. (D7) we can alternatively write Z_{CM} as

$$Z_{\text{CM}} = \int \mathcal{D}\bar{\mathbf{P}}\,\mathcal{D}\bar{\mathbf{Q}}\exp\left[i\int_{t_1}^{t_2}dt[\bar{\mathbf{P}}\dot{\bar{\mathbf{Q}}}-H_+^*]+\int_{t_1}^{t_2}dt\,\bar{\mathbf{Q}}\mathbf{j}\right], \quad (74)$$

where $H_+^*=H_+(\bar{\mathbf{P}},\bar{\mathbf{Q}},Q_1=Q_1^*(\bar{\mathbf{P}},\bar{\mathbf{Q}}),P_a=0,Q_{1+i}=0)$. In passing we may notice that $\bar{P}_a$ and $\bar{Q}_a$ are true canonical variables on the submanifold Γ^* of the initial conditions for Eq. (49). Indeed, in terms of a noncanonical system of variables $\{\zeta_i\}=\{\varphi;\chi;\phi_i;\bar{\mathbf{Q}};\bar{\mathbf{P}}\}$ the Poisson bracket of any two *observable* quantities (say f and g) on the constraint manifold $\mathcal{M}$ is

$$\{f,g\}|_{\mathcal{M}} = \left[\sum_{a,b}\{\zeta_a,\zeta_b\}\frac{\partial f}{\partial\zeta_a}\frac{\partial g}{\partial\zeta_b}\right]\Bigg|_{\mathcal{M}} = \sum_{i,j}\{\bar{P}_i,\bar{Q}_j\}\frac{\partial f^*}{\partial\bar{P}_i}\frac{\partial g^*}{\partial\bar{Q}_j} = \sum_{i,j}\Omega_{ij}\frac{\partial f^*}{\partial\bar{\mathcal{Q}}_i}\frac{\partial g^*}{\partial\bar{\mathcal{Q}}_j}, \quad (75)$$

with $\{\bar{\mathcal{Q}}_j\}=\{\bar{\mathbf{Q}};\bar{\mathbf{P}}\}$ and with

$$f^*(\bar{\mathbf{Q}},\bar{\mathbf{P}}) = f(\varphi=0,\chi=0,\phi_i=0,\bar{\mathbf{Q}},\bar{\mathbf{P}}),$$

$$g^*(\bar{\mathbf{Q}},\bar{\mathbf{P}}) = g(\varphi=0,\chi=0,\phi_i=0,\bar{\mathbf{Q}},\bar{\mathbf{P}})$$

representing the physical quantities on $\mathcal{M}$. The latter depend only on the canonical variables $\bar{\mathbf{Q}}$ and $\bar{\mathbf{P}}$, which are the independent variables on Γ^*. In deriving Eq. (75) we have used the fact that various terms are vanishing on account of Eqs. (49) and (54). So, for instance, $[\{\varphi,\zeta_i\}\partial f/\partial\zeta_i]|_{\mathcal{M}}=0, \{\varphi_i,\bar{P}_j\}=0, \{\varphi_i,\bar{Q}_j\}=0, [\{\chi,\zeta_i\}\partial f/\partial\chi]|_{\mathcal{M}}=0$, etc. The matrix Ω_{ij} stands for the $(2N-2)\times(2N-2)$ symplectic matrix.

Z_{CM} as defined by Eqs. (73) and (74) does not generally represent a (classical) deterministic system. This is because the constraint $\varphi=0$ explicitly breaks the BRST invariance of Z_{CM}, which (as illustrated in Sec. III) is key in preserving the classical nature of the partition function. Indeed, using the relations $\{\chi,\bar{p}_a\}=\{\chi,p_a-\bar{q}_a\}=0$ we immediately obtain

$$\{\chi,\varphi\} = \sum_a\left\{\frac{\partial\chi}{\partial q_a}\left(\frac{\partial\varphi}{\partial p_a}+\frac{\partial\varphi}{\partial\bar{q}_a}\right)-\frac{\partial\chi}{\partial p_a}\frac{\partial\varphi}{\partial q_a}\right\}, \quad (76)$$

which implies that

$$\{\chi,\varphi\}|_{\mathcal{M},\bar{q}_a=\lambda_a} = \sum_a\left\{\frac{\partial\chi^*}{\partial q_a}\frac{\partial\varphi^*}{\partial\lambda_a}-\frac{\partial\chi^*}{\partial\lambda_a}\frac{\partial\varphi^*}{\partial q_a}\right\}\equiv\{\chi^*,\varphi^*\}. \quad (77)$$

Here the notations $\chi^*(\mathbf{q},\lambda)=\chi(\mathbf{q},\mathbf{p}=\lambda,\bar{\mathbf{q}}=\lambda,\bar{\mathbf{p}}=0)$ and $\varphi^*(\mathbf{q},\lambda)=\varphi(\mathbf{q},\lambda,\lambda,0)$ were used. We also took advantage of the fact that $\bar{\mathbf{q}}=\boldsymbol{\lambda}$ as indicated in Sec. III. So the generating functional (73) [or (74)] can be rewritten as

[4] In the literature this phenomenon frequently goes under the name of the Edwards-Gulyaev effect [40].

$$Z_{\mathrm{CM}}[\mathbf{J}=\mathbf{0}] = \int \mathcal{D}\mathbf{q}\,\mathcal{D}\lambda\,\mathcal{D}\bar{\mathbf{c}}\,\mathcal{D}\mathbf{c}\,\exp[i\mathcal{S}]\delta[\varphi^*]\delta[\chi^*] \times|\det\|\{\chi^*,\varphi^*\}\|\|, \tag{78}$$

where the integration over the ghost fields was reintroduced for convenience. By reformulating Z_{CM} in terms of $\mathbf{q},\lambda,\mathbf{c}$ and $\bar{\mathbf{c}}$ we can now easily check the BRST invariance. The BRST transformations (25) imply that

$$\delta_{\mathrm{BRST}}\,\varphi^* = \frac{\partial\varphi^*}{\partial q_i}\bar{\varepsilon}c_i = -\bar{\varepsilon}\pounds_{X_{Q\ \mathrm{BRST}}}\,\varphi^*,$$

$$\bar{\delta}_{\mathrm{BRST}}\,\varphi^* = -\frac{\partial\varphi^*}{\partial q_i}\varepsilon\bar{c}_i = -\bar{\varepsilon}\pounds_{X^-_{Q\ \mathrm{BRST}}}\,\varphi^*. \tag{79}$$

Here $\pounds_{X_{Q\ \mathrm{BRST}}}$ and $\pounds_{X^-_{Q\ \mathrm{BRST}}}$ represent the Lie derivatives with respect to flows generated by the BRST and anti-BRST charges, respectively. Analogous relations hold also for χ^*. Correspondingly, to the lowest order in $\bar{\varepsilon}$ we can write

$$\delta[\chi^*]\to|1-\mathrm{Tr}(\bar{\varepsilon}\pounds_{X_{Q\ \mathrm{BRST}}})|^{-1}\delta[\chi^*],$$

$$|\det\|\{\chi^*,\varphi^*\}\||\to|1-\mathrm{Tr}(\bar{\varepsilon}\pounds_{X_{Q\ \mathrm{BRST}}})||\det\|\{\chi^*,\varphi^*\}\||. \tag{80}$$

The transformations (80) show that the term $\delta[\chi^*]|\det\|\{\chi^*,\varphi^*\}\||$ in Eq. (78) is the BRST invariant (as, of course, are both the integration measure and the effective action $\mathcal{S}$). However, because the variation $\delta_{\mathrm{BRST}}\delta[\varphi^*]$ is not compensated in Eq. (78) we have in general $\delta_{\mathrm{BRST}}Z_{\mathrm{CM}}[\mathbf{J}=\mathbf{0}]\neq 0$. An analogous result applies also to the anti-BRST transformation.

We should note that the condition $\delta_{\mathrm{BRST}}Z_{\mathrm{CM}}[\mathbf{J}=\mathbf{0}]\neq 0$ only indicates that the *classical* path-integral structure is destroyed; it does not, however, ensure that the ensuing Z_{CM} can be recast into a form describing a proper quantum-mechanical generating functional. The straightforward path-integral such as (73) emerges only after the gauge freedom inherent in the "information loss" condition φ is properly fixed via the gauge constraint χ. Let us finally emphasize once more that the partition function (73) [or (74)] has arisen as a consequence of the application of the classical Dirac-Bergmann algorithm for singular systems to the classical path integral of Gozzi *et al.*

V. EXPLICIT EXAMPLES

A. Free particle

Although the preceding construction may seem a bit abstract, its implementation is quite straightforward. Let us now illustrate this with two systems. As a warm-up example we start with the Hamiltonian

$$H = L_3 = xp_y - yp_x, \tag{81}$$

which is known to represent the angular momentum with values unbounded from below. Alternatively, Eq. (81) can be regarded as describing the mathematical pendulum. This is because the corresponding dynamical equation (2) for $\mathbf{q}$ is a plane pendulum equation with the pendulum constant $l/g=1$. The Lagrangian (3) reads

$$\bar{L} = \bar{x}\dot{x} + \bar{y}\dot{y} + \bar{x}y - \bar{y}x. \tag{82}$$

It is well known [42] that the system has two (functionally independent) constants of motion—Casimir functions. For Eq. (81) they read

$$C_1 = x^2 + y^2, \quad C_2 = xp_x + yp_y. \tag{83}$$

The charge C_1 corresponds to the conserved radius of the orbit while C_2 is the Noether charge of dilatation invariance of the Lagrangian (82) under the transformations $(\bar{x},\bar{y},x,y)\mapsto(e^{-s}\bar{x},e^{-s}\bar{y},e^{s}x,e^{s}y)$. As only C_1 is $\mathbf{p}$ independent, the functions F_+ and F_- of this system are according to Eq. (47) chosen as

$$F_+ = \frac{(H+a_1C_1)^2}{4a_1C_1}, \quad F_- = \frac{(H-a_1C_1)^2}{4a_1C_1}. \tag{84}$$

Hence $H_-=0$ implies that $H_+\approx a_1(x^2+y^2)$. Here a_1 is some constant to be specified later. The ensuing first-class constraint is

$$\varphi = xp_y - yp_x - a_1x^2 - a_1y^2 - \bar{p}_{\bar{x}}\bar{y} + 2a_1\bar{p}_{\bar{x}}x + \bar{p}_{\bar{y}}\bar{x} + 2a_1\bar{p}_{\bar{y}}y \approx H - a_1C_1. \tag{85}$$

The gauge condition can then be chosen in the form $\chi=\bar{p}_{\bar{y}}-y$. Indeed, we easily find that

$$\{\chi,\varphi\} = \bar{p}_{\bar{x}} - x \neq 0,$$

$$\{\chi,\phi_i\} = 0,\ i=1,\ldots,4. \tag{86}$$

The advantage of our choice of χ is that it will not run into Gribov ambiguities, i.e.,the equation $\varphi=0$ will have a globally unique solution for Q_1 on Γ^*. This should be contrasted with such choices as, e.g., $\chi=p_x$ or $\chi=p_y$, which also satisfy the conditions (86), but lead to two Gribov copies each.

With the above choice of χ we may directly write the canonical transformations

$$P_1 = \chi = \bar{p}_{\bar{y}} - y, \quad Q_1 = p_y,$$

$$P_2 = p_x - \bar{x}, \quad Q_2 = \bar{p}_{\bar{x}},$$

$$P_3 = p_y - \bar{y}, \quad Q_3 = \bar{p}_{\bar{y}},$$

$$\bar{P} = \bar{p}_{\bar{x}} - x, \quad \bar{Q} = p_x. \tag{87}$$

It might be checked that the transformation Jacobian is indeed 1. In the new canonical variables the Hamiltonian K reads

$$K(\bar{P},\bar{Q},Q_1) = H(\bar{P},\bar{Q},P_a=0,Q_1,Q_2=0,Q_3=0) = -\bar{P}Q_1. \tag{88}$$

The functional δ function (72) has the form

$$\delta[Q_1 - Q_1^*(\overline{P},\overline{Q})] = \delta[Q_1 + a_1\overline{P}], \tag{89}$$

and hence $K^*(\overline{P},\overline{Q}) = H_+^*(\overline{P},\overline{Q}) = a_1\overline{P}^2$. Let us now set $a_1 = 1/2m\hbar$. After changing variables $\overline{Q}(t)$ to $\overline{Q}(t)/\hbar$ we obtain not only the correct "quantum-mechanical" path-integral measure

$$\mathcal{D}\overline{Q}\,\mathcal{D}\overline{P} \approx \prod_i \left(\frac{d\overline{Q}(t_i)d\overline{P}(t_i)}{2\pi\hbar} \right), \tag{90}$$

but also the prefactor $1/\hbar$ in the exponent. So Eq. (74) reduces to the quantum partition function for a free particle of mass m. As the constant a_1 represents the choice of units (or scale factor) for C_1 we see that the quantum scale $\hbar$ is implemented into the partition function via the choice of the "loss of information" constraint.

B. Harmonic oscillator

The system (81) can also be used to obtain the quantized linear harmonic oscillator. This is possible by observing that not only is $C_1 = x^2 + y^2$ a constant of motion for (81) but also $C_1 = x^2 + y^2 + c$ with c being any $\mathbf{q}$- and $\mathbf{p}$-independent constant. So in particular we can choose $c = c(\overline{\mathbf{q}})$. The functional dependence of c on $\overline{\mathbf{q}}$ cannot be, however, arbitrary. The requirement that 't Hooft's constraint should not generate any new (i.e., secondary) constraint represents quite a severe restriction. Indeed, in order to satisfy Eq. (D2) the following condition must hold (see Appendix D):

$$\sum_{i=0}^{2N} e_i\{\phi_i, \overline{H}\} = -\sum_{a,i} a_i\{C_i, \overline{p_a}\}\{p_a, \overline{H}\} = \sum_{i,k,a} a_i \frac{\partial c_i(\overline{\mathbf{q}})}{\partial \overline{q_a}} \overline{q_k} \frac{\partial f_k(\mathbf{q})}{\partial q_a} \tag{91}$$

which for the system in question is weakly zero only if

$$\overline{x}\frac{\partial c(\overline{\mathbf{q}})}{\partial \overline{y}} - \overline{y}\frac{\partial c(\overline{\mathbf{q}})}{\partial \overline{x}} = 0. \tag{92}$$

The latter equation has the solution (modulo an irrelevant additive constant) $c(\overline{\mathbf{q}}) = d^2(\overline{x}^2 + \overline{y}^2)$. Here d^2 represents a multiplicative constant. Hence we have that C_1 has the general form

$$C_1 = x^2 + y^2 + d^2(\overline{x}^2 + \overline{y}^2). \tag{93}$$

It will be further convenient to choose $a_1 = -1/2d$.The resulting first-class constraint then reads

$$\begin{aligned}\varphi &= xp_y - yp_x + \frac{1}{2d}x^2 + \frac{1}{2d}y^2 - \frac{d}{2}\overline{x}^2 - \frac{d}{2}\overline{y}^2 - \overline{y}\overline{p_x} + \overline{x}\overline{p_y} \\ &\quad - \frac{1}{d}x\overline{p_x} - \frac{1}{d}y\overline{p_y} + d\overline{x}p_x + d\overline{y}p_y \\ &\approx\ H + \frac{1}{2d}C_1.\end{aligned} \tag{94}$$

If we choose the gauge condition to be

$$\chi = \overline{p_y} + dp_x - y, \tag{95}$$

it ensures that

$$\{\chi, \varphi\} = 2\overline{p_x} - 2x - 2dp_y \neq\ 0,$$

$$\{\chi, \phi_i\} = 0, \quad i = 1, \ldots, 4. \tag{96}$$

In addition, we shall see that Eq. (95) guarantees the unique global solution of the equation $\varphi = 0$ for Q_1 on Γ^* (hence it avoids the undesired Gribov ambiguity).

The canonical transformation discussed in Sec. IV now takes the form

$$\begin{aligned} &P_1 = \chi = \overline{p_y} + dp_x - y, \quad Q_1 = p_y, \\ &P_2 = p_x - \overline{x}, \quad Q_2 = \overline{p_x}, \\ &P_3 = p_y - \overline{y}, \quad Q_3 = \overline{p_y}, \\ &\overline{P} = \overline{p_x} + dp_y - x, \quad \overline{Q} = p_x, \end{aligned} \tag{97}$$

and the Hamiltonian K reads

$$K(\overline{P},\overline{Q},Q_1) = -\overline{P}Q_1 + dQ_1^2 - d\overline{Q}^2. \tag{98}$$

The functional δ function (72) now has the form

$$\delta[Q_1 - Q_1^*(\overline{P},\overline{Q})] = \delta\left[Q_1 - \frac{1}{2d}\overline{P} \right]. \tag{99}$$

This finally implies that the Hamiltonian on the physical space Γ^* has the form $K^*(\overline{P},\overline{Q}) = H_+^*(\overline{P},\overline{Q}) = -(1/4d)\overline{P}^2 - d\overline{Q}^2$. By choosing $d = -m\hbar/2$ and transforming $\overline{Q} \mapsto \overline{Q}/\hbar$ in the path integral (73) (or (74)) we obtain the quantum partition function for a system described by the Hamiltonian $(1/2m)\overline{P}^2 + (m/2)\overline{Q}^2$, i.e., the linear harmonic oscillator with unit frequency. This is precisely the result which in the context of the system (81) was originally conjectured by 't Hooft in Ref. [14]. Note again that the fundamental scale (suggestively denoted as $\hbar$) was implemented into the theory via the "loss of information" condition.

C. Free particle weakly coupled to Duffing's oscillator

There is no difficulty, in principle, in carrying over our procedure to nonlinear dynamical systems. As an illustration we will consider here the Rössler system. This is a three-dimensional continuous-time chaotic system described by the three autonomous nonlinear equations

$$\begin{aligned} \frac{dx}{dt} &= -y - z, \\ \frac{dy}{dt} &= x + Ay, \\ \frac{dz}{dt} &= B + xz - Cz, \end{aligned} \tag{100}$$

where A,B, and C are adjustable constants. The associated 't Hooft Hamiltonian reads

$$H = -p_x(y+z) + p_y(x+Ay) + p_z(B+xz-Cz), \quad (101)$$

and the Lagrangian (3) has the form

$$\bar{L} = \bar{x}\dot{x} + \bar{y}\dot{y} + \bar{z}\dot{z} + \bar{x}(y+z) - \bar{y}(x+Ay) - \bar{z}(B+xz+Cz). \quad (102)$$

The Rössler system is considered to be the simplest possible chaotic attractor with important applications in far-from-equilibrium chemical kinetics [43]. It also frequently serves as a playground for studying, e.g., period-doubling bifurcation cycles or Feigenbaum's universality theory. For the sake of an explicit analytic solution we will confine ourselves only to the special case when $A=B=C=0$. With such a choice of parameters the Rössler system can be expressed in a scalar form as $\dddot{y} = y\dot{y} + \dot{y}\ddot{y} - \dot{y}$ which ensures its integrability [44]. The latter implies that in this regime Rössler's system does not posses chaotic attractors.

To proceed further, we should realize that because C_i are supposed to be **p** independent their finding is equivalent to specifying the first integrals of the system (100) [i.e., functions that are constant along lines of (x,y,z) satisfying (100)]. In other words, the differential equations (100) represent a characteristic system for the differential equation $\{H, C_i\}=0$. It is simple to see that the first integrals of the above Rössler system are x^2+y^2+2z and ze^{-y}; hence we can identify C_1 and C_2 with

$$C_1 = (x^2+y^2+2z)^2, \quad C_2 = z^2 e^{-2y}. \quad (103)$$

The previous choice provides indeed positive and irreducible charges. The first-class constraint φ then reads

$$\begin{aligned}\varphi = & -p_x(y+z) + p_y x + p_z xz - a_1(x^2+y^2+2z)^2 - a_2 z^2 e^{-2y} \\ & - \bar{p}_x[\bar{y} + \bar{z}z - 4a_1 x(x^2+y^2+2z)] + \bar{p}_y[\bar{x} + 4a_1 y(x^2+y^2 \\ & + 2z) - 2a_2 z^2 e^{-2y}] + \bar{p}_z[\bar{x} - \bar{z}x + 4a_1(x^2+y^2+2z) \\ & + 2a_2 z e^{-2y}] \approx H - a_1 C_1 - a_2 C_2. \end{aligned} \quad (104)$$

Explicit values of a_1 and a_2 will be fixed in footnote [5] below. A little algebra shows that the gauge condition χ can be selected, for instance, as

$$\chi = \bar{p}_x - y. \quad (105)$$

Such a choice satisfies the necessary conditions

$$\{\chi, \varphi\} = \bar{p}_y + \bar{p}_z + x \neq 0, \quad \{\chi, \phi_i\} = 0, \quad i = 1, \ldots, 6. \quad (106)$$

The above χ also allows us to perform the following linear canonical transformation:

$$P_1 = \chi = \bar{p}_x - y, \quad Q_1 = p_y,$$

$$P_2 = p_x - \bar{x}, \quad Q_2 = \bar{p}_x,$$

$$P_3 = p_y - \bar{y}, \quad Q_3 = \bar{p}_y,$$

$$P_4 = p_z - \bar{z}, \quad Q_4 = \bar{p}_z,$$

$$\bar{P}_1 = (\bar{p}_z/d - z/d)/\sqrt{2}, \quad \bar{Q}_1 = (2dp_z - \bar{p}_x/c + x/c)/\sqrt{2},$$

$$\bar{P}_2 = (2cp_x - \bar{p}_z/d + z/d)/\sqrt{2}, \quad \bar{Q}_2 = (x/c - \bar{p}_x/c)/\sqrt{2}. \quad (107)$$

Here c and d represent arbitrary real constants to be specified later. The transformation (107) secures the unique global solution Q_1 for $\varphi=0$ on Γ^*. To show this it is sufficient to observe that $[H-a_1C_1-a_2C_2]|_{\Gamma^*}$ is linear in Q_1. Indeed,

$$\begin{aligned}[H - a_1C_1 - a_2C_2]_{\Gamma^*} = & \sqrt{2}c\, Q_1\bar{Q}_2 - \sqrt{2}c(\bar{Q}_1 - \bar{Q}_2)\bar{Q}_2\bar{P}_1 \\ & + d/c(\bar{P}_1 + \bar{P}_2)\bar{P}_1 - \mathcal{A}(\bar{P}_1)^2 \\ & - \mathcal{B}\,\bar{P}_1(\bar{Q}_2)^2 - \mathcal{C}(\bar{Q}_2)^4, \end{aligned} \quad (108)$$

with $\mathcal{A}=2d^2(4a_1+a_2)$, $\mathcal{B}=-8\sqrt{2}a_1dc^2$, and $\mathcal{C}=4a_1c^4$. As a result

$$K^*(\bar{\mathbf{P}}, \bar{\mathbf{Q}}) = H_+^*(\bar{\mathbf{P}}, \bar{\mathbf{Q}}) = \mathcal{A}(\bar{P}_1)^2 + \mathcal{B}\,\bar{P}_1(\bar{Q}_2)^2 + \mathcal{C}(\bar{Q}_2)^4. \quad (109)$$

Inserting this into Eq. (73) [or Eq. (74)] and integrating over $\bar{P}_1$ and $\bar{P}_2$ we obtain the following chain of identities:

$$\begin{aligned} Z_{\rm CM} = & \int \mathcal{D}\bar{\mathbf{P}}\,\mathcal{D}\bar{\mathbf{Q}} \exp\left\{ i\int_{t_1}^{t_2} dt[\bar{\mathbf{P}}\dot{\bar{\mathbf{Q}}} - \mathcal{A}(\bar{P}_1)^2 \right. \\ & \left. - \mathcal{B}\bar{P}_1(\bar{Q}_2)^2 - \mathcal{C}(\bar{Q}_2)^4 + \bar{\mathbf{Q}}\mathbf{j}] \right\} \\ = & \int \mathcal{D}\bar{Q}_1\mathcal{D}\bar{Q}_2\ \delta[\dot{\bar{Q}}_2]\exp\left\{ i\int_{t_1}^{t_2} dt\left[\frac{1}{4\mathcal{A}}[\dot{\bar{Q}}_1 - \mathcal{B}(\bar{Q}_2)^2]^2 \right.\right. \\ & \left.\left. - \mathcal{C}(\bar{Q}_2)^4 + \bar{\mathbf{Q}}\mathbf{j} \right]\right\} \\ = & \lim_{a\to 0_+} \int \mathcal{D}\bar{Q}_1\mathcal{D}\bar{Q}_2 \exp\left\{ i\int_{t_1}^{t_2} dt\left[\frac{1}{4\mathcal{A}}(\dot{\bar{Q}}_1)^2 + \frac{1}{4a}(\dot{\bar{Q}}_2)^2 \right.\right. \\ & \left.\left. - \frac{\mathcal{B}}{2\mathcal{A}}\dot{\bar{Q}}_1(\bar{Q}_2)^2 \right]\right\} \exp\left\{ i\int_{t_1}^{t_2} dt\left[\left(\frac{\mathcal{B}^2}{4\mathcal{A}} - \mathcal{C}\right) \right.\right. \\ & \left.\left. \times(\bar{Q}_2)^4 + \bar{\mathbf{Q}}\mathbf{j} \right]\right\}. \end{aligned} \quad (110)$$

As an explanatory step we should mention that the formal measure in the second equality of Eq. (110) has the explicit time-sliced form

$$\mathcal{D}\bar{Q}_1\mathcal{D}\bar{Q}_2 \approx \prod_i \left(\frac{d\bar{Q}_1(t_i)}{\sqrt{4\pi i\epsilon\mathcal{A}}} d\bar{Q}_2(t_i) \right), \quad (111)$$

while in the third equality the shorthand notation $\mathcal{D}\bar{Q}_1\mathcal{D}\bar{Q}_2$ stands for

$$\mathcal{D}\bar{Q}_1\mathcal{D}\bar{Q}_2 \approx \prod_i \left(\frac{d\bar{Q}_1(t_i)}{\sqrt{4\pi i \epsilon \mathcal{A}}} \frac{d\bar{Q}_2(t_i)}{\sqrt{4\pi i a \epsilon}} \right). \qquad (112)$$

The symbol ϵ represents the infinitesimal width of the time slicing. During our derivation we have used the Fresnel integral

$$\int_{-\infty}^{\infty} dx\, e^{-iax^2+ix\xi} = \sqrt{\frac{\pi}{a}} e^{i(\xi^2/a-\pi)/4} = \sqrt{\frac{\pi}{ia}} e^{i\xi^2/(4a)}, \quad a>0, \qquad (113)$$

and the ensuing representation of the Dirac δ function:

$$\lim_{a\to 0_+} \sqrt{\frac{1}{4i\pi a}} e^{i\xi^2/(4a)} = \delta(\xi). \qquad (114)$$

In the following we perform the scale transformation $\bar{Q}_2/\sqrt{a} \mapsto \sqrt{2m_2}\bar{Q}_2$ and set $\mathcal{A}=1/(2m_1), \mathcal{B}=1/(\sqrt{m_1 m_2})$, and $\mathcal{C}=1/m_2$. [5] The resulting partition function then reads

$$Z_{\rm CM} = \lim_{g\to 0_+} \int \mathcal{D}\bar{Q}_1\mathcal{D}\bar{Q}_2 \exp\left\{ i\int_{t_1}^{t_2} dt \left[\frac{m_1}{2}(\dot{\bar{Q}}_1)^2 + \frac{m_2}{2}(\dot{\bar{Q}}_2)^2 \right] \right\} \times \exp\left\{ i\int_{t_1}^{t_2} dt \left[g\sqrt{\frac{m_1 m_2}{2}} \dot{\bar{Q}}_1 (\bar{Q}_2)^2 - \frac{m_2 g^2}{4}(\bar{Q}_2)^4 + \bar{\mathbf{Q}}\mathbf{j} \right] \right\}, \qquad (115)$$

where we have set $g=2\sqrt{2}a$. The system thus obtained describes a pure anharmonic (Duffing's) oscillator ($\bar{Q}_2$ oscillator) weakly coupled through the Rayleigh interaction with a free particle ($\bar{Q}_1$ particle). Alternatively, when $m_1 = m_2 = m$ we can interpret the Lagrangian in Eq. (115) as a planar system describing a particle of mass m in a quartic scalar potential $e\Phi(\bar{\mathbf{Q}}) = mg^2/4(\bar{Q}_2)^4$ and a vector potential $e\mathbf{A} = (gm\sqrt{1/2}(\bar{Q}_2)^2, 0)$ (i.e., in the linear magnetic field $B_3 = \epsilon_{3ij}\partial_i A_j = -gm\sqrt{2}\bar{Q}_2/e$).

It is preferable to set $m_1 \mapsto m_1\hbar$ and $m_2 \mapsto m_2/\hbar$. The latter corresponds to the scale factors $a_2 = 1/(2m_1\hbar)$ and $a_1 = 1/(8m_1\hbar)$. After rescaling $\bar{Q}_1(t) \mapsto \bar{Q}_1(t)/\hbar$ the partition function (115) boils down to the usual quantum-mechanical partition function with the path-integral measure

$$\mathcal{D}\bar{\mathbf{Q}} \approx \prod_i \left(\frac{d\bar{Q}_1(t_i)}{\sqrt{2\pi i \epsilon \hbar/m_1}} \frac{d\bar{Q}_2(t_i)}{\sqrt{2\pi i \epsilon \hbar/m_2}} \right) \qquad (116)$$

and with $1/\hbar$ in the exponent. Hence, just as found in the previous two cases, the choice of 't Hooft's condition ensures that the Planck constant enters the partition function (115) in a correct quantum-mechanical manner. In turn, $\hbar$ enters only via the scale factors a_1 and a_2 (the factors d and c are $\hbar$ independent) and hence it represents a natural scale on which the "loss of information" condition operates. In other words, whenever one would be able to "measure" or determine from "first principles" the "loss of information" condition one could, in principle, determine the value of the fundamental quantum scale $\hbar$.

As a final note we mention that the 't Hooft quantization procedure can be straightforwardly extended to other nonlinear systems and particularly to systems possessing chaotic behavior (e.g., strange attractors). In general cases this might be, however, hindered by our inability to find the corresponding first integrals (and hence C_i's) in the analytic form. It is interesting to notice that machinery outlined above allows to find the emergent quantistic system for the configuration-space strange attractors. This is because in 't Hooft's "quantization" one only needs the dynamical equations in the *configuration* space. The latter should be contrasted with the Hamiltonian (or symplectic) systems where strange attractors cannot exist in the *phase space* on account of the Liouville theorem [45].

VI. CONCLUSIONS AND OUTLOOK

In this paper we have attempted to substantiate the recent proposal of 't Hooft in which quantum theory is viewed as not a complete final theory, but as in fact an emergent phenomenon arising from a deeper level of dynamics. The underlying dynamics are taken to be classical mechanics with singular Lagrangians supplied with an appropriate information loss condition. With plausible assumptions about the actual nature of the constraint dynamics, quantum theory is shown to emerge when the classical Dirac-Bergmann algorithm for constrained dynamics is applied to the classical path integral of Gozzi *et al.*.

There are essentially two different tactics for implementing the classical path integrals in 't Hooft's quantization scenario. The first is to apply the configuration-space formulation [26]. This is suited to situations when 't Hooft's systems are phrased through the Lagrangian description. The alternative approach is to start with the phase-space version [27]. The latter provides a natural framework when the Hamiltonian formulation is of interest or where the language of symplectic geometry is preferred. It should be, however, stressed that it is not merely a matter of a computational convenience which method is actually employed. In fact, both approaches are mathematically and conceptually very different (as they are also in conventional quantum mechanics [11,46]). Besides, the methodology for handling singular systems is distinct in Lagrangian and Hamiltonian formulations (see Refs. [39,41] and citation's therein). In passing, we should mention that the currently popular Hamilton-Jacobi

[5]This choice is equivalent to the solution:

$$a_1 = \frac{a_2}{4}, \quad d = \frac{1}{2\sqrt{2a_2 m_1}}, \quad c = \pm\frac{1}{\sqrt[4]{a_2 m_2}}.$$

Without loss of generality we can set $d=1/2$; then

$$a_2 = \frac{1}{2m_1}, \quad a_1 = \frac{1}{8m_1}, \quad c = \pm 2^{3/4}\sqrt[4]{\frac{m_1}{m_2}}.$$

[47] and Legendre-Ostrogradskiĭ [48] approaches for a treatment of constrained systems, though highly convenient in certain cases (e.g., in higher-order Lagrangian systems), have not found as yet any particular utility in the present context.

Throughout this paper we have considered only the configuration-space formulation of classical path integrals. (Incidently, the phase-space path integral which appears in Sec. IV [after Eq. (55)] is not the phase-space path integral of Gozzi, Reuter, and Thacker [27] but rather Gozzi's configuration-path [26] integral with extra degrees of freedom.) By choosing to work within such a framework we have been able to render a number of formal steps more tractable (e.g., BRST analysis is reputed to be simpler in the configuration space, uniqueness proof for 't Hooft systems is easy and transparent in the Lagrange description, etc.). The key advantage, however, lies in two observations. First, the position-space path integral of Gozzi *et al.* provides a conceptually clean starting point in view of the fact that it represents the classical limit of both the stochastic-quantization path integral and the closed-time-path integral for the transition probability of systems coupled to a heat bath. Such a connection is by no means obvious in the canonical path-integral as both the Parisi-Wu stochastic quantization and the Feynman-Vernon formalism (with ensuing closed-time-path integral) are intrinsically formulated in the configuration space. Second, according to 't Hooft's conjecture the "loss of information" condition should operate in the position space where it is supposed to eliminate some of the transient trajectories leaving behind only stable (or near to stable) orbits [14]. Hence working in configuration space may allow one to probe the plausibility of 't Hooft's conjecture. The price that has been paid for this choice is that the configuration space must have been doubled. This is an unavoidable step whenever one wishes to obtain first-order autonomous dynamical equations directly from the Lagrange formulation (a fact well known in the theory of dissipative systems [49]). Our analysis in Appendix B suggests that the auxiliary coordinates $\overline{q_i}$ may be related to relative coordinates on the backward-forward time path in the Feynman-Vernon approach. (Such coordinates also go under the names *fast variables* [50] or *quantum noise variables* [51].) On the formal side, the auxiliary variables $\overline{q_i}$ are nothing but Gozzi's Lagrange multipliers λ_i (in our case denoted as $\overline{\lambda_i}$).

In order to incorporate the "loss of information" into our scheme, we have introduced in Sec. IV an auxiliary momentum integration to go over to the canonical representation. Such a step, though formal, allowed us to treat our constrained system via the standard Dirac-Bergmann procedure. It should be admitted that such a choice is by no means unique, e.g., methodologies for treatment of classical constrained systems in configuration space do exist [39,41]. The decision to apply the Dirac-Bergmann algorithm was mainly motivated by its conceptual simplicity and direct applicability to path integrals. On the other hand, we do not expect that the presented results should undergo any substantial changes when some other scheme would be utilized. It should be further emphasized that while we have established the mathematical link [Eqs. (52) and (D7)] between the "loss of information" condition and first-class constraints, it is not yet clear if this connection has more direct physical interpretation (although various proposals exist in the literature [14,19,24]). Such an understanding would not only help to develop this approach for more complicated physical situations but also allow affiliation in a systematic fashion of a quantum system to an underlying classical dynamics. Work along those lines is currently in progress.

To illustrate the presented ideas we have considered two simple systems; the planar pendulum and the Rössler system. In the pendulum case we have taken advantage of free choice of an additive constant in the charge C_1. This in turn, allowed us to impose't Hooft's constraints in two distinct ways. In the case of Rössler's system two **p**-independent, irreducible charges C_1 and C_2 exist. For definiteness sake we have constructed in the latter case the "loss of information" condition with the additive constant set to zero. With this we were able to convert the corresponding classical path integrals into path integrals describing a quantized free particle, a harmonic oscillator, and a free particle weakly coupled to Duffing's oscillator. As a by-product we could observe that our prescription provides a surprisingly rigid structure with rather tight maneuvering space for the emergent quantum dynamics. Indeed, when the classical dynamics is fixed, the 't Hooft condition is formulated via linear combination of charges C_i which correspond to the first integrals of the autonomous dynamical equations for **q**, i.e., Eq. (2). Due to the explicit form of 't Hooft's Hamiltonian the constraint is of the first class and so we must remove the redundancy in the description by imposing the gauge condition χ. By requiring that the consistency conditions (51) and (54) are satisfied, that the choice of χ does not induce Gribov ambiguity, and that the canonical transformations defined in Sec. IV are linear, we substantially narrowed down the class of possible emergent quantum systems. Note also, that when we start with the N-dimensional classical system (**q** variables), the emergent quantum dynamics has $(N-1)$ dimensions ($\overline{\mathbf{Q}}$ variables). Indeed, by introducing the auxiliary degrees of freedom $\overline{\mathbf{q}}$ we obtain $4N$-dimensional phase space which is constrained by $2N+2$ conditions (ϕ_i, φ, and χ), which leaves behind $(2N-2)$-dimensional phase space $\overline{\mathbf{Q}}, \overline{\mathbf{P}}$. This disparity between the dimensionality of the classical and emergent quantum systems vindicates in part the terminology "information loss" used throughout the text.

An important conclusion of this work is that 't Hooft's quantization proposal seems to provide a tenable scenario which allows for deriving certain quantum systems from classical physics. It should be stressed that although we assumed throughout that the deeper level dynamics is the classical (Lagrangian or Hamiltonian) one, there is in principle no fundamental reason that would preclude starting with more exotic premises. In particular, our conceptual reasoning would go unchanged if we had begun with Lagrangians operating over coordinate superspaces (pseudoclassical mechanics [52]) or with the currently much discussed discrete classical mechanics (i.e., having foam-, fractal-, or crystal-like configuration space) [53], etc. The only prerequisite for such approaches is the possibility of formulating a corresponding variant of Gozzi's path integral, and a method for implementing the "loss of information" constraint in such integrals.

There are many interesting applications of the above method. Applications to chaotic dynamical systems especially seem quite pertinent. After all, central to our reasoning is a (doubled) set of real first-order dynamical equations[6] which, under favorable conditions, may by associated with a chaotic dynamics in the configuration space. We should emphasize that the reader should not confuse the above with the extensively studied but unrelated notion of chaos in Hamiltonian systems—we do not deal here with dynamical equations on symplectic manifolds. This is important, as Hamiltonian systems forbid *per se* the existence of attractive orbits which are otherwise key in ’t Hooft’s proposal. In this respect our approach is parallel with some more conventional approaches. Indeed, a direct “quantization” of the equations of the motion—originally proposed by Feynman [54]—is one of the techniques for tackling quantization of dissipative systems [55,56]. In field theories this line of reasoning was recently extended by Biró, Müller and Matinyan [19] who demonstrated that quantum gauge field theories can emerge in the infrared limit of a higher-dimensional classical (non-Abelian) gauge field theory, known to have chaotic behavior [57].

We finally wish to comment on two more points. First, in cases where one strives for an explicit reparametrization invariance (or general covariance) of the emergent quantum system the presented framework is not very suitable. The absence of explicit covariance in both Dirac-Bergmann and Fadeev-Senjanovic algorithms makes the actual analysis very cumbersome or even impossible. In fact, expressions (68) and (70) are evidently not generally covariant due to the presence of time-independent constraints in the measure. Although generalizations that include covariant constraints do exist [33,58,59] they result in gauge fixing conditions which depend not only on the canonical variables but also on the Lagrange multipliers (or explicit time). Such gauge constraints are, however, incompatible with our Poisson bracket analysis used in Sec. IV and Appendixes A and D. Hence, if the emergent quantum system is supposed to be reparametrization invariant (e.g., relativistic particle, canonical gravity, relativistic string, etc.) a new framework for the path-integral implementation of ’t Hooft’s scheme must be sought. Second, the formalism of functional integrals is sometimes deceptive when taken too literally. The latter is the case, for instance, when gauge conditions are imposed and/or canonical transformations performed. The difficulty involved is known as the Edwards-Gulyaev effect [11,40,46] and it resides in the exact nature of the limiting sequence of the finite dimensional integrals which constitute the path integral. As a result the classical canonical transformation does not leave, in general, the measure of the path integral Liouville invariant but instead induces an anomaly [46,60]. Thus, for our construction to be meaningful it should be shown that the canonical transformations in Sec. IV are unaffected by the Edwards-Gulyaev effect. Fortunately, in cases when the generating function is at most quadratic (making canonical transformations linear) and not explicitly time dependent, it can be shown [29,60,61] that the anomaly is absent. It was precisely for this reason that more general transformations were not considered in the present paper. Clearly, both mentioned points are of key importance for further development of our procedure and, due to their delicate nature, they deserve a separate discussion.

Let us end with the remark that the notorious problem with operator ordering known from canonical approaches has an elegant solution in path integrals. The ordering is there naturally generated by the necessary physical requirement that path integrals must be invariant under coordinate transformations [65].

[6]Nontrivial are only the equations over actual configuration space. The dynamical equations for the auxiliary variables $\overline{q_i}$ are linear and hence they are not relevant in this connection

ACKNOWLEDGMENTS

M.B. and P.J. are grateful to the ESF network COSLAB for funding their stay at FU, Berlin. One of us (P.J.) acknowledges very helpful discussions with R. Banerjee, G. Vitiello, and Y. Satoh, and thanks the Japanese Society for Promotion of Science for financial support.

APPENDIX A

In this appendix we show that the system (1) has no secondary constraints. In contrast to the primary constraints which are a consequence of the noninvertibility of the velocities in terms of the p’s and q’s, secondary constraints result from the equations of motion. To show their absence in ’t Hooft’s system we start with the observation that the time derivative of any function $f(\mathbf{q},\mathbf{p})$ is given by [39]

$$\dot{f} \approx \{f,\overline{H}\} + u^j\{f,\phi_j\}. \tag{A1}$$

Here u^a are the Lagrange multipliers to be determined by the consistency conditions

$$0 \approx \dot{\phi}_i \approx \{\phi_i,\overline{H}\} + u^j\{\phi_i,\phi_j\}. \tag{A2}$$

The latter is nothing but the statement that constraints (as functions of $\mathbf{q}$ and $\mathbf{p}$) must hold at any time. If all u^j could not be determined from the consistency condition (A2) then we would have the so-called secondary constraints. In our case we have

$$\{\phi_1^a,\overline{H}\} = -\frac{\partial\overline{H}}{\partial q_a} \not\approx 0,$$

$$\{\phi_2^a,\overline{H}\} = -f_a(\mathbf{q}) \not\approx 0, \quad \{\phi_1^a,\phi_2^b\} = -\delta_{ab}. \tag{A3}$$

Using the fact that $\{\phi_i,\overline{H}\} \neq 0$ and det $|\{\phi_i,\phi_j\}| = 1$, the inhomogeneous system of linear equations (A2) can be uniquely resolved with respect to u^j, thus implying the absence of secondary constraints.

APPENDIX B

We show here that Gozzi’s configuration-space path integral results from the “classical” limit of the stochastic-quantization partition function, i.e., the limit where the width of a noise distribution tends to zero. For this purpose we start with the form of the partition function for stochastic quantization as written down by Zinn-Justin [34,62]:

$$Z_{\mathrm{SC}}(J) = \int \mathcal{D}\mathbf{q}\, \mathcal{D}\mathbf{c}\, \mathcal{D}\overline{\mathbf{c}}\, \mathcal{D}\lambda\, \exp\left\{ - \mathcal{S}[\mathbf{q},\mathbf{c},\overline{\mathbf{c}},\lambda] + \int \mathbf{J}(x)\mathbf{q}(x)dx \right\}, \tag{B1}$$

where

$$\mathcal{S} \equiv - w(\lambda) + \int \lambda(x)\left(\frac{\partial \mathbf{q}(x)}{\partial \tau} + \frac{\delta \mathcal{A}}{\delta \mathbf{q}(x)}\right)dx - \int dx\, dx'\, \overline{c_a}(x) \times \left(\frac{\partial}{\partial \tau}\delta_{ab}\delta(x-x') + \frac{\delta^2 \mathcal{A}}{\delta q_a(x)\,\delta q_b(x')}\right) c_b(x') \tag{B2}$$

and

$$\exp[w(\lambda)] \equiv \int \mathcal{D}\nu\, \exp\left\{ - \sigma(\nu) + \int dx\, \lambda(x)\nu(x)\right\}, \tag{B3}$$

with $\mathcal{D}\nu \exp[-\sigma(\nu)]$ being the functional measure of noise. Here $x=(t,\tau)$ and $dx=dt\,d\tau$ where τ is the Parisi-Wu fictitious time. The dynamical equation for $\mathbf{q}(x)$ is described by the Langevin equation

$$\frac{\partial \mathbf{q}(x)}{\partial \tau} + \left.\frac{\delta \mathcal{A}[\mathbf{q}]}{\delta \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}(x)} = \nu(x), \tag{B4}$$

with the initial condition $\mathbf{q}(t,0)=\mathbf{q}(t)$. For Gaussian noise of variance $2\hbar$, the noise measure is

$$\mathcal{D}\nu \exp[- \sigma(\nu)] = \prod_{i,x} \frac{d\nu_i(x)}{2\sqrt{\pi\hbar}} \exp\left(- \frac{1}{4\hbar}\int dx\, \nu^2(x)\right), \tag{B5}$$

and (B1) takes the form

$$\begin{aligned} Z_{\mathrm{SC}}(\mathbf{J}) &= \int \mathcal{D}\mathbf{q}\, \mathcal{D}\nu\, \delta\left(\frac{\partial \mathbf{q}}{\partial \tau} + \frac{\delta \mathcal{A}[\mathbf{q}]}{\delta \mathbf{q}} - \nu\right) \det\left\| \frac{\partial}{\partial \tau}\delta_{ab}\delta(x-x') + \frac{\delta^2 \mathcal{A}}{\delta q_a(x)\,\delta q_b(x')}\right\| \exp\left\{ - \sigma(\nu) + \int \mathbf{J}(x)\mathbf{q}(x)dx\right\} \\ &= \int \mathcal{D}\mathbf{q}\, \mathcal{D}\nu\, \delta[\mathbf{q} - \mathbf{q}^{[\nu]}] \exp\left\{ - \sigma(\nu) + \int \mathbf{J}(x)\mathbf{q}(x)dx\right\}, \end{aligned} \tag{B6}$$

where $\delta[f(\mathbf{q})] \equiv \Pi_{t,\tau}\delta(f(\mathbf{q}(t,\tau)))$ and $\mathbf{q}^{[\nu]}(x)$ is a solution of Eq. (B4). Using the representation

$$\delta(x) = \lim_{\hbar \to 0_+} \frac{1}{2\sqrt{\pi\hbar}} e^{-x^2/(4\hbar)}, \tag{B7}$$

we get in the limit of zero distribution width (i.e., $\hbar \to 0_+$) that

$$Z_{\mathrm{SC}}(J,\hbar) \to \int \mathcal{D}\mathbf{q}\, \delta[\mathbf{q} - \mathbf{q}^{[0]}] \exp\left\{ \int \mathbf{J}(x)\mathbf{q}(x)dx \right\}. \tag{B8}$$

Choosing a special source $\mathbf{J}(x)=\mathbf{J}(t)\delta(\tau)$ we can sum in the path integral solely over configurations with $\mathbf{q}(t,0)=\mathbf{q}(t)$ as other configurations will contribute only to an overall normalization constant. Thus we finally obtain

$$\lim_{\hbar \to 0^+} Z_{\mathrm{SC}}(\mathbf{J},\hbar) = \mathcal{Z}_{\mathrm{CM}}(\mathbf{J}). \tag{B9}$$

Next we show that Gozzi's configuration-space partition function (19) results from the "classical" limit of the closed-time-path integral for the transition probability of a system coupled to a thermal reservoir at some temperature T. By the classical limit we mean the high temperature and weak heat bath coupling limit.

The path-integral treatment of systems that are linearly coupled to a thermal bath of harmonic oscillators was first considered by Feynman and Vernon [63]. For our purpose it will be particularly convenient to utilize the so-called Ohmic limit version, as discussed in Refs.[11,64]:

$$\begin{aligned} \mathcal{Z}_{\mathrm{FV}}[\mathbf{J}_+,\mathbf{J}_-] = \int \mathcal{D}\mathbf{q}_+ \mathcal{D}\mathbf{q}_- \exp&\left\{ \frac{i}{\hbar}[\mathcal{A}[\mathbf{q}_+] - \mathcal{A}[\mathbf{q}_-]] + \int dt[\mathbf{J}_+(t)\mathbf{q}_+(t) - \mathbf{J}_-(t)\mathbf{q}_-(t)]\right\} \\ &\times \exp\left\{ - i\frac{m\gamma}{2\hbar}\int dt[\mathbf{q}_+(t) - \mathbf{q}_-(t)] \times [\dot{\mathbf{q}}_+(t) + \dot{\mathbf{q}}_-(t)]^R \right\} \\ &\times \exp\left\{ - \frac{m\gamma}{\hbar^2\beta}\int dt \int dt' [\mathbf{q}_+(t) - \mathbf{q}_-(t)] \times K(t,t')[\mathbf{q}_+(t') - \mathbf{q}_-(t')]\right\}. \end{aligned} \tag{B10}$$

Here the paths $\mathbf{q}_+(t)$ and $\mathbf{q}_-(t)$ are associated with the forward and backward movement of the particles in time. The superscript R indicates a *negative* shift in the time argument of the velocities with respect to positions. The latter ensures the causality of the friction forces [64]. In addition, m represents the particle mass (for simplicity we assume here that all system particles have the same mass), $\beta=1/T$, and γ is the friction constant (or thermal reservoir coupling). The function $K(t,t')$ is the bath correlation function. As argued in [11,64], at high temperatures $K(t,t') \approx \delta(t-t')$. Introducing the new set of variables $\mathbf{q}=[\mathbf{q}_+ + \mathbf{q}_-]/2$ and $\overline{\mathbf{q}}=[\mathbf{q}_+ - \mathbf{q}_-]$ (i.e.,the center-of-mass and *fast* coordinates) we can in the high-temperature case recast (B10) into

$$\mathcal{Z}_{\mathrm{FV}}[\mathbf{J},\overline{\mathbf{J}}] = \int \mathcal{D}\mathbf{q}\,\mathcal{D}\overline{\mathbf{q}}\exp\left\{\frac{i}{\hbar}[\mathcal{A}[\mathbf{q}+\overline{\mathbf{q}}/2]-\mathcal{A}[\mathbf{q}-\overline{\mathbf{q}}/2]] + \int dt[\mathbf{J}(t)\mathbf{q}(t)-\overline{\mathbf{J}}(t)\overline{\mathbf{q}}(t)]\right\} \times\exp\left\{-i\frac{m\gamma}{\hbar}\int dt\,\overline{\mathbf{q}}(t)[\dot{\mathbf{q}}(t)]^R - \frac{m\gamma}{\hbar^2\beta}\int dt\,\overline{\mathbf{q}}^2(t)\right\}. \tag{B11}$$

Here the self-explanatory notation $\mathbf{J}=[\mathbf{J}_+-\mathbf{J}_-]$ and $\overline{\mathbf{J}}=-[\mathbf{J}_++\mathbf{J}_-]/2$ was used. Let us now define $\omega=2m\gamma/\beta$, integrate over $\overline{\mathbf{q}}$, and go to the classical limit $\gamma\to 0$. Then we obtain the following chain of equations:

$$\begin{aligned}\lim_{\gamma\to 0}\mathcal{Z}_{\mathrm{FV}}[\mathbf{J},\overline{\mathbf{J}}] &= \lim_{\gamma\to 0}\int \mathcal{D}\mathbf{q}\,\mathcal{D}\overline{\mathbf{q}}\exp\left\{\frac{i}{\hbar}\int dt\,\overline{\mathbf{q}}(t)\left[\frac{\delta\mathcal{A}}{\delta\mathbf{q}(t)} - m\gamma[\dot{\mathbf{q}}(t)]^R + i\hbar\overline{\mathbf{J}}(t)\right] - \frac{\omega}{2\hbar^2}\int dt\,\overline{\mathbf{q}}^2(t)\right\}\exp\left\{\int dt\,\mathbf{J}(t)\mathbf{q}(t)\right\}\\ &= \lim_{\gamma\to 0}\int \mathcal{D}\mathbf{q}\,\exp\left\{-\frac{1}{2\omega}\int dt\left[\frac{\delta\mathcal{A}}{\delta\mathbf{q}(t)} - m\gamma[\dot{\mathbf{q}}(t)]^R + i\hbar\overline{\mathbf{J}}(t)\right]^2 + \int dt\,\mathbf{J}(t)\mathbf{q}(t)\right\}\\ &= \lim_{\gamma\to 0}\int \mathcal{D}\mathbf{q}\,\mathcal{J}[\mathbf{q}]\exp\left\{-\frac{1}{2\omega}\int dt\left[\frac{\delta\mathcal{A}}{\delta\mathbf{q}(t)} - m\gamma\dot{\mathbf{q}}(t) + i\hbar\overline{\mathbf{J}}(t)\right]^2 + \int dt\,\mathbf{J}(t)\mathbf{q}(t)\right\}\\ &= \int \mathcal{D}\mathbf{q}\,\delta\left[\frac{\delta\mathcal{A}}{\delta\mathbf{q}} + i\hbar\overline{\mathbf{J}}\right]\mathcal{J}[\mathbf{q}]\exp\left\{\int dt\,\mathbf{J}(t)\mathbf{q}(t)\right\}\\ &= \int \mathcal{D}\mathbf{q}\,\delta\left[\mathbf{q}-\mathbf{q}^{[\overline{\mathbf{J}}]}\right]\exp\left\{\int dt\,\mathbf{J}(t)\mathbf{q}(t)\right\}.\end{aligned} \tag{B12}$$

The Jacobian $\mathcal{J}[\mathbf{q}]$ results from transition to the "unretarded" velocities and its explicit form reads [64]

$$\mathcal{J}[\mathbf{q}] = \det\left\|\frac{\partial}{\partial t}\delta_{ab}\delta(t-t') + \frac{\delta^2\mathcal{A}}{\delta q_a(t)\delta q_b(t')}\right\|. \tag{B13}$$

The coordinates $\mathbf{q}^{[\overline{\mathbf{J}}]}$ are solutions of the equation of the motion

$$\frac{\delta\mathcal{A}[\mathbf{q}]}{\delta\mathbf{q}(t)} = -i\hbar\overline{\mathbf{J}}(t). \tag{B14}$$

In the limit $\gamma\to 0$, we find again the Gozzi *et al.* partition function

$$\lim_{\gamma\to 0}\mathcal{Z}_{\mathrm{FV}}[\mathbf{J},\mathbf{0}] = \lim_{\hbar\to 0}\lim_{\gamma\to 0}\mathcal{Z}_{\mathrm{FV}}[\mathbf{J},\overline{\mathbf{J}}] = Z_{\mathrm{CM}}[\mathbf{J}]. \tag{B15}$$

APPENDIX C

In this appendix we prove that Eq. (34) is a special case of the Euler-like functionals (42). Let us first show that Eq. (34) can be replaced by an action of the form (42). Indeed, because of the homogeneity of Eq. (34), we can immediatley replace it by

$$\mathcal{A}[r^{\alpha_i}q_i] = \sum_i \int dt\;\alpha_i r^{\alpha_i}(t)q_i(t)\frac{\delta\mathcal{A}[r^{\alpha_i}q_i]}{\delta r^{\alpha_i}(t)q_i(t)} = \int dt\;r(t)\frac{\delta\mathcal{A}[r^{\alpha_i}q_i]}{\delta r(t)}. \tag{C1}$$

Since this is true for any $r(t)$, we see that

$$\int dtdt'\,r(t)\frac{\delta^2\mathcal{A}[r^{\alpha_i}q_i]}{\delta r(t)\delta r(t')} = 0. \tag{C2}$$

This simply expresses the fact that the functional $\mathcal{A}[r^{\alpha_i}q_i]$ is linear in $r(t)$. The right-hand side of Eq. (C1) has then precisely the Euler form (42).

The reverse direction is proved in the following way: We first recast Eq. (42) in the general form

$$\int dt\;r(t)L(\mathbf{q}(t),\dot{\mathbf{q}}(t)) = \int dt\;L(r^{\alpha_i}(t)q_i(t),d(r^{\alpha_i}(t)q_i(t))/dt). \tag{C3}$$

Applying the variation $\int dt\;\delta/\delta r(t)$ to Eq. (C3) we obtain

$$\mathcal{A}[\mathbf{q}] = \int dt\sum_i \alpha_i r^{\alpha_i-1}q_i(t)\left(\frac{\partial L}{\partial r^{\alpha_i}(t)q_i(t)} - \frac{d}{dt}\frac{\partial L}{\partial[d(r^{\alpha_i}(t)q_i(t))/dt]}\right). \tag{C4}$$

This relation must hold for all $r(t)$, and hence by choosing $r(t)=1$ we arrive at the required result

$$\mathcal{A}[\mathbf{q}] = \int dt\sum_i \alpha_i q_i(t)\frac{\delta\mathcal{A}[\mathbf{q}]}{\delta q_i(t)}. \tag{C5}$$

APPENDIX D

Here we prove the fact that inclusion of the subsidiary constraint (46) in the primary constraints (5) does not produce any secondary constraints. The secondary constraints result from the consistency conditions (A2) or, in other words, occur when existing constraints are incompatible with the equation of motion.

We first observe that the condition $H_- \approx 0$ can be equivalently represented by the condition $(\overline{H} - \Sigma_i a_i C_i) \equiv \phi_0 \approx 0$. If we now add the subsidiary constraint ϕ_0 to the remaining $2N$ constraints ϕ_i and again require that the constraints ϕ_i remain (weakly) zero at all times we have

$$0 \approx \dot{\phi}_i \approx \{\phi_i, \overline{H}\} + u^j \{\phi_i, \phi_j\}, \quad i,j = 0,1\ldots,2N. \quad \text{(D1)}$$

Since there is an odd number of constraints and because $\{\phi_i, \phi_j\}$ is an antisymmetric matrix we have that $\det\|\{\phi_i, \phi_j\}\| = 0$. From the analysis in Appendix A it is clear that the rank of the matrix $\{\phi_i, \phi_j\}$ is $2N$ and hence it has one null eigenvector, say $\mathbf{e}$. Thus, Eq. (D1) implies the constraint

$$\sum_{i=0}^{2N} e_i \{\phi_i, \overline{H}\} \approx 0. \quad \text{(D2)}$$

If the latter represented a new nontrivial constraint (i.e., a constraint that cannot be written as a linear combination of constraints ϕ_i) we would need to include such a new constraint (the so-called secondary constraint) in the list of existing constraints and go again through the consistency condition (D1). Fortunately, the condition (D2) is automatically satisfied and hence it does not constitute any new constraint. Indeed, by choosing

$$\mathbf{e} = \begin{pmatrix} 1 \\ \{\phi_0, \phi_2^a\} \\ \{\phi_1^a, \phi_0\} \\ \{\phi_0, \phi_2^b\} \\ \{\phi_1^b, \phi_0\} \\ \vdots \\ \{\phi_0, \phi_2^N\} \\ \{\phi_1^N, \phi_0\} \end{pmatrix} = \begin{pmatrix} 1 \\ f_a(\mathbf{q}) \\ -\dfrac{\partial \phi_0}{\partial q_a} \\ f_b(\mathbf{q}) \\ -\dfrac{\partial \phi_0}{\partial q_b} \\ \vdots \\ f_N(\mathbf{q}) \\ -\dfrac{\partial \phi_0}{\partial q_N} \end{pmatrix}, \quad \text{(D3)}$$

and using $\{\phi_0, \overline{H}\} = 0$ together with (A3) we obtain

$$\sum_{i=0}^{2N} e_i \{\phi_i, \overline{H}\} = -\sum_{i,a} a_i(t) f_a(\mathbf{q}) \frac{\partial C_i}{\partial q_a} = \sum_{i=1}^{n} a_i(t) \{H, C_i\} = 0. \quad \text{(D4)}$$

As the latter is zero (even strongly) there is no new constraint condition generated by inclusion of ϕ_0 in the original set of (primary) constraints. Note that the key in obtaining (D4) was the fact that the C_i's are $\mathbf{p}$-independent constants of motion.

The rank of $\{\phi_i, \phi_j\}$ being $2N$ means that there is one relation

$$\sum_{i=0}^{2N} e_i \{\phi_i, \phi_j\} \approx 0. \quad \text{(D5)}$$

Any linear combination of the constraints ϕ_i is again a constraint. So, particularly if we define $\varphi = \Sigma_i e_i \phi_i$, we obtain that φ has weakly vanishing Poisson brackets with all constraints, i.e.,

$$\{\varphi, \phi_i\} \approx 0, \quad i = 1, \ldots, 2N. \quad \text{(D6)}$$

Thus, according to Dirac's classification (see, e.g., Ref. [32]) φ is a first-class constraint. The remaining $2N$ constraints (which do not have vanishing Poisson brackets with all other constraints) are of the second class. Note particularly that the explicit form for φ reads

$$\varphi = \sum_{i=0}^{2N} e_i \phi_i = \left(H - \sum_{i=1}^{n} a_i C_i \right) - \sum_{a=1}^{N} \overline{p}_a \frac{\partial \phi_0}{\partial q_a}, \quad \text{(D7)}$$

which is clearly weakly identical to $\underline{H} - \Sigma_i a_i C_i$. Observe that it is H and not $\overline{H}$ that is present in Eq. (D7).

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