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8.B.5

A MODEL FOR DEEP INELASTIC STRUCTURE FUNCTIONS

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Abstract: We calculate the electromagnetic structure functions from an infinite component wave equation which is a relativistic generalization of the Schrödinger equation of the H atom. The elastic form factor has a dipole structure leading to the threshold behavior of $F_2(\xi) \alpha (1-\xi)^3$ at $\xi = 1$. Apart from that, $F_2(\xi)$ differs from the structure function of the proton by being larger and by having a zero at $\xi = 0$.

While our amplitude appears at first sight to be typical resonance model for the structure functions, all properties of the result allow for a direct interpretation in terms of the parton picture.

A specific choice of the mass parameters reduces our model to that of the H atom. In this limiting case we show that the result is almost identical to what one would obtain from the parton approximation.

1. INTRODUCTION

It is well known that both the large magnitude and the scaling property of deep inelastic electron nucleon scattering can be explained by imagining the nucleon to be composed of point like constituents called partons [1]. Appropriate additional assumptions on their instrinsic properties and on their distribution inside the nucleon lead to reasonable fits of most of the available data [2] **.

It has been pointed out by Bloom and Gilman [4], that there is an alternative way of understanding the data in terms of s–channel resonances. The argument is that these resonances should, at least for $\xi \gtrsim \frac{1}{6}$, be responsible for a considerable part of the amplitude. The observed threshold behavior of $(1-\xi)^3$ in νW_2 would then be indicative of an equal fall-off of all transition form factors to higher resonances

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** Note however, the serious disagreement with some of the recent experimental findings, namely

$$\nu W_2^n / \nu W_2^p \approx (1-\xi) ,$$

with the lowest value (for ξ close to 1) being smaller than 0.25 [3]. The quark parton model predicted $\frac{2}{3}$. We shall use the scaling variables $\xi \equiv 1/\omega \equiv -q^2/2m\nu$.

$\approx (q^2)^{-2}$. Also these ideas supplemented with some quite simple ad hoc assumptions are able to reproduce the gross features of the data [5] *.

Recently it was noticed [6] that theories of the electromagnetic form factors constructed on the basis of infinite component wave equations provide a natural framework for obtaining such results without such ad hoc assumptions. Within this framework, the structure functions have been calculated [6] for the Majorana equation [7]**, the Abers, Grodsky, Norton equation [8], and a wave equation with an oscillator type of spectrum describing linearly rising trajectories [9] ***.

All of these models, apart from giving a far too large result for νW_2, fail to account for the characteristic threshold behaviour of $(1-\xi)^3$ close to $\xi = 1$. The reason is that they show the connection between the threshold behavior $\nu W_2 \approx (1-\xi)^{2p-1}$ and the fall-off of the elastic form factors $(q^2)^{-p}$ first discovered in a model by Drell and Yan [10]. But it has been known for a long time that the elastic form factors of the Majorana equation go asymptotically like $(q^2)^{-1}$ [11], those of the Abers Grodsky Norton equation like $(q^2)^{-\frac{1}{2}}$, and those [10] of the oscillator equation like $e^{-\gamma q^2}$. Correspondingly, the structure functions νW_2 go like $(1-\xi)$, const., and zero, respectively †. The way to remedy these defects is well known [12]. First one considers Majorana equations constructed on a unitary representation space of the Lorentz group characterized by a parameter $\nu \neq 0$ ††. For such equations the form factors oscillate asymptotically. This corresponds to having hollow particles concentrated on the surface of a sphere. Then one constructs a physical particle by mixing such states via radial wave functions.

Such a construction can be economized by employing a representation space of a larger group which is able to accommodate such radial wave functions. The mechanism of representation mixing is learned from the study of the non-relativistic H atom. Here the radial wave function describes the mixtures of irreducible representations of the Galilean group which are contained in every state and which generate the form factors. By following this example given by nature, a complete theory of the currents of the baryon resonances has been developed some time ago [13].

* With a similar problem for $\nu W_2^n/\nu W_2^p$ which is predicted to go as

$$\left(\frac{\mu_n}{\mu_p}\right)^2 \approx \tfrac{1}{2}\,.$$

** For a review of this equation see ref. [12] and references therein.

*** At the cost of containing states of negative norm.

† In the third case, the threshold behaviour should be, according to the Drell Yan rule, faster than any power of $(1-\xi)$. In fact, νW_2 is found to be zero up to a point $\xi_0 < 1$.

†† The parameter ν is defined by the invariants of the Lorentz group

$$L^2 - M^2 = j_0^2 - \nu^2 - 1\ , \qquad L\cdot M = -j_0\nu$$

where j_0 is the lowest spin of representation.

These currents satisfy current algebra, give approximately the correct mass spectrum of the observed baryon resonances and describe correctly the known electromagnetic form factors. In particular, these currents supply, up to now, the only explanation of the dipole fit of the nucleon form factors $G_E = G_M/\mu = (1-q^2/m^2)^{-2}$ by relating it to the spectrum of the baryon resonances.

Since the explicit construction of these currents is quite involved we prefer in this note, as a preliminary study, to discuss a slightly less realistic but simpler model. This simpler model incorporates many of the essential properties of the correct currents and has the advantage of yielding directly transparent results.

In our model, internal symmetries have been neglected. It should be mentioned, however, that any of the currents constructed on the basis of an infinite component wave equation excites the nucleon only to non-exotic states. This very general property, shared by all such models, can in principle be tested since it implies relations among the structure functions of electron and neutrino scattering. Such tests involve, however, the knowledge of strangeness changing parts in neutrino proton scattering and therefore will remain out of experimental reach for quite some time.

The same remark certainly holds for any model in which only resonances in the s–channel are exchanged [4,5], since resonances are, as much as we know, non-exotic *.

At first sight the wave equation is a typical example of a resonance model for the structure functions. Closer inspection of the results shows, however, that every property finds a straight-forward interpretation in terms of the parton picture of deep inelastic scattering processes. The reason is that the wave equation as a generalization of the Schrödinger equation retains a typical two body character.

The lesson learned by the study of this model points ways for possible modifications which have to be done with current wave equations in order to permit them to incorporate the typical many parton-aspects necessarily present in elementary particle collisions. In our discussion we have concentrated completely on the good aspects of infinite component wave equations. The deseases which are brought along by the locality of the fields occuring together with the solubility of the model, like space like solutions and states of negative norm, have been thoroughly discussed in the literature and will not be mentioned here [19]. Fortunately, electron proton scattering measures only the form factors of resonances with positive m^2 at negative q^2. It is exactly in this region that wave equations appaer to provide excellent fits to experimental form factors [13].

2. THE MODEL

We shall employ a generalized form of the relativistic wave equation of the H atom. This equation was proposed first some time ago as a possible relativistic exten-

* The author thanks M. Gell-Mann for bringing this point to his attention.

sion of the non-relativistic Schrödinger theory [15]* and has since then been the object of several studies. In particular, several attempts were made to understand the relation of this equation with other relativistic models, like Bethe Salpeter [16] and quasi-potential equations [17].

Consider the Hilbert space generated by applying an equal number of creation operators a_r^+ and $b_r^+ (r = 1,2)$ to the vacuum $|0\rangle$. On this Hilbert space a Lorentz invariant wave equation can be writen down in the form

$$L(p)\, u(p) \equiv [-(m_+ - m_-) p_\mu \Gamma^\mu + (p^2 - m_+ m_-) L_{46} + \kappa]\, u(p) = 0 \ , \tag{1}$$

where

$$\Gamma^\mu \equiv (L_{56}, L_{i6}) \ , \tag{2}$$

and L_{ab} are part of the 15 generators of the group O(4, 2) satisfying the commutation rules

$$[L_{ab}, L_{ac}] = -i g_{aa} L_{bc} , \qquad g_{ab} = \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} . \tag{3}$$

All generators can be written down in terms of the operators a and b (see appendix A). In particular L_{56} is diagonal and counts the number of a and b operators:

$$L_{56} = \tfrac{1}{2} (a^+ a + b^+ b) + 1 \ . \tag{4}$$

Eq. (1) is invariant under Lorentz transformations generated by L_{ij}, L_{i6}. Thus, the spinors $u(\boldsymbol{p})$ can be constructed by Lorentz boosting a spinor at rest $u(\mathbf{0})$:

$$u(\boldsymbol{p}) = \mathrm{e}^{-i\zeta_i L_{i6}}\, u(\mathbf{0}) \ , \qquad \zeta = \operatorname{arsh} \frac{|\boldsymbol{p}|}{m} \boldsymbol{p} \ . \tag{5}$$

These spinors at rest satisfy the equation

$$[-(m_+ - m_-)\, m\, L_{56} + (m^2 - m_+ m_-)\, L_{46} + \kappa]\, u(0) = 0 \ . \tag{6}$$

As long as $(m_+ - m_-) m$ is larger than $m^2 - m_+ m_-$, this equation can be diagonalized by a transformation (a so called "tilt operation")

$$u(0) = \mathrm{e}^{-i\theta L_{45}}\, \tilde{u}(0) \ , \tag{7}$$

* It was proved that in the limit of weak coupling $\alpha \to 0$, which slows down the orbital motion to non-relativistic velocities, all form factors become the same as those of the Schrödinger equation.

with

$$\mathrm{ch}\theta = \frac{b(m^2)}{\kappa}(m_+ - m_-)m, \quad \mathrm{sh}\theta = \frac{b(m^2)}{\kappa}(m^2 - m_+ m_-) \tag{8}$$

where the factor $b(m^2)/\kappa$ is introduced in order to get the normalization right:

$$\mathrm{ch}^2\theta - \mathrm{sh}^2\theta = \left(\frac{b(m^2)}{\kappa}\right)^2 (m^2 - m_-^2)(m_+^2 - m^2) = 1 \tag{9}$$

which can be achieved by choosing

$$\frac{b(m^2)}{\kappa} = [(m^2 - m_-^2)(m_+^2 - m^2)]^{-\frac{1}{2}} . \tag{10}$$

Under the transformation (7), eq. (6) goes over into

$$\widetilde{L}(m^2)\,\widetilde{u}(0) = -\frac{\kappa}{b(m^2)}(L_{56} - b(m^2))\,\widetilde{u}(0) = 0 , \tag{11}$$

which is solved by the basis states

$$|pq\bar{p}\bar{q}> = [p!q!\bar{p}!\bar{q}!]^{-\frac{1}{2}} a_1^{\dagger p} a_2^{\dagger q} b_2^{\dagger \bar{p}} b_2^{\dagger \bar{q}} |0>, \quad p+q = \bar{p} + \bar{q} ,$$

on which L_{56} has the eigenvalue $\frac{1}{2}(p + q + \bar{p} + \bar{q}) + 1 \equiv n$ with masses given by the solutions of

$$n = b(m^2) , \quad n = 1, 2, 3, \ldots . \tag{12}$$

For $(m_+ - m_-)\,m < m^2 - m_+ m_-$, one can rotate eq. (5) into

$$\left(+\frac{\kappa}{\hat{b}(m^2)} L_{46} + \kappa\right)\widetilde{u}(0) = 0 , \quad \kappa/\hat{b}(m^2) \equiv \sqrt{(m^2 - m_-^2)(m^2 - m_+^2)} , \tag{14}$$

if one uses the angle θ determined by

$$\mathrm{ch}\hat{\theta} = \frac{\hat{b}(m^2)}{\kappa}(m^2 - m_+ m_-) , \quad \mathrm{sh}\hat{\theta} = \frac{\hat{b}(m^2)}{\kappa}(m_+ - m_-)m . \tag{15}$$

Here the eigenstates of L_{46} with the continuous eigenvalues $\hat{n} \in [-\infty, \infty]$ provide a solution. The corresponding continuum of masses is given by

$$\hat{n} = -\hat{b}(m^2) . \tag{16}$$

The mass spectrum can be summarized by squaring (12) and (16). One finds

$$n^2 = -\frac{\kappa^2}{(m^2-m_+^2)(m^2-m_-^2)} \tag{17}$$

with discrete values of $n^2 \geqslant 1$ and a continuum of $n^2 \leqslant 0$ allowed, part of which is spacelike (see fig. 1).

The masses are discrete between the values m_-^2 and m_+^2. Above and below there is a continuum. The states of the relativistic H atom are obtained by choosing the specific values

$$m_+ = m_p + m_e\,, \quad m_- = -m_p + m_e\,, \quad \kappa = 2\alpha m_p m_e\,. \tag{18}$$

Then then the mass spectrum becomes

$$m_n = m_+ - \frac{m_p m_e}{m_+}\frac{\alpha^2}{2n^2} + O(\alpha^4) \tag{19}$$

at the upper branch and a continuum above m_+. Besides this there is a lower branch obtained by changing $m_e \to -m_e$. These states can be interpreted as bound states of protons and electrons of negative energy *.

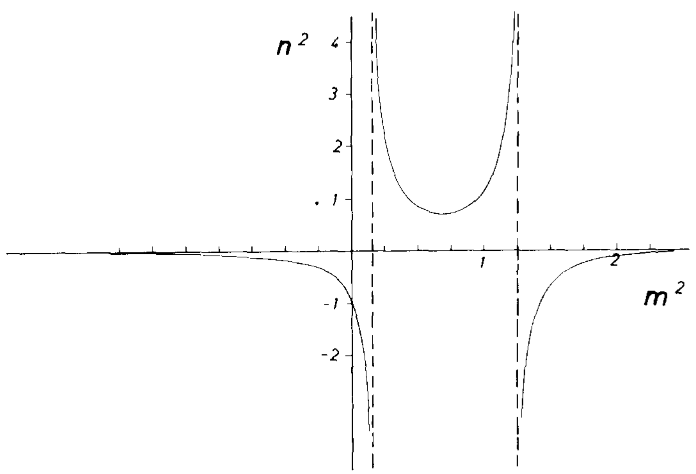

Fig. 1. The mass spectrum of our wave equation is shown for the following values of the mass parameters: $m_+ = 1.12$, $m_- = 0.4$.

* At present it is unclear how to incorporate them into a consistent field theoretic description of the H atom.

Minimal coupling of a photon to eq. (1) provides a conserved electromagnetic current * $J^\mu(p',p) = -(m_+ - m_-)\Gamma^\mu + (p'+p)^\mu L_{46}$ such that

$$\langle p'|j^\mu(0)|p\rangle = \frac{1}{N_{n'}N_n} u^+_{n'}(p') J^\mu(p',p)\, u_n(p) \tag{20}$$

where N_n are normalization factors chosen to provide unit charge for every state. Going to diagonal elements of j^0 in (20) we find

$$N_n^2 = \mathrm{sh}\theta_n - \frac{m_+ - m_-}{2m_n}\, \mathrm{ch}\theta_n \; . \tag{21}$$

With this current our wave equation (1) gives rise to a Compton amplitude for the scattering of a photon on the ground state m_1 of our infinite component multiplet. If we leave out u–channel and seagull contributions, we have

$$T_{\mu\nu} = \frac{1}{N_1^2} u_1^+(p')\, J_\mu(p',p+q) \frac{1}{L(p+q)} J_\nu(p+q,p)\, u_1(p). \tag{22}$$

The contraction of the currents with the polarization vectors $\epsilon^\nu J_\nu$ can be written in the form

$$\epsilon^\nu(q)\, J_\nu(p+q,q) = -\,(m_+ - m_-)\, \epsilon^\nu(q)\Gamma_\nu + \epsilon^\nu(2p+q)_\nu\, L_{46}. \tag{23}$$

At his stage it is useful to introduce a five vector [18]

$$\Gamma^A \equiv [\Gamma^\mu, L_{46}], \quad A = 0,1,\ldots,4 \; , \tag{24}$$

and a similar polarization vector

$$\in^A \equiv [(m_+ - m_-)\epsilon^\mu \; , \epsilon(p+q+q)] \tag{25}$$

such that (23) can be written as the scalar product $-\epsilon^A\Gamma_A$ with a metric [18].

$$g_{AB} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix} .$$

Then the Compton amplitude becomes simply

$$T \equiv \in'^A \in^B T_{AB} = \in'^A \in^B \frac{1}{N_1^2} u_1^+(p')\Gamma_A \frac{1}{L(p+q)} \Gamma_B u_1(p) \; . \tag{26}$$

* The states $|p\rangle$ are normalized according to $\langle p'|p\rangle = 2p_0(2\pi)^3\, \delta^{(3)}(p'-p)$. Our scattering amplitude is defined by $S = 1 - (2\pi)^4 i\delta^{(4)} \epsilon^\mu(q')\, T_{\mu\nu}\epsilon^\nu(q)$.

This expression can be calculated explicitly. For this we simply go to the c.m. frame and choose a c.m. energy $\sqrt{s}$ for which

$$(m_+ - m_-)\sqrt{s} > s - m_+ m_- \ .$$

Then $L(p + q)$ can be diagonalized by writing it in the form

$$\frac{1}{L(p+q)} = e^{-i\theta_s L_{45}} \frac{1}{\tilde{L}(s)} e^{i\theta_s L_{45}} \equiv T_s \frac{1}{\tilde{L}(s)} T_s^{-1} \tag{27}$$

with θ_s given by eq. (8) and m replaced by $\sqrt{s}$. Here we can insert a complete set of intermediate states

$$\sum_n \tilde{u}_n(0)\,\tilde{u}_n^+(0) = 1 \tag{28}$$

such that T_{AB} becomes

$$T_{AB} = -\frac{1}{N_1^2}\frac{b(s)}{\kappa}\sum_n u_1^+(p')\,\Gamma_A T_s \tilde{u}_n(0)\frac{1}{n-b(s)}\tilde{u}_n^+(0) T_s^{-1}\Gamma_B u_1(p) \ . \tag{29}$$

This expression is evaluated in two steps. First one defines the five-vector of the proton:

$$\lambda^A \equiv \frac{b(m_1^2)}{\kappa}\,[(m_+ - m_-)p^\mu \ , m_1^2 - m_+ m_-] \ , \quad b(m_1^2) = 1 \ , \tag{30}$$

with

$$\lambda^2 = \lambda^A \lambda_A = 1 \ ,$$

and notices that due to the wave equation (1), $\lambda^A \Gamma_A\, u_1(p) = u_1(p)$. Therefore one finds

$$\begin{aligned} \lambda'^A \lambda^B T_{AB} &= -\frac{1}{N_1^2}\frac{b(s)}{\kappa}\sum_n u_1^+(p') T_s\, \tilde{u}_n(0)\frac{1}{n-b(s)}\tilde{u}_n^+(0) T_s^{-1} u_1(p) \\ &= -\frac{1}{N_1^2}\frac{b(s)}{\kappa}\sum_n \tilde{u}_1^+ G^+(\zeta')\,\tilde{u}_n(0)\frac{1}{n-b(s)}\tilde{u}_n^+(0) G(\zeta)\, u_1(0) \equiv A \end{aligned} \tag{31}$$

where we have introduced the abbreviation

$$G(\zeta) \equiv e^{i\theta_s L_{45}} e^{-i\zeta_i L_{i5}} e^{-i\theta_1 L_{45}} , \tag{32}$$

Obviously, A is just the scalar amplitude

$$A \equiv -\frac{1}{N_1^2} u_1(p') \frac{1}{L(p+q)} u_1(p) . \tag{33}$$

Second one takes the 5–trace

$$T_A^A = -\frac{1}{N_1^2} \frac{b(s)}{\kappa} \sum \tilde{u}_1^+(0) e^{i\theta_1 L_{45}} e^{i\zeta_i' L_{i5}} \Gamma_A e^{-i\theta_s L_{45}} \tilde{u}_n(0)$$
$$\times \frac{1}{n-b(s)} \tilde{u}_n(0) e^{i\theta_s L_{45}} \Gamma^A e^{-i\zeta_i L_{i5}} e^{-i\theta_1 L_{45}} \tilde{u}_1(0) \tag{34}$$

and moves the operators Γ_A in front of $\tilde{u}_1(0)$ since the scalar product $\Gamma_A \Gamma^A$ is invariant under the transformation $e^{-i\zeta_i L_{i5}} e^{-i\theta_1 L_{i5}}$. Thus one can write for $\zeta' = \zeta$:

$$T_A \overset{A}{=} -\frac{1}{N_1^2} \frac{b(\zeta)}{\kappa} \sum_n \tilde{u}_1^+(0) \Gamma_A G^+(\zeta) \tilde{u}_n(0) \frac{1}{n-b(s)} u_n^+(0) G(\zeta) \Gamma^A \tilde{u}_1(0). \tag{35}$$

But using the creation and annihilation representation of Γ_A, one calculates (see appendix A):

$$\Gamma_A \tilde{u}_1(0) \tilde{u}_1^+(0) \Gamma^A \equiv \Gamma_A |0\rangle\langle 0| \Gamma^A$$
$$= |0\rangle\langle 0| - \tfrac{1}{2} \{ |1010\rangle\langle 1010| + |10101\rangle\langle 0101| + |1001\rangle\langle 1001| + |10110\rangle\langle 0110| \} \tag{36}$$
$$= P^{(1)} - \tfrac{1}{2} P^{(2)}$$

where $P^{(n)}$ denotes the projection onto the subspace of principal quantum number n. Therefore T_A^A becomes

$$T_A{}^A = A + \tfrac{1}{2}\frac{1}{N_1^2}\frac{b(s)}{\kappa}\sum_n \frac{1}{n-b(s)}\,\text{trace}\,[P^{(2)}\,G^+(\xi)\,P^{(n)}\,G(\zeta)]$$
$$\equiv A - B\ . \tag{37}$$

The calculation of A and B can be done according to the general method developed in appendix B of ref. [19]. We give here only the result for forward scattering (see our appendix B)

$$A = -\frac{1}{N_1^2}\frac{b(s)}{\kappa}\sum \frac{4n}{n-b(s)}\left(\frac{x-1}{x+1}\right)^n \frac{1}{x^2-1}\ , \tag{38}$$

$$B = -\frac{1}{N_1^2}\frac{b(s)}{\kappa}\sum \frac{4n}{n-b(s)}\left(\frac{x-1}{x+1}\right)^n \frac{1}{(x^2-1)^2}(x^2-2nx+2n^2-1)\ , \tag{39}$$

where x is the scalar product of the five-vector λ^A of the proton with an analogously defined five-vector of the intermediate resonance

$$\eta^A = \frac{b(s)}{\kappa}\,[(m_+ - m_-)(p+q)^\mu\ , s - m_+m_-]\ . \tag{40}$$

In the c.m. frame λ and η_s take the simple form

$$\lambda^A = [\text{ch}\theta_1\,\text{ch}\zeta, 0, 0, \text{ch}\theta_1\,\text{sh}\zeta, \text{sh}\theta_1]\ ,$$
$$\eta_s^A = [\text{ch}\theta_s\ , 0, 0,\ 0\ , \text{sh}\theta_s]\ ,$$

such that x can be written as

$$x = \text{ch}\theta_s\,\text{ch}\theta_1\,\text{ch}\zeta - \text{sh}\theta_s\,\text{sh}\theta_1\ . \tag{41}$$

In the region $st\ [m_-^2, m_+^2]$, the parameter x is $x \geqslant 1$ implying

$$0 \leqslant \frac{x-1}{x+1} < 1$$

and convergence of the series for A and B. We can continue the amplitude into the whole s plane cut from $-\infty$ to m_-^2 and from m_+^2 to ∞ by performing the standard

Sommerfeld Watson transformation and rewriting *

$$A = -\frac{1}{N_1^2}\frac{4b(s)}{\kappa}\left[\frac{i}{2}\int_C \frac{dn}{\sin \pi n}\frac{n}{n-b(s)}(-\rho)^n\right]\frac{1}{x^2-1}, \quad \rho \equiv \frac{x-1}{x+1}, \tag{42}$$

where the contour C comes from $n = -i\infty$, intersects the real axis between -1 and $+1$, passes to the right of the pole at $n = b(s)$ and then runs up to $n = i\infty$. The phase of $(-\rho)$ has to be taken between $-\pi$ and π. For B we have to multiply the integrand by

$$(x^2 - 2nx + 2n^2 - 1)\frac{1}{x^2-1} . \tag{43}$$

In this form, the discontinuity across the right hand cut is readily evaluated. One just notices that above and below this cut, $b(s)$ is purely imaginary with the values

$$b(s_\pm) = \pm i\,\hat{b}(s) . \tag{44}$$

Therefore $\mathrm{ch}\theta_s$ and $\mathrm{sh}\theta_s$ are

$$\begin{aligned} \mathrm{ch}\theta_{s_\pm} &= \pm i\,\mathrm{sh}\hat{\theta}_s , \\ \mathrm{sh}\theta_{s_\pm} &= \pm i\,\mathrm{ch}\hat{\theta}_s , \end{aligned} \tag{45}$$

and also x is purely imaginary

$$x = \pm i\,(\mathrm{sh}\hat{\theta}_s\,\mathrm{ch}\theta_1\,\mathrm{ch}\zeta - \mathrm{ch}\hat{\theta}_s\,\mathrm{sh}\theta_1) . \tag{46}$$

As we consequence we find that $A(s_-)$ can be written in the same form as $A(s_+)$ except that the contour of integration passes to the left of the pole at $n = i\,\hat{b}(s)$. In the difference $A(s_+) - A(s_-)$ one can contract the contour to a circle around this pole and obtains

$$\mathrm{disc}\,A(s_+) = \frac{1}{N_1^2}\frac{4b(s)}{\kappa}\frac{\pi\,b(s)}{\sin \pi\,b(s)}(-\rho)^{b(s)}\frac{1}{x^2-1} . \tag{47}$$

* Notice that A can also be written in form of hypergeometric function

$$A = -\frac{1}{N_1^2}\frac{b(s)}{\kappa}\frac{4}{(1+x)^2}\frac{1}{1-b}F(2, 1-b, 2-b, \frac{x-1}{x+1}) .$$

The corresponding expression for B is slightly more complicated.

The formula for B contains just one more factor (43) evaluated at the pole $n = i\,b(s)$:

$$\operatorname{disc} B(s_+) = \frac{1}{N_1^2}\,\frac{4b(s)}{\kappa}\,\frac{\pi\,b(s)}{\sin\pi\,b(s)}\,(-\rho)^{b(s)}\,\frac{1}{(x^2-1)^2}\,(x^2-1-2b(x-b)\,) \quad (48)$$

With these results it is just a matter of kinematics to determine the structure functions of the scattering of an electron on the ground state $m_1 \equiv m$ of our infinite multiplet.

3. IDENTIFICATION OF THE STRUCTURE FUNCTIONS

If the forward Compton amplitude is expanded in the form

$$T_{\mu\nu} = -\,(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})\,T_1 + \frac{1}{m^2}(p_\mu - \frac{pq}{q^2}\,q_\mu)(p_\nu - \frac{pq}{q^2}\,q_\nu)\,T_2 \quad (49)$$

then the standard definition for the dimensionless structure functions is *

$$W_{1,2} \equiv -\frac{1}{2\pi}\operatorname{Im} T_{1,2} = \frac{i}{4\pi}\operatorname{disc} T_{1,2}\ . \quad (50)$$

We now show how to determine T_1, T_2 in terms of $\lambda^A\lambda^B\,T_{AB}$ and $T_A{}^A$.

Let us expand T_{AB} in the most general covariant basis:

$$T_{AB} = H_1\,\lambda_A\lambda_B + H_2\,\eta_A\eta_B + \tfrac{1}{2}\,H_3(\lambda_A\eta_B + \lambda_B\eta_A) + H_4\,q_{AB}\ . \quad (51)$$

Current conservation reduces the number of independent amplitudes to two. Contracting our current

$$J^\mu(p+q,p) \equiv -\,(m_+ - m)\,\Gamma^\mu + (2p+q)^\mu\,\Gamma^4\ , \quad (52)$$

with q^μ we find

$$q_\mu J^\mu\,(p+q,p) = -\,(m_+ - m_-)\,q_\mu\,\Gamma^\mu + q_\mu\,(2p+q)^\mu\,\Gamma^\mu \quad (53)$$

* With the cross sections sections for longitudinal and transverse photons given by

$$\sigma_L = \frac{4\pi^2\alpha}{m\left(\nu+\frac{q^2}{2m}\right)}\,W_L \equiv \frac{4\pi^2\alpha^2}{m\left(\nu+\frac{q^2}{2m}\right)}\left[\left(1-\frac{\nu^2}{q^2}\right)W_2 - W_1\right],\quad \sigma_T = \frac{4\pi^2\alpha}{m\left(\nu+\frac{q^2}{2m}\right)}\,W_1\ .$$

which can be written in five-vector language as

$$q_\mu J^\mu(p+q,p) = -Q^A \Gamma_A \tag{54}$$

with

$$Q^A \equiv [(m_+ - m_-)q^\mu, q(2p+q)] \ . \tag{55}$$

By comparison with λ^A and η^A we observe

$$Q^A = \kappa\left(\frac{1}{b(s)}\eta^A - \lambda^A\right) , \tag{56}$$

such that current conservation implies

$$b\lambda^A T_{AB} = \eta^A T_{AB} \ . \tag{57}$$

Inserting the decomposition (51) we find the two constraints on $A_i(x=\lambda\eta)$:

$$\begin{aligned} &H_1(x-b) + \tfrac{1}{2} H_3(1-xb) = H_4 b \ , \\ &H_2(1-xb) + \tfrac{1}{2} H_3(x-b) = -H_4 \ . \end{aligned} \tag{58}$$

On the other hand the expressions $\lambda^A\lambda^B T_{AB}$ and $T_A{}^A$ are the following combinations of H_i:

$$\begin{aligned} \lambda^A\lambda^B T_{AB} &= H_1 + H_2x^2 + H_3x + H_4 \ , \\ T_A{}^A &= H_1 + H_2 + H_3x + 5H_4 \ . \end{aligned} \tag{59}$$

We can invert equations (58) and (59) and calculate $H_1 \ldots H_4$ from $\lambda^A\lambda^B T_{AB}$ and $T_A{}^A$:

$$\begin{aligned} H_1 &= \frac{1}{3(x^2-1)^2}\{(3x^2b^2 - 8xb + b^2 + 4)\lambda^A\lambda^B T_{AB} + (x^2-1)T_A{}^A\} \ , \\ H_2 &= \frac{1}{6(x^2-1)^2}\{(6x^2 - 16xb + 8b^2 + 2)\lambda^A\lambda^B T_{AB} + 2(x^2-1)T_A{}^A\} \ , \\ H_3 &= -\frac{2}{3(x^2-1)^2}\{(-5bx + 4x(1+b^2) - 3b)\lambda^A\lambda^B T_{AB} + x(x^2-1)T_A{}^A\} \\ H_4 &= \frac{1}{3(x^2-1)}\{(b^2 - xb + 1)\lambda^A\lambda^B T_{AB} + (x^2-1)T_A{}^A\} \ . \end{aligned} \tag{60}$$

For the calculation of T_1 and T_2 in terms of $\lambda^A \lambda^B T_{AB}$ and $T_A{}^A$ we simply contract the amplitudes $T_{\mu\nu}$ and T_{AB} with two specific polarization vectors of photons, once transverse

$$\epsilon^\mu(1) = (0, 1, 0, 0)$$

and once longitudinal

$$\epsilon^\mu(0) = (\mathrm{p}, 0, 0, -\mathrm{q}_0)$$

in the c.m. frame where p is the momentum of the proton running along the z direction. From $T_{\mu\nu}$ one has:

$$\epsilon^\mu(1)\, \epsilon^\nu(1)\, T_{\mu\nu} = T_1 \equiv T_\mathrm{T} \ , \tag{61}$$

$$\begin{aligned} \epsilon^\mu(0)\, \epsilon^\nu(0)\, T_{\mu\nu} &= -q^2\left[\left(1 - \frac{\nu^2}{q^2}\right) T_2 - T_1\right] \\ &\equiv -q^2 T_\mathrm{L} \ . \end{aligned} \tag{62}$$

The same amplitudes are obtained from T_{AB} by contracting with the corresponding five-vectors of polarization:

$$\begin{aligned} \in^A(1) &= [(m_+ - m_-)\, \epsilon^\mu(1), \quad \epsilon(1)(2p+q)] = [0, m_+ - m_-, 0, 0, 0] \ , \\ \in^A(0) &= [(m_+ - m_-)\, \epsilon^\mu(0), \quad \epsilon(0)(2p+q)] \\ &= [(m_+ - m_-)p, 0, 0, -(m_+ - m_-)q_0, 2pW] \, . \end{aligned} \tag{63}$$

For the individual covariants

$$\lambda_A \lambda_B, \qquad \eta_A \eta_B, \qquad \lambda_A \eta_B, \qquad g_{AB}$$

we obtain

$$\begin{aligned} &\in^A(1)\in^B(1)\, \lambda_A \lambda_B = \in^A(1)\in^B(1)\, \eta_A \eta_B = \in^A(1)\in^B(1)\, \lambda_A \eta_B = 0 \ , \\ &\in^A(1)\in^B(1)\, g_{AB} = -(m_+ - m_-)^2 \ , \end{aligned} \tag{64}$$

such that

$$T_T = -(m_+ - m_-)^2 H_4 \ . \tag{65}$$

The longitudinal part is more complicated since all amplitudes H_i contribute. From

$$\begin{aligned} \epsilon^A(0)\,\lambda_A &= -2pW N_1^2 \ , \\ \epsilon^A(0)\,\eta_A &= -2pW\, i\hat{b}(s)\left[N_1^2 + \frac{s-m^2}{\kappa}\right] \ , \\ \epsilon^A(0)\epsilon_A(0) &= -(m_+ - m_-)^2 - 4p^2W^2 \ , \end{aligned} \tag{66}$$

we find directly

$$\begin{aligned} -q^2T_L = 4p^2W^2 \{ & N_1^4 H_1 - \hat{b}^2(s)\left[N_1^2 + \frac{s-m^2}{\kappa}\right]^2 H_2 \\ & - iN_1^2\,\hat{b}(s)\left[N_1^2 + \frac{s-m^2}{\kappa}\right] H_3 - H_4\} - (m_+ - m_-)^2 q^2 H_4 \ . \end{aligned} \tag{67}$$

Expressing T_L in terms of T_T and T_2, the last terms drops out and we obtain for T_2

$$\begin{aligned} T_2 = 4m^2 \{ & N_1^4 H_1 - \hat{b}^2\left[N_1^2 + \frac{s-m^2}{\kappa}\right]^2 H_2 \\ & - i N_1^2\,\hat{b}(s)\left[N_1^2 + \frac{s-m^2}{\kappa}\right] H_3 - H_4\} \ . \end{aligned} \tag{68}$$

For large energy, $\hat{b}(s) \to \kappa/s$ such that (68) simplifies to

$$T_2 \xrightarrow[s\to\infty]{} 4m^2 \{N_1^4 H_1 - H_{2+4} - i N_1^2 H_3\} \ . \tag{69}$$

The same results hold for the structure functions W_T and W_2 if one replaces H_i on the right-hand side by

$$-\frac{1}{4\pi}\,\mathrm{disc}\, H_i$$

which are determined from (60), (31), (37), (47) and (48).

4. THE STRUCTURE FUNCTIONS IN THE SCALING LIMIT

Let us now consider our result in the scaling limit (S.L.)

$$\nu \to \infty, \qquad q^2 \to \infty, \qquad \xi = -\frac{q^2}{2m\nu} = \text{fixed}$$

Then x has the form:

$$x = i\,\frac{b(s)}{\kappa}\,2m\nu\,\mathrm{sh}\theta_1\left[-\frac{q^2-\dfrac{\kappa}{\mathrm{sh}\theta_1}}{2m\nu}-\frac{N_1^2}{\mathrm{sh}\theta_1}\right] \underset{\mathrm{S.L.}}{\longrightarrow} i\,\frac{\mathrm{sh}\theta_1}{1-\xi}\left(\xi-\frac{N_1^2}{\mathrm{sh}\theta_1}\right), \tag{70}$$

i.e. it stays finite! From (47) and (48) we see that

$$\mathrm{disc}\,A(s_+)\underset{\mathrm{S.L.}}{\longrightarrow}\mathrm{disc}\,B(s_+)\underset{\mathrm{S.L.}}{\longrightarrow}\frac{i}{N_1^2}\frac{4}{s}\frac{1}{x^2-1}\,. \tag{71}$$

Hence

$$\mathrm{disc}\,\lambda^A\lambda^B\,T_{AB}\underset{\mathrm{S.L.}}{\longrightarrow}\frac{i}{N_1^2}\frac{4}{s}\frac{1}{x^2-1}\;, \tag{72}$$

$$\mathrm{disc}\,T_A{}^A\underset{\mathrm{S.L.}}{\longrightarrow}0\;. \tag{73}$$

Eqs. (60) tell us:

$$\mathrm{disc}\begin{pmatrix}H_1\\ H_2\\ H_3\\ H_4\end{pmatrix}\underset{\mathrm{S.L.}}{\longrightarrow}\begin{pmatrix}\dfrac{4}{3(x^2-1)^2}\\[2ex] \dfrac{x^2+\frac{1}{3}}{(x^2-1)^2}\\[2ex] -\dfrac{8x}{3(x^2-1)^2}\\[2ex] \dfrac{1}{3(x^2-1)}\end{pmatrix}\mathrm{disc}\,\lambda^A\lambda^B T_{AB}\;. \tag{74}$$

Hence that we obtain the results:

$$W_{\mathrm{T}}\underset{\mathrm{S.L.}}{\longrightarrow}\frac{1}{s}\,\frac{1}{\pi N_1^2}\,(m_+-m_-)^2\,\frac{1}{3(x^2-1)^2}\;, \tag{75}$$

$$W_2 \underset{\text{S.L.}}{\rightarrow} -\frac{4m^2}{s\pi}\{N_1^2\tfrac{4}{3}-N_1^{-2}\tfrac{4}{3}x^2+i\tfrac{8}{3}x\}\frac{1}{(x^2-1)^3}\,. \tag{76}$$

The standard scaling functions are defined as

$$\begin{aligned} F_T &\equiv \lim_{\text{S.L.}} W_T\,, \\ F_2 &\equiv \lim_{\text{S.L.}} \frac{\nu}{m} W_2\,. \end{aligned} \tag{77}$$

According to (75) and (76) they have the properties

$$\begin{aligned} F_T &= 0\,, \\ F_2 &= \frac{8}{3\pi}\frac{1}{1-\xi}\{N_1^2 - N_1^{-2}x^2 + 2ix\}\frac{1}{(1-x^2)^3}\,. \end{aligned} \tag{78}$$

The vanishing of the transversal structure function F_T is characteristic for boson targets.

In order to display the behaviour of T_2 as a function of ξ we rewrite $(1-x^2)$ as

$$\begin{aligned} &(1-\xi)^{-2}\left[(1-\xi)^2+\mathrm{sh}^2\theta_1\left(\xi-\frac{N_1^2}{\mathrm{sh}\theta_1}\right)^2\right] \\ &=(1-\xi)^2\,[\mathrm{ch}^2\theta_1\,\xi^2-2\xi\,[1+\mathrm{sh}\theta_1\,N_1^2]+1+N_1^4]\equiv(1-\xi)^{-2}D(\xi)\,, \end{aligned} \tag{79}$$

and the factor in curly brackets in the form

$$\begin{aligned} &(1-\xi)^{-2}\left\{N_1^2(1-\xi)^2+N_1^{-2}\,\mathrm{sh}^2\theta_1\left(\xi-\frac{N_1^2}{\mathrm{sh}\theta_1}\right)^2+2(1-\xi)\mathrm{sh}\theta_1\left(\xi-\frac{N_1^2}{\mathrm{sh}\theta_1}\right)\right\} \\ &=(1-\xi)^{-2}N_1^{-2}(\mathrm{sh}\theta_1-N_1^2)^2\,\xi^2 \equiv (1-\xi)^{-2}\,\xi^2\,C\,. \end{aligned} \tag{80}$$

Then F_2 is simply

$$F_2(\xi) = \frac{8}{3\pi}\,C\,\xi^2(1-\xi)^3\,D^{-3}(\xi) \tag{81}$$

Since $D^{-3}(1)$ is finite, this result has the threshold behaviour $\alpha\,(1-\xi)^3$ at $\xi = 1$

showing the Drell-Yan relation with the dipole shape of the elastic form factor [13]:

$$G_E = \frac{4}{(1+x)^2} = \frac{1}{\left(1 - \frac{\text{ch}^2\theta_1}{4m^2} q^2\right)^2} . \tag{82}$$

At $\xi = 0$, $F_2(\xi)$ vanishes quadratically in ξ. The maximum of F_2 is found pretty close to the minimum of D. By writing D in the form

$$D = \text{ch}^2\theta(\xi - \xi_0)^2 + \left(\frac{m_+ - m_-}{2m}\right)^2 \tag{83}$$

with

$$\xi_0 = \tfrac{1}{2}\left(1 + \frac{m_+ m_-}{m^2}\right) \tag{84}$$

we see that $F_2(\xi)$ peaks at the right or left half of the interval according to whether m_+ and m_- have equal or opposite signs.

This property of the solution can easily be understood on physical grounds. Let us remember that our equation is a generalization of the Schrödinger equation of two particles of masses $m_p = m_+ - m_-$ and $m_e = m_+ + m_-$ with the photon coupling *only* to the constituent of mass m_e. The average momentum of the constituent m_e will be roughly a fraction m_e/m_+ of the total momentum p. In the parton picture, a constituent of momentum xp contributes to $F_2(\xi)$ in the form

$$F_2(\xi) = \frac{\nu}{m}\,\delta\left(\nu - \frac{q^2}{2mx}\right) = \xi\,\delta(\xi - x) . \tag{85}$$

If $f(x)$ denotes the probability distribution for finding the constituent at momentum xp then

$$F_2(\xi) = \xi f(\xi) \tag{86}$$

our "parton" of mass m_e will have a distribution $f(x)$ peaked roughly at m_e/m_+. This is about the same result as that we have found in (84). In fact

$$\xi_0 = \frac{m^2 + m_e^2 - m_p^2}{2m^2} \tag{87}$$

becomes exactly equal to m_e/m_+ for binding energies small compared to m_e and m_p.

The resulting curves are displayed in fig. 2 for several values of m_+, m_-. A typical set of parameters

$$m_+ = \tfrac{1}{2}\sqrt{s} \approx 1.12, m_- = 0 \qquad \text{or } m_e = m_p = 0.56 \tag{88}$$

gives

$$\begin{aligned} \kappa &= \tfrac{1}{2}, \ \mathrm{ch}\theta_1 = \sqrt{5}, \ \mathrm{sh}\theta_1 = 2, \ \left(\text{i.e.} G_E(q^2) = \left(1 - \frac{t}{0.71}\right)^{-2}\right) , \\ N_1^2 &= \tfrac{3}{4} , \\ C &= \tfrac{25}{12}, \ D = 5(\xi^2 - \xi + \tfrac{5}{16}) , \end{aligned} \tag{89}$$

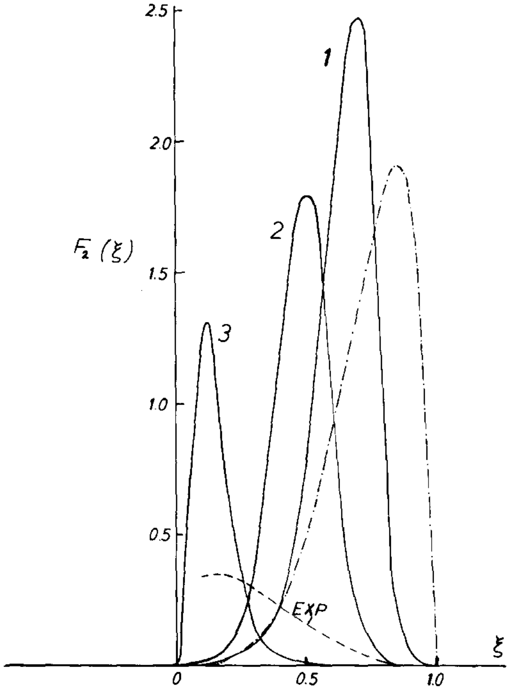

Fig. 2. The structure function $F_2(\xi)$ of our model is displayed for three typical values of the mass parameters: 1) $m_e = 0.76$, $m_p = 0.36$ or $m_+ = 1.12$, $m_- = +0.4$; 2) $m_e = m_p = 0.56$ or $m_+ = 1.12$, $m_- = 0$; 3) $m_e = 0.16$, $m_p = 0.96$ or $m_+ = 1.12$, $m_- = -0.8$. Our curves have the threshold behaviour $(1-\xi)^3$ for $\xi \approx 1$. We see that for decreasing mass m_e of the charged constituent the peak moves left and the area decreases. In the limiting case of the H atom $m_e/m_+ \approx 10^{-3}$, $F_2(\xi)$ shows a high and narrow peak at small $\xi \approx m_e/m_+$. For comparison we show the structure function of the Majorana equation of ref. [6]. (– . – . –) and the experimental $F_2(\xi)$ of the proton (– – – –).

such that

$$F_2 = \frac{2}{45\pi}\,\xi^2(1-\xi)^3\,\frac{1}{(\xi^2-\xi+\frac{5}{16})^3} \quad . \tag{90}$$

It has the maximum almost at $\xi = \frac{1}{2}$ of high $F_2(\frac{1}{2}) \approx 1.8$. If one takes a somewhat smaller m_e by choosing

$$m_+ = 1.12, m_- = -0.8 \quad \text{or } m_e = 0.16, m_p = 0.96 \ ,$$

$$\kappa = 0.303$$

$$\mathrm{ch}\theta_1 = 6.34,\ \ \mathrm{sh}\theta_1 = 6.27,\ \ N_1^2 = 0.174 \ ,$$

one obtains

$$C = 214 \ ,$$

$$D = 40.2\,\xi^2 - 4.18\,\xi + 1.03 \ ,$$

which peaks around $\xi \approx 0.1$ with a value $F_2 \approx 1.26$.

If we compare these results with the experimental structure functions of the proton we see that

(i) The threshold behaviour at $\xi \approx 1$ agrees with experiment since the form factors have the correct falloff in t.

(ii) The longitudinal to transverse ratio is opposite to what is observed experimentally.

(iii) The zero at small ξ is wrong. Experimentally $F_2(\xi)$ seems to become constant ≈ 0.25 for small ξ.

(iv) The size of F_2 is too large by a factor at about 3.

The failure to reproduce the longitudinal to transverse ratio is well known to come from the integer spins contained in our model. This defect will certainly be absent in the realistic model of currents.

The zero at $\xi = 0$ is also observed in the structure functions of the other three infinite component wave equations. In the parton model there is always a zero if the probability $P(N)$ of having N constituents falls faster than N^{-2}. Therefore in order to explain the experimental fact $F_2(0) \approx 0.25$, parton people are forced to invoke the presence of a cloud of quark-antiquark pairs inside the nucleon following a distribution $P_N \approx C/N^2$. Our wave equation obviously remembers that it is just a generalization of a non-relativistic two body problem.

What is the way to correct these defects of the model? When particles described by a wave equation based on a representation space of O(3, 1) turned out to be hollow, we had to introduce mixtures of such representations to construct bulk objects.

This was done by going to a larger group, O(4, 2), which accommodates radial wave functions in a natural manner analogous to the H atom. The resulting particle turns out to be still a relativistic bound state of two constituents. Experimentally there are constituents of very low mass present in a baryon. Therefore the straight-forward generalization of our equation will consist in mixing solutions for many low mass constituents. There should again be a larger representation space taking care of such mixtures in a natural manner by introducing a new quantum number describing the constituent distribution on the mass axis. Analogously to the radial quantum number characterizing the radial mixtures this quantum number could be called "massal". One should study the structure functions in wave equations written down on such larger representation spaces containing such a massal quantum number. This type of procedure would presumably also provide the cure to the exaggeration of size of F_2. We can see from our fig. 2 that by decreasing m_e/m_+ the peak not only moves left but the area under it becomes smaller by the same amount. The reason is that, in the parton approximation, $f(\xi) = F_2(\xi)/\xi$ is a normalized probability distribution for finding the parton of mass m_e in a state of momentum ξp. Clearly, a peak in $F_2(\xi)$ at lower ξ_0 goes along with an area proportional to ξ_0. Thus, composing the proton predominantly out of small mass partons will necessarily depress the $\int_0^1 F_2(\xi)\mathrm{d}\xi$ which is experimentally ≈ 0.16.

4. DEEP INELASTIC SCATTERING OF ELECTRONS WITH HYDROGEN ATOMS

Our result allows us to compute the amplitude for the process

$$e + H \to e + e^- + p$$

with the virtual photon being absorbed by the electron in the H atom. Since the form factors fall off very fast in q^2, on a momentum transfer scale of $q^2 \approx \frac{1}{2}\alpha^2 m_e m_p$, the amplitudes will be very small in most of the deep inelastic region and therefore not be measurable. The structure function F_2 is obtained by inserting *

$$m_+ = m_p + m_e\,, \quad m_- = -m_p + m_e\,, \quad N_1^2 \approx \frac{1}{\alpha}\,,$$
$$\mathrm{ch}\theta_1 \approx \frac{m_+}{\alpha m_e}\,, \quad \mathrm{sh}\theta_1 \approx \frac{m_+}{\alpha m_e}\,, \tag{91}$$

into our formula (81):

* The $\approx$ sign denotes an approximation to lowest order in α^2. The result contains no assumption on the smallness of m_e/m_p.

$$F_2(\xi) = \frac{8}{3\pi}\frac{1}{\alpha}\left(\frac{m_p}{m_e}\right)^2 \xi^2(1-\xi)^3 \left[\left(\frac{m_+}{\alpha m_e}\right)^2\left(\xi - \frac{m_e}{m_+}\right)^2 + \left(\frac{m_p}{m_+}\right)^2\right]^{-3} . \quad (92)$$

We see that F_2 is extremely sharply peaked at very small values of $\xi \approx m_e/m_+ \approx 10^{-3}$. There it takes a considerable size

$$F_2\left(\frac{m_e}{m_+}\right) = \frac{8}{3\pi}\frac{1}{\alpha}\frac{m_+}{m_p} . \quad (93)$$

However, the width of this peak is only

$$\Delta\xi \approx \tfrac{1}{2}\frac{\alpha m_e m_p}{m_+^2} \approx 10^{-5} ,$$

which makes it hard to resolve it experimentally.

Also this result can easily be understood in terms of the parton picture The wave function of the ground state electron in the atom is given by

$$\psi_{100}(p^*) = \frac{\sqrt{8}}{\pi}\frac{1}{\left[\left(\frac{p^*}{\alpha\mu}\right)^2 + 1\right]^2} , \quad \mu \equiv \frac{m_e m_p}{m_+} , \quad (94)$$

where p^* is the momentum of the electron in the c.m. frame. The normalized probability of finding the electron in a state of longitudinal momentum p_L^* is obtained by integrating $|\psi|^2$ over all transverse momenta p_L^*:

$$\begin{aligned} P(p_L^*) &= \int d^2p_T^* \; |\psi_{100}(p^*)|^2/(\alpha\mu)^2 \\ &= \frac{8}{3\pi}\frac{1}{\left[\left(\frac{p_L^*}{\alpha\mu}\right)^2 + 1\right]^3} \end{aligned} \quad (95)$$

Consider the atom at a very high momentum. Then the longitudinal electron momentum is the following fraction of the total momentum p:

$$p_e \approx \frac{m_e}{m_+}\left(1 + \frac{p_L^*}{m_e}\right)p \equiv xp . \quad (96)$$

Therefore

$$p_L^* = m_+\left(x - \frac{m_e}{m_+}\right) . \quad (97)$$

Inserting this into $P(p_L^*)$ and multiplying by a factor $m_+^2/m_e m_p \alpha$ we obtain the normalized probability distribution for the different x values

$$f(x) = \frac{m_+^2}{\alpha m_e m_p} \frac{8}{3\pi} \left[\left(\frac{m_+^2}{\alpha m_e m_p}\right)^2 \left(x - \frac{m_e}{m_+}\right)^2 + 1\right]^{-3} , \tag{98}$$

leading to an $F_2(\xi) = \xi f(\xi)$ of

$$F_2(\xi) = \frac{8}{3\pi} \frac{1}{\alpha} \frac{m_+^2}{m_e m_p} \xi \left(\frac{m_p}{m_+}\right)^6 \left[\left(\frac{m_+}{\alpha m_e}\right)^2 \left(\xi - \frac{m_e}{m_+}\right)^2 + \left(\frac{m_p}{m_+}\right)^2\right]^{-3} . \tag{99}$$

This F_2 is almost exactly the same as the one obtained before. It has an extremely sharp peak centered around m_e/m_+ with a peak value

$$\frac{8}{3\pi} \frac{1}{\alpha} \frac{m_+}{m_p} .$$

It differs from our exact formula only where it is extremely small.

Consider the contribution to the scattering caused by the coupling of the photon to the proton. It is obvious that in the H atom this coupling will be associated with a distribution $f^p(x)$ and a structure function $F_2^p(x)$ which are obtained from (98), (99) by substituting $m_e \leftrightarrow m_p$. This structure-function has a sharp peak at

$$\xi \approx \frac{m_p}{m_+} = 1 - \frac{m_e}{m_+} \approx 1$$

due to the fact that the proton is a parton carrying almost the whole total momentum. The width of the peak is the same as for the electron since the wave function has the same spread in momentum space, The peak height is much larger, though, by a factor of m_p/m_e since $F(\xi)/\xi = f(\xi)$ is normalized to one. In fact, this peak is observable in electron scattering on a hydrogen target and is usually referred to as the elastic peak †.

5. CONCLUSION

The model we have studied seems to provide an interesting link between the parton interpretation of the structure functions and the models based on resonances exchanged in the s–channel. While the explicit form of our amplitude has the characteristic s–channel resonance form, the parameters in it can be interpreted in terms of constituents. Defects of the model and their cures are strongly related to the corresponding problems in the parton model. With respect to the models based on wave equations of the Majorana type, we have been able to improve on the threshold be-

† If one corrects for the proton form factor. In order to avoid confusion we should remind the reader that the true experimental elastic peak does not survive the scaling limit due to the form factor of the proton. In the Schrödinger theory of the H atom the proton is treated as a point-like constituent with the observed electromagnetic structure coming entirely from its orbital motion, so that there the elastic peak is finite in the scaling limit.

haviour for $\xi \approx 1$ since our equation gives a proper description of the fall off of resonance form factors or of the momentum distribution of constituents. The rapid vanishing of F_2 in our model for small ξ and its large size have been shown to be related to the lack of the wave equation to provide for a mixture of constituents of many different low masses. While parton models have to invoke core clouds of quark anti-quark pairs, the approach via wave equations needs an additional quantum number describing the parton mass content in every state. In the resonance picture such a quantum number amounts to introducing a richer spectrum of particles with faster growing multiplicities than those contained in our model.

We hope that the construction of a new wave equation on the representation space of a larger group which accommodates such a quantum number in the desired form will be feasible. Such an equation would be a considerable improvement over presently known equations which all describe at heart a relativistic two body problem.

The author wishes to thank Richard Brandt, M. Breidenbach and Murray Gell-Mann for many enlightening discussions on scale invariance

APPENDIX A: THE REPRESENTATION SPACE

The representation space for our infinite component wave equation is build up by applying an equal number of spin $\frac{1}{2}$ creation operators $a_r^\dagger$ and $b_r^\dagger$ to the vacuum state $|0\rangle$:

$$|pq\bar{p}\bar{q}\rangle \equiv [p!q!\bar{p}!\bar{q}!]^{-\frac{1}{2}}\; a_1^{+p} a_2^{+q} b_1^{+\bar{p}} b_2^{+\bar{q}} |0\rangle\,, \quad p+q=\bar{p}+\bar{q}\;. \tag{A.1}$$

On these states the operators

$$\begin{aligned}
L_{ij} &= \tfrac{1}{2}(a^\dagger\sigma_k a + b^\dagger\sigma_k b) \equiv L_k\;,\\
L_{i4} &= -\tfrac{1}{2}(a^\dagger\sigma_i a - b^\dagger\sigma_i b)\;,\\
L_{i5} &= -\tfrac{1}{2}(a^\dagger\sigma_i C b^\dagger - aC\sigma_i b)\;, \quad C\equiv i\sigma_2\;,\\
L_{45} &= \tfrac{1}{2i}(a^\dagger C b^\dagger - aCb)\;,\\
L_{i6} &= \tfrac{1}{2i}(a^\dagger\sigma_i C b^\dagger + aC\sigma_i b)\;,\\
L_{46} &= \tfrac{1}{2}(a^\dagger C b^\dagger + aCb)\;,\\
L_{56} &= \tfrac{1}{2}(a^\dagger a + b^\dagger b) + 1\;,
\end{aligned} \tag{A.2}$$

form an irreducible representation of the group O(4, 2), the group of orthogonal transformations in a space with the metric

$$g = \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} . \tag{A.3}$$

Their commutation rules are

$$[L_{ab} L_{ac}] = -ig_{aa} L_{bc} \; . \tag{A.4}$$

The operators L_3, L_{34}, and L_{56} are diagonal with eigenvalues $\frac{1}{2}[p + \bar{p} - q - \bar{q}]$, $\frac{1}{2}[q - p - \bar{q} + p]$ and $\frac{1}{2}[p + q + \bar{p} + \bar{q}] + 1$, respectively. In analogy to the H atom, one can also introduce the so called hyperbolic quantum numbers n_1, n_2, m defined by L_3, L_{34} and L_{56} having the eigenvalues m, $n_1 - n_2$ and $n_1 + n_2 + |m| + 1$, i.e.

$$|n_1 n_2 m\rangle \equiv \begin{cases} |n_2 + m, n_1, n_1 + m, n_2\rangle & m \geqslant 0 \\ |n_2, n_1 - m, n_1, n_2 - m\rangle & m \leqslant 0 \; . \end{cases} \tag{A.5}$$

Notice that in spite of this analogy there is no unitary transformation connecting these states $|n_1 n_2 m\rangle$ with the hyperbolic bound states of the H atom. The reason is that our states form a complete Hilbert space while the hyperbolic bound states are just part of the Hilbert space of the H atom. In fact, our wave equation accomodates both, the *bound and the continuum states* on the same Hilbert space $|n_1 n_2 m\rangle$ by employing the non-unitary tilt operation $e^{-i\theta_n L_{45}}$. The non-unitarity comes from the dependence of the tilting angle on n!

The states A are the solutions of the tilted wave equation with a mass given by $b(m^2) = n = \frac{1}{2}(p + q + \bar{p} + \bar{q}) + 1 = 1, 2, 3, \ldots$ Therefore we can identify

$$\tilde{u}_{pq\bar{p}\bar{q}}(0) \equiv |pq\bar{p}\bar{q}\rangle \; , \; p + q = \bar{p} + \bar{q} \tag{A.6}$$

In the text we have simply written $\tilde{u}_n(0)$ leaving just the index characterizing the mass and omitting the degeneracy labels.

APPENDIX B: CALCULATION OF A AND B

In sect. 2 we have obtained simple expressions for the invariant amplitudes $\lambda^A \lambda^B T_{AB}$ and $T_A{}^A$. By employing the basis decribed in detail in appendix A, eq. (31) can be written as

$$A \equiv \lambda'^A \lambda^B T_{AB} = -\frac{1}{N_1^2} \frac{b(s)}{\kappa} \sum_{pq\bar{p}\bar{q}} \frac{1}{n - b(s)} \langle 0|G^\dagger(\zeta)|pq\bar{p}\bar{q}|G(\zeta)|0\rangle \quad , \tag{B.1}$$

where it is understood that $p + q \equiv \bar{p} + \bar{q}$ and $n = \frac{1}{2}(p + q + \bar{p} + \bar{q}) + 1$. Similarly we can rewrite B, from (37), in the form

$$B = \frac{1}{2N_1^2}\frac{b(s)}{\kappa}\sum_{pq\bar{p}\bar{q}}\frac{1}{n - b(s)}\,[\langle 1001|G^\dagger(\zeta')|pq\bar{p}\bar{q}\rangle\langle pq\bar{p}\bar{q}|G(\zeta)|1001\rangle$$

$$+\langle 0110|G^\dagger(\zeta')|pq\bar{p}\bar{q}\rangle\langle pq\bar{p}\bar{q}|G(\zeta)|0110\rangle \tag{B.2}$$

$$+\langle 1010|G^\dagger(\zeta')|pq\bar{p}\bar{q}\rangle\langle pq\bar{p}\bar{q}|G(\zeta)|1010\rangle$$

$$+\langle 0101|G^\dagger(\zeta')|pq\bar{p}\bar{q}\rangle\langle pq\bar{p}\bar{q}|G(\zeta)|0101\rangle]\ .$$

For forward scattering, $\zeta' = \zeta$ and we can choose both protons to run in the Z direction. Then G cannot change the quantum number $m = (p + \bar{p} - q - \bar{q})/2$ and for fixed n, the sum in (B.1) extends just over states of the form $|n_2\, n_1\, n_1\, n_1\rangle$ with $n_1 = 0, 1, 2, \dots n - 1$ and $n_2 = n - 1 - n_1$. In B we see from rotational invariance that the first two and the second two terms are equal among each other. In the first term, $m = 0$ and the intermediate states are the same as before. In the third term all intermediate states have to be $m = 1$. They can be written in the form $|n_2 + 1\ n_1\ n_1 + 1\ n_2\rangle$ with $n_2 = n - 2 - n_1$. Thus we are left with the evaluation of the following matrix elements:

$$\sum_{\substack{n_1 = 0,\dots,n-1\\ n_2 = n-1-n_1}} |\langle 0|G(\zeta)|n_2\, n_1\, n_1\, n_2\rangle|^2 \tag{B.3}$$

$$\sum_{\substack{n_1 = 0,\dots,n-1\\ n_2 = n-1-n_1}} |\langle 1001|G(\zeta)|n_2\, n_1\, n_1\, n_2\rangle|^2 \tag{B.4}$$

$$\sum_{\substack{n_1 = 0,\dots,n-2\\ n_2 = n-2-n_1}} |\langle 1010|G(\zeta)|n_2 + 1, n_1\, n_1 + 1, n_2\rangle|^2 \tag{B.5}$$

Thanks to the simplicity of the representation space, these matrix elements of finite rotations in O(4, 2) can all be expressed in terms of the rotation matrices of O(2, 1).

For this one first simplifies

$$G(\zeta) = \mathrm{e}^{i\theta_{\zeta} L_{45}}\,\mathrm{e}^{-i\xi L_{35}}\,\mathrm{e}^{-i\theta_1 L_{45}} \tag{B.6}$$

by bringing it to Euler angle form

$$G(\zeta) = e^{-i\alpha L_{34}} e^{-i\beta L_{45}} e^{-i\alpha L_{34}} . \tag{B.7}$$

The relation between the angles α, β, γ and θ_s, ζ, θ, is obtained by using the fact that L_{45}, $-L_{35}$ and L_{34} commute just like $\frac{1}{2}i\sigma_1$, $\frac{1}{2}i\sigma_2$, and $\frac{1}{2}\sigma_3$ and one can do the calculation by using these specific two by two matrices. In this manner a direct comparison of (B.6) and (B.7) yields

$$\begin{aligned}
\cos\tfrac{1}{2}(\alpha+\gamma) &= \mathrm{ch}\,\tfrac{1}{2}(\theta_s-\theta_1)\ \mathrm{ch}\,\tfrac{1}{2}\zeta/\mathrm{ch}\,\tfrac{1}{2}\beta\ ,\\
\sin\tfrac{1}{2}(\alpha+\gamma) &= -\,\mathrm{sh}\,\tfrac{1}{2}(\theta_s+\theta_1)\,\mathrm{sh}\,\tfrac{1}{2}\zeta/\mathrm{ch}\,\tfrac{1}{2}\beta\ ,\\
\cos\tfrac{1}{2}(\alpha-\gamma) &= -\,\mathrm{sh}\,\tfrac{1}{2}(\theta_s-\theta_1)\,\mathrm{ch}\,\tfrac{1}{2}\zeta/\mathrm{sh}\,\tfrac{1}{2}\beta\ ,\\
\sin\tfrac{1}{2}(\alpha-\gamma) &= -\,\mathrm{ch}\,\tfrac{1}{2}(\theta_s+\theta_1)\,\mathrm{sh}\,\tfrac{1}{2}\zeta/\mathrm{sh}\,\tfrac{1}{2}\beta\ ,
\end{aligned} \tag{B.8}$$

which can be solved for α, β, γ as

$$\begin{aligned}
&\sin\alpha = -\,\mathrm{ch}\theta_1\,\mathrm{sh}\zeta/\mathrm{sh}\beta, &&\cos\alpha = [\mathrm{ch}\theta_s\,\mathrm{sh}\theta_1 - \mathrm{sh}\theta_s\,\mathrm{ch}\theta_1\,\mathrm{ch}\zeta]/\mathrm{sh}\beta\ ,\\
&\sin\gamma = \mathrm{ch}\theta_s\,\mathrm{sh}\zeta/\mathrm{sh}\beta, &&\cos\gamma = [-\,\mathrm{sh}\theta_s\,\mathrm{ch}\theta_1 + \mathrm{ch}\theta_s\,\mathrm{sh}\theta_1\,\mathrm{ch}\zeta]/\mathrm{sh}\beta\ ,
\end{aligned} \tag{B.9}$$

$$\begin{Bmatrix}\mathrm{sh}\frac{1}{2}\beta\\ \\ \mathrm{ch}\frac{1}{2}\beta\end{Bmatrix} = [\tfrac{1}{2}(\mathrm{ch}\theta_s\,\mathrm{ch}\theta\,\mathrm{ch}\zeta - \mathrm{sh}\theta_s\,\mathrm{sh}\theta \mp 1)]^{\frac{1}{2}}\ .$$

Using our scalar product $x \equiv \lambda^A\eta_A$ of eq. (41), we see

$$\begin{aligned}
\mathrm{sh}^2\,\tfrac{1}{2}\beta &= \tfrac{1}{2}(x-1)\ ,\\
\mathrm{ch}^2\,\tfrac{1}{2}\beta &= \tfrac{1}{2}(x+1)\ .
\end{aligned} \tag{B.10}$$

In the matrix elements (B.3) - (B.5) the rotations by $e^{-i\alpha L_{34}}$ and $e^{-i\gamma L_{34}}$ can be dropped since L_{34} is diagonal. Only $e^{-i\beta L_{45}}$ remains. The method to calculate this is the following. The basis states of fixed m support the representations of two commuting O(2, 1) groups which we denote by M_i and N_i:

$$\begin{aligned}
&M_+ \equiv M_1 + iM_2 \equiv -a_2^+b_1^+\ , \quad M_- \equiv M_1 - iM_2 \equiv -a_2\,b_1,\quad M_3 \equiv \tfrac{1}{2}(a_2^+a_2 + b_1^+b_1 + 1)\ ,\\
&N_+ \equiv N_1 + iN_2 \equiv a_+^+b_2^+\ , \quad N_- \equiv N_1 - iN_2 \equiv a_1b_2\ , \quad N_3 \equiv \tfrac{1}{2}(a_1^+a_1 + b_2^+b_2 + 1)\ .
\end{aligned} \tag{B.11}$$

The eigenvalues of M_3 and N_3 are $\frac{1}{2}(q+\bar{p}+1)$ and $\frac{1}{2}(p+\bar{q}+1)$, respectively. Applying raising and lowering operators $M_\pm$ and $N_\pm$, one preserves the differences $|p - \bar{q}|$ and $|p - \bar{q}|$ respectively. The corresponding lowest eigenvalues which completely characterize the representation are therefore $\frac{1}{2}(|q - \bar{p}| + 1)$ and $\frac{1}{2}(|p - \bar{q}| + 1)$. The generator L_{45} can be written as the sum $M_2 + N_2$. Since M_2 and N_2 commute,

$$e^{-i\beta L_{45}} \equiv e^{-i\beta M_2} e^{-i\beta N_2} . \tag{B.12}$$

If we match this product between two arbitrary states,

$$\langle p'q'\bar{p}'\bar{q}'| \, e^{-i\beta M_2} e^{-i\beta N_2} |pq\bar{p}\bar{q}\rangle \tag{B.13}$$

only one intermediate state can contribute: $|p'q\bar{p}\bar{q}'|$. The reason is that M_2 cannot change p and $\bar{q}$ while N_2 leaves q and $\bar{p}$ invariant. This brings (B.13) to the form:

$$\langle p'q'\bar{p}'\bar{q}'| \, e^{-i\beta M_2} |p'q\bar{p}\bar{q}'\rangle\langle p'q\bar{p}\bar{q}'| \, e^{-i\beta N_2} |pq\bar{p}\bar{q}\rangle , \tag{B.14}$$

which can directly be written as a product of representation functions V^k_{mn} of the representations D^k_+ of O(2, 1):

$$V^{\frac{1}{2}(|q - \bar{p}| + 1)}_{\frac{1}{2}(q' + \bar{p}'),\, \frac{1}{2}(q + \bar{p} + 1)}(\beta) \; V^{\frac{1}{2}(|p - \bar{q}| + 1)}_{\frac{1}{2}(p' + \bar{q}' + 1),\, \frac{1}{2}(p + \bar{q} + 1)}(-\beta) , \tag{B.15}$$

where V^k_{mn} are given for $m \geqslant n$ by

$$V^k_{mn} = \theta^k_{mn} \frac{\mathrm{sh}^{m-n}\frac{1}{2}\beta}{\mathrm{ch}^{m+n}\frac{1}{2}\beta} F_{2,1}(k - n, 1 - n - k, 1 + m - n, -\mathrm{sh}^2\tfrac{1}{2}\beta) ,$$
$$\theta^k_{mn} \equiv \frac{1}{(m-n)!} \sqrt{\frac{(m-k)!(m+k-1)!}{(n-k)!(n+k-1)!}} . \tag{B.16}$$

For $m < n$ we have to use $(-)^{m-n}(V^{\mathrm{T}})_{mn}$. In evaluating (B.3)–(B.5) we run only into the particular matrix elements:

$$V^{\frac{1}{2}}_{n_1 + \frac{1}{2}, \frac{1}{2}} = \frac{\mathrm{sh}^{n_1}\frac{1}{2}\beta}{\mathrm{ch}^{n_1+1}\frac{1}{2}\beta} ,$$
$$V^{1}_{n_1 + 1, 1} = \sqrt{n_1 + 1} \, \frac{\mathrm{sh}^{n_1}\frac{1}{2}\beta}{\mathrm{ch}^{n_1+2}\frac{1}{2}\beta} , \tag{B.17}$$
$$V^{\frac{1}{2}}_{n_2 + \frac{1}{2}, \frac{3}{2}} = n_2 \frac{\mathrm{sh}^{n_2-1}\frac{1}{2}\beta}{\mathrm{ch}^{n_2+2}\frac{1}{2}\beta} \left(1 - \frac{1}{n_2}\mathrm{sh}^2\tfrac{1}{2}\beta\right) , \quad n_2 \geqslant 1 ,$$

Performing the sums over n, at fixed n we obtain

$$\sum_{\substack{n_1=0,\dots,n-1\\ n_2=n-1-n_1}} |\langle 0|G(\zeta)|n_2\, n_1\, n_1\, n_2\rangle|^2 = n\, \mathrm{th}^{2n-2}\tfrac{1}{2}\beta\, \mathrm{ch}^{-4}\tfrac{1}{2}\beta$$

$$= n\left(\frac{x-1}{x+1}\right)^{n-1} \frac{4}{(x+1)^2}\,, \qquad \text{(B.18)}$$

$$\sum_{\substack{n_1=0,\dots,n-1\\ n_2=n-1-n_1}} |\langle 0|G(\zeta)|n_2\, n_1\, n_1\, n_2\rangle|^2$$

$$= n\, \mathrm{th}^{2n-4}\tfrac{1}{2}\beta\, \mathrm{ch}^{-8}\tfrac{1}{2}\beta\, [\tfrac{1}{6}(2n^2-3n+1) - (n-1)\mathrm{sh}^2\tfrac{1}{2}\beta + n\,\mathrm{sh}^4\tfrac{1}{2}\beta]\,,$$

$$\sum_{\substack{n_1=0,\dots,n-2\\ n_2=n-2-n_1}} |\langle 0|G(\zeta)|n_2+1, n_1\, n_1+1\, n_2\rangle|^2 \qquad \text{(B.19)}$$

$$= n\, \mathrm{th}^{2n-4}\tfrac{1}{2}\beta\, \mathrm{ch}^{-8}\tfrac{1}{2}\beta\, [\tfrac{1}{6}(n^2-1)]\,.$$

Adding the last two terms up we find

$$n\, \mathrm{th}^{2n-2}\, \mathrm{ch}^{-4}\tfrac{1}{2}\beta\, \{\mathrm{sh}^{-2}\tfrac{1}{2}\beta\, \mathrm{ch}^{-2}\tfrac{1}{2}\beta\, [\tfrac{1}{2}n(n-1) - (n-1)\mathrm{sh}^2\tfrac{1}{2}\beta + \mathrm{sh}^4\tfrac{1}{2}\beta]\}$$

$$= n\left(\frac{x-1}{x+1}\right)^{n-1} \frac{4}{(x+1)^2}\left\{\frac{1}{x^2-1}[x^2-2nx+2n^2-1]\right\}\,. \qquad \text{(B.20)}$$

Finally, inserting both terms into (B.1) and (B.2) we arrive exactly at the result given in eqs. (36) and (37) of sect. 2.

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