# Quantum behavior of deterministic systems with information loss: Path integral approach

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't Hooft's derivation of quantum from classical physics is analyzed by means of the classical path integral of Gozzi *et al.*. It is shown how the key element of this procedure – the loss of information constraint — can be implemented by means of Faddeev-Jackiw's treatment of constrained systems. It is argued that the emergent quantum systems are identical with systems obtained in [Phys.Rev. A**71** (2005) 052507] through Dirac-Bergmann's analysis. We illustrate our approach with two simple examples – free particle and linear harmonic oscillator. Potential Liouville anomalies are shown to be absent.

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## I. INTRODUCTION

The idea of quantum mechanics as the low-energy limit of some more fundamental deterministic dynamics [1, 2] has been revived recently by G.'t Hooft [3, 4], in the attempt for a radical solution of the so-called holographic paradox, originally formulated in the context of black-hole thermodynamics [5, 6].

There is a widespread negative attitude towards the possibility of deriving quantum from classical physics which relies on Bell's inequalities [7]. However, although being clear that quantum mechanics at laboratory scales violates these inequalities, a common prejudice is that Bell's theorem should be true at all scales. As observed by 't Hooft [3], this need not be the case because such fundamental concepts as rotational symmetry, isospin or even Poincaré invariance — on which the usual forms of the Bell inequalities are based — may simply cease to exist at Planck scale.

By resorting to simple dynamical systems, 't Hooft has shown that an appropriate constraining procedure applied to the deterministic system, can reduce the physical degrees of freedom so that quantum mechanics emerges. Such a reduction of the degrees of freedom may be physically implemented by a mechanism of information loss (dissipation). This idea has been further developed by several authors [4, 8–13], and it forms the basis also of this paper.

Our aim is to study 't Hooft's quantization procedure by means of path integrals, along the line of what done in our previous work [8]. However, in contrast to Ref. [8] here we treat 't Hooft's constrained dynamics by means of the Faddeev-Jackiw technique [14]. The constrained dynamics enters into 't Hooft's scheme twice: first, in the classical starting Hamiltonian which is of first order in the momenta and thus singular in the Dirac-Bergmann sense [15]. Second, in the information loss condition that one has to enforce in order to achieve quantization [8]. In our previous paper [8] we have adopted the customary Dirac-Bergmann technique, which is often cumbersome. Here, we want to point out the simplifications arising from the alternative Faddeev-Jackiw method, which turns out to admit a clearer exposition of the basic concepts.

The paper is organized as follows: In Section II, we briefly discuss the main features of 't Hooft's scheme. By utilizing the Faddeev-Jackiw procedure we present in Section III a Lagrangian formulation of 't Hooft's system, which allows us to quantize 't Hooft's system via path integrals in configuration space. It is shown that the fluctuating system produces a classical partition function. In Section IV, we make contact with Gozzi's superspace path integral formulation of classical mechanics. In Section V, we introduce 't Hooft's constraint which accounts for information loss. This is again handled by means of Faddeev-Jackiw analysis. Central to this analysis is the fact that 't Hooft's condition breaks the BRST symmetry and allows to recast the classical generating functional into a form representing a genuine quantum-mechanical partition function. In Section VI, we present two simple applications of our formalism. Associated technical details of the anomaly cancelation are relegated to Appendix A. A final discussion is given in Section VII.

## II. 'T HOOFT'S QUANTIZATION PROCEDURE

In this section we briefly review the main aspects of 't Hooft's quantization procedure [4, 12] to be used in this work. The basic idea is that there exists a simple class of classical systems that can be described by means of Hilbert space techniques without loosing their deterministic character. Only after enforcing certain constraints expressing information loss, one obtains *bona fide* quantum systems. Thus, the quantum states of actually observed degrees of freedom (*observables*) can be identi-

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fied with equivalence classes of states that span the original (primordial) Hilbert space of truly existing degrees of freedom (*be-ables*).

Such a scheme is realized in certain model quantum cases where one may indeed identify the primordial systems of *be-ables* that are entirely deterministic. In discrete-time systems this scenario has been successfully applied, e.g., to cellular automata with embedded information loss [3] where the equivalence classes were invoked to obtain a unitary evolution operator with a genuine quantum mechanical Hamiltonian. Further examples of discrete-time systems can be found, e.g., in Refs. [9, 13].

In the continuous cases the equivalence classes are tightly linked with the loss of information condition — that is represented by a suitably chosen first-class primary constraint – and ensuing gauge freedom. As only the continuous times will be of concern here let us briefly address the key points thereof. We begin by observing that classical systems of the form

$$H(\boldsymbol{p},\boldsymbol{q}) = f^a(\boldsymbol{q})p_a\,, \tag{1}$$

with repeated indices summed, evolve deterministically even after quantization [12]. This happens since in the Hamiltonian equations of motion

$$\dot{q}^a \;=\; \{q^a, H\} = f^a(\boldsymbol{q})\,, \tag{2}$$

$$\dot{p}_a \;=\; \{p_a, H\} = -p_a\frac{\partial f^a(\boldsymbol{q})}{\partial q^a}\,, \tag{3}$$

the equation for the $q^a$ does not contain $p_a$, making the $q^a$ *be-ables*. Because of the autonomous character of the dynamical equations (2) we can always decide to define a formal Hilbert space spanned by the states $\{|\boldsymbol{q}\rangle\}$, and define the associated momenta $\hat{p}_a = -i\partial/\partial q^a$. The quantum mechanical “Hamiltonian” generating (2) is then $\hat{H} = f^a(\hat{\boldsymbol{q}})\hat{p}_a$. Indeed, due to linearity of $\hat{H}$ in $\hat{p}_a$ we have that $\hat{q}^a(t+\Delta t) = F^a[\hat{\boldsymbol{q}}(t),\Delta t]$ ($F^a$ is some function) and hence $[\hat{q}^a(t),\hat{q}^b(t')] = 0$ for any $t$ and $t'$. This in turn implies that the Heisenberg equation of the motion for $\hat{q}^a(t)$ in the $\boldsymbol{q}$-representation is identical with the $c$-number dynamical equation (2).

The basic physical problem with systems described by the Hamiltonian (1) is that they are not bounded from below. This defect can be repaired in the following way [12]: Let $\rho(\hat{\boldsymbol{q}})$ be some positive function of $\hat{q}_a$ with $[\hat{\rho},\hat{H}] = 0$. Then, we perform splitting

$$\hat{H} = \hat{H}_+ - \hat{H}_-\,,$$
$$\hat{H}_+ = \frac{1}{4}\hat{\rho}^{-1}\left(\hat{\rho}+\hat{H}\right)^2, \quad \hat{H}_- = \frac{1}{4}\hat{\rho}^{-1}\left(\hat{\rho}-\hat{H}\right)^2, \tag{4}$$

where $\hat{H}_+$ and $\hat{H}_-$ are positive definite operators satisfying

$$[\hat{H}_+,\hat{H}_-] \;=\; [\hat{\rho},\hat{H}] = 0\,. \tag{5}$$

We may now employ the Dirac canonical quantization of constrained systems and enforce a lower bound upon the Hamiltonian by imposing the restriction

$$\hat{H}_-|\psi\rangle = 0\,, \tag{6}$$

on the Hilbert space of *be-ables*. The resulting *physical* state space, i.e. the space of *observables* has the energy eigenvalues that are trivially positive owing to

$$\hat{H}|\psi\rangle = \hat{H}_+|\psi\rangle = \hat{\rho}|\psi\rangle\,. \tag{7}$$

Concomitantly, in the Schrödinger picture the equation of motion

$$\frac{d}{dt}|\psi_t\rangle = -i\hat{H}_+|\psi_t\rangle\,, \tag{8}$$

has only positive frequencies on physical states. Note that due to condition (5) ’t Hooft’s constraint (6) is a first-class constraint. It is well known in the theory of constrained dynamics [17] that first-class conditions generate gauge transformation and thus not only restrict the full Hilbert space but also produce equivalence classes of states. It should be noticed that above equivalence classes are generally non-local, in the sense that two states belong to the same class if they can be transformed into each other by gauge transformation with the generator $\hat{H}_-$. If, in addition, the ensuing fiber bundle structure is non-trivial one may encounter signatures of this through the emergence of geometric phases.

’t Hooft proposed in Ref. [12] that in cases when the dynamical equations (2) describe the configuration-space chaotic dynamical system, the equivalent classes could be related to its stable orbits (e.g., limit cycles). The mechanism responsible for clustering of trajectories to equivalence classes was identified by ’t Hooft as information loss — after while one cannot retrace back the initial conditions of a given trajectory, one can only say at what attractive trajectory it will end up. As the mechanism of equivalent classes is embodied in Eq.(6) we shall henceforth refer to it as *information loss condition*. Applications of the the outlined “canonical” scenario were given, e.g., in Refs. [13].

As Feynman’s path integrals represent a legitimate alternative to canonical quantization it is intriguing to formulate ’t Hooft’s procedure in the language of path integrals. Apart from the fact that path integrals have a close proximity to classical physics, they have also the additional advantage that they can incorporate constraints in a straightforward manner. In this respect, the Faddeev-Jackiw treatment of constrained systems is an interesting option which we are going to explore in the following.

## III. PATH INTEGRAL QUANTIZATION OF ’T HOOFT’S SYSTEM

We now consider the path integral quantization [16] of the class of systems described by Hamiltonians of the type (1). Because of the absence of a leading kinetic term quadratic in the momenta $p_a$, the system can be viewed as *singular* and the ensuing quantization can be achieved through some standard technique for quantization of constrained systems.

Particularly convenient is the technique proposed by Faddeev and Jackiw [14]. There one starts by observing that a Lagrangian for 't Hooft's equations of motion (2) and (3) can be simply taken as

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}, \dot{\boldsymbol{p}}) \;=\; \boldsymbol{p}\cdot\dot{\boldsymbol{q}} - H(\boldsymbol{p}, \boldsymbol{q})\,, \tag{9}$$

with $\boldsymbol{q}$ and $\boldsymbol{p}$ being *Lagrangian variables* (in contrast to phase space variables). Note that $L$ does not depend on $\dot{\boldsymbol{p}}$. It is easily verified that the Euler-Lagrange equations for the Lagrangian (9) indeed coincide with the Hamiltonian equations (2) and (3). Thus given 't Hooft's Hamiltonian (1) one can always construct a first-order Lagrangian (9) whose configuration space coincides with the Hamiltonian phase space. By defining $2N$ configuration-space coordinates as

$$\begin{aligned} \xi^a &= p_a, \quad a = 1, \ldots, N\,,\\ \xi^a &= q^a, \quad a = N+1, \ldots, 2N\,, \end{aligned} \tag{10}$$

the Lagrangian (9) can be cast into the more expedient form, namely (summation convention understood)

$$L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) \;=\; \tfrac{1}{2}\xi^a \boldsymbol{\omega}_{ab}\dot{\xi}^b - H(\boldsymbol{\xi})\,. \tag{11}$$

Here $\boldsymbol{\omega}$ is the $2N \times 2N$ symplectic matrix

$$\boldsymbol{\omega}_{ab} \;=\; \begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix}_{ab}\,, \tag{12}$$

which has an inverse $\boldsymbol{\omega}^{-1}_{ab} \equiv \boldsymbol{\omega}^{ab}$. The equations of the motion read

$$\dot{\xi}^a \;=\; \boldsymbol{\omega}^{ab}\frac{\partial H(\boldsymbol{\xi})}{\partial \xi^b}\,, \tag{13}$$

indicating that there are no constraints on $\boldsymbol{\xi}$. Thus the Faddeev-Jackiw procedure makes the system unconstrained, so that the path integral quantization may proceed in the standard way. The time evolution amplitude is simply [16]

$$\langle \boldsymbol{\xi}_2, t_2|\boldsymbol{\xi}_1, t_1\rangle = \mathcal{N}\int_{\boldsymbol{\xi}(t_1)=\boldsymbol{\xi}_1}^{\boldsymbol{\xi}(t_2)=\boldsymbol{\xi}_2} \mathcal{D}\boldsymbol{\xi}\; \exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} dt\; L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})\right], \tag{14}$$

where $\mathcal{N}$ is some normalization factor, and the measure can be rewritten as

$$\mathcal{N}\int_{\boldsymbol{\xi}(t_1)=\boldsymbol{\xi}_1}^{\boldsymbol{\xi}(t_2)=\boldsymbol{\xi}_2} \mathcal{D}\boldsymbol{\xi} = \mathcal{N}\int_{\boldsymbol{q}(t_1)=\boldsymbol{q}_1}^{\boldsymbol{q}(t_2)=\boldsymbol{q}_2} \mathcal{D}\boldsymbol{q}\mathcal{D}\boldsymbol{p}\,. \tag{15}$$

Since the Lagrangian (9) is linear in $\boldsymbol{p}$, we may integrate these variables out and obtain

$$\langle \boldsymbol{q}_2, t_2|\boldsymbol{q}_1, t_1\rangle = \mathcal{N}\int_{\boldsymbol{q}(t_1)=\boldsymbol{q}_1}^{\boldsymbol{q}(t_2)=\boldsymbol{q}_2} \mathcal{D}\boldsymbol{q}\; \prod_a \delta[\dot{q}^a - f^a(\boldsymbol{q})]\,, \tag{16}$$

where $\delta[\boldsymbol{f}] \equiv \prod_t \delta(\boldsymbol{f}(t))$ is the functional version of Dirac's $\delta$-function. Hence, the system described by the Hamiltonian (1) retains its deterministic character even after quantization. The paths are squeezed onto the classical trajectories determined by the differential equations $\dot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q})$. The time evolution amplitude (16) contains a sum over only the classical trajectories — there are no quantum fluctuations driving the system away from the classical paths, which is precisely what should be expected from a deterministic dynamics.

The amplitude (16) can be brought into more intuitive form by utilizing the identity

$$\delta\left[\boldsymbol{f}(\boldsymbol{q}) - \dot{\boldsymbol{q}}\right] \;=\; \delta[\boldsymbol{q} - \boldsymbol{q}_{\rm cl}]\;(\det M)^{-1}\,, \tag{17}$$

where $M$ is a functional matrix formed by the second functional derivatives of the action $\mathcal{A}[\boldsymbol{\xi}] \equiv \int dt\, L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$:

$$M_{ab}(t, t') \;=\; \left.\frac{\delta^2\mathcal{A}}{\delta\xi^a(t)\;\delta\xi^b(t')}\right|_{\boldsymbol{q}=\boldsymbol{q}_{\rm cl}}. \tag{18}$$

The Morse index theorem ensures that for sufficiently short time intervals $t_2 - t_1$ (before the system reaches its first focal point), the classical solution with the initial condition $\boldsymbol{q}(t_1) = \boldsymbol{q}_1$ is unique. In such a case, Eq. (16) can be brought to the form

$$\langle \boldsymbol{q}_2, t_2|\boldsymbol{q}_1, t_1\rangle \;=\; \tilde{\mathcal{N}}\int_{\boldsymbol{q}(t_1)=\boldsymbol{q}_1}^{\boldsymbol{q}(t_2)=\boldsymbol{q}_2} \mathcal{D}\boldsymbol{q}\;\delta\left[\boldsymbol{q} - \boldsymbol{q}_{\rm cl}\right]\,, \tag{19}$$

with $\tilde{\mathcal{N}} \equiv \mathcal{N}/(\det M)$. Remarkably, the Faddeev-Jackiw treatment bypasses completely the discussion of constraints, in contrast with the conventional Dirac-Bergmann method [15, 17] where $2N$ (spurious) second-class primary constraints must be introduced to deal with 't Hooft's system, as done in [8].

## IV. EMERGENT SUSY — SIGNATURE OF CLASSICALITY

We now turn to an interesting implication of the result (19). If we had started in Eq.(16) with an external current

$$\tilde{L}(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) + i\hbar\boldsymbol{J}\cdot\boldsymbol{q}\,, \tag{20}$$

integrated again over $\boldsymbol{p}$, and took the trace over $\boldsymbol{q}$, we would end up with a generating functional

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \tilde{\mathcal{N}}\int \mathcal{D}\boldsymbol{q}\;\delta[\boldsymbol{q} - \boldsymbol{q}_{\rm cl}]\exp\left[\int_{t_1}^{t_2} dt\;\boldsymbol{J}\cdot\boldsymbol{q}\right]. \tag{21}$$

This coincides with the path integral formulation of classical mechanics postulated by Gozzi *et al.* [18, 19]. The same representation can be derived from the classical limit of a closed-time path integral for the transition probabilities of a quantum particle in a heat bath [8, 16], The path integral (21) has an interesting mathematical structure. We may rewrite it as

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] \;=\; \tilde{\mathcal{N}}\int \mathcal{D}\boldsymbol{q}\;\delta\left[\frac{\delta\mathcal{A}}{\delta\boldsymbol{q}}\right]\;\det\left|\frac{\delta^2\mathcal{A}}{\delta q_a(t)\;q_b(t')}\right| \times\;\exp\left[\int_{t_1}^{t_2} dt\;\boldsymbol{J}\cdot\boldsymbol{q}\right]. \tag{22}$$

By representing the delta functional in the usual way as a functional Fourier integral

$$\delta\left[\frac{\delta\mathcal{A}}{\delta\boldsymbol{q}}\right] = \int\mathcal{D}\boldsymbol{\lambda}\;\exp\left(i\int_{t_1}^{t_2} dt\;\boldsymbol{\lambda}(t)\frac{\delta\mathcal{A}}{\delta\boldsymbol{q}(t)}\right),$$

and the functional determinant as a functional integral over two real time-dependent Grassmannian *ghost variables* $c_a(t)$ and $\bar{c}_a(t)$,

$$\det\left|\frac{\delta^2\mathcal{A}}{\delta q^a(t)\;\delta q^b(t')}\right| = \int\mathcal{D}\mathbf{c}\mathcal{D}\bar{\mathbf{c}}\;\exp\left[\int_{t_1}^{t_2} dtdt'\;\bar{c}_a(t)\frac{\delta^2\mathcal{A}}{\delta q^a(t)\;\delta q^b(t')}\;c_b(t')\right],$$

we obtain

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int\mathcal{D}\boldsymbol{q}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\boldsymbol{c}\mathcal{D}\bar{\boldsymbol{c}}\exp\left[i\mathcal{S}+\int_{t_1}^{t_2} dt\;\boldsymbol{J}\cdot\boldsymbol{q}\right], \tag{23}$$

with the new action

$$\mathcal{S}[\boldsymbol{q},\bar{\boldsymbol{c}},\boldsymbol{c},\boldsymbol{\lambda}] \equiv \int_{t_1}^{t_2} dt\;\boldsymbol{\lambda}(t)\frac{\delta\mathcal{A}}{\delta\boldsymbol{q}(t)} - i\int_{t_1}^{t_2} dt\int_{t_1}^{t_2} dt'\;\bar{c}_a(t)\frac{\delta^2\mathcal{A}}{\delta q^a(t)\;\delta q^b(t')}\;c_b(t'). \tag{24}$$

Since $\mathcal{Z}_{\rm CM}[\boldsymbol{J}]$ can be derived from the classical limit of a closed-time path integral for the transition probability, it comes to no surprise that $\mathcal{S}$ exhibits BRST (and anti-BRST) symmetry. It is simple to check [8] that $\mathcal{S}$ does not change under the symmetry transformations

$$\delta_{\rm BRST}\,\boldsymbol{q} = \bar{\varepsilon}\boldsymbol{c}\,,\quad \delta_{\rm BRST}\,\bar{\boldsymbol{c}} = -i\bar{\varepsilon}\boldsymbol{\lambda}\,,\quad \delta_{\rm BRST}\,\boldsymbol{c} = 0\,,\quad \delta_{\rm BRST}\,\boldsymbol{\lambda} = 0\,, \tag{25}$$

where $\bar{\varepsilon}$ is a Grassmann-valued parameter (the corresponding anti-BRST transformations are related to (25) by charge conjugation). As noted in [19], the ghost fields $\bar{\boldsymbol{c}}$ and $\boldsymbol{c}$ are mandatory at the classical level as their rôle is to cut off the fluctuations *perpendicular* to the classical trajectories. On the formal side, $\bar{\boldsymbol{c}}$ and $\boldsymbol{c}$ may be identified with Jacobi fields [19, 20]. The corresponding BRST charges are related to Poincaré-Cartan integral invariants [21].

By analogy with the stochastic quantization the path integral (23) can be rewritten in a compact form with the help of a superfield [16, 18, 22]

$$\Phi_a(t,\theta,\bar{\theta}) = \; q_a(t) + i\theta c_a(t) - i\bar{\theta}\bar{c}_a(t) + i\bar{\theta}\theta\lambda_a(t)\,, \tag{26}$$

in which $\theta$ and $\bar{\theta}$ are anticommuting coordinates extending the configuration space of $\boldsymbol{q}$ variables to a superspace. The latter is nothing but the degenerate case of supersymmetric field theory in $d=1$ in the superspace formalism of Salam and Strathdee [23]. In terms of superspace variables we see that

$$\int d\bar{\theta}d\theta\;\mathcal{A}[\boldsymbol{\Phi}] = \int dtd\bar{\theta}d\theta\;L(\boldsymbol{q}(t)+i\theta\boldsymbol{c}(t)-i\bar{\theta}\bar{\boldsymbol{c}}(t)+i\bar{\theta}\theta\boldsymbol{\lambda}(t)) = -i\mathcal{S} \tag{27}$$

To obtain the last line we Taylor expanded $L$ and used the standard integration rules for Grassmann variables. Together with the identity $\mathcal{D}\boldsymbol{\Phi} = \mathcal{D}\boldsymbol{q}\mathcal{D}\boldsymbol{c}\mathcal{D}\bar{\boldsymbol{c}}\mathcal{D}\boldsymbol{\lambda}$ we may therefore express the classical partition functions (21) and (22) as a supersymmetric path integral with fully fluctuating paths in superspace

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] \;=\; \int\mathcal{D}\boldsymbol{\Phi}\;\exp\left\{-\int d\theta d\bar{\theta}\;\mathcal{A}[\boldsymbol{\Phi}](\theta,\bar{\theta})\right\} \times\;\exp\left\{\int dtd\theta d\bar{\theta}\;\boldsymbol{\Gamma}(t,\theta,\bar{\theta})\boldsymbol{\Phi}(t,\theta,\bar{\theta})\right\}.$$

Here we have introduced the supercurrent $\boldsymbol{\Gamma}(t,\theta,\bar{\theta}) = \bar{\theta}\theta\boldsymbol{J}(t)$.

Let us finally add that under rather general assumptions it is possible to prove [8] that 't Hooft's deterministic systems are the *only* systems with the peculiar property that their full quantum properties are classical in the Gozzi *et al.* sense. Among others, the latter also indicates that the Koopman-von Neumann operator formulation of classical mechanics [24] when applied to 't Hooft systems must agree with their canonically quantized counterparts.

## V. INCLUSION OF INFORMATION LOSS

As observed in Section II, the Hamiltonian (1) is not bounded from below, and this is clearly true for any function $f^a(\boldsymbol{q})$. Hence, no deterministic system with dynamical equations $\dot{q}^a = f^a(\boldsymbol{q})$ can describe a stable *quantum world*. To deal with this situation we now employ 't Hooft's procedure of Section II. We assume that the system (1) has $n$ conserved irreducible charges $C^i$, i.e.,

$$\{C^i, H\} = 0\,,\quad i = 1,\ldots,n\,. \tag{28}$$

Then, we enforce a lower bound upon $H$, by imposing the condition that $H_-$ is zero on the physically accessible part of a phase space.

The splitting of $H$ into $H_-$ and $H_+$ is conserved in time provided that $\{H_-,H\} = \{H_+,H\} = 0$, which is ensured if $\{H_+,H_-\} = 0$. Since the charges $C^i$ in (28) form an irreducible set, the Hamiltonians $H_+$ and $H_-$ must be functions of the charges and $H$ itself. There is a certain amount of flexibility in finding $H_-$ and $H_+$. For convenience take the following choice:

$$H_+ \;=\; \frac{(H+a_iC^i)^2}{4a_iC^i}\,,\;\; H_- \;=\; \frac{(H-a_iC^i)^2}{4a_iC^i}\,, \qquad (29)$$

where $a_i(t)$ are $\boldsymbol{q}$ and $\boldsymbol{p}$ independent. The lower bound is reached by choosing $a_i(t)C^i$ to be non-negative. We shall select a combination of $C^i$ which is $\boldsymbol{p}$-independent [this condition may not necessarily be achievable for general $f^a(\boldsymbol{q})$].

In the Dirac-Bergmann quantization approach used in our previous paper [8], the information loss condition (6) was a first-class primary constraint. In the Dirac-Bergmann analysis, this signals the presence of a gauge freedom — the associated Lagrange multipliers cannot be determined from dynamical equations alone [15]. The time evolution of observable quantities, however, should not be affected by the arbitrariness of Lagrange multipliers. To remove this superfluous freedom one must choose a gauge. For details of this more complicated procedure see [8].

In the Faddeev-Jackiw approach, Dirac's elaborate classification of constraints to first or second class, primary or secondary is avoided. It is therefore worthwhile to rephrase the entire development in Ref. [8] once more in this approach. The information loss condition may now be introduced by simply adding to the Lagrangian (11) a term enforcing

$$H_-(\boldsymbol{\xi}) \;=\; 0\,, \qquad (30)$$

by means of a Lagrange multiplier. More in general we can take instead of $H_-$ any function $\phi(\boldsymbol{\xi})$, such that $\phi(\boldsymbol{\xi}) = 0$ implies $H_-(\boldsymbol{\xi}) = 0$. In this way we obtain

$$L(\boldsymbol{\xi},\dot{\boldsymbol{\xi}}) = \tfrac{1}{2}\xi^a\boldsymbol{\omega}_{ab}\dot{\xi}^b - H(\boldsymbol{\xi}) - \eta\,\phi(\boldsymbol{\xi})\,, \qquad (31)$$

In Faddeev-Jackiw method one directly applies the constraint and thus eliminates one of $\xi^a$, say $\xi^1$, in terms of the remaining coordinates. This reduces the dynamical variables to $2N-1$. Apart from an irrelevant total derivative, the canonical term $\xi^a\boldsymbol{\omega}_{ab}\dot{\xi}^b$ changes to $\xi^i\boldsymbol{f}_{ij}(\hat{\boldsymbol{\xi}})\dot{\xi}^j$, with

$$\boldsymbol{f}_{ij}(\hat{\boldsymbol{\xi}}) = \boldsymbol{\omega}_{ij} - \left[\boldsymbol{\omega}_{1i}\frac{\partial\xi^1}{\partial\xi^j} - (i\leftrightarrow j)\right]. \qquad (32)$$

Here $i,j = 2,\ldots,2N$, and $\hat{\boldsymbol{\xi}} = \{\xi^2,\ldots,\xi^{2N}\}$. Eliminating $\xi^1$ also in the Hamiltonian $H$ we obtain the reduced Hamiltonian $H_R(\hat{\boldsymbol{\xi}})$, so that we are left with the reduced Lagrangian

$$L_R(\hat{\boldsymbol{\xi}},\dot{\hat{\boldsymbol{\xi}}}) \;=\; \tfrac{1}{2}\xi^i\boldsymbol{f}_{ij}(\hat{\boldsymbol{\xi}})\dot{\xi}^j - H_R(\hat{\boldsymbol{\xi}})\,. \qquad (33)$$

At this point one must worry about the notorious operator-ordering problem, not knowing in which temporal order $\hat{\boldsymbol{\xi}}$ and $\dot{\hat{\boldsymbol{\xi}}}$ must be taken in the kinetic term. A path integral in which the kinetic term is coordinate-dependent can in general only be defined perturbatively, in which all anharmonic terms are treated as interactions. The partition function is expanded in powers of expectation values of products of these interactions which, in turn, are expanded into integrals over all Wick contractions, the Feynman integrals. Each contraction represents a Green function. For the Lagrangian of the form (33), the contractions of two $\xi^i$'s contain a Heaviside step function, those of one $\xi^i$ and one $\dot{\xi}^i$ contain a Dirac $\delta$-function, and those of two $\dot{\xi}^i$'s contain a function $\dot{\delta}(t-t')$. Thus, the Feynman integrals run over products of distributions and are mathematically undefined. Fortunately, a unique definition has recently been found. It is enforced by the necessary physical requirement that path integrals must be invariant under coordinate transformations [25].

The Lagrangian is processed further with the help of Darboux's theorem [26]. This allows us to perform a non-canonical transformation $\xi^i \mapsto (\zeta^s, z^r)$ which brings $L_R$ to the canonical form

$$L_R(\boldsymbol{\zeta},\dot{\boldsymbol{\zeta}},\boldsymbol{z}) \;=\; \tfrac{1}{2}\zeta^s\boldsymbol{\omega}_{st}\dot{\zeta}^t - H'_R(\boldsymbol{\zeta},\boldsymbol{z})\,, \qquad (34)$$

where $\boldsymbol{\omega}_{st}$ is the canonical symplectic matrix in the reduced $s$-dimensional space. Darboux's theorem ensures that such a transformation exists at least locally. The variables $z^r$ are related to zero modes of the matrix $\boldsymbol{f}_{ij}(\hat{\boldsymbol{\xi}})$ which makes it non-invertible. Each zero mode corresponds to a constraint of the system. In Dirac's language these would correspond to the secondary constraints. Since there is no $\dot{z}^r$ in the Lagrangian, the variables $z^r$ do not play any dynamical rôle and can be eliminated using the equations of motion

$$\frac{\partial H'_R(\boldsymbol{\zeta},\boldsymbol{z})}{\partial z^r} \;=\; 0\,. \qquad (35)$$

In general, $H'_R(\boldsymbol{\zeta},\boldsymbol{z})$ is a nonlinear function of $z^{r_1}$. One now solves as many $z^{r_1}$ as possible in terms of remaining $z$'s, which we label by $z^{r_2}$, i.e.,

$$z^{r_1} \;=\; \varphi^{r_1}(\boldsymbol{\zeta}, z^{r_2})\,. \qquad (36)$$

If $H'_R(\boldsymbol{\zeta},\boldsymbol{z})$ happens to be linear in $z^{r_2}$, we obtain the constraints

$$\varphi_{r_2}(\boldsymbol{\zeta}) \;=\; 0\,. \qquad (37)$$

Inserting the constraints (36) into (34) we obtain

$$L_R(\boldsymbol{\zeta},\dot{\boldsymbol{\zeta}},\boldsymbol{z}) \;=\; \tfrac{1}{2}\zeta^s\boldsymbol{\omega}_{st}\dot{\zeta}^t - H''_R(\boldsymbol{\zeta}) - z^{r_2}\varphi_{r_2}(\boldsymbol{\zeta})\,, \quad (38)$$

with $z^{r_2}$ playing the rôle of Lagrange multipliers. We now repeat the elimination procedure until there are no more $z$-variables. The surviving variables represent the true physical degrees of freedom. In the Dirac-Bergmann approach, these would span the *reduced* phase space $\Gamma^*$.

Let us follow the procedure in more detail if there is just one variable $z$ in (35) and only equation (36) holds. As in Ref. [8], we can pass to the new set of canonical variables $\boldsymbol{\xi} \mapsto (\boldsymbol{\zeta}, z, p_z)$ with $p_z = \phi$. Let us define the function

$$\chi(\boldsymbol{\zeta}, z) \equiv \frac{\partial H'_R(\boldsymbol{\zeta}, z)}{\partial z} = \frac{\partial H_+(\xi^1(\hat{\boldsymbol{\xi}}), \hat{\boldsymbol{\xi}})}{\partial z} = \{H_+, \phi\}|_{p_z=0} = 0\,. \quad (39)$$

Its derivative is given by the Poisson bracket

$$\frac{\partial \chi(\boldsymbol{\zeta}, z)}{\partial z} = \{\chi(\boldsymbol{\zeta}, z), p_z\} = \{\chi, \phi\} \neq 0\,.. \quad (40)$$

Because (40) is different from zero on account of (36) we can identify the function $\chi(\boldsymbol{\zeta}, z)$ with the implicit gauge fixing condition of the Faddeev-Jackiw analysis.

Let us now see how we can include the constraints (30) and (39) into the path integral (21) for $\mathcal{Z}_{\rm CM}[\boldsymbol{J}]$. This cannot simply be done by inserting $\delta$-functionals $\delta[\phi]$ and $\delta[\chi]$ into the integrand, since $\phi$ and $\chi$ may not be independent. Allowing for this, the path integral reads (see Ref.[8])

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int \mathcal{D}\boldsymbol{\xi}\ \delta[\phi]\delta[\chi]|\det\|\{\phi, \chi\}\|| \times \exp\left[i\int_{t_i}^{t_f} dt\ L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) + \int_{t_i}^{t_f} dt\ \boldsymbol{J}\boldsymbol{\xi}\right]. \quad (41)$$

Assuming that $\xi^1$ can be eliminated globally from (31), we obtain

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int \mathcal{D}\hat{\boldsymbol{\xi}}\ \delta[\chi]|\det\|\{\phi, \chi\}\|| \left|\det\left\|\frac{\delta\phi}{\delta\xi^1}\right\|\right|^{-1}_{\xi^1=\xi^1(\hat{\boldsymbol{\xi}})} \times \exp\left[i\int_{t_i}^{t_f} dt\ L_R(\hat{\boldsymbol{\xi}}, \dot{\hat{\boldsymbol{\xi}}}) + \int_{t_i}^{t_f} dt\ \boldsymbol{J}\boldsymbol{g}(\hat{\boldsymbol{\xi}})\right]. \quad (42)$$

After the Darboux transformation, this becomes

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int \mathcal{D}\boldsymbol{\zeta}\mathcal{D}z\ \delta[z - \varphi(\boldsymbol{\zeta})] \times \exp\left[i\int_{t_i}^{t_f} dt\ L_R(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, z) + \int_{t_i}^{t_f} dt\ \boldsymbol{J}\boldsymbol{g}(\boldsymbol{\zeta}, z)\right], \quad (43)$$

where we have used the functional relation

$$\delta[\chi]|\det\|\{\phi, \chi\}\|| = \delta\left[\frac{\delta H_+}{\delta z}\right]\left|\det\left\|\frac{\delta^2 H_+}{\delta z(t)\delta z(t')}\right\|\right| = \delta[z - \varphi(\boldsymbol{\zeta})]\,, \quad (44)$$

together with Jacobi-Liouville equality

$$\frac{\partial(\xi^2, \ldots, \xi^{2N})}{\partial(\zeta^1, \ldots, \zeta^{2N-2}, z)} = \frac{\partial(\xi^2, \ldots, \xi^{2N}, p_z)}{\partial(\zeta^1, \ldots, \zeta^{2N-2}, z, p_z)} \frac{\partial(\zeta^1, \ldots, \zeta^{2N-2}, z, p_z)}{\partial(\xi^2, \ldots, \xi^{2N}, \xi^1)} = \left(\frac{\partial p_z}{\partial \xi^1}\right)_{\hat{\boldsymbol{\xi}}} = \left(\frac{\partial \phi}{\partial \xi^1}\right)_{\xi^1=\xi^1(\hat{\boldsymbol{\xi}})}.. \quad (45)$$

With the notation $H^*_+(\boldsymbol{\zeta}) = H_+(\boldsymbol{\zeta}, z = \varphi(\boldsymbol{\zeta}), p_z = 0)$, this can be rewritten as

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int \mathcal{D}\boldsymbol{\zeta} \exp\left[i\int_{t_i}^{t_f} dt\ \zeta^t \boldsymbol{\omega}_{ts}\dot{\zeta}^s\right] \times \exp\left[-i\int_{t_i}^{t_f} dt\ H^*_+(\boldsymbol{\zeta}) + \int_{t_i}^{t_f} dt\ \boldsymbol{J}\boldsymbol{g}^*(\boldsymbol{\zeta})\right]. \quad (46)$$

At this point we note that the result (46) is equivalent to the result derived in Ref. [8]. In fact, when $\chi$ in [8] coincides with the the form (39) and we set $\boldsymbol{\zeta} = (\bar{\boldsymbol{Q}}, \bar{\boldsymbol{P}})$, $z = Q_1$, and $p_z = P_1$, then $\mathcal{Z}_{\rm CM}[\boldsymbol{J}]$ from Ref. [8] reduces exactly to the form (46). In general cases, however, the gauge fixing condition of the Dirac-Bergmann procedure can be chosen in a different way with respect to the natural choice implicit in the Faddeev-Jackiw analysis. In such a situation the resulting reduced Lagrangians do not coincide but are connected via a canonical transformation.

Important simplification happens when $H'_R$ is independent of $z$ (e.g., when $\phi = H_-$). In Dirac-Bergmann's language this indicates that there is no secondary constraint. In such a case the gauge fixing can be enforced by choosing $\chi = z$ (see L. Faddeev in Ref. [27]), and the procedure outlined in steps (41)- (46) is streamlined by the fact that $|\det\|\{\phi, \chi\}\|| = 1$. The corresponding coordinate basis $\{\boldsymbol{\zeta}, \chi, \phi\}$ is known as the Shouten-Eisenhart basis [17].

## VI. EXAMPLES OF EMERGENT QUANTUM SYSTEMS

### A. Free particle

We conclude our presentation by exhibiting how our quantization method works for a simple system described by 't Hooft's Hamiltonian

$$H(\boldsymbol{q}, \boldsymbol{p}) = xp_y - yp_x\,. \quad (47)$$

Formally, this represents the $z$ component of the angular momentum, whose spectrum is unbounded from below. Alternatively, one can regard (47) as describing the mathematical pendulum. This is because the corresponding dynamical equation (2) for $\boldsymbol{q}$ is a plane pendulum equation with the pendulum constant $l/g = 1$. The Lagrangian (9) then reads

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}, \dot{\boldsymbol{p}}) = p_x\dot{x} + p_y\dot{y} - xp_y + yp_x\,. \quad (48)$$

Here, indeed, the $L$ is $\dot{\boldsymbol{p}}$-independent, as discussed in Section III. It is well-known [28] that the system (47) has two independent constants of motion - the Casimir functions:

$$C_1 = x^2 + y^2\,, \quad C_2 = xp_x + yp_y\,. \quad (49)$$

Only $C_1$ is $\boldsymbol{p}$-independent, so that 't Hooft's constraint $\rho(\boldsymbol{q})$ acquires the form: $\rho(\boldsymbol{q}) = a_1C_1(\boldsymbol{q})$, with the constant $a_1$ to be determined later.

The Faddeev-Jackiw procedure is based on the reduced Lagrangian

$$L_R(\hat{\boldsymbol{\xi}},\dot{\hat{\boldsymbol{\xi}}}) \;=\; \dot{y}p_y + \frac{\dot{x}}{y}(p_y x - a_1(x^2+y^2)) - a_1(x^2+y^2)$$
$$= \sqrt{(x^2+y^2)}\,\frac{d}{dt}\left[-2a_1\sqrt{(x^2+y^2)}\;\mathrm{arct}\!\left(\frac{x}{y}\right)\right.$$
$$\left. - \frac{xp_x+yp_y}{\sqrt{(x^2+y^2)}}\right] - a_1(x^2+y^2)\,. \quad (50)$$

We can diagonalize the symplectic structure by means of the Darboux transformation

$$p_\zeta \;=\; \sqrt{x^2+y^2}\,,$$
$$\zeta \;=\; -2a_1\sqrt{(x^2+y^2)}\;\mathrm{arct}\!\left(\frac{x}{y}\right) - \frac{xp_x+yp_y}{\sqrt{(x^2+y^2)}}\,. \quad (51)$$

Thus, up to a total derivative, the reduced Lagrangian (50) goes over into

$$L_R(\boldsymbol{\zeta},\dot{\boldsymbol{\zeta}},z) \;=\; \tfrac{1}{2}\zeta^s\boldsymbol{\omega}_{st}\dot{\zeta}^t \;-\; a_1(p_\zeta)^2\,, \quad (52)$$

with the symplectic notation $\zeta \equiv \zeta^1$ and $p_\zeta \equiv \zeta^2$. The reduced Hamiltonian is $z$-independent and thus $\chi = z$. Note that (51) together with

$$z = -\mathrm{arct}\!\left(\frac{x}{y}\right), \quad \text{and} \quad p_z^2 = \phi^2 = 4a_1p_\zeta^2\,H_-\,, \quad (53)$$

constitute the canonical transformation $\boldsymbol{\xi} \mapsto (\boldsymbol{\zeta}, z, p_z)$.

Due to a non-linear nature of the canonical transformation (51) and (53) we must check up the path integral measure for a potential anomaly. In Appendix A, we show that although the anomaly is indeed generated, it gets cancelled due to the presence of the constraining $\delta$–functionals in the measure. In other words, the Liouville anomaly is not present in the reduced phase space. In addition, because (52) is cyclic in $\zeta$, it can be argued [31] that no new (non-classical) corrections to the Hamiltonian are generated in the action after the above canonical transformation is performed.

Let us now set $a_1 = 1/2m\hbar$. After changing in the path integral the variable $\zeta(t)$ to $\zeta(t)/\hbar$ we obtain the path integral measure of quantum systems:

$$\mathcal{D}\boldsymbol{\zeta} \;\approx\; \prod_i\left[\frac{d\zeta(t_i)dp_\zeta(t_i)}{2\pi\hbar}\right]. \quad (54)$$

In addition, the prefactor $1/\hbar$ in the exponent emerges correctly. Thus, the classical partition function of Gozzi *et al.* turns into the quantum partition function for a free particle of mass $m$. As the constant $a_1$ represents the choice of units (or scale factor) for $C_1$ we see that the quantum scale $\hbar$ is implemented into the partition function via the choice of the information loss condition.

A free particle can emerge also from another class of 't Hooft's systems. Such systems can be obtained by modifying slightly the previous discussion and considering instead the Hamiltonian

$$H \;=\; xp_y - yp_x + \lambda(x^2+y^2)\,, \quad (55)$$

where $\lambda$ is a constant. 't Hooft's information loss condition and $\rho(\boldsymbol{q})$ remain clearly the same as in the previous case. The reduced Lagrangian then reads

$$L_R(\hat{\boldsymbol{\xi}},\dot{\hat{\boldsymbol{\xi}}}) = \sqrt{(x^2+y^2)}\,\frac{d}{dt}\left[-2a_1\sqrt{(x^2+y^2)}\;\mathrm{arct}\!\left(\frac{x}{y}\right)\right.$$
$$\left. - \frac{xp_x+yp_y}{\sqrt{(x^2+y^2)}}\right] - a_1^*(x^2+y^2)\,. \quad (56)$$

with $a_1^* = a_1 + \lambda$. Identical reasonings as in the preceding situation lead again to a proper quantum-mechanical partition function for a free particle.

### B. Harmonic oscillator

In a previous paper that utilized the Dirac-Bergmann treatment [8], it was shown that the system (47) can be also used to obtain the quantized linear harmonic oscillator. This is because there is a certain ambiguity in imposing 't Hooft's condition. This will be illustrated with $\phi = xp_y - yp_x - a_1(x^2+y^2)$ used in Eq.(53). The constraint $\phi = 0$ can be equivalently written as

$$\phi \;=\; \boldsymbol{x}\wedge\boldsymbol{A} \;=\; 0\,, \quad (57)$$

with $\boldsymbol{x} = (x,y)$ and $\boldsymbol{A} = (p_x + a_1y, p_y - a_1x)$. The solution of $\phi = 0$ is formally given by

$$x \;=\; \alpha\,(p_x + a_1y)\,, \qquad y \;=\; \alpha\,(p_y - a_1x)\,, \quad (58)$$

where $\alpha$ is an arbitrary real number. Note that $\alpha = 0$ and $\alpha = \infty$ also cover the singular cases $|\boldsymbol{x}| = 0$ and $|\boldsymbol{A}| = 0$, respectively. Inasmuch, instead of one first-class condition $\phi = 0$ we can consider two second-class constraints

$$\phi_1 \;=\; \left(p_x - \frac{x}{\alpha} + a_1y\right) \;=\; 0$$
$$\phi_2 \;=\; \left(p_y - \frac{y}{\alpha} - a_1x\right) \;=\; 0\,, \quad (59)$$

($\{\phi_1,\phi_2\} = 2a_1 \neq 0$). Equivalently one may view $\phi_1$ as a primary first-class constraint and $\phi_2$ as the gauge fixing condition. To make contact with the Faddeejev-Jackiw procedure we chose the second scenario. The corresponding reduced Lagrangian is then

$$L_R(\hat{\boldsymbol{\xi}},\dot{\hat{\boldsymbol{\xi}}}) \;=\; \dot{y}p_y + \dot{x}\left(\frac{x}{\alpha} - a_1y\right) - xp_y + y\left(\frac{x}{\alpha} - a_1y\right)$$
$$= -\frac{1}{2a_1}\left(p_y + a_1x - \frac{y}{\alpha}\right)\frac{d}{dt}\left(p_x + a_1y - \frac{x}{\alpha}\right)$$
$$-\; xp_y + y\left(\frac{x}{\alpha} - a_1y\right). \quad (60)$$

At this point we can perform Darboux's transformation

$$
\begin{aligned}
p_\zeta &= \frac{1}{\sqrt{2}}\left(p_y + a_1 x - \frac{y}{\alpha}\right) \\
\zeta &= -\frac{1}{\sqrt{2}a_1}\left(p_x - \frac{x}{\alpha} - a_1 y\right) \\
z &= \phi_2/2a_1 \;=\; \frac{1}{2a_2}\left(p_y - \frac{y}{\alpha} - a_1 x\right). \qquad (61)
\end{aligned}
$$

The reduced Lagrangian (60) then becomes

$$
L_R(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, z) = \tfrac{1}{2}\zeta^s \boldsymbol{\omega}_{st}\dot{\zeta}^t - \frac{1}{2a_1}p_\zeta^2 - \frac{a_1}{2}\left(\zeta^2 - 2z^2\right), \quad (62)
$$

($\zeta \equiv \zeta^1, p_\zeta \equiv \zeta^2$). The stabilization condition $\chi(\boldsymbol{\zeta}, z) = 0$ in this case yields the gauge fixing condition

$$
\chi(\boldsymbol{\zeta}, z) \;=\; \frac{\partial H'_R(\boldsymbol{\zeta}, z)}{\partial z} \;=\; -2a_1 z \;=\; 0\,. \qquad (63)
$$

By plugging $z = 0$ into Eq.(62) (i.e., by enforcing a gauge constraint) we eliminate the variable $z$ and obtain a non-degenerate reduced Lagrangian

$$
L_R(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}) \;=\; \tfrac{1}{2}\zeta^s \boldsymbol{\omega}_{st}\dot{\zeta}^t - \frac{1}{2a_1}p_\zeta^2 - \frac{a_1}{2}\zeta^2\,. \qquad (64)
$$

The canonical transformation $\boldsymbol{\xi} \mapsto (\boldsymbol{\zeta}, z, p_z)$ is completed by identifying

$$
p_z \;=\; -\phi_1 \;=\; -p_x - a_1 y + \frac{x}{\alpha}\,. \qquad (65)
$$

Note that, similarly as in the previous case, $\{p_\zeta, \zeta, z, p_z\}$ can be identified with the Shouten-Eisenhart basis.

By choosing $a_1 = 1/m\hbar$ and rescaling $\zeta(t) \mapsto \zeta(t)/\hbar$ in the path integral (46) we obtain the quantum partition function for the linear harmonic oscillator with a unit frequency. One can again observe that the fundamental scale (suggestively denoted as $\hbar$) enters the partition function in a correct quantum mechanical manner. This is precisely the result which 't Hooft conjectured for the system (47) in Ref. [12].

Because the canonical transformation $\boldsymbol{\xi} \mapsto (\boldsymbol{\zeta}, z, p_z)$ is in this case linear it does not induce anomaly in the path integral measure nor in the action (see also Appendix A).

In the framework of the Dirac-Bergmann treatment both results discussed above were already derived in Ref. [8]. It is clear that other emergent quantum systems can be generated in an analogous manner. For instance, in Ref.[8] free particle weakly coupled to Duffing's oscillator was obtained from the Rössler system. Further development in this direction is presently in progress.

## VII. SUMMARY

Let us summarize the novel elements of this paper in comparison with our previous work [8]. Here, we have utilized the Faddeev-Jackiw treatment of singular Lagrangians [14] which entirely obviates the need for the Dirac-Bergmann distinction between first and second class, primary and secondary constraints used in [8]. Both approaches, however, require a doubling of configuration space degrees of freedom. Apart from formulating the path integral for singular Hamiltonians, the Faddeev-Jackiw method is convenient in imposing 't Hooft's information loss condition. In the Dirac-Bergmann scheme, this condition represents a first-class subsidiary constraint which has to be supplemented by a gauge fixing condition [8]. In the Faddeev-Jackiw procedure the degrees of freedom are reduced before quantization. This seems at first sight simpler than the Dirac-Bergmann method, but it can be complicated in practice. In particular, the change of coordinates (Darboux coordinates) from the pre-symplectic to a symplectic form plus nondynamical $\boldsymbol{z}$-variables may be involved, or even impossible. A detailed discussion of such difficulties can be found, for instance, in Ref. [29].

In the Dirac-Bergmann procedure, the reduction to physical degrees of freedom is performed by dividing the constraints into first-class and second class. Second-class constraints are removed via Dirac's brackets machinery while the first-class constrains can be imposed only after the gauge fixing procedure. In the Faddeev-Jackiw treatment one does not need to classify constraints and perform gauge fixing. Any possible gauge conditions are taken care of implicitly by the reduction procedure. If the implicit gauge conditions are global, it is possible to show [30] that both the Faddeev-Jackiw treatment and Dirac-Bergmann procedure leads to the same reduced system. If they are only local, this equivalence between the two schemes may be obstructed by unwanted Gribov ambiguities. Thus, under the assumption that there exists a global Darboux transformation we have shown that 't Hooft's quantization program performed with the Dirac-Bergmann and the Faddeev-Jackiw procedure lead to equivalent path integral representations of emergent quantum systems.

Another problem may come from the the specific form of the Darboux transformation. Although it is essentially non-canonical, it shows up as a canonical transformation in the original configuration space in which the constraints are embedded. Under normal circumstances, the path integral measure is not Liouville-invariant under canonical transformations, often developing an anomaly [31, 32]. This may invalidate our formal path integral manipulations in Section V. Fortunately, the forewarned is also forearmed: if the canonical transformations are linear it can be argued [31] that an anomaly is not present. This strategy seems to be simpler to utilize in the Dirac-Bergmann than in Faddeev-Jackiw approach. This is because in the Dirac-Bergmann analysis the gauge constraint is introduced by hand (provided it is admissible) and hence one can try to choose it in a way that the resulting canonical transformation is linear or at least free of the anomaly.

In the Dirac-Bergmann approach it seems also easier to handle the ordering problem mentioned in Section V. This is because 't Hooft's constraint is there implemented directly via the linear canonical transformation in the extended phase space. Due to the fact that

$$\int_{t_1}^{t_2} dt\ (\boldsymbol{p}\dot{\boldsymbol{q}} - H(\boldsymbol{p},\boldsymbol{q})) = \int_{t_1}^{t_2} dt\ (\boldsymbol{P}\dot{\boldsymbol{Q}} - H^*(\boldsymbol{P},\boldsymbol{Q}))\,,$$

under a canonical transformation (modulo total derivative) there is no explicit coordinate dependence in the term $\dot{\boldsymbol{Q}}\boldsymbol{P}$. Thus the path integral is in this case well defined even globally. This should be contrasted with the Faddeev-Jackiw method where the phase space is not extended and 't Hooft's constraint is imposed directly through a non-canonical transformation. Although the latter is only a halfway step toward an ultimately canonical transformation it causes the path integral to be well defined only perturbatively at these stages.

Note finally that according to analysis in Section V, when we start with the $N$–dimensional classical system ($\boldsymbol{q}$ variables) then the emergent quantum dynamics has $N-1$ dimensions ($\boldsymbol{\zeta}$ variables). This reduction of dimensionality vindicates in part the terminology "information loss" used throughout the text.

## Appendix A

In the operator approach to quantum mechanical systems any non-trivial change of variables is complicated by the ordering and non-commutativity of the constituent operators that occur in expressions. Due to $c$-number nature of path integrals such difficulties are not immediately apparent. However, a careful analysis of time-sliced representations of path integrals reveals that complications related with canonical transformations are hidden in two places [16, 31, 32]. **a)** the path-integral phase space measure cannot be viewed as a product of Liouville measures and, as a rule, canonical transformations often produce anomaly — the Jacobian is not unity. **b)** the time sliced canonical transformation may generate in the action additional terms that are of order $O(\Delta\boldsymbol{P})$ and $O(\Delta\boldsymbol{Q})$ ($\boldsymbol{P}$ and $\boldsymbol{Q}$ are new variables, $\Delta X$ stands for $X(t_{i+1})-X(t_i)$), i.e., terms that need not vanish in the continuous limit (i.e, when $\Delta t \equiv \epsilon \to 0$). It is purpose of this appendix to show that neither **a)** nor **b)** are hampering conclusions of Section VI.

As for **a)**, it can be shown [31] that to the lowest order the anomalous inverse Jacobian for our canonical transformation $(x, y, p_x, p_y) \equiv \boldsymbol{\xi} \mapsto (\boldsymbol{\zeta}, z, p_z)$ can be written as

$$J^{-1} = \prod_{j=1}^{N}(1 + A_j^{\zeta}\Delta\zeta_j + A_j^{z}\Delta z_j + B_j^{\zeta}\Delta p_j^{\zeta} + B_j^{z}\Delta p_j^{z})\,,$$

where $\lim N \to \infty$ is understood and

$$\begin{aligned}
A_j^{\zeta} &= \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^b \partial p_j^c \partial \zeta_j}\frac{\partial p_j^b}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial p_j^c} + \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^b \partial p_j^c \partial z_j}\frac{\partial p_j^b}{\partial \zeta_j}\frac{\partial z_j}{\partial p_j^c}\\
&+ \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^x \partial \zeta_j \partial \zeta_j}\frac{\partial \zeta_j}{\partial x_j} + \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^x \partial z_j \partial \zeta_j}\frac{\partial \zeta_j}{\partial y_j}\\
&+ \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^y \partial \zeta_j \partial \zeta_j}\frac{\partial z_j}{\partial x_j} + \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^y \partial z_j \partial \zeta_j}\frac{\partial z_j}{\partial y_j}\,,\\
A_j^{z} &= \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^b \partial p_j^c \partial \zeta_j}\frac{\partial p_j^b}{\partial z_j}\frac{\partial \zeta_j}{\partial p_j^c} + \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^b \partial p_j^c \partial z_j}\frac{\partial p_j^b}{\partial z_j}\frac{\partial z_j}{\partial p_j^c}\\
&+ \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^x \partial \zeta_j \partial z_j}\frac{\partial \zeta_j}{\partial x_j} + \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^x \partial z_j \partial z_j}\frac{\partial \zeta_j}{\partial y_j}\\
&+ \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^y \partial \zeta_j \partial z_j}\frac{\partial z_j}{\partial x_j} + \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^y \partial z_j \partial z_j}\frac{\partial z_j}{\partial y_j}\,,\\
B_j^{\zeta} &= \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^b \partial p_j^c \partial \zeta_j}\frac{\partial p_j^b}{\partial p_j^{\zeta}}\frac{\partial \zeta_j}{\partial q_j^c} + \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^b \partial p_j^c \partial z_j}\frac{\partial p_j^b}{\partial p_j^{\zeta}}\frac{\partial z_j}{\partial q_j^c}\,,\\
B_j^{z} &= \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^b \partial p_j^c \partial \zeta_j}\frac{\partial p_j^b}{\partial p_j^{z}}\frac{\partial \zeta_j}{\partial q_j^c} + \tfrac{1}{2}\frac{\partial^3 F_j}{\partial p_j^b \partial p_j^c \partial z_j}\frac{\partial p_j^b}{\partial p_j^{z}}\frac{\partial z_j}{\partial q_j^c}\,.
\end{aligned} \tag{66}$$

Here $F_j$ represents the classical generating function of the third kind $F(p_x, p_y, \zeta, z)$ at the sliced time $t_j$. The new variables are determined by solving the system of equations

$$\begin{aligned}
x &= -\frac{\partial F(p_x,p_y,\zeta,z)}{\partial p_x}, \;\; y = -\frac{\partial F(p_x,p_y,\zeta,z)}{\partial p_y},\\
p^{\zeta} &= -\frac{\partial F(p_x,p_y,\zeta,z)}{\partial \zeta}, \;\; p^{z} = -\frac{\partial F(p_x,p_y,\zeta,z)}{\partial z}.
\end{aligned} \tag{67}$$

Indices $b, c$ in (66) run from 1 to 2 and summation convention is assumed. It should be stressed that the higher order contributions to the inverse Jacobian involve third and higher order derivatives of $F(p_x, p_y, \zeta, z)$.

Straightforward but tedious calculations reveal that for the canonical transformation (51), (53) we obtain

$$\begin{aligned}
A_j^{z} &= \frac{\partial^3 F_j}{\partial p_j^x \partial z_j \partial z_j}\frac{\partial \zeta_j}{\partial y_j} + \frac{\partial^3 F_j}{\partial p_j^y \partial z_j \partial z_j}\frac{\partial z_j}{\partial y_j}\\
&= -\left[(1 + 2a_1\, z\, p_{\zeta})\cos z + \left(p_z p_{\zeta}^{-1} - a_1 p_{\zeta}\right)\sin z\right]\\
&\times \left.\frac{\sin z}{2}\right|_{t=t_j}\,,\\
A_j^{\zeta} &= B_j^{\zeta} = B_j^{z} = 0\,.
\end{aligned} \tag{68}$$

The non-trivial contribution from $A_j^z$ is however zero at the physical subspace because of the presence of $\delta[z]$ functional in the path integral measure of (41).

To complete the proof we must show that the leading-order form for $J^{-1}$ is sufficient and that there is no need to go to higher orders. This can be seen, for instance, from the exponentiated form of the Jacobian:

$$
\begin{aligned}
J^{-1} &= e^{\sum_{j=1}^{N} \ln(1+A_j^\zeta \Delta\zeta_j + A_j^z \Delta z_j + B_j^\zeta \Delta p_j^\zeta + B_j^z \Delta p_j^z)} \\
&\approx e^{\sum_{j=1}^{N} (A_j^\zeta \Delta\zeta_j + A_j^z \Delta z_j + B_j^\zeta \Delta p_j^\zeta + B_j^z \Delta p_j^z)} . \qquad (69)
\end{aligned}
$$

From the $\delta$-functionals in the measure we immediately have that $\Delta z_j = \Delta p_j^z = 0$. On the other hand, from the cyclicity of the Hamiltonian in $\zeta$ follows [31] that $\Delta p_j^\zeta = 0$ and $\Delta\zeta_j = O(\epsilon)$, i.e., the Hölder continuity index is 1 rather than 1/2 . So, although there exists a contribution that can potentially bestow a finite quantity on the action, namely

$$
\exp\left[\sum_{j=1}^{N} A_j^\zeta \Delta\zeta_j\right] \to \exp\left[\int_{t_i}^{t_f} dt \; A^\zeta \; \dot{\zeta}\right],
$$

this term is trivial because $A_j^\zeta = 0$ for all $j$.

Similar analysis can be done for the canonical transformation (61), (65). Since the transformation is linear, $F(p_x, p_y, \zeta, z)$ must be quadratic and hence (66) implies that

$$
A_j^\zeta = A_j^z = B_j^\zeta = B_j^z = 0 . \qquad (70)
$$

In this case the Hölder continuity index is 1/2 as usual. So by taking into account the constraints we have $\Delta p_j^z = \Delta z_j = 0$ and $\Delta p_j^\zeta = \Delta\zeta_j = O(\sqrt{\epsilon})$. In general case we would need to consider also terms of order $O(\epsilon)$ since the original Hamiltonian also carries a factor of $\epsilon$ in the action (for our system are higher orders in $\epsilon$ clearly irrelevant and can be omitted). Fortunately, as already mentioned, higher order terms in $J^{-1}$ come from third (and higher) derivatives of $F(p_x, p_y, \zeta, z)$ and hence are identically zero for any linear canonical transformation. Inasmuch the transformation (61), (65) does not produce any Liouville anomaly.

As for **b)**, the situation is simpler in the case of a transforation (51), (53). This is because the transformation yields the Hamiltonian that is cyclic in $\zeta$ and $z$ which by itself ensures [31] that any potential pieces generated in the canonical transformation due to a finite time slicing are of order $O(\epsilon^2)$ and hence disappear in the path integral in the continuous limit.

In the case of transformation (61), (65) the generating function reads

$$
\begin{aligned}
&F(p_x, p_y, \zeta, z) \\
&= \frac{1}{2(a_1^2\,\alpha^2 - 1)} \left[p_x^2\,\alpha + p_y^2\,\alpha + 2\,a_1\,p_x\,\alpha\;(2\,a_1\,z\,\alpha \right. \\
&- p_y\,\alpha + \sqrt{2}\,\zeta\Big) - 2\,a_1\,p_y\,\alpha\,\left(2\,z + \sqrt{2}\,a_1\,\alpha\,\zeta\right) \\
&\left. + 2\,a_1\,\left(\sqrt{2}\,z\,\zeta + \sqrt{2}\,a_1^2\,z\,\alpha^2\,\zeta + a_1\,\alpha\,\left(2\,z^2 + \zeta^2\right)\right)\right] .
\end{aligned}
$$

The new momenta and coordinates then fulfil symmetrized equations [31]

$$
\begin{aligned}
p_j^z &= \frac{a_1}{(1-\,a_1^2\,\alpha^2)}\left[2a_1\alpha^2\,p_x - 2\alpha\,p_y + 4a_1\alpha\,z \right. \\
&+ \left.\left(1+a_1^2\alpha^2\right)\sqrt{2}\,\zeta\right] \\
&- \frac{a_1}{2(1-\,a_1^2\,\alpha^2)}\left[4a_1\alpha\,\Delta z + \sqrt{2}\left(1+a_1^2\alpha^2\right)\,\Delta\zeta\right] , \\
p_j^\zeta &= \frac{\sqrt{2}a_1}{(1-\,a_1^2\,\alpha^2)}\left[\alpha\,p_x - a_1\alpha^2 p_y + \left(1+a_1^2\alpha^2\right)z \right. \\
&+ \left.\sqrt{2}a_1\alpha\,\zeta\right] \\
&- \frac{a_1}{\sqrt{2}(1-\,a_1^2\,\alpha^2)}\left[\left(1+a_1^2\alpha^2\right)\Delta z + \sqrt{2}a_1\alpha\,\Delta\zeta\right] , \\
x_j &= \frac{\alpha}{(1-\,a_1^2\,\alpha^2)}\left[p_x + a_1\left(2a_1\alpha\,z - \alpha\,p_y + \sqrt{2}\,\zeta\right)\right] \\
&+ \frac{\alpha}{2(1-\,a_1^2\,\alpha^2)}\left[a_1\alpha\,\Delta p_y - \Delta p_x\right] , \\
y_j &= \frac{\alpha}{(1-\,a_1^2\,\alpha^2)}\left[p_y - a_1\alpha\,p_x - a_1\left(2\,z + \sqrt{2}a_1\alpha\,\zeta\right)\right] \\
&+ \frac{\alpha}{2(1-\,a_1^2\,\alpha^2)}\left[a_1\alpha\,\Delta p_x - \Delta p_y\right] . \qquad (71)
\end{aligned}
$$

Relations (71) yield $p_x$ and $p_y$ in terms of the new variables. We can now utilize the leading order Taylor expansions

$$
\begin{aligned}
\Delta p_x &= -\tfrac{1}{2}\,\Delta p^z + \frac{1}{\sqrt{2}a_1\alpha}\,\Delta p^\zeta - \frac{1}{\alpha}\,\Delta z - \frac{a_1}{\sqrt{2}}\,\Delta\zeta , \\
\Delta p_y &= -\frac{1}{2a_1\alpha}\,\Delta p^z + \frac{1}{\sqrt{2}}\,\Delta p^\zeta + a_1\,\Delta z + \frac{1}{\sqrt{2}\alpha}\,\Delta\zeta ,
\end{aligned}
$$

and substitute (71) into the old Hamiltonian. After imposing the constraints $z_j = \Delta z_j = p_j^z = \Delta p_j^z = 0$ we obtain

$$
\begin{aligned}
&(xp_y - yp_x)_j \\
&\quad\to \frac{1}{2a_1}\,(p_j^\zeta)^2 + \frac{a_1}{2}\,\zeta_j^2 - \frac{1}{4a_1\alpha}(\alpha\,p_j^\zeta + \zeta_j)\Delta p_j^\zeta \\
&\quad + \left(\frac{a_1}{4}\,\zeta_j - \frac{1}{4a_1\alpha}\frac{(1+7a_1^2\alpha^2)}{(a_1^2\alpha^2-1)}p_j^\zeta\right)\Delta\zeta_j + O(\epsilon) . \qquad (72)
\end{aligned}
$$

Because $\Delta p_j^\zeta = \Delta\zeta_j = O(\sqrt{\epsilon})$, the contribution of the correction terms to the action is of order $O(\epsilon^{3/2})$ which means that such terms are suppressed in the continuous limit.

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