Thermally induced rotons in two-dimensional dilute Bose gases

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Abstract

We show that roton-like excitations are thermally induced in a two-dimensional dilute Bose gas as a consequence of the strong phase fluctuations in two dimensions. At low momentum, the roton-like excitations lead for small enough temperatures to an anomalous phonon spectrum with a temperature dependent exponent reminiscent of the Kosterlitz-Thouless transition. Despite the anomalous form of the energy spectrum, it is shown that the corresponding effective theory of vortices describes the usual Kosterlitz-Thouless transition. The possible existence of an anomalous normal state in a small temperature interval is also discussed.

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I. INTRODUCTION

Superfluidity in interacting Bose systems has been a fascinating research topic for several decades. After the achievement of Bose-Einstein condensation in dilute atomic gases,[1] many of the remarkable properties of superfluid ^{4}He were also observed in these weakly interacting systems, such as vortices and Josephson oscillations.[2] In order to understand their superfluid properties, it is necessary to clarify the role of the elementary excitations in determining these properties. According to a celebrated criterion due to Landau, a Bose gas without interactions cannot be a superfluid, since its spectrum $\varepsilon_p = p^2/2m$ makes it indifferent to Galilei transformations. A superfluid, on the other hand, is resistant to a slowdown of the molecules due to the loss of Galilei invariance. A Bose system with the relativistic looking excitation spectrum $E_p = cp$ has this property, thus being a superfluid by Landau's argument. Thus, phonon-like excitation spectra are an essential part of a superfluid.

In superfluid helium, which is a strongly interacting Bose system, the interatomic potential has an attractive short-range part over a distance a_0, the average interatomic distance. This sets the scale of a further important set of excitations. Scattering experiments of neutrons show that around a momentum $p_0 \approx 1/a_0$ the spectrum behaves like $E_p = \Delta + (p - p_0)^2/2m^*$. These elementary excitations are called rotons.

Feynman[3] was the first to recognize the importance of the rotons for a superfluid. In modern language, Feynman's theory describes rotons as a result of large *quantum* phase fluctuations at low temperatures. These produce small vortex loops of size $\sim a_0$. At higher temperatures where thermal fluctuations take over, the vortex loops combine to larger *thermally* excited vortex loops which profit from the high configurational entropy of line-like excitations. At the critical λ-point, these loops become infinitely long and destroy the order of the superfluid.[4]

In two dimensions, phase fluctuations are so strong that they destroy the long-range order at any temperature.[5] As noted first by Kosterlitz and Thouless (KT),[6] there still exists a phase transition driven by phase fluctuations. At low temperature, a film of superfluid ^{4}He contains vortex- antivortex pairs bound by Coulomb attraction, whose unbinding causes the KT transition. One of the most important predictions of this theory is the universal jump to zero of the superfluid stiffness at the critical temperature.[7] This destroys the excitations of energy $E = cp$, and thus the superfluidity.

The increased relevance of phase fluctuations in two dimensions suggests that Feynman's rotons are more abundant than in three dimensions. We want to argue that this is indeed the case: in spite of the weak interactions, a dilute Bose gas possesses roton-like excitations which are the precursors of the high-temperature vortices that lead to a KT phase transition. This is quite remarkable since, contrary to superfluid helium, the bare interaction does not contribute with an attractive part to fix the size of p_0.

The plan of the paper is as follows. In Sect. II we briefly review the known results of the t-matrix formalism in d dimensions for dilute Bose gases. Sect. III contains most of the main results of the paper. There we apply the so called dielectric formalism[8,9] to the two-dimensional dilute Bose gas. We will essentially work out a RPA approximation in two dimensions. Since the dielectric formalism is well known in three-dimensional applications, we will not discuss its derivation here, referring the reader instead to the literature. The focus will be in the application of the method to two dimensions. From our calculations a new, temperature dependent, excitation spectrum emerges, namely,

$$E_p = \sqrt{\varepsilon_p^2 + 2g_2 n\varepsilon_p \left[1 - \frac{Tm}{\pi n}\ln(pa)\right]}. \tag{1}$$

We will show that the above spectrum allows for thermally excited roton-like excitations and that its low momentum behavior exhibits an anomalous power behavior with a temperature dependent exponent $\eta_0(T)$. Remarkably, in the two-dimensional dilute Bose gas at finite temperature, the RPA analysis will lead to closed form analytic results including roton-like excitations. In Sect. IV we discuss the effect of phase fluctuations in the system. Our analysis in this Section will allow us to derive the approximate critical temperature of the system as $\eta_0(T_c) = 1/4$. The actual critical temperature is determined in the usual way following the Kosterlitz-Thouless vortex unbinding mechanism.[6] It will also be shown that a crossover temperature T_* exists, above which our anomalous spectrum becomes unstable. Sect. V concludes the paper. The whole discussion is supplemented with two Appendices containing calculational details.

II. THE EFFECTIVE INTERACTION AT ZERO TEMPERATURE

In general the effective interaction for any dimension $d < 4$ depends on a momentum scale $\bar{p}$ associated to the energy of the scattered particles. At zero temperature the effective interaction in d dimensions reads

$$g(\bar{p}) = \frac{4\pi^{d/2}a^{d-2}/m}{2^{2-d}\Gamma(1-d/2)(\bar{p}a)^{d-2} + \Gamma(d/2-1)}, \tag{2}$$

where m is the reduced mass of the scattered particles and a is the s-wave scattering length. The above result is essentially exact at zero temperature and corresponds to a geometric sum of ladder diagrams which gives the only nonvanishing contribution to the vertex function.[10]

From Eq. (2) we derive the exact renormalization group (RG) β-function for the dimensionless coupling $\tilde{g}(\bar{p}) = m\bar{p}^{d-2}g(\bar{p})$

$$\beta(\tilde{g}) \equiv \bar{p}\frac{\partial \tilde{g}}{\partial \bar{p}} = (d-2)\left[\tilde{g} + \frac{d\,\Gamma(-d/2)}{2^{d+1}\pi^{d/2}}\tilde{g}^2\right]. \tag{3}$$

For $2 \le d < 4$ the only fixed point is $\tilde{g}_* = 0$. On the other hand, for $d < 2$ a nontrivial fixed point located at

$$\tilde{g}_* = -\frac{2^{d+1}\pi^{d/2}}{d\Gamma(-d/2)} \tag{4}$$

exists. Thus, $d = 2$ is the upper critical dimension for the dilute Bose gas. Writing $d = 2 - \epsilon$ and expanding Eq. (4) for small ϵ, we obtain

$$\tilde{g}_* \approx 2\pi\epsilon. \tag{5}$$

In the dimension interval $2 < d < 4$ we can easily take the limit $\bar{p}a \to 0$ in Eq. (2) to obtain

$$g_0 = \frac{4\pi^{d/2}a^{d-2}}{\Gamma(d/2-1)m}. \tag{6}$$

For $d = 3$ the above equation reproduces the familiar formula $g_0 = 4\pi a/m$. For $d = 2$, however, we cannot set $\bar{p}a = 0$. In two dimensions Eq. (2) becomes

$$g_{2\text{D}}(\bar{p}) = \frac{2\pi/m}{\ln 2 - \gamma - \ln(\bar{p}a)}, \tag{7}$$

where γ is the Euler constant. Eq. (7) is in agreement with previous work.[11,12]

The first theories for the two-dimensional dilute Bose gas were developed by Popov[13] and Schick.[14] More recently, an improved version of Popov's theory was presented by Stoof and collaborators.[15,16] In Popov's approach,[13] the bare interaction $g_0\delta(\mathbf{x})$ is replaced by an effective interaction $g\delta(\mathbf{x})$ determined by a t-matrix, leading in d dimensions with $d \in [2, 4)$ to[17]

$$g_d = \frac{4\pi^{d/2}a^{d-2}/m}{2^{2-d}\Gamma(1-d/2)(na^d)^{d-2} + \Gamma(d/2-1)}. \tag{8}$$

where n is the density and a the s-wave scattering length. The above interaction corresponds precisely to the one given in Eq. (2), where we have set $\bar{p}a = na^d$. For $d \to 2$ we obtain the two-dimensional coupling constant of the Popov-Schick theory[13,14]

$$g_2 \equiv \lim_{d\to 2} g = -\frac{2\pi/m}{\ln(e^\gamma na^2/2)}, \tag{9}$$

where γ is the Euler-Mascheroni constant. The logarithm in the denominator implies that the effective repulsion decreases only very slowly with decreasing density.[17] Fisher and Hohenberg[17] have shown within the Popov-Schick theory that the dilute limit $na^d \ll 1$ of the $d > 2$ theory must be replaced for $d = 2$ by $\ln\ln(1/na^2) \gg 1$.

III. THE DIELECTRIC FORMALISM IN $d = 2$

The t-matrix result (2) incorporates, via a Lippmann-Schwinger integral equation, the sum of all ladder diagrams. In this Section we take into account the sum of bubble diagrams of the plasmon type,[18] which are nonvanishing at finite temperature. This corresponds to the random phase approximation (RPA), which sums geometrically the diagrams shown in Fig. 1. This approximation has often been applied in $d = 3$ dimensions.[8,9] In RPA the vertex function containing the effects of both resummations is given explicitly in d dimensions by

$$\Gamma(i\omega, \mathbf{p}) = \frac{g_d}{1 - g_d\widetilde{\Pi}(i\omega, \mathbf{p})}, \tag{10}$$

where the polarization bubble is given by

$$\widetilde{\Pi}(i\omega, \mathbf{p}) = \frac{1}{\beta}\sum_n \int \frac{d^d q}{(2\pi)^d} G_0(i\omega + i\omega_n, \mathbf{p} + \mathbf{q}) G_0(i\omega_n, \mathbf{q}), \tag{11}$$

with $G_0(i\omega, \mathbf{p}) = 1/(i\omega - \varepsilon_p)$. The chemical potential is canceled by the Hartree contribution.[17] The denominator of (11) determines the *regular* part $\epsilon_r(i\omega, \mathbf{p})$ of the dielectric function $\epsilon(i\omega, \mathbf{p})$ renormalizing the density correlation function, i.e.,

$$\chi_{\rho\rho}(i\omega, \mathbf{p}) = \frac{\Pi(i\omega, \mathbf{p})}{\epsilon(i\omega, \mathbf{p})} \tag{12}$$

where

$$\epsilon(i\omega, \mathbf{p}) = 1 - g_d[\Pi_0(i\omega, \mathbf{p}) + \widetilde{\Pi}(i\omega, \mathbf{p})], \tag{13}$$

with

$$\Pi_0(i\omega, \mathbf{p}) = \frac{2n\varepsilon_p}{(i\omega)^2 - \varepsilon_p^2}, \tag{14}$$

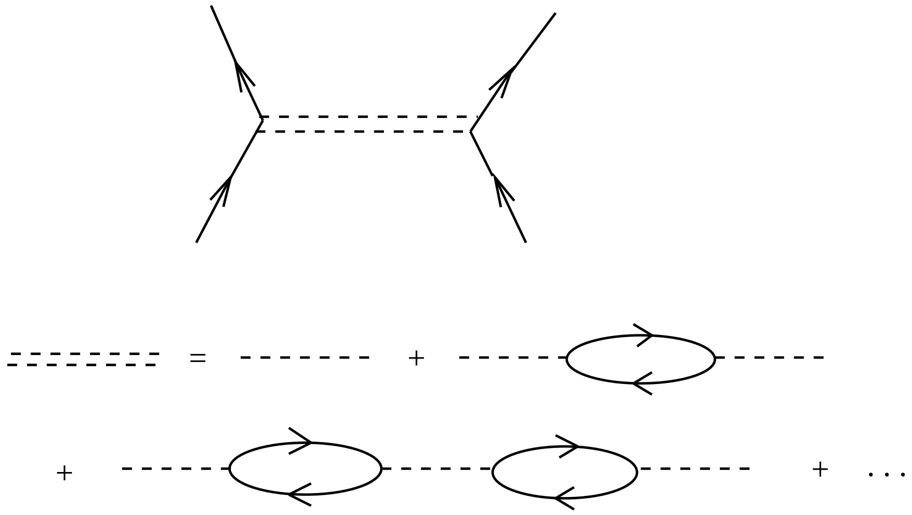

FIG. 1: Feynman diagram representation of the vertex function Eq. (10). The dashed line represents the bare interaction. The vertex function is obtained as a geometric series of polarization bubbles.

contributing to the singular part of the dielectric function.[9] The regular contribution, being given by a particle-hole diagram, vanishes at zero temperature. We will treat $\widetilde{\Pi}(i\omega, \mathbf{p})$ in the so called classical approximation, where the Bose-Einstein distribution function $n_B(x) \equiv 1/(e^{\beta x} - 1)$ is replaced by $n_B(x) \approx T/x$. Thus, the classical approximation is valid in the limit $p\lambda_T \ll 1$, where $\lambda_T = (2mT)^{-1/2}$ is the thermal wavelength. Therefore, we obtain in the limit $p\lambda_T \ll 1$ the result

$$\widetilde{\Pi}(i\omega, \mathbf{p}) = -A_d m^2 T p^{d-4} \left[e^{i\pi(d-2)/2} \left(1 - i\omega/\varepsilon_p\right)^{d-3} + e^{-i\pi(d-2)/2} \left(1 + i\omega/\varepsilon_p\right)^{d-3} \right], \qquad (15)$$

with $A_d = 2^{2-d}\pi^{-d/2}\Gamma(d/2-1)\Gamma(3-d)$. A derivation of Eq. (15) using Feynman parameters is given in the Appendix. For $d = 3$, this reduces to the well-known result[8,9]

$$\widetilde{\Pi}(i\omega, \mathbf{p}) \stackrel{d=3}{=} -i\frac{Tm^2}{2\pi p} \ln\left(\frac{i\omega + \varepsilon_p}{i\omega - \varepsilon_p}\right). \qquad (16)$$

In $d = 2$ dimensions Eq. (15) has a pole associated with a logarithmic short distance divergence. We will remove this divergence using the s-wave scattering length as short-distance cutoff. This leads to

$$\widetilde{\Pi}(i\omega, \mathbf{p}) \stackrel{d=2}{=} -\frac{2mT}{\pi} \frac{\varepsilon_p \ln(pa)}{(i\omega)^2 - \varepsilon_p^2}. \qquad (17)$$

The spectrum of elementary excitations is obtained from the poles of $\chi_{\rho\rho}(i\omega, \mathbf{p})$.[9,19,20] At zero temperature, only $\Pi_0(i\omega, \mathbf{p})$ contributes, and we recover the usual Bogoliubov spectrum.

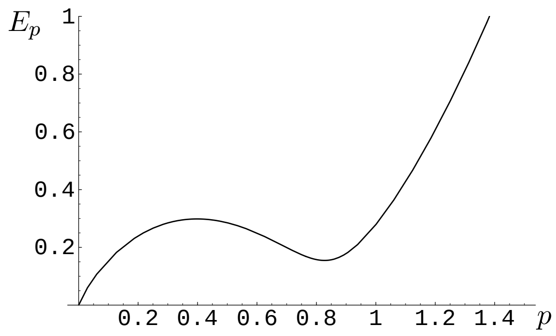

FIG. 2: The RPA corrected excitation spectrum of Popov-Schick theory exhibiting a roton-like minimum. The parameters in a system of units such that $k_B = \hbar = 1$ are $m = 0.5$ cm^{-2} sec, $n = 0.01$ cm^{-2}, $T = 1.2$ sec^{-1}, and $a = 2$ cm.

The excitation spectrum is obtained from the pole of the density correlation function, which corresponds to the vanishing of the dielectric constant (13). In this way we obtain the excitation spectrum announced in the Introduction:

$$E_p = \sqrt{\varepsilon_p^2 + 2g_2 n \varepsilon_p \left[1 - \frac{Tm}{\pi n}\ln(pa)\right]}. \tag{18}$$

The above energy spectrum possesses a thermally induced roton-like minimum. The excitation spectrum is shown in Fig. 2 for a suitable set of parameters.

The approximate position of the roton minimum can be determined in the following way using the Landau criterion for superfluidity. According to Landau criterion, the critical velocity above which excitations appear in the fluid is given by the minimal value of the ratio E_p/p. This minimum value corresponds to the point $p = p_0$ for which

$$\frac{dE_p}{dp} = \frac{E_p}{p}. \tag{19}$$

On the other hand, the energy spectrum (18) satisfies the equation:

$$\frac{dE_p}{dp} = \frac{E_p}{p} + \frac{\varepsilon_p}{pE_p}\left(\varepsilon_p - \frac{Tmg_2}{\pi}\right). \tag{20}$$

The point p_0 for which Eqs. (19) and (20) coincide is determined by the vanishing of the second term on the right-hand side of Eq. (20), i.e.,

$$p_0(T) = \frac{1}{a_T} = m\sqrt{\frac{2Tg_2}{\pi}}. \tag{21}$$

The above value of p corresponds approximately to the position of the roton-like minimum of the excitation spectrum (18). The T-dependent length scale a_T nearly replaces the a_0 of bulk ^{4}He.

The consistency of the calculation leading to the spectrum (18) can be checked by computing the spectrum also from the pole of the anomalous propagator. It is well known that the pole of the propagator should be the same as the one from the density correlation function,[9,20] although most approximations fail to fulfill this requirement. After analytically continuing to real frequencies, we obtain that the pole of the propagator is given by the solution of the equation[8,9]

$$\omega^2 - \varepsilon_p^2 - 2n\varepsilon_p\Gamma(\omega, \mathbf{p}) = 0. \tag{22}$$

The above equation is a generalization of the Bogoliubov result for the excitation spectrum. Indeed, it corresponds to an improvement of the Bogoliubov result where the coupling constant g_2 is replaced by the vertex function (10).[8] Solution of Eq. (22) with the polarization bubble (17) gives precisely the energy spectrum (18).

The excitation spectrum (18) has another interesting property in the low-momentum regime, which for $mg_2/\pi < 1$ is defined by

$$p^2 \ll p_0^2(T) = \frac{2m^2Tg_2}{\pi} < \frac{1}{\lambda_T^2} = 2mT \ll 4\pi n. \tag{23}$$

In the above low-momentum regime the logarithm gives rise to an *anomalous* power behavior

$$E_p \approx \sqrt{\frac{g_2 n}{m}} a^{-Tm/2\pi n} p^{1-Tm/2\pi n}. \tag{24}$$

The exponent of p in (24) can be rewritten as

$$\sigma(T) \equiv 1 - \frac{Tm^2}{2\pi\rho_s(0)} = 1 - \eta_0(T), \tag{25}$$

where $\rho_s(0) = mn$ is the superfluid mass density at zero temperature, and $\eta_0(T) \equiv Tm^2/2\pi\rho_s(0)$. Interestingly, precisely the same exponent arises in the spin wave treatment of *classical* phase fluctuations in the study of the KT transition. We will revisit this analysis in the next Section, where it will also be shown that despite the anomalous scaling of the spectrum, the KT transition occurs as usual at higher temperatures, in agreement with the discussion of Ref. 16. This is to be expected, since the KT transition actually occurs through a vortex-antivortex unbinding mechanism.

The elementary excitations described by Eq. (24) are stable only for $\eta_0(T) < 1$, i.e., for $T < 2\pi\rho_s(0)/m^2 \equiv T_*$. Note that the low-momentum inequality (23) already requires $T \ll T_*$. But this condition can be softened to just $T < T_*$, in which case Eq. (24) is still approximately valid up to logarithmic corrections. The role of the temperature T_* will be discussed later.

It is instructive to calculate the low-momentum contribution to the superfluid density following from (24). Since vortices are not included in the above calculation, the result will show only the anomalous phonon contribution to the so called background superfluid density, ρ_{s0}. The calculation is based on the Landau prescription,[2] according to which the superfluid density is $\rho_{s0}^{\rm ph}(T) = \rho - \rho_{n0}^{\rm ph}(T)$, where $\rho_{n0}^{\rm ph}(T)$ is the anomalous phonon contribution to the normal fluid density. From the Landau formula the normal background fluid density is given by

$$\rho_{n0} = \frac{\beta}{d}\int \frac{d^dp}{(2\pi)^d}\frac{p^2 e^{\beta E_p}}{(e^{\beta E_p}-1)^2}. \qquad (26)$$

Now we set $d = 2$ and insert the anomalous phonon spectrum (24) in Eq. (26) to obtain

$$\rho_{n0}^{\rm ph}(T) \approx \frac{T^{[4-\sigma(T)]/\sigma(T)}}{8\pi^2}\frac{4-\sigma(T)}{\sigma(T)\tilde{c}^{\;4/\sigma(T)}}\Gamma\left[\frac{4-\sigma(T)}{\sigma(T)}\right]\zeta\left[\frac{4-\sigma(T)}{\sigma(T)}\right], \qquad (27)$$

where $\tilde{c} \equiv (g_2 n/m)^{1/2} a^{-\eta_0(T)}$. By expanding this in powers of T, we obtain to leading order the usual phonon contribution, which in two dimensions is proportional to T^3. In deriving Eq. (27) we have assumed the usual hydrodynamic limit where the upper cutoff — here $p_0(T)$ — is taken to be infinity.

IV. PHASE FLUCTUATIONS

By integrating out small density fluctuations in the hydrodynamic limit, we obtain the following effective action for the phase fluctuations:

$$S_{\rm eff} = \int_0^\beta d\tau \left[\frac{1}{2g_2}\int d^2r(\partial_\tau\theta)^2 + H_{\rm eff}\right], \qquad (28)$$

where the effective Hamiltonian contains a local and a nonlocal interaction between the superfluid velocities $\mathbf{v}_s(\tau,\mathbf{r}) = \nabla\theta(\tau,\mathbf{r})/m$:

$$H_{\rm eff} = H_{\rm eff}^{\rm local} + H_{\rm eff}^{\rm non-local}, \qquad (29)$$

where

$$H_{\text{eff}}^{\text{local}} = \frac{mn}{2} \int d^2r \mathbf{v}_s^2(\tau, \mathbf{r}), \tag{30}$$

and

$$H_{\text{eff}}^{\text{non-local}} = \frac{1}{2} \int d^2r \int d^2r' \mathcal{M}(\mathbf{r} - \mathbf{r}')\ \mathbf{v}_s(\tau, \mathbf{r}) \cdot \mathbf{v}_s(\tau, \mathbf{r}'), \tag{31}$$

with

$$\mathcal{M}(\mathbf{r} - \mathbf{r}') = \frac{mn(2a)^{-2\eta_0(T)}\Gamma[1 - \eta_0(T)]}{\pi\Gamma[\eta_0(T)]|\mathbf{r} - \mathbf{r}'|^{2[1-\eta_0(T)]}}, \tag{32}$$

being a bilocal mass density, which is obtained from the Fourier transform of $\mathcal{M}(p) = nm(pa)^{-2\eta_0(T)}$. In the folowing discussion we will show that at high temperatures the non-local part contributes only in a small temperature interval above T_c. Below T_c the usual effective Hamiltonian for the phase fluctuations given by Eq. (30) dominates the critical behavior and the KT transition obtains. The arguments to be described below consider the scaling behavior of the spin wave theory for $H_{\text{eff}}^{\text{non-local}}$ and the field theory for the vortices, which consists of a generalized sine-Gordon theory.

Let us consider first the case without the RPA correction, i.e., in the absence of the non-local effective Hamiltonian. This just corresponds to the usual phonon spectrum. In such a situation the effective Hamiltonian is given simply by Eq. (30). The spin wave analysis is in this case well known.[6,21] However, it is useful to review it here in order to compare with the non-local spin wave regime.

At higher temperatures we can neglect the higher Matsubara modes so that $\partial_\tau\theta = 0$ and the effective action becomes

$$S_{\text{eff}} = \frac{n}{2mT} \int d^2r [\nabla\theta(\mathbf{r})]^2, \tag{33}$$

and the problem is essentially a classical one. Let us recall the computation of the correlation function $\langle\psi(\mathbf{r})\psi^*(\mathbf{r}')\rangle$ in this regime and in the absence of vortices (spin-wave theory).[6,21] In such a case we just have to compute the correlation between the phases:

$$\langle e^{i[\theta(\mathbf{r})-\theta(\mathbf{r}')]}\rangle = \frac{1}{Z} \int \mathcal{D}\theta e^{-\int d^2r''\left\{\frac{n}{2mT}[\nabla\theta(\mathbf{r}'')]^2 + iJ(\mathbf{r}'')\theta(\mathbf{r}'')\right\}}, \tag{34}$$

where $J(\mathbf{r}'') = \delta^2(\mathbf{r}'' - \mathbf{r}) - \delta^2(\mathbf{r}'' - \mathbf{r}')$. The Gaussian integral is straightforward and yields

$$\langle e^{i[\theta(\mathbf{r})-\theta(\mathbf{r}')]}\rangle = \exp\left\{\frac{mT}{n}[G(\mathbf{r} - \mathbf{r}') - G(0)]\right\}, \tag{35}$$

where

$$G(\mathbf{r}) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p^2}. \tag{36}$$

In order to evaluate the above Green function we introduce the regularized Green function

$$G_{M^2}(\mathbf{r}) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p^2 + M^2}, \tag{37}$$

where M is a regularizing mass to be sent to zero at the end of the calculations. Evaluating the integral explicitly, we obtain

$$G_{M^2}(\mathbf{r}) = \frac{1}{2\pi} K_0(Mr), \tag{38}$$

where $K_0(y)$ is the modified Bessel function of the second kind. On the other hand, we have

$$G_{M^2}(0) = \frac{1}{2\pi} \ln\left(\frac{\Lambda}{M}\right), \tag{39}$$

where $\Lambda = \pi/a$ is the ultraviolet cutoff. Now we can safely take the limit $M \to 0$:

$$G(\mathbf{r}) - G(0) = \lim_{M\to 0} [G_{M^2}(\mathbf{r}) - G_{M^2}(0)] = -\frac{1}{2\pi} \ln\left(\frac{r}{a}\right) + \text{const.} \tag{40}$$

Therefore,

$$\langle e^{i[\theta(\mathbf{r})-\theta(\mathbf{r}')]}\rangle \sim \frac{1}{|\mathbf{r}-\mathbf{r}'|^{\eta_0(T)}}. \tag{41}$$

Note that precisely the exponent $\eta_0(T)$ that we have defined in Eq. (25) arises in the above equation. However, the mechanism that generates the anomalous behavior in the above classical spin-wave theory is completely different from the quantum case discussed in the previous Section. Indeed, there the anomalous scaling of the spectrum arises due to interaction effects, while in the above calculation it follows from the analytic properties of the Green function of a Gaussian classical theory in two dimensions.

Let us now study the classical problem associated to the effective Hamiltonian at large distances. In this case (31) dominates, since the corresponding power of p is smaller than two. Once more we neglect the higher Matsubara modes to obtain the effective action for the classical problem as

$$S_{\text{eff}} = \frac{1}{2T} \int d^2r \int d^2r' \mathcal{M}(\mathbf{r}-\mathbf{r}')\ \mathbf{v}_s(\mathbf{r}) \cdot \mathbf{v}_s(\mathbf{r}'), \tag{42}$$

where $\mathbf{v}_s(\mathbf{r}) \equiv \mathbf{v}_s(0,\mathbf{r})$. The correlation function between the phases has again the form (35), except that the Green function is now given by

$$G(\mathbf{r}) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p^2\mathcal{M}(p)}. \tag{43}$$

We have

$$G(\mathbf{r}) = \frac{a^{2\eta_0(T)}}{(2\pi)^2}\int_0^{2\pi} d\phi \int_0^\infty dp \frac{e^{ipr\cos\phi}}{p^{1-2\eta_0(T)}}$$
$$= \frac{a^{2\eta_0(T)}}{2\pi}\int_0^\infty dp \frac{J_0(pr)}{p^{1-2\eta_0(T)}}, \quad (44)$$

where $J_0(y)$ is a Bessel function of the first kind. As before, we use a ultraviolet cutoff $\Lambda = \pi/a$ to evaluate $G(0)$:

$$G(0) = \frac{\pi^{2\eta_0(T)}}{4\pi\eta_0(T)}. \quad (45)$$

We obtain finally

$$G(\mathbf{r}) - G(0) = \frac{1}{4\pi}\left\{\left(\frac{2a}{r}\right)^{2\eta_0(T)} \frac{\Gamma[\eta_0(T)]}{\Gamma[1-\eta_0(T)]} - \frac{\pi^{2\eta_0(T)}}{\eta_0(T)}\right\}, \quad (46)$$

which for small $\eta_0(T)$ can be rewritten as

$$G(\mathbf{r}) - G(0) \approx \frac{1}{4\pi\eta_0(T)}\left[\left(\frac{2a}{r}\right)^{2\eta_0(T)} - 1\right], \quad (47)$$

and in the $\eta_0(T) \to 0$ limit Eq. (40) is obviously reproduced.

For arbitrary values of $0 < \eta_0(T) < 1$ it is not obvious to see how the KT theory is recovered when the anomalous phonon spectrum is taken into account. Recall that the logarithmic behavior in Eq. (40) is crucial in the KT argument in the presence of vortices. Indeed, a simple scaling argument with free energy of the vortices combined with the results of spin-wave theory allows us to determine the value of $\eta_0(T)$ at the critical temperature T_c.[6] A more elaborate argument[7,21] using the RG shows that the stiffeness is renormalized and $\eta_0(T)$ becomes $\eta(T) = Tm^2/2\pi\rho_s(T)$. However, the value of $\eta(T)$ at T_c is the same as the value of $\eta_0(T)$ at T_c. Such an RG analysis led to the cellebrated prediction of a universal jump of $\rho_s(T)$ as T_c is approached from below.[7] At first sight we may think that an anomalous phonon spectrum would disrupt the whole argumentation due to the form of the corresponding non-local hydrodynamics (31). In the following we will show that this *is not* the case. To this end it is necessary to proceed in two steps. First, we make an analysis about the validity of the hydrodynamic description given by (31), which will provide us with a lower bound for $\eta_0(T)$. Second, we consider a careful analysis of the vortex field theory associated to this problem. We will see that the anomalous contribution to the vortex field theory becomes irrelevant at large distances and that, effectively, the *same* vortex field theory as in the KT case holds. This result will in turn provide us with a physical interpretation of the lower bound for $\eta_0(T)$.

Let us then start by discussing in more detail the effective Hamiltonian (31). In order to define the mass density (32) we have to be able to Fourier transform $\mathcal{M}(p) = nm(pa)^{-2\eta_0(T)}$. Such a Fourier transformation is only possible if $1/4 < \eta_0(T) < 1$. To see this we consider the integral

$$\begin{aligned} I(r) &= \frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \int_0^{\infty} dp\, \frac{p\, e^{ipr\cos\phi}}{p^{2\eta_0(T)}} \\ &= \frac{1}{2\pi}\int_0^{\infty} dp \frac{J_0(pr)}{p^{2\eta_0(T)-1}} \\ &= \frac{1}{2\pi\, r^{2[1-\eta_0(T)]}}\int_0^{\infty} dy \frac{J_0(y)}{y^{2\eta_0(T)-1}}. \end{aligned} \tag{48}$$

The power of r in the second line of Eq. (48) must be positive in order to make the above result well defined at large distances. This gives us once more the upper bound for $\eta_0(T)$, i.e., $\eta_0(T) < 1$. The lower bound for $\eta_0(T)$ follows from the asymptotic behavior of the Bessel function for y large,

$$J_0(y) \sim \sqrt{\frac{2}{\pi y}}\cos(y - \pi/4). \tag{49}$$

Thus, the integrand in Eq. (48) behaves for large y like $\sim 1/y^{2\eta_0(T)-1/2}$ and it follows that in order to avoid a power-like ultraviolet divergence we must have $2\eta_0(T) - 1/2 > 0$, which leads to $\eta_0(T) > 1/4$. Therefore, in order to have a well defined mass density $\mathcal{M}(\mathbf{r}-\mathbf{r}')$ the inequality $1/4 < \eta_0(T) < 1$ has to be fulfilled. Note that the upper bound is associated to a large distance (infrared) divergence while the lower bound is to short distance (ultraviolet) divergence. We have already seen that saturation of the upper bound leads to the determination of the temperature T_*. On the other hand, we will give arguments below showing that saturation of the lower bound determines the actual critical temperature of the system.

At higher temperatures, near the phase transition, we are allowed to keep only the zero Matsubara mode, such that the field theory becomes two-dimensional. The vortices are introduced in the standard way,[4,7] and a duality transformation gives the following generalized sine-Gordon action for the dual field theory of vortices:

$$S_{\rm dual} = \int d^2r \int d^2r' \left[\frac{T}{8\pi^2}\Gamma^{-1}(\mathbf{r}-\mathbf{r}')\partial_{\mathbf{r}}\varphi(\mathbf{r})\cdot\partial_{\mathbf{r}'}\varphi(\mathbf{r}') - z\,\delta^2(\mathbf{r}')\cos\varphi(\mathbf{r})\right], \tag{50}$$

where $\Gamma(\mathbf{r}-\mathbf{r}') = mn\delta^2(\mathbf{r}-\mathbf{r}') + \mathcal{M}(\mathbf{r}-\mathbf{r}')$ and z is the fugacity of the gas of point vortices. A derivation of the action (50) is given in Appendix B. Nonlocal gradient terms in sine-Gordon theory have recently been discussed in a different context,[22,23] where a renormalization group (RG) analysis shows that a local gradient term $(\nabla\varphi)^2$ with a *positive* coefficient is generated.[23] The nonlocal contributions can be rewritten as a sum of higher-gradient terms which are all irrelevant at large distances. Let us state this in more simple terms. In momentum space the non-local gradient has the form

$$\frac{a^{2\eta_0(T)}T}{8\pi^2 nm}\int\frac{d^2p}{(2\pi)^2}\,p^{2[1+\eta_0(T)]}\varphi(\mathbf{p})\varphi(-\mathbf{p}). \tag{51}$$

Thus, as a local gradient term is generated, we have that for small p (large distances) the p^2-term dominates over the $p^{2[1+\eta_0(T)]}$ one and the anomalous contribution can be neglected. This argument shows that the dual theory (50) actually has a KT transition, since the fluctuation generated p^2-term is dominant in the infrared, leading effectively to a sine-Gordon theory of the usual type. Thus, our theory will ultimately be in agreement with the analysis of Ref. 16 where no anomalous dimension arising from RPA corrections is considered.

In the low-temperature phase, where $0 < \eta_0(T) \leq 1/4$, the hydrodynamic description via the effective Hamiltonian (31) breaks down, since the inverse Fourier transform of $\mathcal{M}(p)$ is no longer defined. This regime is equivalent to the one in the dual theory where the non-local gradient term becomes irrelevant. This means that the effective Hamiltonian of the classical theory in this range of temperatures is actually given by (30). In other words, when the anomalous sine-Gordon theory becomes at large distances effectively the usual sine-Gordon theory (remember that a local gradient term is generated by fluctuations), we can dualize it back to obtain the *effective* original theory as given by Eq. (30).

In a KT transition the critical temperature T_c is determined from the equation $\eta(T_c) = 1/4$, or equivalently, $T_c = \pi\rho_s(T_c)/(2m^2)$.[7] Note that $\eta_0(T)$ corresponds to a low-temperature approximation to $\eta(T)$. Thus, we can interpret that saturating the lower bound in the spin-wave inequality $1/4 < \eta_0(T) < 1$ leads to an approximate value of the actual critical temperature for the KT transition. Therefore, we have that

$$T_c \approx \frac{\pi\rho_s(0)}{2m^2}. \tag{52}$$

We want to emphasize that the agreement of the value of $\eta_0(T)$ at the lower bound of the inequality $1/4 < \eta_0(T) < 1$ with the value $\eta(T_c) = 1/4$ obtained from the KT theory is not a simple coincidence. It follows from the fact that the temperature at which $\eta_0(T) = 1/4$ corresponds to the onset of the regime where the non-local term of the generalized sine-Gordon theory becomes irrelevant.

We have obtained $\eta_0(T)$ as a correction to the power of the excitation spectrum at finite temperature by using a *classical* approximation to evaluate the density correlation function. Thus, we have simply accounted for a classical effect in a quantum calculation, i.e., our calculation of the spectrum at finite temperature is actually a semi-classical one. In the KT theory the value $\eta(T) = 1/4$ is follows due to a conspiracy between spin-wave theory and the statistical mechanics of vortices.[6] In our case, a similar result obtains: our bound for $\eta_0(T)$ is derived by analysing the non-local spin-wave theory, while the conclusion that $\eta_0(T) = 1/4$ determines the critical temperature needs additional analysis involving the vortex field theory (50). A more technical explanation follows by recalling that density and *longitudinal* phase fluctuations are related through the Ward identities.[20] More precisely, in Ref. 20 it is shown that the density correlation function is related through an exact identity to the longitudinal component of the current correlation function, which corresponds to the response of the system to the longitudinal phase fluctuations or, in other words, spin-waves. Since $\eta_0(T)$ is determined by the density correlation function, by taking the classical limit of this identity we obtain a relation between $\eta_0(T)$ and the spin-wave response. The vortices, on the other hand, are related to the *transverse* phase fluctuations. It can be shown[20] that the transverse component of the current correlation function decouples from all other correlation functions. This result holds also in the classical limit and that is the reason why the classical statistical mechanics of vortices alone can determine the phase transition.

From the lower and upper bound to $\eta_0(T)$ we obtain that the actual T_c obtained from the KT theory is smaller than T_*. Indeed, we have that

$$T_c = \frac{\pi \rho_s(T_c)}{2m^2} < \frac{\pi \rho_s(0)}{2m^2} < \frac{2\pi \rho_s(0)}{m^2} = T_*. \tag{53}$$

Since $T_c < T_*$, it seems to exist a region $T_c < T < T_*$ where the system is not a superfluid, while not being a normal fluid either, due to the anomalous exponent. This region would be actually very small: from the inequality (23) we see that a more correct statement would be to claim the existence of a temperature region $T_c < T \ll T_*$ where an anomalous normal state occurs. This state can be thought as a quasi-condensate *without* phase coherence. In this respect, it bears some resemblance with the pseudogap state in high-T_c superconductors,[24] in which the phase fluctuations above the superconducting critical temperature play a similar role.[25,26] In the pseudogap phase the spectrum is not like the one of a normal metal, although the system is not in a superconducting phase. However, we should warn the reader that the actual physics of the pseudogap state is likely much more complex and that such an analogy must be considered with utmost caution.

V. CONCLUSION

In this paper we have shown that temperature effects induce roton-like excitations in a dilute two-dimensional Bose gas. From this result it followed that for nonzero low temperatures the spectrum *is not* of the phonon type and has an anomalous scaling with temperature dependent exponent. Thus, we have obtained in the quantum regime a situation which is reminiscent of the Kosterlitz-Thouless transition, namely, a continuously varying exponent.

It would be interesting to extend this analysis to strongly-coupled two-dimensional Bose systems such as films of ^{4}He, where experimental data have so far been fitted by a superfluid density whose vortex-independent background contribution is calculated from the Bogoliubov spectrum.[27] It was found that the phonon contribution alone with its T^3 behavior (recall that for a phonon spectrum we find in general a contribution behaving like T^{d+1} in d dimensions) is not consistent with the temperature dependence of the data, thus calling for an improvement of the theory. In order to apply our approach to ^{4}He films we must derive the t-matrix for the actual atomic interaction in helium beyond the dilute limit. Preliminary work has been done some time ago,[28] and phonon as well as roton excitations of the spectrum were obtained. However, these works concentrate only on the low-temperature properties. We believe that thermally induced roton-like excitations should also occur in this case, though in a more complicate manner. Certainly, such strong-coupling problems require more powerful calculation methods, for example, field-theoretic variational perturbation theory as developed in Ref. 29, which has led to the most accurate predictions of critical exponents so far.[31]

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APPENDIX A: CLASSICAL LIMIT OF THE POLARIZATION BUBBLE

Eq. (11) can be rewritten as

$$\widetilde{\Pi}(i\omega,\mathbf{p}) = 4m^2 \int \frac{d^dq}{(2\pi)^d} n_B\left(\frac{q^2}{2m}\right)\left(\frac{1}{2mi\omega - p^2 - 2\mathbf{p}\cdot\mathbf{q}} - \frac{1}{2mi\omega + p^2 - 2\mathbf{p}\cdot\mathbf{q}}\right). \tag{A1}$$

In the classical approximation we write $n_b(x) \approx T/x$ and the polarization bubble can be rewritten as

$$\widetilde{\Pi}(i\omega,\mathbf{p}) = 4m^2T(I_+ - I_-), \tag{A2}$$

where

$$I_\pm = -i\int \frac{d^dq}{(2\pi)^d}\frac{1}{2m\omega + i(2\mathbf{p}\cdot\mathbf{q}\pm p^2)}\frac{1}{q^2}. \tag{A3}$$

The integrals $I_\pm$ can be evaluated using the Feynman parameters,[30]

$$I_\pm = -i\int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \frac{d^dq}{(2\pi)^d} e^{-\lambda_1(2m\omega \pm ip^2 + 2i\mathbf{p}\cdot\mathbf{q})} e^{-\lambda_2 q^2}. \tag{A4}$$

After evaluating the Gaussian integral over $\mathbf{q}$ we obtain

$$\begin{aligned} I_\pm &= -\frac{i}{(2\pi)^d}\int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \left(\frac{\pi}{\lambda_2}\right)^{d/2} e^{-\lambda_1(2m\omega\pm ip^2)} e^{-p^2\lambda_1^2/\lambda_2} \\ &= \mp\frac{(\pm i)^{d-2}}{2^d\pi^{d/2}}\Gamma(d/2-1)\Gamma(3-d)p^{d-4}\left(1\mp\frac{2mi\omega}{p^2}\right)^{d-3}. \end{aligned} \tag{A5}$$

Substituting the above expression back into (A2) we obtain Eq. (15).

APPENDIX B: DERIVATION OF THE ANOMALOUS SINE-GORDON ACTION

At sufficiently high temperatures the Matsubara time dependence of the phase $\theta(\tau,\mathbf{r})$ can be neglected and the effective action (28) can be written simply as

$$S_{\text{eff}} \approx \frac{1}{2T}\int d^2r \int d^2r' \mathcal{M}(\mathbf{r}-\mathbf{r}')\ \mathbf{v}_s(\mathbf{r})\cdot\mathbf{v}_s(\mathbf{r}'). \tag{B1}$$

Although the duality transformation can also be performed in the continuum, a more technically correct analysis is obtained in the lattice formalism.[4] The lattice version of the above action suitable for a duality transformation is given by the Villain form

$$S_L = \frac{1}{2m^2T}\sum_{i,j}\mathcal{M}_{ij}(\nabla\theta_i - 2\pi\mathbf{n}_i)\cdot(\nabla\theta_j - 2\pi\mathbf{n}_j), \tag{B2}$$

where we have set the lattice spacing to unit and the components of $\nabla\theta_i$ are understood as lattice derivatives. The field $\mathbf{n}_i$ is an integer field defined on the lattice. The partition function is then given by

$$Z = \sum_{\{\mathbf{n}_i\}}\int_{-\pi}^{\pi}\prod_i \frac{d\theta_i}{2\pi}\exp(-S_L). \tag{B3}$$

The first step in the duality transformation is the introduction of an auxiliary field through a Gaussian completion, i.e.,

$$\begin{aligned}&\exp\left[-\frac{1}{2m^2T}\sum_{i,j}\mathcal{M}_{ij}(\nabla\theta_i - 2\pi\mathbf{n}_i)\cdot(\nabla\theta_j - 2\pi\mathbf{n}_j)\right]\\ &\propto \int_{-\infty}^{\infty}\prod_{k,\mu} db_{k\mu}\exp\left[\frac{m^2T}{2}\sum_{i,j}\mathbf{b}_i(\mathcal{M}^{-1})_{ij}\mathbf{b}_j - \mathrm{i}\sum_j \mathbf{b}_j\cdot(\nabla\theta_j - 2\pi\mathbf{n}_j)\right].\end{aligned} \tag{B4}$$

Next we apply the Poisson formula

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dx f(x) e^{\mathrm{i}2\pi nx} = \sum_{m=-\infty}^{\infty} f(m), \tag{B5}$$

in order to convert the integral over the real auxiliary $\mathbf{b}_i$ field into a sum over an integer field $\mathbf{N}_i$. After this manipulation the periodic field θ_i can be easily integrated out after a summation by parts. This leads to the constraint $\nabla\cdot\mathbf{N}_i = 0$. Up to unimportant overall factors the partition function becomes

$$Z = \sum_{\{\mathbf{N}_i\}}\delta_{\nabla\cdot\mathbf{N}_i,0}\exp\left[-\frac{m^2T}{2}\sum_{i,j}\mathbf{N}_i(\mathcal{M}^{-1})_{ij}\mathbf{N}_j\right]. \tag{B6}$$

In two dimensions the constraint is solved through

$$N_{i\mu} = \varepsilon_{\mu\nu}\nabla_\nu l_i. \tag{B7}$$

The resulting partition function will be the one of a neutral Coulomb gas in the lattice. By applying the Poisson formula once more, we obtain

$$Z = \sum_{\{s_i\}}\int_{-\infty}^{\infty}\prod_k d\varphi_k \exp\left[-\frac{m^2T}{2}\sum_{i,j}\nabla_\mu\varphi_i(\mathcal{M}^{-1})_{ij}\nabla_\mu\varphi_j - \mathrm{i}\sum_j 2\pi s_j\varphi_j\right]. \tag{B8}$$

The integer fields s_j represent the point vortices of the theory. We can now introduce the vortex fugacity z to build a grand-canonical ensemble of vortices.[21] The most relevant vortex configurations correspond to $s_j = \pm 1$. After the rescaling $\varphi_j \to \varphi_j/2\pi$ and taking the continuum limit of the grand-canonical ensemble theory, we obtain Eq. (50).

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