Field Transformations to Multivalued Fields

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Abstract. Changes of field variables may lead to multivalued fields which do not satisfy the Schwarz integrability conditions. Their quantum field theory needs special care as is shown in an applications to the superconducting phase transitions.

1. Introduction

Changes of coordinates or field variables must not change the physical content of a theory. This trivial requirement automatically guaranteed in quantum field theories. As a simple example consider consider the path integral of a harmonic oscillator

$$Z_\omega = \int \mathcal{D}x\, e^{-\mathcal{A}_\omega[x]} = \exp\left[-\frac{D}{2}\mathrm{Tr}\log(-\partial^2+\omega^2)\right] \equiv e^{-\beta F_\omega}. \qquad (1) \quad \{\texttt{pix@mq2}\}$$

with an action

$$\mathcal{A}_\omega[x] = \frac{1}{2}\int_0^\beta d\tau\,\left[\dot{x}^2(\tau)+\omega^2 x^2(\tau)\right], \qquad (2) \quad \{\texttt{pix@m1}\}$$

and a free energy $F=\beta^{-1}\log\sinh\bar{\omega}/2$. Let us subject this path integral to a simple coordinate transformation such as

$$x = x_\eta(q) = q - \eta q^3/3. \qquad (3) \quad \{\texttt{@}\}$$

where η is some expansion parammeter. The tranformed path integral

$$Z = \int \mathcal{D}q(\tau)\, e^{-\mathcal{A}_\omega[q]-\mathcal{A}^{\mathrm{int}}[q]-\mathcal{A}_J[q]} \equiv e^{-\beta F} \qquad (4) \quad \{\texttt{pix@m2}\}$$

has an interaction $\mathcal{A}_{\mathrm{int}}[q]=\mathcal{A}^{\mathrm{ah}}[q]+\mathcal{A}_J[q]$, consisting of the anharmonic part of the transformed action

$$\mathcal{A}^{\mathrm{ah}}[q] = \int_0^\beta d\tau\left\{-\eta\left[q^2(\tau)\dot{q}^2(\tau)+\frac{\omega^2}{3}q^4(\tau)\right]+\eta^2\left[\frac{1}{2}q^4(\tau)\dot{q}^2(\tau)+\frac{\omega^2}{18}q^6(\tau)\right]+\mathcal{O}(\eta^3)\right\}, \qquad (5) \quad \{\texttt{pix@m7}\}$$

and a an anharmonic part due to the Jacobian $\mathcal{D}x/\mathcal{D}q=\exp[\delta(0)\log\partial x(q)/\partial q]$:

$$\mathcal{A}_J[q] = -\delta(0)\int d\tau\,\log\frac{\partial x_\eta(q)}{\partial q} = -\delta(0)\int_0^\beta d\tau\left[-\eta q^2(\tau)-\frac{\eta^2}{2}q^4(\tau)+\ldots+\mathcal{O}(\eta^3)\right] \quad . \qquad (6) \quad \{\texttt{pix@m4}$$

The transformed path integral (4) can no longer be solved exactly but only perturbatively as an expansion in powers of the parameter η:

$$\beta F = \beta F_\omega + \langle \mathcal{A}_{\rm int} \rangle_c - \frac{1}{2!} \langle \mathcal{A}_{\rm int}^2 \rangle_c + \dots \; = \beta F_\omega + \beta \sum_{n=1}^{\infty} \eta^n \, F_n . \tag{7}$$ {chap10@2@m

In order to guarantee coordinate invariance, all coefficients F_n have to vanish.

The Feynman diagrams contributing to F_n consist of vertices and three kinds of lines representing the one-dimensional versions of the correlation functions

$$G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle q_\mu(\tau) q_\nu(\tau') \rangle = \text{———} \, , \tag{8}$$ {pix@p1}

$$\partial_\tau G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle \dot q_\mu(\tau) q_\nu(\tau') \rangle = \text{—····} \, , \tag{9}$$ {pix@p2}

$$\partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle q_\mu(\tau) \dot q_\nu(\tau') \rangle = \text{···—} \, , \tag{10}$$ {pix@p2r}

$$\partial_\tau \partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle \dot q_\mu(\tau) \dot q_\nu(\tau') \rangle = \text{·······} \, . \tag{11}$$ {pix@p3}

These contain distributions $\Theta(\tau-\tau')$ and $\delta(\tau-\tau')$ (see Fig. 1), so that the Feynman integrals

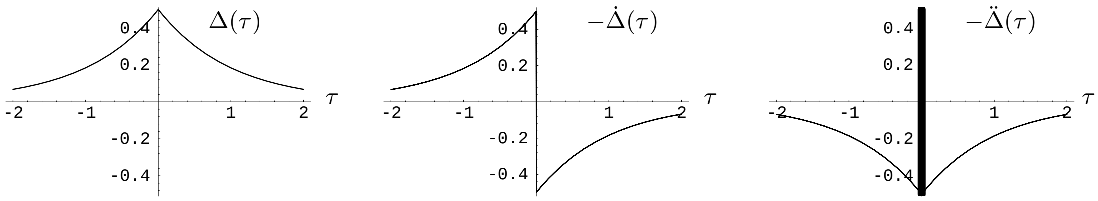

Figure 1. Green functions for perturbation expansions in curvilinear coordinates in natural units with $\omega = 1$. The second contains a Heaviside function $\Theta(\tau)$, the third a Dirac δ-function at the origin. {@f1}

run over products of ditributions which in the standard theory of generalized functions are undefined. Recently, however, it is been shown that there is a way of defiening products of distributions in such a way that all F_n vansih, i.e., that coordinate invariance can be maintained [1].

The situation becomes much more involved if the transformed coordinates $q(\tau)$ are multivalued fields in D dimensions. This happens, for instance, if a complex field $\psi(\mathbf{x})$ in a functional integral is replaced by its radial and azimuthal parts of $\rho(\mathbf{x}) \equiv |\psi(\mathbf{x})|$ and $\theta(\mathbf{x}) \equiv \arctan[{\rm Im}\psi(\mathbf{x})/{\rm Re}\psi(\mathbf{x})]$. A good example is the Landau-Pitaevski energy density of superfluid helium near the critical point:

$$\mathcal{H}_{\rm He}(\psi, \boldsymbol{\nabla}\psi, \mathbf{A}, \boldsymbol{\nabla}\mathbf{A}) = \frac{1}{2}\Big\{ |\boldsymbol{\nabla}\psi|^2 + \tau|\psi|^2 + \frac{g}{2}|\psi|^4 \Big\} , \tag{12}$$ {@ner}

The parameter $\tau \equiv T/T_c^{\rm MF} - 1$ is a reduced temperature measuring the distance from the characteristic temperature $T_c^{\rm MF}$ at which the $|\psi|^2$-term changes sign. Under a field tranformation $\psi(x) \to \rho(x)e^{\imath\theta(x)}$, the energy density *cannot* be simply replaced by

$$\mathcal{H}_1 = \frac{\rho^2}{2}(\boldsymbol{\nabla}\theta)^2 + \frac{1}{2}(\boldsymbol{\nabla}\rho)^2 + \frac{\tau}{2}\rho^2 + \frac{g}{4}\rho^4 . \tag{13}$$ {@ner1}

as we might be tempted to do following the naive Leibnitz rule

$$D\psi = (i\boldsymbol{\nabla}\theta\rho + \boldsymbol{\nabla}\rho)e^{i\theta}, \tag{14}$$ {@tr}

This rule is no longer valid. Since $\theta(\mathbf{x})$ and $\theta(\mathbf{x})+2\pi$ correspond to the same complex field $\psi(\mathbf{x})$, the corrected Leibnitz rule reads

$$D\psi = [i(\boldsymbol{\nabla}\theta - 2\pi\boldsymbol{\theta}^v)\rho + \boldsymbol{\nabla}\rho]e^{i\theta}, \tag{15}$$ {@trc}

The cyclic nature of the scalar field $\theta(\mathbf{x})$ requires the presence of a vector field $\boldsymbol{\theta}^v(\mathbf{x})$ called *vortex gauge field*. This field is a sum of δ-functions on Volterra surfaces across which $\theta(\mathbf{x})$ has jumps by 2π. The boundary lines of the surfaces are vortex lines. They are found from the vortex gauge field $\boldsymbol{\theta}^v(\mathbf{x})$ by forming the curl

$$\boldsymbol{\nabla}\times\boldsymbol{\theta}^v(\mathbf{x}) = \mathbf{j}^v(\mathbf{x}), \tag{16}$$ {@VD}

where $\mathbf{j}^v(\mathbf{x})$ is the *vortex density*, a sum over δ-functions $\boldsymbol{\delta}(L;\mathbf{x}) \equiv \int_L d\bar{\mathbf{x}}\,\delta(\mathbf{x}-\bar{\mathbf{x}})$ along the vortex lines L.

Vortex gauge transformations correspond to deformations of the surfaces at fixed boundary lines which add to $\boldsymbol{\theta}^v(\mathbf{x})$ pure gradients of the form $\boldsymbol{\nabla}\delta(V;\mathbf{x})$, where $\delta(V;\mathbf{x}) \equiv \int_V d^3\bar{x}\,\delta(\mathbf{x}-\bar{\mathbf{x}})$ are δ-functions on the volumes V over which the surfaces have swept. The theory of these fields has been developed in the textbook [2] and the Cambridge lectures [3]. Being a gauge field, $\boldsymbol{\theta}^v(\mathbf{x})$ may be modified by a further gradient of a smooth function to make it purely transverse, $\boldsymbol{\nabla}\cdot\boldsymbol{\theta}^v_T(\mathbf{x}) = 0$, as indicated by the subscript T.

Since the vortex gauge field is not a gradient, it cannot be absorbed into the vector potential by a gauge transformation. Hence it survives in the last term in Eq. (13), and the correct partition function is

$$Z_{\rm He} \approx \int \mathcal{D}\boldsymbol{\theta}^v_T \int \mathcal{D}\rho\rho\,\mathcal{D}\mathbf{A}\exp\left[-\frac{\rho^2}{2}(\boldsymbol{\nabla}\theta)^2 - \frac{1}{2}(\boldsymbol{\nabla}\rho)^2 - \frac{\tau}{2}\rho^2 - \frac{g}{4}\rho^4 - \frac{4\pi^2\rho^2}{2}{\boldsymbol{\theta}^v_T}^2\right]. \tag{17}$$ {@XY1}

The symbol $\int\mathcal{D}\boldsymbol{\theta}^v_T$ does not denote an ordinary functional integral. It is defined as a sum over any number and all shapes of Volterra surfaces S in $\boldsymbol{\theta}^v_T(\mathbf{x})$, across which the phase jumps by 2π [3].

The important observation is now that due to the fluctuations of the vortex gauge field $\boldsymbol{\theta}^v_T(\mathbf{x})$, the partition function (17) possesses a second-order phase transition, the famous λ-transition observed in in superfluid helium at 2.18 K. The critical exponents of this transition are in the same universality as those of the so-called XY-model, which describes only interacting phase angles $\theta(\mathbf{x}) \in (0,2\pi)$ on a lattice.

At the mean-field level, the λ-transition of (17) takes place if τ drops below zero where the pair field $\psi(x)$ acquires the nonzero expectation value $\langle\psi(x)\rangle = \rho_0 = \sqrt{-\tau/g}$, the order parameter of the system. The ρ-fluctuations around this value have a *coherence length* $\xi = 1/\sqrt{-2\tau}$.

For a long time it has been a debate whether this transition persists if a fluctuating vector potential $\mathbf{A}(\mathbf{x})$ is coupled minimally to the field $\psi(\mathbf{x})$ in (18), which then becomes the Ginzburg-Landau Hamiltonian density of superconductivity

$$\mathcal{H}_{\rm sc}(\psi,\boldsymbol{\nabla}\psi,\mathbf{A},\boldsymbol{\nabla}\mathbf{A}) = \frac{1}{2}\Big\{[(\boldsymbol{\nabla}-iq\mathbf{A})\psi]^2 + \tau|\psi|^2 + \frac{g}{2}|\psi|^4\Big\} + \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A})^2. \tag{18}$$ {@ner}

Now $\psi(\mathbf{x})$ is the field describing Cooper pairs of charge $q = 2e$. The theory needs gauge fixing, which may be done by absorbing the gradient of the phase $\theta(\mathbf{x})$ of the field $\psi(\mathbf{x})$ in the vector potential, so that we can replace $\psi(\mathbf{x}) \to \rho(\mathbf{x})$. The transverse vortex gauge field $\boldsymbol{\theta}^v_T(\mathbf{x})$, however, cannot be absorbed and it interacts with the vector potential $A(\mathbf{x})$. This has a partial partition function

$$Z_A[\rho] \equiv \int \mathcal{D}\boldsymbol{\theta}^v_T\mathcal{D}\mathbf{A}\exp\left\{-\frac{1}{2}\int d^3x(\boldsymbol{\nabla}\times\mathbf{A})^2 - \frac{1}{2}\int d^3x\,\rho^2(e\mathbf{A}-2\pi\boldsymbol{\theta}^v_T)^2\right\}. \tag{19}$$ {@EFF}

Without the vortex gauge field $\boldsymbol{\theta}^v_T(\mathbf{x})$, the partition function (19) describes free bosons of space-dependent mass $\rho^2(\mathbf{x})$. (13). If we ignore $\boldsymbol{\theta}^v_T(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$, the total partition function has the same form as in (17) and describes a second order phase transition.

Let us now admit the vector partition function (19), but still ignore vortices by setting $\boldsymbol{\theta}^v_T(\mathbf{x}) \equiv 0$, and ignoring the space dependence of $\rho(\mathbf{x})$. Then the second term in (19) in the condensed phase with $\rho_0 \neq 0$ generates a Meissner-Higgs mass term. This gives rise to a finite *penetration depth* of the magnetic field $\lambda = 1/m_A = 1/\rho_0 q$. The ratio of the two length scales $\kappa \equiv \lambda/\sqrt{2}\xi$ (which for historic reasons carries a factor $\sqrt{2}$) is the Ginzburg parameter whose mean field value is $\kappa_{\rm MF} \equiv \sqrt{g/q^2}$. Type I superconductors have small values of κ, type-II superconductors have large values. At the mean-field level, the dividing line lies at $\kappa = 1/\sqrt{2}$.

Let us now allow for $\mathbf{A}(\mathbf{x})$-fluctuations [still ignoring the vortex gauge field $\boldsymbol{\theta}^v_T(\mathbf{x})$. At very smooth $\rho(\mathbf{x})$, they can be integrated out in (19) which becomes

$$Z^0_A[\rho] = \exp\left[\int d^3x \frac{e^3\rho^3}{6\pi}\right] \qquad (20)$$ {@Z=}

This adds the energy density (13) a cubic term $-e^3\rho^3/6\pi$. Such a term makes the transition first-order. The feee energy has now a minimum at

$$\tilde{\rho}_0 = \frac{c}{2g}\left(1+\sqrt{1-\frac{4\tau g}{c^2}}\right). \qquad (21)$$ {@newmin}

If τ decreases below

$$\tau_1 = 2c^2/9g. \qquad (22)$$ {@tricr}

the new minimum lies *lower* than the one at the origin (see Fig. 2), so that the order parameter jumps from zero to

$$\rho_1 = 2c/3g \qquad (23)$$ {@RHO1}

in a phase transition. At this point, the coherence length of the ρ-fluctuations $\xi = 1/\sqrt{\tau + 3g\rho^2 - 2c\rho}$ has the finite value

$$\xi_1 = \frac{3}{c}\sqrt{\frac{g}{2}}, \qquad (24)$$ {@xi1}

this being the same as at $\rho = 0$. The jump from $\rho = 0$ to ρ_1 implies a phase transition of first-order [6].

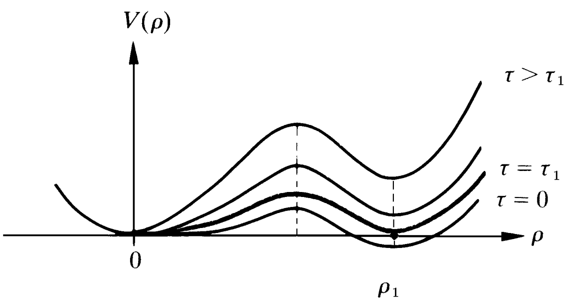

Figure 2. Potential for the order parameter ρ with cubic term. At τ_1, the order parameter jumps from $\rho = 0$ to ρ_1, corresponding to a phase transition of first-order. {@p}

However, this result is reliable only under the assumption of a smooth $\rho(\mathbf{x})$. This is applicable only in the type-I regime where vortex lines are absent. In the type-II regime, such lines can be excited thermally and we can no longer ignore the vortex gauge field $\boldsymbol{\theta}^v_T(\mathbf{x})]$. This invalidates the above conclusion and gives rise to a second-order transition (of the same XY-universality class as in the superfluid) if the Ginzburg parameter κ is sufficiently large. Integrating now out the $\mathbf{A}$-field and obtain

$$Z_A[\rho] = \exp\left[\int d^3x \frac{e^3\rho^3}{6\pi}\right] \int \mathcal{D}\boldsymbol{\theta}^v_T \exp\left[\frac{4\pi^2\rho^2}{2}\int d^3x \left(\frac{1}{2}\boldsymbol{\theta}^{v\,2}_T - \boldsymbol{\theta}^v_T \frac{\rho^2q^2}{-\boldsymbol{\nabla}^2+\rho^2q^2}\boldsymbol{\theta}^v_T\right)\right],$$

rather than (20). The second integral can be simplified to

$$\frac{4\pi^2\rho^2}{2}\int d^3x \left(\boldsymbol{\theta}^v_T \frac{-\boldsymbol{\nabla}^2}{-\boldsymbol{\nabla}^2+\rho^2q^2}\boldsymbol{\theta}^v_T\right). \tag{25}$$ {@EFF2p}

Integrating this by parts, and replacing $\nabla_i\boldsymbol{\theta}^v_T\,\nabla_i\boldsymbol{\theta}^v_T$ by $(\boldsymbol{\nabla}\times\boldsymbol{\theta}^v_T)^2 = \mathbf{j}^{v\,2}$, since $\boldsymbol{\nabla}\cdot\boldsymbol{\theta}^v_T = 0$, the partition function (25) takes the form

$$Z_A[\rho] \;=\; \exp\left[\int d^3x \frac{e^3\rho^3}{6\pi}\right] \int \mathcal{D}\boldsymbol{\theta}^v_T \exp\left[-\frac{4\pi^2\rho^2}{2}\int d^3x \left(\mathbf{j}^v \frac{1}{-\boldsymbol{\nabla}^2+\rho^2q^2}\mathbf{j}^v\right)\right]. \tag{26}$$ {@EFF3}

This is the partition function of a grand-canonical ensemble of closed fluctuating vortex lines L described by the δ-functions over lines in $\mathbf{j}^v(\mathbf{x})$. The interaction between them has a finite range equal to the penetration depth $\lambda = 1/\rho q$. It is well-known how to compute pair and magnetic fields of the Ginzburg-Landau theory for a single straight vortex line from the extrema of the energy density (18). In an external magnetic field, there exist triangular and various other regular arrays of vortex lattices and various phase transitions. In the core of each vortex line, the pair field ρ goes to zero over a distance ξ. If we want to sum over grand-canonical ensemble of fluctuating vortex lines of any shape in the partition function (17), the space dependence of ρ causes complications. These can be avoided by an approximation, in which the system is placed on a simple-cubic lattice of spacing $a = \alpha\,\xi$, with α of the order of unity, and replacing the variable $\rho(\mathbf{x})$ by a *fixed* $\rho = \tilde{\rho}_0$ given by Eq. (21). Thus we replace the partial partition function (26) approximately by

$$Z_2[\tilde{\rho}_0] \;=\; \sum_{\{\mathbf{l};\boldsymbol{\nabla}\cdot\mathbf{l}=0\}} \exp\left[-\frac{4\pi^2\tilde{\rho}_0^2 a}{2}\sum_{\mathbf{x}} \mathbf{l}(\mathbf{x})v_{\tilde{\rho}_0 e}(\mathbf{x}-\mathbf{x}')\mathbf{l}(\mathbf{x}')\right]. \tag{27}$$ {@EFF4}

The sum runs over the discrete versions of the vortex density in (16). These are integer-valued vectors $\mathbf{l}(\mathbf{x}) = (l_1(\mathbf{x}), l_2(\mathbf{x}), l_3(\mathbf{x}))$ which satisfy $\boldsymbol{\nabla}\cdot\mathbf{l}(\mathbf{x}) = 0$, where $\boldsymbol{\nabla}$ denotes the lattice derivative. This condition restricts the sum over all $\mathbf{l}(\mathbf{x})$-configurations in (27) to all non-selfbacktracking integer-valued closed loops. The function

$$v_m(\mathbf{x}) \;=\; \prod_{i=1}^{3}\int \frac{d^3(ak_i)}{(2\pi)^3} \frac{e^{i(k_1x_1+k_2x_2+k_3x_3)}}{2\sum_{i=1}^{3}(1-\cos ak_i)+a^2m^2} = I_{x_1}(2s)I_{x_2}(2s)I_{x_3}(2s). \tag{28}$$ {@YP}

is the lattice Yukawa potential.

The lattice partition function (27) is known to have a second-oder phase transition in the universality class of the XY-model. This can be seen by a comparison with the Villain approximation [4] to the XY model, whose partition function is a lattice version of

$$Z_V[\rho] = \int \mathcal{D}\theta \int \mathcal{D}\boldsymbol{\theta}^v_T \exp\left[-\frac{b}{2}\int d^3x\;(\boldsymbol{\nabla}\theta - \boldsymbol{\theta}^v_T)^2\right]. \tag{29}$$ {@EFF7}

After integrating out $\theta(\mathbf{x})$, this becomes

$$Z_V[\rho] = \mathrm{Det}^{-1/2}(-\boldsymbol{\nabla}^2) \int \mathcal{D}\boldsymbol{\theta}_T^v \exp\left(-\frac{b}{2}\int d^3x\, \boldsymbol{\theta}_T^{v\,2}\right), \tag{30}$$ {@EFF7}

and we can replace $\boldsymbol{\theta}_T^{v\,2}$ by $(\boldsymbol{\nabla}\times\boldsymbol{\theta}_T^v)(-\boldsymbol{\nabla}^2)^{-1}(\boldsymbol{\nabla}\times\boldsymbol{\theta}_T^v) = \mathbf{j}^v(-\boldsymbol{\nabla}^2)^{-1}\mathbf{j}^v$. By taking this expression to a simple-cubic lattice we obtain the partition function (27), but with $\tilde{\rho}_0^2 a$ replaced by $\beta_V \equiv ba$, and the Yukawa potential $v_{\tilde{\rho}_0 e}(\mathbf{x})$ replaced by the Coulomb potential $v_0(\mathbf{x})$.

The partition function (27) has the same transition at roughly the same place as its local approximation

$$Z_2[\tilde{\rho}_0] \;\approx\; \sum_{\{\mathbf{l};\boldsymbol{\nabla}\cdot\mathbf{l}=0\}} \exp\left[-\frac{4\pi^2\tilde{\rho}_0^2 a}{2} v_{\tilde{\rho}_0 e}(\mathbf{0})\sum_{\mathbf{x}}\mathbf{l}^2(\mathbf{x})\right]. \tag{31}$$ {@EFF5}

A similar approximation holds for the Villain model with $v_0(\mathbf{x})$ instead of $v_{\tilde{\rho}_0 e}(\mathbf{x})$, and $\tilde{\rho}_0^2 a$ replaced by $\beta_V \equiv ba$.

The Villain model is known to undergo a second-order phase transition of the XY-model type at $\beta_V = r/3$ with $r \approx 1$, where the vortex lines become infinitely long [4, 7]. Thus we conclude that also the partition function (31) has a second-order phase transition of the XY-model type at $\tilde{\rho}^2 v_{\tilde{\rho}_0 e}(\mathbf{0})a \approx v_0(\mathbf{0})/3$. The potential (28) at the origin has the hopping expansion [8]

$$v_m(\mathbf{0}) = \sum_{n=0,2,4} \frac{H_n}{(a^2m^2+6)^{n+1}},\; H_0 = 1, H_2 = 6, \dots . \tag{32}$$ {@}

To lowest order, this yields the ratio $v_m(\mathbf{0})/v_0(\mathbf{0}) \equiv 1/(m^2/6+1)$. A more accurate numerical fit to the ratio $v_m(\mathbf{0})/v_0(\mathbf{0})$ which is good up to $m^2 \approx 10$ (thus comprising all interesting κ-values since m^2 is of the order of $3/\kappa^2$) is $1/(\sigma\, m^2/6+1)$ with $\sigma \approx 1.38$. Hence the transition takes place at

$$\frac{\tilde{\rho}_0^2 a}{(\sigma\, a^2\tilde{\rho}_0^2 q^2/6+1)} \approx \frac{r}{3} \;\text{ or }\; \tilde{\rho}_0 \approx \frac{1}{\sqrt{3a}}\sqrt{\frac{r}{1-\sigma r a q^2/18}}. \tag{33}$$ {@rho0}

The important point is now that this transition can occur only until $\tilde{\rho}_0$ reaches the value $\rho_1 = 2c/3g$ of Eq. (23). From there on, the transition will no longer be of the XY-model type but occur discontinuously as a first-order transition.

Replacing in (33) a by $\alpha\xi_1$ of Eq. (24), and $\tilde{\rho}_0$ by ρ_1, we find the equation for the mean-field Ginzburg parameter $\kappa_{\mathrm{MF}} = \sqrt{g/q^2}$:

$$\kappa_{\mathrm{MF}}^3 + \alpha^2\sigma\frac{\kappa_{\mathrm{MF}}}{3} - \frac{\sqrt{2}\alpha}{\pi r} = 0. \tag{34}$$ {@}

Inserting $\sigma \approx 1.38$ and choosing $\alpha \approx r \approx 1$, the solution of this equation yields the tricritical value

$$\kappa_{\mathrm{MF}}^{\mathrm{tric}} \approx 0.81/\sqrt{2}. \tag{35}$$ {@}

In spite of the roughness of the approximations, this result is very close to the value $0.8/\sqrt{2}$ derived from the dual disorder field theory [5]. The approximation has three uncertainties. First, the identification of the effective lattice spacing $a = \alpha\xi$ with $\alpha \approx 1$; second the associated neglect of the $\mathbf{x}$-dependence of ρ and its fluctuations, and third the localization of the critical point of the XY-model type transition in Eq. (33).

Our goal has been achieved: We have shown the existence of a tricritical point in a superconductor directly within the fluctuating Ginzburg-Landau theory, by taking the vortex fluctuations into account. This became possible after correcting the covariant derivative (14) of $\psi = \rho e^{i\theta}$ to (15). For $\kappa > 0.81/\sqrt{2}$, vortex fluctuations give rise to an XY-model type second-order transition before the cubic term becomes relevant. This happens for $\kappa < 0.81/\sqrt{2}$ where the cubic term causes a discontinuous transition.

These examples show that the subtleties of functional integration over multivalued fields are crucial for understanding important physical phenomena such a phase transitions. Similar considerations are necessary in the context of elasticity theory where the energy is usually expressed in terms of the strain $u_{ij} = \partial_i u_j(\mathbf{x})$ of the displacement field $u_i(\mathbf{x})$ of the atoms from their rest position. Such a descrition is also false since the displacement field $u_i(\mathbf{x})$ is a multivalued field. It is defined only up to multiples of the lattice vectors. This multivaluedness mut be taken into account with the help of a *defect gauge field* similar to $\boldsymbol{\theta}^v(\mathbf{x})$, one for each lattice direction. Its fluctuations give rise to the melting transition, as has been shown in the textbook [9].

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