

Uncertainty Relation on World Crystal and Applications to Micro Black Holes

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I. INTRODUCTION

The geometry of Einstein and Einstein-Cartan spaces can be considered as being a manifestation of the defect structure of a world crystal whose lattice spacing ℓ_0 is of the order of the Planck length ℓ_p [1–3]. Curvature is due to rotational defects, torsion to translational defects. The elastic deformations do not alter the defect structure, i.e., the geometry is invariant under elastic deformations. If we assume these to be controlled by a second-gradient elastic action, the forces between local rotational defects, i.e., between curvature singularities, are the same as in Einstein's theory [4]. Moreover, the elastic fluctuations of the displacement fields possess logarithmic correlation functions at long distances, so that the memory of the crystalline structure is lost over large distances. In other words, the Bragg peaks of the world crystal are not δ -function-like, but display the typical behavior of a quasi-long-range order, similar to the order in a Kosterlitz-Thouless transition in two-dimensional superfluids [5].

The purpose of this note is to study the generalised uncertainty principle (GUP) associated with the quantum field theory on the world crystal and to derive physical consequences from it. We shall find that the GUP implies that quantum physics tends to the classical limit at the scale of the lattice spacing. Our results have interesting implications for the physics of micro black holes to be discussed in Section V.

There exists also a close contact with two interesting mainstreams of contemporary research. One is 't Hooft's approach to deterministic quantum mechanics, the other deformed (or double) special relativity.

On a more phenomenological side, our version of GUP provides a nice way out of two long standing puzzling situations: the final Hawking temperature of a decaying micro black hole remains *finite*, in contrast to the infinite temperature of the standard result where the Heisenberg's uncertainty principle operates. On the world crystal, the final mass of the evaporation process is zero, thus avoiding the problems caused by the existence of massive

black hole remnants. Such a decay behavior has the advantage that it is more likely to be observed experimentally in the LHC when it will restarts operations in the near future.

II. DIFFERENTIAL CALCULUS ON A LATTICE

On a lattice of spacing ϵ in one dimension, the lattice sites lie at $x_n = n\epsilon$ where n runs through all integer numbers. There are two fundamental derivatives of a function $f(x)$:

$$\begin{aligned} (\nabla f)(x) &= \frac{1}{\epsilon} [f(x + \epsilon) - f(x)] , \\ (\bar{\nabla} f)(x) &= \frac{1}{\epsilon} [f(x) - f(x - \epsilon)] . \end{aligned} \quad (1)$$

They obey the generalized Leibnitz rule

$$\begin{aligned} (\nabla fg)(x) &= (\nabla f)(x)g(x) + f(x + \epsilon)(\nabla g)(x) , \\ (\bar{\nabla} fg)(x) &= (\bar{\nabla} f)(x)g(x) + f(x - \epsilon)(\bar{\nabla} g)(x) . \end{aligned} \quad (2)$$

On a lattice, integration is performed as a summation:

$$\int dx f(x) = \epsilon \sum_x f(x) , \quad (3)$$

where x runs over all x_n .

For periodic functions on the lattice or for functions vanishing at the boundary of the world crystal, the lattice derivatives can be subjected to the lattice version of integration by parts:

$$\sum_x f(x) \nabla g(x) = - \sum_x g(x) \bar{\nabla} f(x) , \quad (4)$$

$$\sum_x f(x) \bar{\nabla} g(x) = - \sum_x g(x) \nabla f(x) . \quad (5)$$

One can also define the lattice Laplacian as

$$\nabla \bar{\nabla} f(x) = \bar{\nabla} \nabla f(x) = \frac{1}{\epsilon^2} [f(x + \epsilon) - 2f(x) + f(x - \epsilon)] \quad (6)$$

which reduces in the continuum limit to an ordinary Laplace operator ∂^2 . Note that the lattice Laplacian can

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also be expressed in terms of the difference of the two lattice derivatives:

$$\nabla\bar{\nabla}f(x) = \frac{1}{\epsilon} [\nabla f(x) - \bar{\nabla}f(x)]. \quad (7)$$

The calculus can be easily be extended to any number D of dimensions [6].

III. POSITION AND MOMENTUM OPERATORS

Consider now the quantum mechanics on a 1D lattice in a Schrödinger-like picture. Wave function are square-integrable complex functions on the lattice, where “integration” means here summation, and scalar products are defined by

$$\langle\psi_1|\psi_2\rangle = \epsilon \sum_x \psi_1^*(x)\psi_2(x). \quad (8)$$

It follows from Eq. (4) that

$$\langle f|\nabla g\rangle = -\langle\bar{\nabla}f|g\rangle, \quad (9)$$

so that $(i\nabla)^\dagger = i\bar{\nabla}$, and neither $i\nabla$ nor $i\bar{\nabla}$ are hermitian operators. The lattice Laplacian (6), however, is hermitian.

The position operator \hat{X}_ϵ acting on wave functions of x is defined by simple multiplication with x :

$$(\hat{X}_\epsilon f)(x) = xf(x). \quad (10)$$

Similarly we define the lattice momentum operator \hat{P}_ϵ . In order to assure hermiticity we shall relate it to the symmetric lattice derivative [8–10]. Using (9) we have

$$\begin{aligned} (\hat{P}_\epsilon f)(x) &= \frac{1}{2i}[(\nabla f)(x) + (\bar{\nabla}f)(x)] \\ &= \frac{1}{2i\epsilon}[f(x+\epsilon) - f(x-\epsilon)]. \end{aligned} \quad (11)$$

For small ϵ , this reduces to the ordinary momentum operator $\hat{p} \equiv -i\hbar\partial_x$:

$$\hat{P}_\epsilon = \hat{p} + O(\epsilon). \quad (12)$$

The “canonical” commutator between \hat{X}_ϵ and \hat{P}_ϵ on the lattice reads

$$\begin{aligned} ([\hat{X}_\epsilon, \hat{P}_\epsilon]f)(x) &= \frac{i}{2}[f(x+\epsilon) + f(x-\epsilon)] \\ &\equiv i(\hat{I}_\epsilon f)(x). \end{aligned} \quad (13)$$

The last line defines a lattice-version of the unit operator as the average over the two neighboring sites. All three operators \hat{X}_ϵ , \hat{P}_ϵ , and \hat{I}_ϵ are hermitian under the scalar product (8).

It was noted in [9] that the operators \hat{X}_ϵ , \hat{P}_ϵ and \hat{I}_ϵ generate the Euclidean algebra $E(2)$ in 2D. Indeed, setting $\hat{M} = \epsilon\hat{X}_\epsilon$, $\hat{P}_1 = \epsilon\hat{P}_\epsilon$ and $\hat{P}_2 = \hat{I}_\epsilon$ we obtain

$$[\hat{M}; \hat{P}_1] = i\hat{P}_2, \quad [\hat{M}; \hat{P}_2] = -i\hat{P}_1, \quad [\hat{P}_1; \hat{P}_2] = 0.$$

The generator \hat{M} corresponds to a rotation, while \hat{P}_1 and \hat{P}_2 represent two translations. In the limit $\epsilon \rightarrow 0$, the Lie algebra of $E(2)$ contracts to the standard Weyl–Heisenberg algebra: $\hat{X}_\epsilon \rightarrow \hat{x}$, $\hat{P}_\epsilon \rightarrow \hat{p}$, $\hat{I}_\epsilon \rightarrow 1$. Thus mathematically ordinary QM is obtained from lattice QM by a contraction of the $E(2)$ algebra via the limit $\epsilon \rightarrow 0$ of the deformation parameter ϵ .

All functions on the lattice can be Fourier-decomposed with wave numbers in the Brillouin zone:

$$f(x) = \int_{-\pi/\epsilon}^{\pi/\epsilon} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx}, \quad (14)$$

with the coefficients

$$\tilde{f}(k) = \epsilon \sum_x f(x) e^{-ikx}. \quad (15)$$

This implies the good-old de Broglie relation

$$(\hat{p}\tilde{f})(k) = \hbar k \tilde{f}(k), \quad (16)$$

and its lattice version

$$(-i\nabla\tilde{f})(k) = K\tilde{f}(k), \quad (-i\bar{\nabla}\tilde{f})(k) = \bar{K}\tilde{f}(k) \quad (17)$$

with the eigenvalues

$$K \equiv (e^{ik\epsilon} - 1)/i\epsilon = \bar{K}^*. \quad (18)$$

From (17) we find the Fourier transforms of the operators \hat{X}_ϵ , \hat{P}_ϵ , \hat{I}_ϵ :

$$(\hat{X}_\epsilon\tilde{f})(k) = i\frac{d}{dk}\tilde{f}(k), \quad (19)$$

$$(\hat{P}_\epsilon\tilde{f})(k) = \frac{1}{\epsilon} \sin(k\epsilon)\tilde{f}(k), \quad (20)$$

$$(\hat{I}_\epsilon\tilde{f})(k) = \cos(k\epsilon)\tilde{f}(k), \quad (21)$$

With the help of (16) and (21) we can rewrite the commutation relation (13) as

$$([\hat{X}_\epsilon, \hat{P}_\epsilon]f)(x) = i\hbar \cos(\epsilon\hat{p}/\hbar) f(x). \quad (22)$$

IV. UNCERTAINTY RELATIONS ON LATTICE

We are now prepared to derive the generalized uncertainty relation implied by the previous commutators. We shall define the uncertainty of an observable A in a state ψ by the standard deviation

$$(\Delta A)_\psi \equiv \sqrt{\langle\psi|(\hat{A} - \langle\psi|\hat{A}|\psi\rangle)^2|\psi\rangle}. \quad (23)$$

Following the conventional procedure (see, e.g., Ref. [11–13]), we derive the inequality

$$\begin{aligned} \Delta X_\epsilon \Delta P_\epsilon &\geq \frac{1}{2} |\langle\psi|[\hat{X}_\epsilon, \hat{P}_\epsilon]|\psi\rangle| = \frac{1}{2} |\langle\psi|\hat{I}_\epsilon|\psi\rangle| \\ &= \frac{\hbar}{2} |\langle\psi|\cos(\epsilon\hat{p}/\hbar)|\psi\rangle|. \end{aligned} \quad (24)$$

Let us study two different extremal regimes, first for long wavelengths where $\langle \hat{p} \rangle_\psi \rightarrow 0$, and second near the boundary of the Brillouin zone where $\langle \hat{p} \rangle_\psi \rightarrow \pi\hbar/2\epsilon$. To this end we first rewrite $\langle \cos(\epsilon\hat{p}/\hbar) \rangle_\psi$ as

$$\langle \cos(\epsilon\hat{p}/\hbar) \rangle_\psi = \sum_{n=0}^{\infty} \int_0^{\infty} dp \varrho(p) (-1)^n \frac{(\epsilon p/\hbar)^{2n}}{(2n)!}, \quad (25)$$

where $\varrho(p) \equiv |\psi(p)|^2$. In the first case, $\varrho(p)$ is peaked around $p \simeq 0$, so that the relation (25) becomes approximately

$$\langle \cos(\epsilon\hat{p}/\hbar) \rangle_\psi = 1 - \frac{\epsilon^2 p^2}{2\hbar^2} + \mathcal{O}(p^4). \quad (26)$$

where $p^2 \equiv \langle \hat{p}_x^2 \rangle_\psi$. If we now apply the identity

$$\langle \hat{A}^2 \rangle_\psi = (\Delta A)^2 + \langle \hat{A} \rangle_\psi^2, \quad (27)$$

we obtain from (24)

$$\begin{aligned} \Delta X_\epsilon \Delta P_\epsilon &\gtrsim \frac{\hbar}{2} \left| 1 - \frac{\epsilon^2 p^2}{2\hbar^2} \right| \\ &= \frac{\hbar}{2} \left| 1 - \frac{\epsilon^2}{2\hbar^2} [(\Delta p)^2 + \langle \hat{p} \rangle_\psi^2] \right|. \end{aligned} \quad (28)$$

For mirror-symmetric states where $\langle \hat{p} \rangle_\psi = 0$ this implies

$$\Delta X_\epsilon \Delta P_\epsilon \gtrsim \frac{\hbar}{2} \left(1 - \frac{\epsilon^2}{2\hbar^2} (\Delta p)^2 \right). \quad (29)$$

Here we have substituted $|\dots|$ with (\dots) since we assume that $\epsilon \simeq \ell_p$ (Planckian lattice) and that Δp is close to zero (this is our original assumption).

For Planckian lattices we can neglect in (29) higher orders in ϵ and write

$$\Delta X_\epsilon \Delta P_\epsilon \gtrsim \frac{\hbar}{2} \left(1 - \frac{\epsilon^2}{2\hbar^2} (\Delta P_\epsilon)^2 \right). \quad (30)$$

In the second case, where $\langle \hat{p} \rangle_\psi \rightarrow \hbar\pi/2\epsilon$, we are going to examine the behavior of (24) at the border of the first Brillouin zone, where the averaged momentum takes its maximum value

$$\langle \hat{p} \rangle_\psi = \frac{\pi\hbar}{2\epsilon}. \quad (31)$$

we expand on the right-hand side of (24):

$$\begin{aligned} \langle \cos[\pi/2 + (\epsilon\hat{p}/\hbar - \pi/2)] \rangle_\psi &= \langle \sin(\pi/2 - \epsilon\hat{p}/\hbar) \rangle_\psi \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} dp \varrho(p) (-1)^n \frac{(\pi/2 - \epsilon\hat{p}/\hbar)^{2n+1}}{(2n+1)!}, \end{aligned} \quad (32)$$

Under the assumption that $\varrho(p)$ is centered around $p \simeq \pi\hbar/2\epsilon$, the first term in the expansion (32) is dominant, and the uncertainty relation becomes

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{\hbar}{2} \left| \frac{\pi}{2} - \frac{\epsilon}{\hbar} \langle \hat{p} \rangle \right|. \quad (33)$$

Since k lies always inside the Brillouin zone, implying that $\langle \hat{p} \rangle \leq \pi\hbar/2\epsilon$, see that the long-wavelength GUP (30) changes close to the boundary of the Brillouin zone to

$$\Delta X_\epsilon \Delta P_\epsilon \gtrsim \frac{\hbar}{2} \left(\frac{\pi}{2} - \frac{\epsilon}{\hbar} \langle \hat{P}_\epsilon \rangle_\psi \right). \quad (34)$$

As the momentum reaches the boundary of the Brillouin zone, the right-hand side vanishes so that the lattice quantum mechanics at short wavelengths shows classical behavior.

It is worth noting that the uncertainty relation (34) has the same form of that found, on different grounds, by Magueijo and Smolin in [17]. In particular the deforming term is also there linear in the momentum. It results that in this model the world becomes "classical" at the Planck scale, in the sense that no blurred quantities show up in the measure of fundamental dynamical variables, as similarly devised by 't Hooft in models on "deterministic" quantum mechanics [18, 19].

V. IMPLICATIONS FOR PHOTONS

The vector potential of a photon in the Lorentz gauge in 1+1 dimensions satisfies the wave equation

$$\frac{1}{c^2} \partial_t^2 A^\mu(x, t) = \partial_x^2 A^\mu(x, t) \quad (35)$$

A plane wave solution $A^\mu(x) = \epsilon^\mu \exp[i(kx - \omega(k)t)]$ possesses the well know linear dispersion relation

$$\omega(k) = ck, \quad (36)$$

where ϵ^μ is a polarization vector. On a one-dimensional lattice, the operator ∂_x^2 is replaced by the lattice Laplacian $\bar{\nabla}\bar{\nabla}$, and the spectrum becomes, on account of Eq. (6) and (17),

$$\frac{\omega(k)}{c} = \sqrt{K\bar{K}} = \frac{\sqrt{[2 - 2\cos(k\epsilon)]}}{\epsilon} = \frac{2}{\epsilon} \sin\left(\frac{k\epsilon}{2}\right), \quad (37)$$

which reduces to (36) for $\epsilon \rightarrow 0$. Denoting the energy on the lattice $\hbar\omega$ by E_ϵ , we obtain the dispersion relation

$$\frac{E_\epsilon}{\hbar c} = \frac{2}{\epsilon} \sin\left(\frac{p\epsilon}{2\hbar}\right) \quad (38)$$

For small momenta p , this has the expansion

$$E_\epsilon = cp - c \frac{\epsilon^2 p^3}{24\hbar^2} + \mathcal{O}(\epsilon^4). \quad (39)$$

The correction to the continuum dispersion relation $E = pc + \mathcal{O}(\epsilon^2)$, so that we have $\Delta P_\epsilon c \approx \Delta E_\epsilon$, which allows us to rephrase (30) as

$$\Delta X_\epsilon \Delta E_\epsilon \geq \frac{\hbar c}{2} \left[1 - \frac{\epsilon^2}{2\hbar^2 c^2} (\Delta E_\epsilon)^2 \right]. \quad (40)$$

VI. APPLICATIONS TO MICRO BLACK HOLES PHYSICS

Let us see which consequences the modified uncertainty relations (30) and (40) have for black holes. In particular we want to study how mass-temperature relations is modified at short distances near the lattice spacing ϵ . For simplicity, we follow here the treatment of Refs. [20–25].

Small distances can be explored by high-momentum collisions. The size of the smallest detectable details by photons of energy E is $\delta x = \hbar c/2E$. The lattice version of this is, from Eq. (40):

$$\delta X_\epsilon \simeq \frac{\hbar c}{2E_\epsilon} \left(1 - \frac{\epsilon^2}{2\hbar^2 c^2} E_\epsilon \right) \quad (41)$$

We now suppose that the lattice spacing is not precisely equal to the Planck length, but merely proportional to it by a factor $a > 0$ with some nonextreme value of a . Let us denote the Planck energy by

$$\mathcal{E}_p = \hbar c/2\ell_p. \quad (42)$$

Then we can write (41) as

$$\delta X_\epsilon \simeq \frac{\hbar c}{2E_\epsilon} - \beta \ell_p \frac{E_\epsilon}{\mathcal{E}_p} \quad (43)$$

where $\beta = a^2/8$.

We now imagine having found a black hole on the lattice as a Schwarzschild solution. If the Schwarzschild radius is much larger than the lattice spacing ϵ , this will not look much different from the well-known continuum solution. We must avoid too small black holes, for otherwise, completely new physics will set in near the center, due to a pileup of defects. These will cause the melting of the world crystal at a critical defect density [26], and the emerging general relativity on the world crystal will look completely different from Einstein's theory.

Consider now, as in Refs.[20–25], a photon of Hawking radiation just outside the event horizon. The position uncertainty of such a photon is of the order of the Schwarzschild radius R_S , i.e., $\delta X_\epsilon \simeq 2R_S$. This, in turn, is equal to $2\ell_p m$, where m is the reduced mass of the black hole, $m = M/M_p$. According to the above arguments, this must be assumed to be much larger than unity, in order to avoid the melting transition. In this regime we can rephrase (43) as

$$2m = \frac{\mathcal{E}_p}{E_\epsilon} - \beta \frac{E_\epsilon}{\mathcal{E}_p}. \quad (44)$$

The energy E_ϵ , associated with this limiting photon via (41) sets the scale for the thermal energy, so that we obtain the temperature T from the relation

$$E_\epsilon = \pi k_B T. \quad (45)$$

Defining the Planck temperature T_p from $\mathcal{E}_p = \frac{1}{2}k_B T_p$, and measuring the temperature in terms of Planck units

as a reduced temperature $\Theta = T/T_p$, we can rewrite Eq. (44) as

$$2m = \frac{1}{2\pi\Theta} - \beta 2\pi\Theta. \quad (46)$$

In the continuum limit $\epsilon \rightarrow 0$ where $\beta \rightarrow 0$ and (43) reflects the ordinary uncertainty principle, this reduces to

$$m = \frac{1}{4\pi\Theta} \quad (47)$$

which is the dimensionless version of Hawking's formula for large black holes:

$$T = \frac{\hbar c}{4\pi k_B R_S}. \quad (48)$$

It is instructive to compare our mass-temperature relation (46) with that suggested by the so-called stringy version of the GUP :

$$2m = \frac{1}{2\pi\Theta} + \beta 2\pi\Theta. \quad (49)$$

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The physical consequences of our relation (46) are quite different from those of the stringy result (49), due to the opposite signs of the deformation term. In Fig. 1 we compare the two results, and add also the curve for the ordinary Hawking relation (47). Considering m and

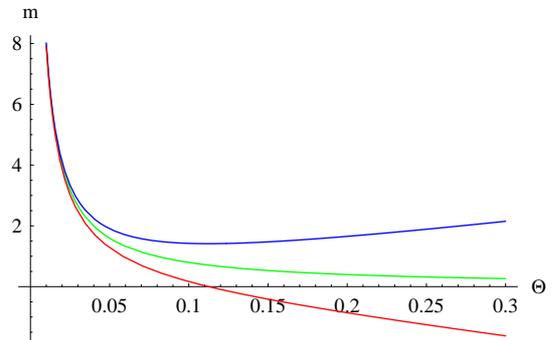


FIG. 1: Diagrams for the three Mass - Temperature relations, ours (red), Hawking's (green), and stringy UPS result (blue), with $\beta = 2$, as an example. As a consequence of lattice uncertainty principle the evaporation ends at a finite temperature.

Θ as functions of time, we can follow the evolution of a micro black hole from the curves in Fig. 1. For the stringy GUP, the blue line predicts a maximum temperature

$$\Theta_{\max} = \frac{1}{2\pi\sqrt{\beta}}, \quad (50a)$$

and a minimum mass

$$m_{\min} = \sqrt{\beta}. \quad (50b)$$

The end of the evaporation process is reached after a finite time, the final temperature is finite, and there is a remnant of finite rest mass (see Refs.[20–25]).

From the standard Heisenberg uncertainty principle we find the green curve, representing the usual Hawking formula. Here the evaporation process ends, after a finite time, with a zero mass and a worrisome infinite temperature.

In contrast to these results, our lattice GUP predicts the red curve. This yields a finite end temperature

$$\Theta_{\max} = \frac{1}{2\pi\sqrt{\beta}} \quad (51)$$

with a zero-mass remnant.

In the literature, the undesirable infinite final temperature prediction of Hawking’s formula has so far been cured only with the help of the stringy GUP, which brings the final temperature to a finite value. This result is, however, also questionable since implies the existence of finite-mass remnants in the universe. Some people like them as candidates for dark matter [27]. But it has also been pointed out that their existence would create other difficult problems in context with the entropy/information problem [cite...], the issue of their detectability, and their possibly too large abundance in the early universe [cite...].

Our result (46) coming from the lattice GUP formula (30) solves all these problems by predicting an end to the evaporation process at a finite final temperature with zero-mass remnants, which thus seems to be a desirable result.

A comment is useful concerning the relation (34) at the largest momentum near the boundary of the Brillouin zone. The relation (45) relating the temperature to the energy of the emitted photons is deduced from the semiclassical Hawking argument, which is a long-wavelength argument. Here we are confronted with the opposite limit of short wavelengths. In fact, Eq. (38) tell us that in this limit

$$E \simeq \frac{\sqrt{2}}{\epsilon} \hbar c \quad (52)$$

which for a world lattice of Planckian spacing $\epsilon = a l_p$ is of the order of the Planck energy, $E = (2\sqrt{2}/a) \mathcal{E}_p$.

Considering the associations of the previous paragraph, $\Delta X_\epsilon \sim m$, $\Delta P_\epsilon \sim E_\epsilon \sim T$ (where m and T are mass and temperature of the micro black hole), we see that close to the border of the Brillouin zone, where $\langle P_\epsilon \rangle \simeq \pi\hbar/(2\epsilon)$, the energy is $E_\epsilon \simeq \mathcal{E}_p$, and

$$m \cdot T \simeq 0. \quad (53)$$

Since the temperature in this regime approaches the Planck temperature, $T \simeq T_p$, we conclude that the mass of the micro black hole must go to zero. This is consistent with the previous result (46) deduced from the

long-wavelength relation Eqs. (45). The reason is simply that for small Θ , the term proportional to β in (46) becomes irrelevant.

VII. COMPARISON WITH 'T HOOFT'S MODEL

The quantum properties of the model proposed by 't Hooft’s to derive quantum physics from a deterministic system [18, 19] displays quite similar properties to our lattice quantum mechanics.

Let us first recapitulate the relevant points of 't Hooft’s scenario. He starts with the *deterministic* system consisting of a finite set of states, $\{(0), (1), \dots, (N-1)\}$ equidistantly distributed on a circle. Time evolution is implemented by means of discrete clockwise time jumps of length τ):

$$t \rightarrow t + \tau \quad : \quad (\nu) \rightarrow (\nu + 1 \bmod N) \quad (54)$$

On a basis spanned by the states (ν) , the evolution operator introduced by 't Hooft is

$$U(\Delta t = \tau) = e^{-iH\tau} = e^{-i\frac{\pi}{N}\tau} \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}. \quad (55)$$

With the hindsight of quantum mechanics, 't Hooft introduced the phase factor $-\pi/N$ which provides the 1/2-contribution to the energy spectrum of H . Indeed, if we diagonalize the previous matrix we obtain that the diagonal entries are $e^{i2\pi n/N}$ with $n = 0, \dots, N-1$. By denoting the corresponding eigenstates as $|n\rangle$ we obtain the energy spectrum:

$$\frac{H}{\omega} |n\rangle = (n + \frac{1}{2}) |n\rangle, \quad \omega \equiv \frac{2\pi}{N\tau}. \quad (56)$$

Because of this spectrum, H seems at first sight equal to the Hamiltonian of the 1D harmonic oscillator. However, there are two major differences. First, the eigenvalues of 't Hooft’s H have an upper bound implied by the finite value N value. Only in the continuum limit, i.e., for $N \rightarrow \infty$, one can expect a complete correspondence with the harmonic oscillator. However, even that is not true since in a the harmonic oscillator, the contribution 1/2 to the ground state energy is a consequence of a “geometric phase” (recall that in the semiclassical Bohr–Sommerfeld quantization it originates directly from the Morse index [29]) and as such it reflects a nontrivial global structure of the theory. So the ground state cannot be decided by any ad hoc choice of a phase factor in an evolution operator. It is only at the moment when 't Hooft’s model is able to predict in the large N limit the same geometric phase as the usual 1D harmonic oscillator when both theories match at $N \rightarrow \infty$.

From the group theoretical reasonings 't Hooft was able to find an explicit realization for quantum mechanical analogues of position and momenta operators. His new operators \hat{x} and \hat{p} fulfilled the “deformed” commutation rules

$$[\hat{x}, \hat{p}] = i \left(1 - \frac{\tau}{\pi} H \right), \quad (57)$$

and the Hamiltonian could be recast into the form

$$H = \frac{1}{2} \omega^2 \hat{x}^2 + \frac{1}{2} \hat{p}^2 + \frac{\tau}{2\pi} \left(\frac{\omega^2}{4} + H^2 \right). \quad (58)$$

In the continuous limit the Hamiltonian goes to the one of the harmonic oscillator, and the \hat{x} and \hat{p} commutator goes to the canonical one. In that limit the original state space (finite N) changes becoming the infinite dimensional state space.

To show the consistency of 't Hooft's model we try to re-derive it in the framework of the non-commutative differential calculus with a particular emphasis on the geometric phase. *****

We now want to investigate what is the geometric phase (if any) for the above model: we will then have an unambiguous definition of the ground state energy for the harmonic oscillator. Let us consider the eigenstates of the unitary operator Eq. (55). This is easily

obtained by considering that the above matrix (including the phase) should satisfy the condition $U^N = \mathbf{1}$, we get the eigenstates of the form $\langle q_l | l \rangle = \frac{1}{\sqrt{N}} e^{i\pi(2l+1)/N}$ with $l = 0, 1, \dots, N$.

Applying the definition of geometric phase, we obtain (the first line is the definition for the continuous case):

$$\begin{aligned} \Gamma &= i \int_0^T \langle \phi(t) | \frac{d}{dt} | \phi(t) \rangle \\ &= i \sum_l \sum_l \epsilon \langle l | q_l \rangle \vec{\partial} \langle q_l | l \rangle \\ &= i \sum_l \sum_l \epsilon \frac{1}{N} e^{-i\pi(2l+1)/N} \vec{\partial} e^{i\pi(2l+1)/N} \end{aligned}$$

with $x = l\epsilon$

VIII. CONCLUSIONS

IX. ACKNOWLEDGEMENTS

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