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Broken Scale Invariance

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I. Introduction

When the avelanche of papers on current algebra came to rest [*1*], many people's philosophy on the $SU(3) \times SU(3)$ properties of the world had settled on roughly the following ideas [*2*]:

1) The time components of the vector and axial vector currents observed in electromagnetic and weak interactions form the algebra $SU(3) \times SU(3)$. Under this algebra, a large portion of the energy density $\Theta_{00}(x)$ of the world is invariant.
2) When the divergence of the axial vector current $\partial A_\pi(x)$ appears in any n-point function, this n-point function is almost zero except for the low mass region[1]), where it is dominated by a pion pole (PCAC).
3) The $SU(3) \times SU(3)$ symmetry breaking term in $\Theta_{00}(x)$ is approximately $SU(2) \times SU(2)$ symmetric. However, the vacuum is not. As a consequence

[1]) Here one refers to the analytic continuation of the n-point function in the momentum squares q^2 of the local operator $\partial A_\pi(x)$.

the pion appears as an almost massless Goldstone boson. This mechanism generates the masses of the baryons and explains the smallness of higher mass contributions to $\partial A_\pi(x)$.

While the first point has become the basic hypothesis of much of the recent work on symmetries, points 2) and 3) are somewhat controversial. The discussion was basically started up by some unexplained experimental data on K_{l3} decay. Indeed, some quite plausible theoretical arguments can be found to raise doubts about the presence of such a universal pion pole dominance as stated in 2). The alternative statements which were proposed to replace 2) and 3) are [3]:

2′) Only the single particle matrix elements of ∂A_π are pion pole dominated. For arbitrary n-point function the dominance depends on the detailed dynamics (Weak PCAC).
3′) The $SU(3) \times SU(3)$ symmetry breaking term in $\Theta_{00}(x)$ is approximately $SU(3)$ invariant. This explains the rather small $SU(3)$ mass splittings of all particle multiplets except for the pseudoscalar octet.

The *principal* weakness of this scheme is that the small mass of the pion, and pion dominance of single particle matrix elements of ∂A_π remain unexplained dynamical accidents. The *practical* weakness consists in its smaller predictive power. Our point of view is that the first scheme apperas up to now to be sufficiently flexible to accommodate present experimental data and many of those to appear in the future. We shall therefore keep sticking to the more aesthetical ideas 1)—3) until they are definitely proven to be wrong. We think, however, that for the sake of understanding the significance of our assumptions it is worthwile to keep the other possibility in mind and to compare the results of both schemes whenever we can.
From the practical point of view, the fashion of current algebra has provided us with a considerable amount of techniques. These techniques have been developed in order to exploit the physical consequences of the following type of hypotheses:

1) There exist some currents $j_i^\mu(x)$.
2) Their charges $Q_i(x_0) = \int d^3x j_i^0(x)$ generate some well defined group transformations when applied to certain local fields at equal time

$$i[Q(x_0), \varphi(x)] = \delta\varphi(x). \tag{1.1}$$

3) Their divergences $\Delta(x) \equiv \partial j(x)$ are local fields dominated by a single meson.

In current algebra itself, assumption 2) was formulated in a stronger form. Among the other local fields $\varphi(x)$, there had to be necessarily also the currents $j_i^\mu(x)$ themselves, and the time components of the currents had to form the Lie algebra of the group transformations $\delta\varphi$. For the general techniques, to be applied below, this stronger form is not needed, though.
The principal consequence of the first two assumptions consist in the following statement.
The $N+1$ point functions

$$\tau^\mu(\varphi; x_1 \cdots x_N) = \langle 0 | T\big(j^\mu(y)\ \varphi^1(x_1) \cdots \varphi^N(x_N)\big) | 0 \rangle \tag{1.2}$$

satisfy the Ward identities

$$
\begin{aligned}
i\,\partial_\mu^{y}\tau^{\mu}(y;x_1\cdots x_N) = {} & i\,\langle 0|\,T\big(\Delta(y)\,\varphi^1(x_1)\cdots\varphi^N(x_N)\big)|0\rangle \\
& + \delta(y-x_1)\,\langle 0|\,T\big(\delta\varphi^1(x_1)\,\varphi^2(x_2)\cdots\varphi^N(x_N)\big)|0\rangle \\
& + \vdots \\
& + \delta(y-x_N)\,\langle 0|\,T\big(\varphi^1(x_1)\,\varphi^2(x_2)\cdots\delta\varphi^N(x_N)\big)|0\rangle .
\end{aligned} \tag{1.3}
$$

The terms on the right hand side can conveniently be re-written as

$$
\begin{aligned}
& i\Delta(y;x_1\cdots x_N) + \delta(y-x_1)\,\delta^1\sigma(x_1\cdots x_N) \\
& \quad + \vdots \\
& \quad + \delta(y-x_N)\,\delta^N\sigma(x_1\cdots x_N).
\end{aligned} \tag{1.4}
$$

If one goes to the Fourier transforms

$$
\begin{aligned}
& (2\pi)^4\delta^4(q+\Sigma p_i)\,\tau(q;p_1\cdots p_N) \\
& \equiv \int dy\,dx_1\cdots dx_N\,e^{i\left(qy+\sum_1^N p_i x_i\right)}\,\tau(y;x_1,\cdots,x_N) \quad \text{etc.}
\end{aligned} \tag{1.5}
$$

the Ward identity (WI) takes the form

$$
q^{\mu}\tau_\mu(q;p_1\cdots p_N) = i\Delta(q;p_1\cdots p_N) + \sum_{r=1}^{N}\delta^r\sigma(p_1,\cdots,p_r+q_1,\cdots p_N). \tag{1.6}
$$

There are very few cases where the Ward identity can directly be tested by experiment[2]). If only the function τ_μ is unknown, it may be eliminated by going to the point $q^\mu = 0$[3]). Here one obtains the low-energy theorem (LET)

$$
0 = i\Delta(0;p_1\cdots p_N) + \sum_{r=1}^{N}\delta^{(r)}\sigma(p_1,\cdots,p_N). \tag{1.7}
$$

However, also the amplitude $\Delta(0;p_1\cdots p_N)$ is usually hard to measure. It is for this reason, that assumption 3) is introduced. With assumption 3) the value $\Delta(0;p_1\cdots p_N)$ is the off shell continuation of an amplitude involving a physical meson of mass μ with the q^2 dependence given by a simple pole term $q^2-\mu^2$. In many cases, the amplitudes occurring in (1.6) refer to processes which are hard to perform in any laboratory. For those cases there is another way of obtain-

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[2]) The most famous example is the Ward identity relating the amplitude of two axial vector currents between nucleon states $\tau^{ba}_{\mu\nu} = i\int dx\,e^{iq'x} < N(p)|\,T\big(A_\mu{}^b(x)A_\nu{}^a(0)\big)|\,N(p) >$ to the corresponding amplitude of the divergences $\tau^{ba}_{\partial A\,\partial A}$ and to the matrix elements of vector current and Σ term $\Sigma^{ba} = i/2\left([Q^b\,\partial A^a] + (ba)\right)$ by $q^{\mu'}q^{\nu}\tau^{ba}_{\mu\nu} = \tau^{ba}_{\partial A\,\partial A} - if^{bac}[(q'+q)/2]\,\langle N(p)|\,V_c|\,N(p)\rangle + \langle N(p)|\,\Sigma^{ba}\,|N(p)\rangle$. The isospin odd part of this relation is directly measurable in neutrino and electron scattering on nuclei. Recall $T^{ba}_{\pi N\to\pi N} = \lim\limits_{\substack{q'^2\to\mu^2\\ q^2\to\mu^2}}(\mu^2-q^2)(\mu^2-q^2)/f_\pi{}^2\mu^4\,\tau^{ba}_{\partial A\,\partial A}$.

[3]) A possible pole at $q^\mu = 0$ can always be eliminated by infinitesimally modifying some internal masses. For example, the single nucleon pole in the Ward identity[2]) for πN scattering disappears upon taking the electromagnetic mass difference into account.

ing physical consequences from the Ward identity, called the hard-meson technique. One simply parametrizes the amplitudes in terms of vertex functions and propagators of particles which one expects to be prominent in the low-energy region. Then WI and LET provide us with relations among these parameters [4].
This technique has been shown to be completely equivalent to the method of effective Lagrangians [5]. Here one introduces separate fields for all those particles whose properties one would like to relate by means of Ward identities. Then one constructs a Lagrangian involving these fields.
The field transformations $\delta\varphi$ are introduced and a current $j^\mu(x)$ with the property (1.1) is found following standard Lagrangian methods. The only technical problem arises in satisfying assumption 3): In order that $\partial j(x)$ is dominated by a single particle only, the Lagrangian has to be chosen approximately invariant under $\delta\varphi$. This can be done by standard group theoretic techniques. Then any n-point functions involving $j^\mu(x)$, calculated via standard Feynman graphs, will satisfy the correct Ward identities (1.6).
While all this technical apparatus was becoming common knowledge among theoreticians, experimentalists made the important observation that the structure functions of electron proton scattering scale in the deep inelastic limit [6]. For large energy ν and virtual mass q^2 of the exchanged photon the cross section turned out to depend only on the ratio $\xi = -q^2/2m\nu$. This scaling property was immediately taken as an indication of an approximate scale invariance of the world. Kinematically, deep inelastic scattering probes the singularity structure of the product of two currents when their relative distance becomes light-like [7]. It is an attractive hypothesis that the Hamiltonian of the world is possibly scale invariant with a symmetry breaking of such a soft type that products of local observables do not notice it at light-like distances. It is worth mentioning that this is by no means the only explanation for the scaling phenomena. It is easy to find counter-examples: There are Lagrangian models with a strongly non-soft scale braking which yield scale invariant structure functions [8]. As usual in such a situation, one prefers assuming the world to be maximally symmetric and changes one's mind only after some clash with experiment.
In these lectures we shall investigate what physical consequences can be derived from the assumption of an approximate scale invariance of the world by using the techniques described above. In order to do so we shall introduce a current $\mathscr{D}_\mu(x)$ generating certain scale transformations on local fields, whose divergence is dominated by a single scalar meson called σ. Ward identities will be derived, parametrized in terms of particles and relations will be obtained for coupling constants involving this σ meson. Due to the equivalence of this approach to that of effective Lagrangians [5] we shall illustrate most of our statements by comparing with the situation in some definite Lagrangian models. We shall not talk about light cone aspects of broken scale invariance which have been discussed in great detail in the Literature.

II. The Dilatation Currents and its Basic Properties

A dilatation is a transformation in space time

$$x_\mu \to e^{-\alpha} x_\mu. \tag{2.1}$$

Accordingly, we shall call any representation of (2.1) in the physical Hilbert space a dilatation by e^α, if it transforms every local observable $O(x)$ into another local observable $O_\alpha(x)$ evaluated at $e^\alpha x$:

$$O(x) \to O_\alpha(e^\alpha x). \tag{2.2}$$

A vector $\mathscr{D}_\mu(x)$[4] is called a dilatation current, if its charge $D(x_0) \equiv \int d^3x \mathscr{D}_0(x)$ is the infinitesimal generator of all such dilatations:

$$e^{iD(x_0)\alpha}\, O(x)\, e^{-iD(x_0)} = O_\alpha(e^\alpha x). \tag{2.3}$$

By taking α infinitesimal, one finds the commutator

$$i[D(x_0), O(x)] = x\,\partial O(x) + O'(x) \tag{2.4}$$

where $O'(x)$ is again a local field $(\equiv \partial/\partial\alpha\, O_\alpha(x)\,|_{\alpha=0})$. If $O'(x)$ is a multiple of $O(x)$:

$$O'(x) = d\,O(x), \tag{2.5}$$

then $O(x)$ is said to have a definite dimension d.

From (2.4) we can immediately see an important property of the dilatation charge: The derivative with respect to the explicit dependence on x_μ[5]), $\tilde{\partial}_\mu D(x_0)$, satisfies the commutator:

$$i[\tilde{\partial}_\mu D(x_0), O(x)] = \partial_\mu O(x). \tag{2.6}$$

Since this is supposed to hold for all local observables of the theory, we conclude

$$\tilde{\partial}_\mu D(x_0) \equiv P_\mu. \tag{2.7}$$

From the equation of motion we therefore find[6])

$$i[D(x_0)H] = H - \frac{d}{dt} D(x_0) = H - \int d^3x\, \partial\mathscr{D}(x) \tag{2.8}$$

$$i[D(x_0)\, P_i] = P_i. \tag{2.9}$$

These equations allow us to prove an important low-energy theorem for diagonal matrix elements of $\partial\mathscr{D}(x)$ without using the general formalism described above[7]).

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[4]) Notice that we want $\mathscr{D}_\mu(x)$ to satisfy vector commutation rules with the Lorentz generators $M_{\mu\nu}$:

$$i[M_{\mu\nu}\, \mathscr{D}_\lambda(x)] = (x_\mu\partial_\nu - x_\nu\partial_\mu)\, \mathscr{D}_\lambda(x) + g_{\mu\lambda}\, \mathscr{D}_\nu(x) - g_{\nu\lambda}\, \mathscr{D}_\mu(x)$$

even though it will turn out to depend explicitly on x_μ i. e.

$$\partial_\mu\, \mathscr{D}_\lambda(x) \neq i[P_\mu\, \mathscr{D}_\lambda(x)].$$

[5]) Recall that the derivative with respect to the *explicit* dependence on x_μ of an operator $A(x)$ is that part of the total derivative $\partial_\mu A(x)$ *not* obtained by commuting with P_μ: $\tilde{\partial}_\mu A(x) \equiv \equiv \partial_\mu A(x) - i[P_\mu A(x)]$. A local operator $O(x)$ satisfies $\partial_\mu O(x) = i[P_\mu O(x)]$ and has not explicit dependence on x_μ.

[6]) In many Lagrangian theories one can define a local energy momentum tensor $\Theta_{\mu\nu}(x)$ such that dilatations are generated by $\mathscr{D}_\mu(x) = x^\nu\Theta_{\mu\nu}(x)$. In these theories, (2.8) is trivially satisfied since $\tilde{\partial}_\nu\, \mathscr{D}_\mu(x_0) = \Theta_{\mu\nu}$. In addition one has $\partial\,\mathscr{D}(x) = \Theta(x)$ (see Sect. VII). Our derivation is more general, though.

[7]) This low-energy theorem could certainly be proved by the methods leading to (1.7). For this particular case of elastic matrix elements we prefer, however, this more direct proof.

If $|\boldsymbol{p}\alpha)$ denotes any state of total momentum $\boldsymbol{p}$, with all tother quantum numbers collected in the index α, which is normalized by[8])

$$\langle \boldsymbol{p}'\alpha' | p\alpha) = 2p_0 (2\pi)^3 \, \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \, \delta_{\alpha'\alpha} N_\alpha \tag{2.10}$$

then

$$\langle \boldsymbol{p}\alpha | \partial \mathscr{D} | \boldsymbol{p}\alpha \rangle = 2\boldsymbol{p}^2 N_\alpha . \tag{2.11}$$

For a proof we simply take (2.8) between two diffent states and find

$$\begin{aligned} i(p_0 - p_0') \langle \boldsymbol{p}'\alpha' | D(x_0) | \boldsymbol{p}\alpha \rangle &= 2p_0{}^2 (2\pi)^3 \, \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \, \delta_{\alpha'\alpha} N_\alpha \\ &\quad - (2\pi)^3 \, \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \langle \boldsymbol{p}'\alpha' | \partial \mathscr{D} | \boldsymbol{p}\alpha \rangle . \end{aligned} \tag{2.12}$$

In this equation, momentum conservation makes sure that $\boldsymbol{p}'$ and $\boldsymbol{p}$ are close to each other. Therefore we can expand

$$p_0 - p_0' \approx \frac{1}{2p_0} \, (\boldsymbol{p}^2 - \boldsymbol{p}'^2) \tag{2.13}$$

and the left hand side of (2.12) can be rewritten as

$$\frac{i}{2p_0} \langle \boldsymbol{p}'\alpha' | \, [D, \boldsymbol{P}^2] \, | \boldsymbol{p}\alpha \rangle . \tag{2.14}$$

But using eqn. (2.9) we have

$$i[D(x_0), \boldsymbol{P}^2] = 2\boldsymbol{P}^2 \tag{2.15}$$

such that (2.14) becomes

$$2\boldsymbol{p}^2 (2\pi)^3 \, \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \, \delta_{\alpha'\alpha} N_\alpha .$$

Inserting this back into (2.12) we obtain indeed (2.11).

This proof is only valid if the dilatation current is not able to produce scalar mesons of mass zero. The reason is that in such a case a pole is present in the matrix elements of $\mathscr{D}_\mu$ at $q^2 = 0$ and the definition of $\tilde{\partial}_\mu$:

$$\tilde{\partial}_\mu \equiv \partial_\mu - i[P_\mu, \ ] \tag{2.16}$$

ceases to coincide with the naive derivative with respect to the explicit dependence on x_μ. One can roughly describe the situation in the following way: In the matrix elements of $D(x_0)$, the local parts of $\mathscr{D}_\mu$ contribute like $\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})$, the parts with linear x_μ dependence like $\partial_i \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})$, etc. If the local part has a $1/q^2 \ \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})$ pole, then some part of it will be attributed by $\tilde{\partial}_\mu$ of formula (2.16) to the second term $\partial_i \, \delta^3(\boldsymbol{p}' - \boldsymbol{p})$. Since we shall not be interested in a world containing such a massless particle, we shall not elaborate much more on this point. Only later, when we get to specific models some more comments will be in place.

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[8]) For many particle states, α contains continuous labels like the relative momenta and $\delta_{\alpha'\alpha}$ denotes continuous δ-functions. For single baryon and meson states we shall use the normalization $N_\alpha = 1/2\,m$ and 1, respectively.

III. Is $\partial \mathscr{D}_\mu$ Dominated by a Single Scalar Meson?

Being equipped with a dilatation current we can now embark on writing down Ward identities. Since n-point functions containing $\partial \mathscr{D}(x)$ are hard to measure in general[9]), assumption 3) of meson dominance of $\partial \mathscr{D}$ is necessary to derive physical consequences. The meson would have to be a scalar of isospin zero. The particle one tentatively accepts for this purpose is the broad s-wave resonance $\sigma(700)$ of width $\Gamma_{\sigma\pi\pi} \approx 400$ MeV which appears to be present in $\pi\pi$ scattering. Evidence for the existence of this particle is rather indirect. Theoreticians have kept needing it either to explain phenomenological fits of data or to make sum rules come out right. Or they have predicted it by unitarizing the $\pi\pi$ scattering amplitude. Among the many examples one could give here we just mention

1) Dispersion theoretic treatments of the processes $\pi\pi \to \pi\pi$ and $\pi\pi \to \mathrm{N}\overline{\mathrm{N}}$ prefer a σ-resonance at [*9*, *10*]

$$m_\sigma \approx 750 \pm 100 \text{ MeV}, \quad \Gamma_{\sigma\pi\pi} \approx 300 \pm 200 \text{ MeV}. \tag{3.1}$$

The corresponding $\sigma\pi\pi$ coupling is:[10])

$$|g_{\sigma\pi\pi}| \approx 3.4 \pm 1.\rangle. \tag{3.2}$$

In addition, the ratio $g_{\sigma\pi\pi}/g_{\sigma\mathrm{NN}}$ can be estimated as [*9*]

$$\frac{g_{\sigma\pi\pi}}{g_{\sigma\mathrm{NN}}} \approx (.9 \pm .25)\,\frac{m_\sigma}{\mu}. \tag{3.3}$$

The mass factor appears explicitely since the ratio $g_{\sigma\pi\pi}\mu/g_{\sigma\mathrm{NN}}m_\sigma$.

2) In backward πN scattering, a t-channel $\pi\pi$ resonance of 700 MeV would have to couple with a strength [*11*]

$$g_{\sigma\pi\pi}g_{\sigma\mathrm{NN}} \approx 69 \pm 4 \tag{3.4}$$

in order to explain the energy dependence of the amplitude close to threshold.
3) The low energy phase shift analysis of nucleon-nucleon scattering requires the exchange of at least one scalar particle. The determinations of $g^2{}_{\sigma\mathrm{NN}}$ vary from 31 ± 16 to 190 [*12*].
4) Constructions of low energy $\pi\pi$ amplitudes satisfying approximately crossing, analyticity, and unitarity and fitting the experimental ρ-shape predict a pole around 420 MeV with $\Gamma_{\sigma\pi\pi} \approx 400$ MeV [*13*].
5) The Adler Weisberger relation for $\pi\pi$ scattering is saturated with the observed ρ (765), $\Gamma_{\rho\pi\pi} \approx 125$ and f (1260), $\Gamma_{\mathrm{f}\pi\pi} \approx 150$ resonances by only 60%. Assuming that the remainder is due to a single s-wave resonance,[11]) this sum rule

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[9]) Except for diagonal matrix elements, for which a balance is sufficient! In theories where $\mathscr{D}_\mu = x^\nu \Theta_{\mu\nu}$ and $\partial\mathscr{D} = \Theta$, gravitational interactions would in principle do. Prof. Weber informs me that he does not have enough resolutions as yet.

[10]) The $\sigma\pi\pi$ and σNN couplings are defined by $\mathscr{L} = g_{\sigma\pi\pi}\,(m_\sigma/2)\,\sigma\pi^2 + g_{\sigma NN}\,\sigma\overline{N}N$ such that $\Gamma_{\sigma\pi\pi} = (3/4)\,(g^2{}_{\sigma\pi\pi}/4\pi)\,q$.

[11]) That the missing part is of positive parity, can be concluded from a combination of forward and backward dispersion relation written down for the amplitude at threshold (see Banerjee et al., Phys. Rev. **D2**, 2141 (1970)).

reads ($f_\pi \approx 0.095$ BeV):

$$f_\pi{}^2 \left[\left(\frac{g_{\rho\pi\pi}}{m_\rho} \right)^2 + \left(\frac{g_{\sigma\pi\pi}}{m_\sigma} \right)^2 + \frac{1}{24} \left(\frac{g_{f\pi\pi}}{m_f} \right)^2 \right] = 1. \tag{3.5}$$

The famous KSFR relation $g_{\rho\pi\pi} \approx m_\rho/\sqrt{2} f_\pi$ gives for the ρ-contribution 50% while the experimental width of f makes this contribution roughly 10%:

$$|g_{f\pi\pi}| \approx 23.3 \approx \sqrt{3}\, \frac{m_f}{f_\pi}. \tag{3.6}$$

As a consequence, $g_{\sigma\pi\pi}$ is about of the size

$$|g_{\sigma\pi\pi}| \approx \frac{m_\sigma}{\sqrt{2}\, f_\pi} \approx 5 \tag{3.7}$$

corresponding to a width of 400 MeV at $m_\sigma \approx 700$ MeV.
Combining this estimate with eqn. (3.4) we conclude that $g_{\sigma\mathrm{NN}}$ is not much different from $g_{\pi\mathrm{NN}}$ $(= 13.5)$:

$$|g_{\sigma\mathrm{NN}}| \approx 15. \tag{3.8}$$

This result was predicted in the σ-model, [*14*] in which the σ plays the role of being the chiral partner of the pion (see Sect. VIII).
Suppose this σ-particle dominates the divergence $\partial\mathscr{D}$. In analogy to PCAC, one calls this hypothesis PCDC (partial conservation of dilatation current).
In this case our low-energy theorem (2.11) allows for a direct experimental consequence in form of a Goldberger Treiman type of relation[12]).
If $m_\pi{}^3/\gamma$ denotes the direct coupling of σ to $\partial\mathscr{D}$ (analogous to $\langle 0 | \partial A | \pi \rangle = \mu^2 f_\pi$)

$$\langle 0 | \partial\mathscr{D}(0) | 0 \rangle \equiv \frac{m_\sigma{}^3}{\gamma} \tag{3.9}$$

we find for matrix elements between pions

$$\langle \pi(p') | \partial\mathscr{D}(0) | \pi(p) \rangle = \frac{m_\sigma{}^3}{\gamma} \frac{g_{\sigma\pi\pi} m_\sigma}{m_\sigma{}^2 - q^2}, \quad q \equiv p' - p \tag{3.10}$$

and between nucleons

$$\langle N(p') | \partial\mathscr{D}(0) | N(p) \rangle = \frac{m_\sigma{}^3}{\gamma} \frac{g_{\sigma\mathrm{NN}}}{m_\sigma{}^2 - q^2}. \tag{3.11}$$

Comparing with (2.11) at $q^2 = 0$ we conclude

$$g_{\sigma\pi\pi} = \gamma\, \frac{2\mu^2}{m_\sigma{}^2} \tag{3.12}$$

$$g_{\sigma\mathrm{NN}} = \gamma\, \frac{m}{m_\sigma}. \tag{3.13}$$

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12) Recall: There the matrix element of ∂A between nucleon states is $\langle N(p) | \partial A^3 | N(p) \rangle = m g_A$ while pion dominance gives $f_\pi g_{\pi NN}$.

Since γ is unknown we can only test the ratio

$$\frac{g_{\sigma\pi\pi}}{g_{\sigma NN}} = \frac{2\mu^2}{m m_\sigma} \approx .06 \tag{3.14}$$

which is experimentally $\approx 1/3$.

Thus one or both of the matrix elements (3.10) and (3.11) *cannot* be dominated by a single σ-meson.
In the following section we shall show that the assumption of σ-dominance for $\partial\mathscr{D}$ between pions is in conflict with the idea that pions are the Goldstone besons of the chiral symmetry. This property of pions enforces a subtraction in the matrix element (3.10). This saves us from a clash with experiment but destroys one prediction.
Since in this philosophy the role of the pions is a rather special one may hope that most other single particle matrix elements are still unsubtracted and derive predictions from this assumption. For example, the vertex $\partial\mathscr{D}\varrho\varrho$ defined by

$$\langle \varrho(p', \varepsilon') \mid \partial\mathscr{D}(0) \mid \varrho(p, \varepsilon)\rangle \equiv G(q^2) m_\rho \varepsilon' \varepsilon - \frac{2}{m_\rho} H(q^2)\, p\varepsilon'\, p\varepsilon' \tag{3.15}$$

has by σ-dominance the form factors[13]).

$$G(q^2) = \frac{m_\sigma{}^3}{\gamma} \frac{g_{\sigma\rho\rho} m_\rho}{m_\sigma{}^2 - q^2} \tag{3.16}$$

$$H(q^2) = \frac{m_\sigma{}^3}{\gamma} \frac{h_{\sigma\rho\rho} m_\rho}{m_\sigma{}^2 - q^2}. \tag{3.17}$$

Comparing the diagonal elements with (2.11), we find

$$-\frac{m_\sigma}{\gamma} g_{\sigma\rho\rho}\, m_\rho = 2 m_\rho{}^2 \tag{3.18}$$

or

$$g_{\sigma\rho\rho} = -2\frac{m_\rho}{m_\sigma}\gamma . \tag{3.19}$$

No restriction is imposed upon $h_{\sigma\rho\rho}$. Similarly, for photons the gauge invariant vertex reads

$$\langle \gamma(k', \varepsilon') \mid \partial\mathscr{D}(0) \mid \gamma(k, \varepsilon)\rangle = F(q^2)\,(k_\mu' k_\nu - g_{\mu\nu}\, k'k)\, \varepsilon_\mu' \varepsilon_\nu \tag{3.20}$$

with[14])

$$F(q^2) = -\frac{m_\sigma{}^3}{\gamma} e^2 \frac{g_{\sigma\gamma\gamma}}{m_\sigma{}^2 - q^2} \frac{2}{m_\sigma}. \tag{3.21}$$

From (2.11), the diagonal matrix elements have to vanish. But this is true for any $g_{\sigma\gamma\gamma}$. A popular method of obtaining anyhow results on $h_{\sigma\rho\rho}$ and $g_{\sigma\gamma\gamma}$ proceeds

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[13]) For the coupling constants see Sec. V.

[14]) We use $\mathscr{L}_{\sigma\gamma\gamma} = e^2 g_{\sigma\gamma\gamma}\, 1/m_\sigma\, (\partial_\mu A^\nu \partial^\mu A_\nu - \partial_\mu A^\nu \partial_\nu A^\mu)$ such that $\Gamma_{\sigma\gamma\gamma} = m_\sigma/4 \times$ $\times\, e^4 g_{\sigma\gamma\gamma}{}^2/4\pi \approx .11\, g_{\sigma\gamma\gamma}^2\, Me\,V$.

by postulating *maximal smoothness* of vertices: All free constants parametrizing a vertex are assumed to vanish except for those determined by low energy theorems (or Ward Identities). In this case we have $h_{\sigma\varrho\varrho} = 0$, $g_{\sigma\gamma\gamma} = 0$, and there is no radiative decay $\sigma \to 2\gamma$.[15]
The latter statement can be tested in principle by photoproduction of two pions on heavy nuclei via photon exchange (Primakoff effect).
At present, only phenomenological arguments are available about the strength of this coupling. A finite-energy sum rule analysis [16] of pion Compton scattering estimates $\Gamma_{\sigma\gamma\gamma} \approx 22$ keV corresponding to $g_{\sigma\gamma\gamma} \approx .47$. However, the analysis contains many sources of uncertainties. Another estimate is obtained from the combined application of forward and backward dispersion relations to nucleon Compton Scattering. [17] Here $g_{\sigma\gamma\gamma}$ comes out zero confirming the assumption of maximal smoothness. We think the latter estimate to be more reliable.[15])
If the first estimate was true, the σ-meson should be produced via the Primakoff effect with a cross section of

$$\sigma \approx 16\pi\alpha Z^2 \frac{\Gamma_{\sigma\pi\pi}}{m_\sigma{}^3} \ln\left(\frac{p_L}{m_\sigma}\right) \approx 8.5 \times 10^{-6}\, Z^2 \ln\left(\frac{p_L}{m_\sigma}\right) \mathrm{mb} \tag{3.22}$$

Unfortunately, a very high angular resolution is necessary to pick up the events of very small t which stick out above the strong interaction background (peak at $\Theta_L \approx \Delta$ of width 2Δ, where $\Delta \approx m_\sigma{}^2/2p_L{}^2 \ll 1$).

IV. Ward Identities for the $\partial\mathcal{D}\pi\pi$ Vertex and the Theorem about a Subtraction in q^2

The statement about the necessity of a substraction [18] in the $\partial\mathcal{D}\pi\pi$ vertex is basically due to the fact that from PCAC the pion can be continued smoothly off mass shell by using the divergence of the axial vector current ∂A as an interpolating field. All information on the $\partial\mathcal{D}\pi\pi$ system is certainly contained in the vertex $(k \equiv -q - p)$:

$$\tau(q^2; p^2, k^2) \equiv \int dx\, dy\, e^{i(qy+px)} \langle 0 \,|\, T\big(\partial\mathcal{D}(y)\, \partial A^\pi(x)\, \partial A^\pi 0)\big) \,|\, 0\rangle. \tag{4.1}$$

The crucial assumption which will be the basis of all the future discussion is that ∂A^π has the definite dimension d. Then the vertex τ is subject to a low energy theorem (1.7)

$$0 = i\tau(0; p^2, p^2) + \int dx e^{ipx} \langle 0 \,|\, T\big((d + x\partial)\, \partial A(x)\, \partial A(0)\big) \,|\, 0\rangle$$
$$+ \int dx e^{ipx} \langle 0 \,|\, T\big(\partial A(x)\, d\, \partial A(0)\big) \,|\, 0\rangle. \tag{4.2}$$

By defining a propagator of the field ∂A

$$i\Delta(p^2) \equiv \int e^{ipx} \langle 0 \,|\, T\big(\partial A(x)\, \partial A(0)\big) \,|\, 0\rangle, \tag{4.3}$$

[15]) Since the input information is quite well known from the analyses of photoproduction on nucleons.

this low-energy theorem becomes explicitly

$$-\tau(0; p^2, p^2) = \left(2d - 4 - p\frac{\partial}{\partial p}\right) \Delta(p^2) \tag{4.4}$$

$$= (2d - 4)\, \Delta(p^2) - 2p^2 \dot{\Delta}(p^2) \tag{4.5}$$

where $\bullet \equiv \partial/\partial p^2$.

One conveniently introduces a reduced vertex function by dividing the propagator $\Delta(p^2)$ ot of τ:

$$\Gamma(q^2; p^2, k^2) = -\Delta^{-1}(p^2)\, \Delta^{-1}(k^2)\, \tau(q^2; p^2, k^2). \tag{4.6}$$

It satisfies

$$\Gamma(0; p^2, p^2) = (2d - 4)\, \Delta^{-1}(p^2) - 2p^2 \Delta^{-2}(p^2)\, \dot{\Delta}(p^2). \tag{4.7}$$

Due to the assumption of PCAC, $\Delta(p^2)$ is, to a good approximation, given by

$$\Delta(p^2) = \frac{f_\pi^2 \mu^4}{p^2 - \mu^2}, \tag{4.8}$$

where f_π is the decay constant of the pion ($f_\pi \approx .095$ MeV).

If one goes to the pion pole, one finds from (4.7)

$$\Gamma(0; \mu^2, \mu^2) = 2\mu^2 \frac{1}{f_\pi^2 \mu^4}. \tag{4.9}$$

This is nothing else but the low-energy theorem (2.11), since from the LSZ reduction formulas,[16] the on-shell matrix element $\langle \pi(p') \mid \partial \mathcal{D} \mid \pi(p) \rangle$ is just given by

$$\langle \pi(p') \mid \partial \mathcal{D} \mid \pi(p) \rangle = \lim_{\substack{q'^2 \to \mu^2 \\ q^2 \to \mu^2}} f_\pi^2 \mu^4 \Delta^{-1}(p'^2)\, \Delta^{-1}(p^2) \times \tau(q^2; q'^2, p^2) \tag{4.10}$$

$$= f_\pi^2 \mu^4 \Gamma(q^2; \mu^2, \mu^2). \tag{4.11}$$

The factor $f_\pi^2 \mu^4$ appears in front since the properly normalized interpolating field for the pion is $\pi \equiv \partial A / f_\pi \mu^2$.

Due to the PCAC assumption (4.8), the p^2 dependence of (4.7) is completely determined. We find

$$f_\pi^2 \mu^4\, \Gamma(0; p^2, p^2) = 2\mu^2 + 2(d - 1)(p^2 - \mu^2). \tag{4.12}$$

Since PCAC is expected to be a good approximation only for q^2 much smaller than some characteristic mass M^2 (maybe $\approx 10\,\mu^2$), the quality of the statement (4.12) will decrease when p^2 leaves the mass shell. Correction terms of the order $O(|p^2 - \mu^2| / M^2)$ are expected to turn up.

Relation (4.12) tells us something quite interesting. It says that the strong form of PCAC implies that the dimension of ∂A is necessarily one. If it were three, as suggested by light cone discussions based on the quark model, the vertex would

[16]) For a nice exposition of this subject see the text book by S. Gasiorowicz, Elementary Particle Physics, John Wiley & Sons, N. Y. (1967).

behave as

$$f_\pi^2 \mu^4 \Gamma(0; p^2, p^2) = 4p^2 - 2\mu^2 \tag{4.13}$$

showing a rapid relative variation when p^2 runs from zero to μ^2.
However, we should not take PCAC that literally. We think that the *known* amount of violation of PCAC in the Goldberger Treiman relation

$$\begin{gathered} m g_A = g_{\pi NN} f_\pi - O\left(\frac{\mu^2}{M^2}\right) \\ (1.16 = 1.28 - .12)\ \mathrm{GeV} \end{gathered} \tag{4.14}$$

should rather be seen as setting the scale of *absolute* variations of amplitudes of the dimension of mass when going from $q^2 = \mu^2$ to $q^2 = 0$. Therefore, it is not surprising if small amplitudes of the size of a pion mass show rapid *relative* changes in magnitude. It is easy to demonstrate this point by means of a simple Lágrangian model (see, for example, Ch. VIII).
The result (4.13) is all the information that can be obtained by using the assumption of ∂A having a definite dimension.
The nice thing about the vertex $\partial\mathscr{D}\ \partial A\ \partial A$ is, however, that additional information can be derived by considering the properties of $\partial\mathscr{D}$ under the chiral $SU(2) \times SU(2)$ group. Clearly, if the commutators $[Q^5(x_0)\ \partial\mathscr{D}(x)]$ and $[Q^5(x_0)\ \partial A(x)]$ were known, an additional Ward identity could be derived by multiplying the vertex

$$\tau_\mu(q, p) \equiv i \int dx\, dy\, e^{i(qy+px)} \langle 0 \,|\, T\big(\partial\mathscr{D}(y)\, A_\mu(x)\, \partial A(0)\big) \tag{4.15}$$

with p_μ.
What do we know about these two commutators?
For the first the answer can be given if one makes only the very mild assumption that, apart from ∂A, also A_0 has a definite dimension. From current algebra this dimension is necessarily equal to 3. We can then show that

$$i[\partial\mathscr{D}(x), Q^5(x_0)] = (4 - d)\, \partial A(x). \tag{4.16}$$

The proof is based on a straight-forward use of the Jacobi identity. Since it is rather lengthy it will be given in the Appendix A.
The second commutator is the famous Σ-term occurring in many current algebra calculations:

$$\Sigma(x) \equiv i[Q_5{}^\pi(x_0), \partial A^\pi(x)]. \tag{4.17}$$

It usually is assumed to be a member of a $(1/2, 1/2)$ representation of $SU(2) \times \times SU(2)$, together with ∂A^π, i.e.

$$i[Q_5{}^\pi(x_0), \Sigma(x)] = -\partial A^\pi(x). \tag{4.18}$$

However, this point will not be of importance at this place.
With these commutators we find

$$\begin{aligned} \partial_\mu{}^x \langle 0 \,|\, T\big(\partial\mathscr{D}(y)\, A^{\pi\mu}(x)\, \partial A^\pi(0)\big) \,|\, 0\rangle &= \\ = \langle 0 \,|\, T\big(\partial\mathscr{D}(y)\, \partial A^\pi(x)\, \partial A^\pi(0)\big) \,|\, 0\rangle &- \\ - i\,\delta(x)\, \langle 0 \,|\, T\big(\partial\mathscr{D}(y) \textstyle\sum(0)\big) \,|\, 0\rangle &+ \\ + i(4 - d)\, \delta(y - x)\, \langle 0 \,|\, T\big(\partial A(x)\, \partial A(0)\big) \,|\, 0\rangle&. \end{aligned} \tag{4.19}$$

Hence τ obeys the Ward Identity

$$-p_\mu \tau^\mu(q, p) = \tau(q^2; p^2, k^2) + \Delta_{\partial\mathscr{D},\Sigma}(q^2) - (4-d)\,\Delta\big((q+p)^2\big) \qquad (4.20)$$

where $\Delta_{\partial\mathscr{D},\Sigma}$ is the propagator $-i\int e^{iqx}\langle 0\,|\,T\big(\partial\mathscr{D}(x)\,\Sigma(0)\big)\,|0\rangle$. The corresponding low-energy theorem at $p = 0$ yields an equation for τ:

$$\tau(q^2; 0, q^2) = (4-d)\,\Delta(q^2) - \Delta_{\partial\mathscr{D},\Sigma}(q^2) \qquad (4.21)$$

implying for the reduced vertex $\Gamma(q^2; 0, q^2)$

$$\Gamma(q^2; 0, q^2) = -(4-d)\,\Delta^{-1}(0) + \Delta^{-1}(0)\,\Delta^{-1}(q^2)\,\Delta_{\partial\mathscr{D};\Sigma}(q^2). \qquad (4.22)$$

Notice that at the point where all arguments are zero, comparison of (4.22) and (4.7) yields

$$\Delta_{\partial\mathscr{D},\Sigma}(0) = d\Delta(0). \qquad (4.23)$$

This result could have been arrived at as well by equating the low energy theorems for the two point functions

$$\langle 0\,|\,T\big(A_\mu(x)\;\partial A(0)\big)\,|\,0\rangle \qquad (4.24)$$

and

$$\langle 0\,|\,T\big(\mathscr{D}_\mu(x)\;\Sigma(0)\big)\,|\,0\rangle \qquad (4.25)$$

and by observing that due to (4.17), $\sum$ has the same dimension d as ∂A. Obviously, if we want to derive any consequences we have to parametrize the propagator $\Delta_{\partial\mathscr{D},\Sigma}$. If $\partial\mathscr{D}$ is dominated by a single particle, this propagator should have the form

$$\Delta_{\partial\mathscr{D},\Sigma}(q^2) = \frac{c}{q^2 - m_\sigma^2}. \qquad (4.26)$$

As can be seen in models, this assumption actually appears to be somewhat weaker than that of σ-dominance of $\partial\mathscr{D}$. The reason is that the symmetry breaker $\sum$ is expected to be a smoother operator than $\partial\mathscr{D}$. We shall come back to this point later in Sect. X. Using equ. (4.23), the comparison of (4.26) with the PCAC form of $\Delta(0)$, equ. (4.8), determines (4.26) completely

$$\Delta_{\partial\mathscr{D},\Sigma}(q^2) = d\,\frac{m_\sigma^2}{\mu^2}\,\frac{f_\pi^2\mu^4}{q^2 - m_\sigma^2}. \qquad (4.27)$$

From (4.22) we obtain an explicit form for the Γ vertex

$$f_\pi^2\mu^4\Gamma(q^2 0 q^2) = \frac{1}{q^2 - m_\sigma^2}\Big[\big((4-d)\,\mu^2 - d m_\sigma^2\big)q^2 - (4-2d)\,\mu^2 m_\sigma^2\Big]. \qquad (4.28)$$

This result can be combined with equ. (4.12). Since $\Gamma(q^2; p_1^2, p_2^2)$ has to be a symmetric function in p_1^2, p_2^2, we find [18]

$$f_\pi^2\mu^4\Gamma(q^2; p_1^2, p_2^2) = \frac{m_\sigma^2}{q^2 - m_\sigma^2}\,[a(q^2 - m_\sigma^2) + b m_\sigma^2] \qquad (4.29)$$

with

$$a = -1 + (4 - d) \frac{\mu^2}{m_\sigma{}^2} \tag{4.30}$$

$$b = -1 + (2 - d) \frac{\mu^2}{m_\sigma{}^2} + (1 - d)(p_1{}^3 + p_2{}^2 - 2\mu^2)/m_\sigma{}^2.$$

With the direct coupling strength $m_\sigma{}^3/\gamma$ between $\partial\mathscr{D}$ and σ introduced before, this result amounts to an on shell $\sigma\pi\pi$ coupling constant of [*18*]

$$g_{\sigma\pi\pi} = \gamma \left(1 + (d - 2) \frac{\mu^2}{m_\sigma{}^2}\right). \tag{4.31}$$

The subtraction constant of

$$f_\pi{}^2 \mu^4 \Gamma(\infty; p_1{}^2, p_2{}^2) = -\left(1 + (d - 4) \frac{\mu^2}{m_\sigma{}^2}\right) m_\sigma{}^2 \tag{4.32}$$

is a measure for the breakdown of exact σ-dominance of $\partial\mathscr{D}$ between pions, just as the term $(1 - d)(p_1{}^2 + p_2{}^2 - 2\mu^2)/m_\sigma{}^2$ breaks PCAC.
Only for $d = 1$ and $m_\sigma{}^2 = 3m_\pi{}^2$ are both hypotheses exactly verified. In referring back to our earlier discussion on the significance of the condition of exact PCAC we just note that as soon as q^2 is somewhat away from the pion mass and the matrix element $f_\pi{}^2\mu^4\Gamma(q^2; p_1{}^2, p_2{}^2)$ is not close to zero any more, the relative PCAC breaking becomes extremely small.
For example, close to the σ-mass shell, $q^2 \approx m_\sigma{}^2$, Γ is essentially proportional to b itself:

$$f_\pi{}^2 \mu^4 \Gamma(q^2; p_1{}^2, p_2{}^2) \approx \frac{m_\sigma{}^4}{q^2 - m_\sigma{}^2} [-1 + (2 - d) + (1 - d)(p_1{}^2 + p_2{}^2 - 2\mu^2)/m_\sigma{}^2]. \tag{4.33}$$

This vertex is independent of $p_1{}^2$, $p_2{}^2$ within the range of a few μ^2 for any reasonable d between 0 and 4.
The hypothesis of σ-dominance of $\partial\mathscr{D}$ in the $\partial\mathscr{D}\ \partial A\ \partial A$ vertex, however, cannot be saved by such an argument, due to the large mass of σ.
At this place we should like to comment on the usage of the term "subtraction" to denote a power of q^2 or p^2 in the reduced vertex functions. It is important to stress that within the hard-meson method we are pursuing, this term has nothing to do with the true subtractions necessary at high energy when dispersing in the corresponding variables. In the hard meson method, Ward identities are merely enforcing certain powers of q^2, p^2, etc. in the reduced vertex functions *as far as the low energy region is concerned.* These powers can very well be a reflection of higher mass singularities in an *unsubstracted* dispersion relation in q^2, p^2, etc. The Ward identity tells us nothing about the physical origin of these powers.

V. The Question of the Size of the Σ-Term

While our hope for universal σ-dominance of $\partial\mathscr{D}$ has been destroyed in the last last section, some models suggest that the Σ-commutator appearing in equ. (4.16) may still be dominated by a single pole at $q^2 = m_\sigma{}^2$. If this is so, then the propagator (4.29) determines the strength of the direct coupling

$$\langle 0\,|\Sigma(0)|\,\sigma\rangle = \gamma d f_\pi^2 \frac{\mu^2}{m_\sigma}. \tag{5.1}$$

This formula allows us to express the famous Σ-term of pion nucleon scattering (for the definition, see footnote in on p. 3) in the following form [*19*]:

$$\langle N(p)\,|\Sigma(0)|\,N(p)\rangle = g_{\sigma\mathrm{NN}}\gamma d \frac{f_\pi^2\mu^2}{m_\sigma^3}. \tag{5.2}$$

But from our last result (4.32) we can substitute γ by $g_{\sigma\pi\pi}$ with only a few percent error

$$\langle N(p)\,|\Sigma(0)|\,N(p)\rangle \approx g_{\sigma\mathrm{NN}} g_{\sigma\pi\pi} d \frac{f_\pi^2\mu^2}{m_\sigma^3} \approx g_{\sigma\mathrm{NN}} g_{\sigma\pi\pi} d \times .5\ \mathrm{MeV}. \tag{5.3}$$

If one uses the estimate (3.4) for the σ-couplings:

$$g_{\sigma\mathrm{NN}} g_{\sigma\pi\pi} \approx 69 \pm 4 \tag{5.4}$$

one finds

$$\langle N(p)\,|\Sigma(0)|\,N(p)\rangle \approx d \times 35\ \mathrm{MeV}.$$

Unfortunately, the value of the Σ-term cannot be measured directly. An off-mass shell continuation is necessary in order to arrive from the the physical pion nucleon scattering amplitude at the point where both pion momenta are zero. For this purpose, Fubini and Furlan have developed a dispersion theoretic method [*20*]. However, the discontinuities of the dispersion integrals are not well known and require further approximations. Applying this method to πN and pion nucleus scattering leads to the estimates[17]:

$$\langle N(p)\,|\Sigma(0)|\,N(p)\rangle \approx \begin{Bmatrix} 25 \\ 35 \end{Bmatrix} \begin{matrix} [21] \\ [22] \end{matrix}. \tag{5.5}$$

It has been argued by Cheng and Dashen [*24*], that within the PCAC approximation, the value of the Σ-term appears with the reversed sign at the on-mass shell point $\nu = 0$, $t = 2\mu^2$ of the πN amplitude. The argument is briefly the following: Current algebra gives a low-energy theorem for the amplitude

$$T^+(\nu, t, q'^2, q^2) = A^+(\nu, t, q'^2, q^2) + \nu B^+(\nu, t, q'^2, q^2) \tag{5.6}$$

$$\equiv A^+ + \frac{g(q'^2)\,g(q^2)}{m}\frac{\nu^2}{\nu_B^2 - \nu^2} + \nu\tilde{B}^+ \equiv \tilde{T}^+ + \frac{g(q'^2)\,g(q^2)}{m}\frac{\nu_B^2}{\nu_B^2-\nu^2} \tag{5.7}$$

where $\nu_B \equiv -\,qq'/2m = t - q'^2 - q^2/4m$, $g(q^2)$ is the off-mass shell continuation of the πNN coupling constant (via the interpolating field $\pi = \partial A^\pi/f_\pi\mu^2$), and $\tilde{T}^+$, $\tilde{B}^+$ are the amplitudes free of the *ps* nucleon pole. This low-energy theorem is

$$\tilde{T}^+(0000) = A^+(0000) - \frac{g^2(0)}{m} = -\frac{1}{f_\pi^2}\langle N(p)\,|\Sigma|\,N(p)\rangle. \tag{5.8}$$

---

[17]) For a criticism of the results of Ref. [*21*] see Ref. [*23*].

On the other hand, any amplitude emitting a pseudoscalar particle of $q^\mu = 0$ via the field ∂A vanishes. As a consequence, $\tilde{T}^+$ has the so called Adler zeros:[1])

$$\tilde{T}^+(0, \mu^2, 0, \mu^2) = A^+(0\ \mu^2\ 0\ \mu^2) - \frac{g(0)\, g(\mu^2)}{m} = 0. \tag{5.9}$$

The crucial assumption is now, that from PCAC [*24*] the amplitude

$$\tilde{T}^+(0, q'^2 + q^2, q'^2, q^2) = A(0, q'^2 + q^2, q'^2, q^2) - \frac{g(q'^2)\, g(q^2)}{m} \tag{5.10}$$

is a smooth function in q'^2, q^2. If we allow for only linear variation in q'^2, q^2, the Adler zeros enforce the on shell value

$$\tilde{T}^+(0, 2\mu^2, \mu^2, \mu^2) = \frac{1}{f_\pi^2} \langle N(p)\, |\Sigma|\, N(p)\rangle. \tag{5.11}$$

We do think that this argument is convincing. Certainly we cannot exclude that $\tilde{T}^+(0, q'^2 + q^2, q'^2, q^2)$ may in fact be a smooth function of q'^2, q^2 . However, the principle of PCAC can certainly not be invoked for a proof. As a counter-example consider the amplitude in the linear σ-model which incorporates *exact* PCAC:

$$\tilde{T}^+(\nu, t, q'^2, q^2) = -g_{\sigma\pi\pi} g_{\sigma\mathrm{NN}} \frac{m_\sigma}{t - m_\sigma^2} - \frac{g^2_{\pi\mathrm{NN}}}{m} \tag{5.12}$$

with

$$g_{\sigma\pi\pi} = -\frac{m_\sigma}{f_\pi}\left(1 - \frac{\mu^2}{m_\sigma^2}\right), \quad g_{\sigma\mathrm{NN}} = -g_{\pi\mathrm{NN}} = -\frac{m}{f_\pi}. \tag{5.13}$$

Due to PCAC, this amplitude has the Adler zero

$$\tilde{T}^+(0, \mu^2, 0, \mu^2) = 0 \tag{5.14}$$

and it is independent of q'^2, q^2.
However,

$$\tilde{T}^+(0, q'^2 + q^2, q'^2, g^2) \tag{5.15}$$

picks up a q'^2, q^2 dependence from the t-channel singularities. If σ was very low in mass, the non-smoothness of $\tilde{T}^+(0, q'^2, q^2, q'^2, q^2)$ would be arbitrarily large in spite of exact PCAC. Since the particle is not very low-lying, one may argue that one can include its effect by means of an expansion linear in q'^2, q^2. However, as long as we do not have a definite idea about the breaking of PCAC we consider this procedure as quite dangerous. If such breaking terms are turned in, the expansion of $\tilde{T}^+$ up to first order corrections to PCAC is

$$\tilde{T}^+(0; t, q'^2, q\text{-}) = \left[\left(a_0 + a_0' \frac{q'^2 + q^2}{\mu^2}\right) + \left(a, + a',\ \frac{q'^2 + q^2}{\mu^2}\right) \frac{t}{\mu^2}\right] \frac{1}{\mu} \tag{5.16}$$

where the Σ-term is contained in a_0:

$$\langle N(p)\, |\Sigma|\, N(p)\rangle = -\frac{f_\pi^2}{\mu^2}\, a_0 \approx -65 a_0\, \mathrm{MeV}. \tag{5.17}$$

On shell, $\tilde{T}^+(0t\mu^2\mu^2)$ has been determined in πN analyses. Let [25—28]

$$\tilde{T}^+(0t\mu^2\mu^2) \equiv \left(A_0 + A_1 \frac{t}{\mu^2}\right)\frac{1}{\mu}. \tag{5.18}$$

Now the point is that A_0 and A_1 are both of the same size[18])

$$A_0^{\exp} \approx -1.4 \pm .6 \tag{5.19}$$

$$A_1^{\exp} \approx 1.13. \tag{5.20}$$

There is no a priori reason why A_1 should have a weaker q^2 dependence than A_0 in a theory with a slight breaking of PCAC[19]). If one takes the condition of the Adler zeros into account, one can eliminate only one of the four parameters. If we leave a_1' as an unknown we can express the others in terms of a_1' and the experimental quantities A_0 and A_1 in the form.

$$\begin{aligned} a_0 &= 2a_1' - 2A_1 - A_0 \\ a_1 &= -2a_1' + A_1 \\ a_0' + a_1' &= A_0 + A_1. \end{aligned} \tag{5.21}$$

We see that only *the sum* of the PCAC breakers $a_0' + a_1'$ is determined by experiment.
How the breaking distributes among a_0' and a_1' is completely model dependent.
Let ϱ be the content of breaking in a_1', i.e.

$$\begin{aligned} a_1' &= \varrho(A_1 + A_0) \\ a_0' &= (1-\varrho)(A_1 + A_0). \end{aligned} \tag{5.22}$$

Then the Σ-term is obviously

$$\begin{aligned} \langle N(p)\,|\Sigma|\,N(p)\rangle &= \frac{f_\pi^2}{\mu}\left(A_0 + 2A_1 - 2\varrho(A_0 + A_1)\right) \\ &= f_\pi^2[(1-\varrho)\,T(0, 2\mu^2, \mu^2, \mu^2) - \varrho\, T(0, 0, \mu^2, \mu^2)]. \end{aligned} \tag{5.23}$$

The ad-hoc assumption $\varrho = 0$ reduces to the result of CHENG and DASHEN

$$\langle N(p)\,|\Sigma|\,N(p)\rangle = f_\pi^2\, \tilde{T}^+(0, 2\mu^2, \mu^2, \mu^2) = 65\,(.9 \pm .6)\ \text{MeV}.^{20)} \tag{5.24}$$

For arbitrary ϱ we have

$$\begin{aligned} \langle N(p)\,|\Sigma|\,N(p)\rangle &= \frac{f_\pi^2}{\mu}\left(1 \pm .6 - 2\varrho\,(-.3 \pm .6)\right) \\ &= 65\,[(1 + .6\varrho) \pm .6\,(1 - 2\varrho)]\ \text{MeV}^{20)}. \end{aligned} \tag{5.25}$$

---

[18]) Due to the nucleon contribution $g^2/m = 27.3/\mu$ cancelling almost completely $A^+(00\mu^2\mu^2) \approx \approx 26.1/\mu$.
[19]) This can be checked for example by adding a PCAC breaking term to the σ-model.
[20]) Our numbers are obtained by inspection of the data points of Ref. [26] since Cheng and Dashen dont quote any errors.

About the value of ϱ we do not know much. One should think, though, that due to PCAC, a_0' and a_1' should be smaller than a_0 and a_1, respectively. Therefore we expect a_0' and a_1' to have opposite signs and ϱ could lie somewhat below zero or above one.

Obviously, the large error bars leave the result (5.25) completely consistent with the off shell extrapolation values of eqn. (5.5)

A word of caution is in place here. The argument raised before concerning the accuracy of the pole dominance approximation for $\partial\mathscr{D}$ can certainly be applied to the matrix elements of Σ as well. In the beginning of this section we justified this assumption for Σ by saying that models suggest Σ to be a smoother operator than $\partial\mathscr{D}$. Unfortunately, due to the small factor μ^2/m_σ appearing in the coupling strength of Σ with σ (eqn. (5.1)) inclusion of PCAC breaking may make this argument completely irrelevant. In the σ-model, for example, one can show that a very small PCAC breaking term (like $-m_0\overline{\Psi}\Psi$ in the Lagrangian with $m_0 \approx \approx 110$ MeV) whose magnitude is chosen to correct the defect of the Goldberger Treiman relation, can yield a subtraction constant of -110 MeV in the Σ-term which is much larger than the contribution of the σ-pole (See Sect. IX for the detailed argument). For the divergence $\partial\mathscr{D}$, on the other hand, the coupling with σ is so large that between nucleons m_0 can be completely neglected.

VI. The Dimensional Properties of the Hamiltonian Density $\Theta_{00}(x)$

Until now we have investigated the consequences of broken scale invariance as they follow from the assumption of certain fields having definite dimensions. We have not, as yet, assumed anything about the detailed mechanism of scale breaking except that $\partial\mathscr{D}$ should be dominated by a single σ-meson. In this section we shall try to find out wether any simple breaking structure in the Hamiltonian is compatible with experiment.

Consider the Hamiltonian density of the world $\Theta_{00}(x)$. If it had a dimension four, the action would be invariant under dilatation and $\partial\mathscr{D}$ would vanish. In this case, consideration of elastic matrix elements of eqn. (2.8) teaches us that all particles have to be massless.[21])

Since the real world is massive, there is at least one term of dimension different from 4 which breaks the scale symmetry of $\Theta_{00}(x)$. Is there any connection of this symmetry breaker with the chiral breaking term? If we accept the standard ideas about $SU(2) \times SU(2)$ breaking in Θ_{00}, then Θ_{00} can be split in an $SU(2) \times SU(2)$ conserving term $\tilde{\Theta}_{00}$ and a local scalar operator $\tilde{\Sigma}(x)$ belonging to a $(1/2, 1/2)$ representation of $SU(2)$. i.e.

$$\left[Q_5^\pi[Q_5^\pi\, \tilde{\Sigma}(x)]\right] = \tilde{\Sigma}(x). \tag{6.1}$$

But then one can show that field $\tilde{\Sigma}(x)$ is identical with the commutator term $\Sigma(x)$ introduced in eqn. (4.18). To see this one uses the equation of motion

$$i[Q_5^\pi H] = i[Q_5^\pi, d^3x\, \tilde{\Sigma}(x)] = -\dot{Q}_5^\pi = -\int d^3x\, \partial A^\pi \tag{6.2}$$

---

[21]) Excluding a zero-mass pole in $\mathscr{D}_\mu$ for physical reasons. We shall discuss the problem associated with a Goldstone way of breaking scale symmetry in Sects. VII and VIII.

and commutes with one more $-iQ_5^{\pi}$:

$$\int d^3x \big[Q_5^{\pi}[Q_5^{\pi}\tilde{\Sigma}(x)]\big] = \int d^3x\, \Sigma(x). \tag{6.3}$$

From the (1/2 1/2) assumption (6.1) one obtains

$$\int d^3x \big(\tilde{\Sigma}(x) - \Sigma(x)\big) = 0. \tag{6.4}$$

If the integrand could be shown to be Lorentz invariant, we would have from a theorem [29] in field theory:

$$\tilde{\Sigma}(x) \equiv \Sigma(x). \tag{6.5}$$

Now $\tilde{\Sigma}(x)$ is a scalar by assumption. That $\Sigma(x)$ is also a scalar can only be seen after a little work if we make the very mild assumption that the commutator $[\partial A^{\pi}(x), \partial A^{\pi}(y)]_{x_0=y_0}$ vanishes[22] (see App. A). As a consequence, Θ_{00} can be written as

$$\Theta_{00}(x) = \tilde{\Theta}_{00}(x) + \Sigma(x). \tag{6.6}$$

If we assume ∂A to have a definite dimension $d \neq 4$, then the term $\Sigma(x)$ has the same dimension and is necessarily one of the breakers of scale invariance. Are there any more? If the chiral structure is supposed to extend also to the group $SU(3) \times SU(3)$ then this is certainly true.

It has been proposed that there exists a whole set of 18 local scalar and pseudoscalar operators u_a and v_a $(a = 0, \ldots, 8)$ transforming according to the $(\bar{3}, 3) \times (3, \bar{3})$ representation of $SU(3) \times SU(3)$ $(i = 1, \ldots, 8)$ [30]

$$\begin{aligned}[Q_5^i(x_0), u^b(x)] &= -i\, d^{ibc} v^c(x) \\ [Q_5^i(x_0), v^b(x)] &= i\, d^{ibc} u^c(x).\end{aligned} \tag{6.7}$$

In terms of these operators, the breaking of $SU(3) \times SU(3)$ symmetry is supposed to be of the form

$$\Theta_{00_{SB}}(x) \equiv u \equiv u_0 + c\, u_8 \tag{6.8}$$

where c is some number. Now $\Theta_{00_{SB}}(x)$ can be split into an $SU(2) \times SU(2)$ symmetric term

$$S = \frac{1}{\sqrt{3}}\left(1 - \sqrt{2}\, c\right) \frac{1}{\sqrt{3}}\left(u_0 - \sqrt{2}\, u_8\right) \tag{6.9}$$

and the (1/2, 1/2) type of term

$$\Sigma = \frac{1}{\sqrt{3}}\left(\sqrt{2} + c\right) \frac{1}{\sqrt{3}}\left(\sqrt{2}\, u_0 + u_8\right) \equiv W_{\pi}(c) \frac{1}{\sqrt{3}}\left(\sqrt{2}\, u_0 + u_8\right). \tag{6.10}$$

From our assumption that Σ has a definite dimension d and from the commutators (6.7) it follows that all components u_a, v_a have the same dimension. We conclude that the whole term u has a definite dimension d. It is an attractive hypothesis

[22]) Actually we need only the lowest Schwinger term to be absent. This is fulfilled in any Lagrangian model where ∂A is equal to the canonical pion field.

that there is no other operator breaking scale symmetry except for the terms S and Σ of definite dimensions d breaking chiral $SU(3) \times SU(3)$ invariance.[29][31]
Let us see the consequences of this hypothesis. Suppose Θ_{00} can be decomposed into a term Θ_{00}^* of dimension four and a term u of dimension $d \neq 4$ breaking scale symmetry. For the sake of the argument, let us also suppose that, in addition, there is a scalar term $\delta(x)$ present of dimension $d_\delta \neq 4$. Our hope is that δ can be a trivial c-number term of $d_\delta = 0$.
As a first consequence of this assumption one obtains the theorem that the divergence is completely determined (by what is called the virial theorem).

$$\partial \mathscr{D}(x) = (4 - d_\delta)\, \delta(x) + (4 - d)\, u(x). \tag{6.11}$$

The proof uses equ. (2.8):

$$i[D(x_0)\, H] = H - \int d^3x\, \partial \mathscr{D} \tag{6.12}$$

inserting $H = \int d^3x (\Theta_{00}^* + \delta + u)$ gives on the left hand side

$$\begin{aligned}&\int d^3x[(4 + x\,\partial)\, \Theta_{00}^* + (d_\delta + x\,\partial)\, \delta + (d + x\,\partial) u] \\ &\quad = \int d^3x [\Theta_{00}^* + (d_\delta - 3)\, \delta + (d - 3) u] + x_0 \partial_0 \int d^3x\, \Theta_{00}\end{aligned} \tag{6.13}$$

and on the right hand side

$$\int d^3x\, \{[\Theta_{00}^* + \delta + u] - \partial \mathscr{D}(x)\}. \tag{6.14}$$

The operator Θ_{00}^* drops out and for the remaining integral over scalar operators we can again use the theorem of field theory quoted before[23]) to show (6.11).
Equ. (6.11) has a simple consequence. Since from equ. (2.11) ∂D between vacuum states is zero we find [*32*].

$$(4 - d_\delta)\, \langle 0 \,|\, \delta \,|\, 0 \rangle + (4 - d)\, \langle 0 \,|\, u \,|\, 0 \rangle = 0. \tag{6.15}$$

Notice that complete absence of any δ-term would imply $\langle 0 | u | 0 \rangle = 0$. In addition, equ. (6.11) implies a simple low-energy theorem for the propagator

$$-i \Delta_{\partial\mathscr{D}\, \partial\mathscr{D}}(q^2) = \int dx\, e^{iqx} \langle 0 \,|\, T\big(\partial \mathscr{D}(x)\, \partial \mathscr{D}(0)\big) \,|\, 0 \rangle. \tag{6.16}$$

For this consider the vacuum expectation value

$$\langle 0 \,|\, T\big(\mathscr{D}_\mu(x)\, \partial \mathscr{D}(0)\big) \,|\, 0 \rangle \tag{6.17}$$

apply the derivative with respect to x_μ and use

$$i[D(x_0),\, \partial \mathscr{D}(0)] = (4 - d_\delta)\, d_\delta \delta + (4 - d)\, d u. \tag{6.18}$$

This yields:

$$\Delta_{\partial\mathscr{D}\, \partial\mathscr{D}}(0) = (4 - d_\delta)\, d_\delta\, \langle 0 \,|\, \delta \,|\, 0 \rangle + (4 - d)\, d\, \langle 0 \,|\, u \,|\, 0 \rangle \tag{6.19}$$

or, using (6.15),

$$\Delta_{\partial\mathscr{D}\, \partial\mathscr{D}}(0) = (4 - d)\, (d - d_\delta)\, \langle 0 \,|\, u \,|\, 0 \rangle. \tag{6.20}$$

---

[23]) In going from equ. (6.4) to (6.5).

If the propagator $\Delta_{\partial\mathcal{D}\,\partial\mathcal{D}}(0)$ is σ-dominated, the left hand side can be expressed as

$$\frac{m_\sigma{}^4}{\gamma^2} \approx \frac{m_\sigma{}^4}{g_{\sigma\pi\pi}{}^2}$$

and is observable.

As a first result we can now conclude that in addition to u there is necessarily a term δ (with $d_\delta \neq 4$), since otherwise equ. (6.15) would force $\langle 0 \,|\, u \,|\, 0\rangle = 0$ in contradiction with experiment. Taking the existence of δ for granted, the term on the right-hand side is known from standard chiral low-energy theorems for the amplitude $\langle 0 \,|\, T\big(A_\mu{}^\pi(x)\, \partial A^\pi(0)\big) \,|\, 0\rangle$. By using the commutators of equ. (6.7), one finds [32]

$$\Delta^i(0) = \langle 0 \,|\, \Sigma^{ii} \,|\, 0\rangle \equiv a^i \langle 0 \,|\, S \,|\, 0\rangle + b^i \langle 0 \,|\, \Sigma \,|\, 0\rangle \qquad (6.21)$$

where

$$a^i = \begin{Bmatrix} 0 \\ \dfrac{r+1}{r+2} \\ \dfrac{4}{3} \end{Bmatrix}; \qquad b^i = \begin{Bmatrix} 1 \\ \dfrac{r+1}{2r} \\ \dfrac{1}{3} \end{Bmatrix} \qquad \text{for } i = \begin{Bmatrix} 1,2,3 \\ 4,5,6,7 \\ 8 \end{Bmatrix} \qquad (6.22)$$

and

$$r \equiv -\frac{2}{3}\,\frac{c+\sqrt{2}}{\sqrt{2}}. \qquad (6.23)$$

These equations imply for S and Σ:

$$\langle 0 \,|\, S \,|\, 0\rangle = (r+2)\left(\frac{1}{r+1}\,\Delta^{\mathrm{K}}(0) - \frac{1}{2r}\,\Delta^\pi(0)\right) \qquad (6.24)$$

$$\langle 0 \,|\, \Sigma \,|\, 0\rangle = \Delta^\pi(0) \qquad (6.25)$$

and for $u = S + \Sigma$

$$\langle 0 \,|\, u \,|\, 0\rangle = \frac{r-2}{2r}\,\Delta^\pi(0) + \frac{r+2}{r+1}\,\Delta^{\mathrm{K}}(0). \qquad (6.26)$$

Putting (6.26) together with (6.20) we find the result [32][24])

$$\Delta_{\partial\mathcal{D}\,\partial\mathcal{D}}(0) = (4-d)\,(d-d_\delta)\left[\frac{r-2}{2r}\,\Delta^\pi(0) + \frac{r+2}{r+1}\,\Delta^{\mathrm{K}}(0)\right]. \qquad (6.27)$$

Since $\Delta^\pi(q^2)$ and $\Delta^{\mathrm{K}}(q^2)$ are usually assumed to be π and K dominated, the expression in square brackets can be rewritten as

$$\left[\frac{r-2}{2r}\,\Delta^\pi(0) + \frac{r+2}{r+1}\,\Delta^{\mathrm{K}}(0)\right] = -\left[\frac{r+2}{2r}\,f_\pi{}^2 m_\pi{}^2 + \frac{r+2}{r+1}\,f_{\mathrm{K}}{}^2 m_{\mathrm{K}}{}^2\right]. \qquad (6.28)$$

---

[24]) Notice that this result can be written in form of a spectral function sum rule for the propagators involved, since

$$\Delta(0) \equiv \int \frac{\varrho(\mu^2)\, d\mu^2}{-\mu^2}.$$

With the standard assumption about the Goldstone nature of π and K mesons discussed in the introduction, the parameter r is determined approximately by

$$r \approx \frac{m_\pi^2}{m_K^2 - m_\pi^2} \tag{6.29}$$

giving (6.28) the form

$$\approx \left[\frac{m_\pi^2}{2}(3f_\pi^2 - 2f_K^2) + m_K^2(2f_K^2 - f_\pi^2)\right]. \tag{6.30}$$

While the value for f_π is quite well known to be $\approx .095$ GeV, the ratio f_K/f_π could be anything between one and 5/4. These particular two values make (6.30) come out as

$$\approx m_K^2 f_\pi^2 \begin{Bmatrix} 1 \\ \frac{34}{16} \end{Bmatrix} \approx \begin{Bmatrix} 2.2 \\ 4.7 \end{Bmatrix} 10^{-3}\,\mathrm{GeV}^4;\ f_K/f_\pi = \begin{Bmatrix} 1 \\ \frac{5}{4} \end{Bmatrix}. \tag{6.31}$$

The left hand side, on the other hand gives with

$$g_{\sigma\pi\pi} \approx 5,\ \ m_\sigma \approx 100:$$

$$\frac{m_\sigma^4}{\gamma^2} \approx 10 \times 10^{-3}\,\mathrm{GeV}^4.$$

In order to obtain agreement with experiment, the factor in front has to be

$$(d - d_\delta)(4 - d) \approx \begin{Bmatrix} 4.5 \\ 2.1 \end{Bmatrix} \text{ for } \frac{f_K}{f_\pi} = \begin{Bmatrix} 1 \\ \frac{5}{4} \end{Bmatrix}. \tag{6.32}$$

We note that a c-number δ-term with $d_\delta = 0$ is certainly compatible with experiment provided d is equal to 1, 2, or 3, for which the factor (6.31) takes the values 3, 4, or 3, respectively.

For completeness we would like to mention also the consequences if one assumes only the validity of the framework of weak PCAC. In this case the parameter r has been determined from other considerations to be [3]

$$r \approx 3.3.$$

We find, instead of (6.32)

$$(d - d_\delta)(4 - d) \approx \begin{Bmatrix} 4.7 \\ 3.0 \end{Bmatrix} \text{ for } \frac{f_K}{f_\pi} = \begin{Bmatrix} 1 \\ \frac{5}{4} \end{Bmatrix} \tag{6.33}$$

We see that also in this case the absence of an operator δ-term is compatible with $d = 1, 2$, or 3.

Notice that if we were dealing only with $SU(2) \times SU(2)$ symmetry, we could obtain similar relations by assuming the energy density to have the form

$$\Theta_{00}(x) = \Theta^*_{00}(x) + \delta(x) + \Sigma(x) \tag{6.34}$$

with Θ^*_{00}, δ, and Σ having the dimension 4, d_δ, and d, respectively. In this case our equation (6.11) would read

$$\partial \mathscr{D} = (4 - d_\delta)\,\delta + (4 - d)\,\Sigma \tag{6.35}$$

and the low-energy theorem (6.27) would become

$$\Delta_{\partial\mathscr{D}\,\partial\mathscr{D}}(0) = (4 - d)\,(d - d_\delta)\,\langle 0\,|\Sigma|\,0\rangle = (4 - d)\,(d - d_\delta)\,\Delta^\pi(0) \tag{6.36}$$

or, saturated with single particles,

$$\frac{m_\sigma{}^4}{\gamma^2} = (4 - d)\,(d - d_\delta)\,f_\pi{}^2\mu^2. \tag{6.37}$$

Here the right hand side would be very much too small compared with $m_\sigma{}^4/\gamma$ due to the absence of K-masses.

Physically this means that breaking of scale and chiral symmetry can never be attributed to the same source at the level of $SU(2) \times SU(2)$, since the σ-mass is much heavier than the pion mass.

The average meson mass within the pseudoscalar octet is, on the other hand, comparable with m_σ such that within $SU(3) \times SU(3)$ the δ-term could very well be a c-number.

Some time ago, an argument was forwarded in the literature trying to prove the necessity of an operator term δ. This argument was based on the determination of the Σ-term of von Hippel and Kim

$$\langle N(p)\,|\Sigma(0)|\,N(p)\rangle = 25 \text{ MeV} \tag{6.38}$$

and went as follows: [*33—35*]

If one calculates Σ for πN scattering by using equations (6—7) one finds

$$\langle N(p)\,|\Sigma(0)|\,N(p)\rangle = \frac{1}{\sqrt{3}}\,W_\pi(c)\,\big[\sqrt{2}\,\langle N|u_0|N\rangle + \langle N|u_8|N\rangle\big]. \tag{6.39}$$

For the value of $c \approx -1.25$ this gives

$$\approx .073\,[\langle N|u_0|N\rangle + .71\,\langle N|u_8|N\rangle]. \tag{6.40}$$

One now assumes that one can do lowest order $SU(3)$ perturbation theory in the Hamiltonian density

$$\Theta_{00} = \Theta^*_{00} + \delta + u_0 + c\,u_8. \tag{6.41}$$

Then the matrix element of $c\,u_8$ is determined from the octet mass splittings as

$$\langle N|c\,u_8|N\rangle = m_{\mathrm{N}} - \frac{1}{2}\,m_\Sigma - \frac{1}{2}\,m_\Lambda = -.215 \text{ GeV}. \tag{6.42}$$

Inserting this, together with the estimate (6.38) into (6.40) we recover

$$\langle N|u_0|N\rangle \approx .2 \text{ GeV} \tag{6.43}$$

or

$$\langle N|u_0 + c\,u_8|N\rangle \approx 0. \tag{6.44}$$

On the other hand, if δ is a c-number, equ. (6.11) between nucleon states gives

$$\langle N | \partial \mathscr{D} | N \rangle = (4 - d) \langle N | u_0 + c u_8 | N \rangle \tag{6.45}$$

which is incompatible with (6.44) for any value of $d \leq 4$.
Fortunately, this argument fails.[25]) The reason is that due to the Goldstone mechanism of chiral symmetry breaking and PCAC, $SU(3)$ perturbation theory is not valid. We shall illustrate this point in a specific chiral Lagrangian model in Sect. X. In this model the $SU(3)$ mass splittings of the baryons do not originate at all in the term $u_8{}^+$) of the Hamiltonian density Θ_{00}[26]).
Instead, they are completely due to an $SU(3) \times SU(3)$ symmetric baryon meson interaction. This symmetric interaction term generates *all* of the baryon masses via the non-zero vacuum expectation values of the meson fields.
In the formula (6.44), on the other hand, both baryon and meson masses, with their $SU(3)$ splittings are generated by u_0 alone$^{+)}$.

VII. Lagrangian Models for Scale Invariance

A welcome illustration for any theorem on broken scale invariance derived from Ward identities is provided by effective Lagrangian models in the tree graph approximation. We shall not go into the details of proving the equivalence of both methods [*5*]. The mechanism will become transparent when we discuss some specific simple models.
Let us first remark that given any Lagrangian $\mathscr{L}(\varphi, \partial_\mu \varphi)$ as a function of abitrary fields φ and $\partial_\mu \varphi$, we can always define a canonical scale current[27]) [*36*]

$$\mathscr{D}_\mu(x) \equiv \pi_\mu(x)\, d\varphi(x) + x^\nu \Theta^c_{\mu\nu}(x) \tag{7.1}$$

with $\Theta^c_{\mu\nu}$ being the canonical energy momentum tensor

$$\Theta^c_{\mu\nu}(x) = \pi_\mu(x)\, \partial_\nu \varphi(x) - g_{\mu\nu}\, \mathscr{L}(x). \tag{7.2}$$

After quantization, the charge of this current has the property of assigning to any dynamically independent component of φ the definite dimension d via the commutator:

$$\delta\varphi(x) = i[D(x_0),\ \varphi(x)] = (x\partial + d)\,\varphi(x). \tag{7.3}$$

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[25]) Ref. [*42*] made also this claim. However, the paper is quite misleading. While the author *did* notice that $SU(3)$ perturbation theory might fail due to the existence of a σ -meson, he illustrated the point by a model that contradicts the ideas of chiral symmetry. His model contains BB terms directly in $u_0 + c\, u_3$ (his equ. (9)). These terms violate PCAC and invalidate the Goldberger Treiman relation. See Sect. X for a proper model.
[26]) Assuming $SU(3)$ invariance of the vacuum.
[27]) The field φ stands representative for any set of different fields. The derivative $\partial\mathscr{L}/\delta\partial^\mu\varphi$ is conveniently written as $\pi_\mu(x)$ such that $\pi_0 = \pi$ is the canonical momentum of $\varphi(x)$.

The choice of d is, at this level, *completely arbitrary*. If we form the divergence of $\mathscr{D}_\mu(x)$ and use the Euler Lagrange equations we find

$$\partial \mathscr{D}(x) = \partial^\mu(\pi_\mu d\varphi + x^\nu \pi_\mu \partial_\nu \varphi - x_\mu \mathscr{L}) =$$

$$= \frac{\delta \mathscr{L}}{\delta \varphi} d\varphi + \pi_\mu (d+1)\, \partial^\mu \varphi - \partial^\mu (x_\mu \mathscr{L}) + x^\nu \left(\frac{\delta \mathscr{L}}{\delta \varphi} \partial_\nu \varphi + \pi_\mu \partial^\mu \partial_\nu \varphi \right) =$$

$$= \frac{\delta \mathscr{L}}{\delta \varphi} d\varphi + \frac{\delta \mathscr{L}}{\delta\, \partial^\mu \varphi} (d+1)\, \partial^\mu \varphi - 4\mathscr{L}. \tag{7.4}$$

Obviously, the first two terms do nothing but indicate the dimension contained in any expression involving the fields φ and $\partial_\mu \varphi$. If $\mathscr{L}$ has the form

$$\mathscr{L} = \sum_n \mathscr{L}_n \tag{7.5}$$

where $\mathscr{L}_n$ are pieces of dimension d_n, then (7-4) yields:

$$\partial \mathscr{D} = \sum_n (d_n - 4) \mathscr{L}_n. \tag{7.6}$$

If all terms in $\mathscr{L}$ have dimension 4, the dilatation current is conserved.
It is obvious that the current $\mathscr{D}_\mu$ is a non-local operator. It has the property that the derivative $\tilde{\partial}_\nu$ with respect to the *explicit* time dependence is

$$\tilde{\partial}_\nu \mathscr{D}_\mu(x) = \Theta^c_{\mu\nu}(x). \tag{7.7}$$

The canonical expression (7.1) for $\mathscr{D}_\mu(x)$ has a form completely analogous to the non-local canonical current of the total angular momentum

$$\mathfrak{M}^\mu_{\lambda\varkappa} = -i\pi^\mu \Sigma_{\lambda\varkappa}\, \varphi + x_\lambda\, \Theta^{c\mu}{}_\varkappa - x_\varkappa \Theta^{c\mu}{}_{\lambda\rangle} \tag{7.8}$$

where $\Sigma_{\lambda\varkappa}$ are the Lorentz generators in the spin space of the fields φ.[28]) The angular momenta

$$M_{\lambda\varkappa} \equiv \int d^3x\, \mathfrak{M}^0_{\lambda\varkappa}(x) \tag{7.9}$$

generate Lorentz transformations via the commutation rule

$$i[M_{\lambda\varkappa}, \varphi(x)] = (x_\lambda \partial_\varkappa - x_\varkappa \partial_\lambda - i\Sigma_{\lambda\varkappa})\, \varphi(x). \tag{7.10}$$

At this point one may recall that BELINFANTE [*37*] has constructed a modified energy momentum tensor $\Theta^B_{\mu\nu}$ which has the advantage of being symmetric and of allowing to bring equ. (7,8) to the more aesthetic form

$$\mathfrak{M}^\mu_{\lambda\varkappa} = x_\lambda \Theta^{B\mu}{}_\varkappa - x_\varkappa \Theta^{B\mu}{}_{\lambda\rangle}. \tag{7.11}$$

---

[28]) They commute like:

$$i[\Sigma_{\lambda\varkappa}, \Sigma_{\lambda\tau}] = g_{\lambda\lambda}\, \Sigma_{\varkappa\tau}; \quad g_{00} = 1, \quad g_{ii} = -1$$

i.e. in the same way as the Lorentz generators $M_{\lambda\varkappa}$. For Dirac particles $\Sigma_{\lambda\varkappa} = i/4\, [\gamma_\lambda, \gamma_\varkappa]$, for vector mesons $(\Sigma_{\lambda\varkappa})_{\alpha\beta} = i(g_{\lambda\alpha} g_{\varkappa\beta} - g_{\lambda\beta}\, g_{\varkappa\alpha})$.

This tensor is defined by[29])

$$\Theta^B_{\mu\nu} = \Theta^c_{\mu\nu} + \partial^\varrho X_{\varrho\mu\nu} \tag{7.12}$$

where

$$X_{\varrho\mu\nu} = -\frac{i}{2}\left[\pi_\varrho \Sigma_{\mu\nu}\varphi - \pi_\mu \Sigma_{\varrho\nu}\varphi - \pi_\nu \Sigma_{\varrho\mu}\varphi\right] \tag{7.13}$$

is antisymmetric in ϱ and μ. For this reason, Θ^B_{oi} and Θ^c_{ci} differ only by a divergence and possess the same spatial integrals

$$P_\mu = \int d^3x\, \Theta^B_{0\mu} = \int d^3x\, \Theta^c_{0\mu}. \tag{7.14}$$

It is natural to ask whether it is possible to find an energy momentum tensor which allows us to write not only the generators of the Lorentz group as (7.11), but also the dilatation current in the simple form

$$\mathscr{D}_\mu = x^\nu \Theta_{\mu\nu}. \tag{7.15}$$

For this one rewrites (7.1) in the form

$$\begin{aligned} \mathscr{D}_\mu &= x^\nu \Theta^B_{\mu\nu} + \pi_\mu\, d\varphi + X^\varrho_{\varrho\mu} - \partial^\varrho (X_{\varrho\mu\lambda} x^\lambda) \\ &= x^\nu \Theta^B_{\mu\nu} + v_\mu - \partial^\varrho (X_{\varrho\mu\lambda} x^\lambda) \end{aligned} \tag{7.16}$$

where

$$v_\mu(x) \equiv \pi_\mu(x)\, d\varphi(x) - i\, \pi^\nu(x)\, \Sigma_{\mu\nu}\, \varphi(x) \tag{7.17}$$

is called the *field-virial*[30]) [*36*].

Suppose now that the field virial v_μ can be written as the divergence of some tensor $\sigma^{\mu\nu}(x)$:

$$v_\mu(x)(x) = \partial^\nu \sigma_{\mu\nu}(x). \tag{7.18}$$

This is the case for a large class of Lagrangians. For example, if a Lagrangian containing scalar, spin 1/2 and vector particles with no derivative couplings, spinors and vectors have $\sigma^{\mu\nu} = 0$ while scalar particles satisfy (7.18) with $\sigma_{\mu\nu} = g_{\mu\nu}\varphi^2/2$.[31])

Let $\sigma^+_{\mu\nu}$ be the symmetric part of $\sigma_{\mu\nu}$ $\left(\sigma^\pm_{\mu\nu} \equiv 1/2\,(\sigma_{\mu\nu} \pm \sigma_{\nu\mu})\right)$. Then one may define a new improved energy momentum tensor

$$\Theta_{\mu\nu} \equiv \Theta^B_{\mu\nu} + \frac{1}{2}\, \partial^\lambda \partial^\varrho X_{\lambda\varrho\mu\nu} \tag{7.19}$$

where

$$\begin{aligned} X_{\lambda\varrho\mu\nu} &= g_{\lambda\varrho}\sigma^+_{\mu\nu} - g_{\lambda\mu}\sigma^+_{\varrho\nu} - g_{\lambda\nu}\sigma^+_{\mu\varrho} + g_{\mu\nu}\sigma^+_{\lambda\varrho} \\ &\quad - \frac{1}{3} g_{\lambda\varrho} g_{\mu\nu} \sigma^{+\varkappa}_{\varkappa} + \frac{1}{3} g_{\lambda\mu} g_{\varrho\nu} \sigma^{+\varkappa}_{\varkappa}. \end{aligned} \tag{7.20}$$

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[29]) Using the equations of motion one can bring (7.12) to the manifestly symmetric form

$$\Theta^B_{\mu\nu} = \frac{1}{2}\,(\Theta^c_{\mu\nu} + \Theta^c_{\nu\mu}) + \frac{i}{2}\,\partial^\varrho\,[\pi_\mu \Sigma_{\varrho\nu}\,\varphi + \pi_\nu \Sigma_{\varrho\nu}\,\varphi].$$

[30]) For all the details of this calculation see Ref. [*36*].

It can easily be checked that $\Theta_{\mu\nu}$ can be used instead of $\Theta^B_{\mu\nu}$ to construct the whole Poincaré algebra P_μ and $M_{\lambda\varkappa}$. None of the space integrals are influenced by the additional term in (7.19).
This energy momentum tensor has indeed the desired property of allowing for $\mathscr{D}_\mu$ the representation (7.15). By inserting $\Theta_{\mu\nu}$ in the expression (7.16) for the dilatation charge one finds

$$\mathscr{D}_\mu = x^\nu \Theta_{\mu\nu} - \frac{1}{2}\, \partial^\lambda \partial^\varrho (X_{\lambda\varrho\mu\nu} x^\nu) - \partial^\nu \sigma^-_{\mu\nu} - \partial^\varrho (X_{\varrho\mu\lambda} x^\lambda). \tag{7.21}$$

But none of the terms on the right hand side contributes to the dilatation charge[32]. Therefore one can use (7.15) as a new dilatation current.
The advantage of this energy momentum tensor is that the divergence of the scale current becomes

$$\partial^\mu \mathscr{D}_\mu = \Theta_\mu{}^\mu (x) \equiv \Theta(x) \tag{7.22}$$

such that the trace $\Theta_\mu{}^\mu$ signalizes directly whether a theory is scale invariant or not.
Also theorems like (2.11) come out naturally in this case. From conservation of $\Theta_{\mu\mu}$ it follows that for a state at rest the so called self stresses all vanish [*38*]

$$\langle 0\alpha \,|\, \Theta_{ii}(0) \,|\, 0\alpha\rangle = 0; \quad i = 1, 2, 3. \tag{7.23}$$

Therefore the trace has necessarily the same elastic matrix elements as Θ_{00} between states at rest

$$\begin{aligned} \langle \boldsymbol{p}\alpha \,|\, \Theta_\mu{}^\mu \,|\, \boldsymbol{p}\alpha\rangle &= \langle 0\alpha \,|\, \Theta_{00} \,|\, 0\alpha\rangle - \sum_i \langle 0\alpha \,|\, \Theta_{ii} \,|\, 0\alpha\rangle. = \\ &= 2\mu^2 N_\alpha . \end{aligned} \tag{7.24}$$

In addition one can see more transparently how the Goldstone mechanism of scale symmetry breaking operates. Consider for example $\Theta_{\mu\nu}$ for a scalar particle π:

$$\langle \pi(p') \,|\, \Theta_{\mu\nu} \,|\, \pi(p)\rangle = \frac{1}{2}\, \Sigma_\mu \Sigma_\nu F_1(q^2) + (g_{\mu\nu} q^2 - q_\mu q_\nu)\, F_2(q^2) \tag{7.25}$$

---

[31] Proof. The different contributions to v_μ are for scalars:

$$\pi_\mu = \partial_\mu \varphi, \; d = 1, \; \Sigma = 0$$

Hence:

$$v_\mu = (\partial_\mu \varphi)\varphi = \frac{1}{2}\, \partial^\nu (g_{\mu\nu} \varphi^2)$$

Spinors:

$$\pi_\mu = i\overline{\Psi}\,\gamma_\mu, \; d = \frac{3}{2}, = \Sigma_{\mu\nu}\, \frac{i}{4}\, [\gamma_\mu, \gamma_\nu]$$

Hence:

$$v_\mu = \frac{3}{2}\, i\,\overline{\Psi}\,\gamma_\mu\,\Psi + \frac{i}{4}\,\overline{\Psi}\,\gamma^\nu [\gamma_\mu \gamma_\nu] \Psi$$

Vectors:

$$\pi_\mu = -F_{\mu\nu}, \; d = 1, \; (\Sigma_{\mu\nu})_{\lambda\varkappa} = i(g_{\mu\lambda} g_{\nu\varkappa} - g_{\mu\varkappa} g_{\nu\lambda})$$

Hence:

$$v_\mu = -F_{\mu\nu}\, \varphi^\nu + i F^{\nu\lambda} (\Sigma_{\mu\nu})_{\lambda\varkappa}\, \varphi^\varkappa = 0$$

[32] We exclude the unphysical case that $X_{\lambda\varrho\mu\nu}$ contains a scalar pole of mass zero, in such a case surface terms could not be neglected in partial integrations and $\int d^3x \Theta_{0\mu}(x)$ would *not* give the energy momentum operator $P_\mu \equiv \int d^3x \Theta^c_{0\mu}(x)$.

where

$$\Sigma_\mu \equiv (p' + p)_\mu; \quad q_\mu \equiv (p' - p)_\mu .$$

The mass normalization

$$\langle \pi(\mathbf{0}) \,|\, \Theta_{00} \,|\, \pi(\mathbf{0}) \rangle = 2\mu^2 \tag{7.26}$$

forces

$$F_1(0) = 1 \tag{7.27}$$

The trace of (7.25) gives

$$\langle \pi(\boldsymbol{p}') \,|\, \Theta_\mu{}^\mu \,|\, \pi(\boldsymbol{p}) \rangle = \frac{1}{2}\,(4\mu^2 - q^2)\, F_1(q^2) + 3q^2 F_2(q^2) \tag{7.28}$$

verifying theorem (2.11) at $q^2 = 0$.
Suppose $\Theta_\mu{}^\mu$ is equal to zero.
Then

$$F_2(q^2) = -\frac{4\mu^2 - q^2}{6q^2}\, F_1(q^2). \tag{7.29}$$

This equation suggests that we can have a scale invariant world with massive particles if there is a pole in $F_2(q^2)$ at $q^2 = 0$. This pole is usually ascribed to a Goldstone particle of mass zero. However, for scale invariance a somewhat delicate problem arises. By going back to (7.25) we notice that $\Theta_{\mu\nu}$ has several deseases, due to the fact that the matrix elements

$$\langle \pi(\boldsymbol{p}') \,|\, \Theta_{\mu\nu} \,|\, \pi(\boldsymbol{p}) \rangle$$

are not uniquely defined when right and left hand momenta go to zero. In particular all self-stresses do not vanish any more. For example, if $\boldsymbol{p}'$ and $\boldsymbol{p}$ approach zero along the z-direction, we find that the energy density does not show any more the value $2\mu^2$ between states at rest but

$$\langle \pi(\mathbf{0}) \,|\, \Theta_{00} \,|\, \pi(\mathbf{0}) \rangle = \frac{4}{3}\,\mu^2. \tag{7.30}$$

Second, among the self-stresses [*38*], only $\langle \pi(\mathbf{0}) \,|\, \Theta_{33} \,|\, \pi(\mathbf{0}) \rangle$ vanishes as follows also from the original proof of JAUCH and ROHRLICH.[33] [*38*]
For the matrix elements $\langle \pi(\mathbf{0}) \,|\, \Theta_{\substack{11\\22}} \,|\, \pi(\mathbf{0}) \rangle$, however, we find instead $-2/3\,\mu^2$ which is necessary to achieve

$$\langle \pi(\boldsymbol{p}') \,|\, \Theta_\mu{}^\mu \,|\, \pi(\boldsymbol{p}) \rangle = 0.$$

These diseases of the "improved" energy momentum tensor are not unexpected. We noticed before that $\Theta_{\mu\nu}$ can be shown to produce the correct energy momentum operator P_μ only if the surface terms when partially integrating the second-term in (7.19) can be neglected. This, however, is impossible due to the long-range correlations caused by a pole of mass zero in the matrix elements.[34]
We mention this point since people have repeatedly argued that there are problems with a spontaneous breakdown of scale symmetry.[35] Any argument involving

[33]) Based on the conservation of $\Theta_{\mu\nu}$!
[34]) Example: Let $\langle p' | O(x) | p \rangle$ have a pole at $q^2 = 0$.
Then $\int d^3x \, \langle p' | \, \partial_i O(x) \, | p \rangle \propto i \int d^3x \, q_i/q^2 \, e^{iqx} = i(2\pi)^3 \, q_i/q^2 \, \delta^3(q) \neq 0$.
[35]) The author likes to thank J. Katz for bringing these arguments to his attention. See also Ref. [*39*].

$\Theta = 0$ uses the diseased "improved" energy momentum tensor and must be discarded. Other arguments will be mentioned when models are at our disposition to illustrate their defects.
As we said before, we shall always, for physical reasons, assume some scale breaking to be present moving the pole at $q^2 = 0$ to some nonzero $q^2 = m_\sigma^2$. We shall call a scalar particle in a broken scale invariant world a Goldstone particle of scale breaking, or a dilation, if it appears as a dominant pole in the same form factor that would need a massless pole for $\Theta = 0$.

VIII. Scale Properties of the Linear σ-Model

This model was constructed a long time ago for the purpose of exhibiting a set of vector and axial vector currents commuting like $SU(2) \times SU(2)$ and having the divergence ∂A dominated by a single pion. The Lagrangian of this model contains a nucleon field $\Psi(x)$ and scalar and pseudoscalar fields $\sigma(x)$ and $\boldsymbol{\pi}(x)$:

$$\mathcal{L} = \frac{1}{2}\,\overline{\Psi}\, i\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - g\overline{\Psi}(\sigma - i\gamma_5\boldsymbol{\pi}\,,\boldsymbol{\tau})\,\Psi + \frac{1}{2}\,[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2] -$$

$$- \frac{\mu_0^2}{2}\,(\sigma^2 + \boldsymbol{\pi}^2) + \frac{\lambda}{4}\,(\sigma^2 + \boldsymbol{\pi}^2)^2 + f_\pi\mu^2\sigma - c\,.^{36)} \tag{8.1}$$

Here c is a constant which is in general necessary to make the vacuum expectation value of $\mathcal{L}$ vanish[37]).
Except for the term $f_\pi\mu^2\sigma$, this Lagrangian is invariant under isospin transformations

$$\delta\sigma = 0, \quad \delta\pi = \boldsymbol{\alpha}\times\boldsymbol{\pi}$$

$$\delta\Psi = -i\,\boldsymbol{\alpha}\,\frac{\boldsymbol{\tau}}{2}\,\Psi, \quad \delta\overline{\Psi} = \overline{\Psi}\,\frac{\boldsymbol{\tau}}{2}\,\boldsymbol{\alpha} i \tag{8.2}$$

and axial transformations

$$\bar{\delta}\sigma = \boldsymbol{\alpha}\cdot\boldsymbol{\pi}, \quad \bar{\delta}\boldsymbol{\pi} = -\boldsymbol{\alpha}\sigma$$

$$\bar{\delta}\Psi = -i\boldsymbol{\alpha}\gamma_5\,\frac{\boldsymbol{\tau}}{2}\,\Psi, \quad \delta\overline{\Psi} = -\,\overline{\Psi}\,\frac{\boldsymbol{\tau}}{2}\,\gamma_5\boldsymbol{\alpha} i \tag{8.3}$$

generated by the vector and axial vector currents

$$\boldsymbol{V}^\mu \equiv \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\boldsymbol{\alpha}} = \overline{\Psi}\gamma^\mu\,\frac{\boldsymbol{\tau}}{2}\,\Psi + \boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi} \tag{8.4}$$

$$\boldsymbol{A}^\lambda \equiv \frac{\bar{\delta}\mathcal{L}}{\bar{\delta}\,\partial_\mu\boldsymbol{\alpha}} = \overline{\Psi}\gamma^\mu\gamma_5\,\frac{\boldsymbol{\tau}}{2}\,\Psi + \boldsymbol{\pi}\overleftrightarrow{\partial}^\mu\sigma. \tag{8.5}$$

---

36) The parameter μ means the numerical value of the pion mass.
37) In order to make $\langle 0|\Theta_{00}(x)|0\rangle = 0$.

The term $f_\pi \mu^2 \sigma$ breaks axial symmetry and gives rise to the PCAC relation

$$\partial \boldsymbol{A}(x) = \frac{\bar{\delta} \mathcal{L}}{\bar{\delta} \boldsymbol{\alpha}} = f_\pi \mu^2 \pi(x) \tag{8.6}$$

which shows that f_π is the pion decay constant ($\approx .095$ BeV).
Due to the occurrence of the terms $f_\pi \mu^2 \sigma$ and $\lambda/4\ (\sigma^2 + \pi^2)^2$, the potential minimum for the σ-field will not be at zero but at a value σ_0 determined by

$$\mu_0^2 \sigma_0 - \lambda \sigma_0^3 = f_\pi \mu^2. \tag{8.7}$$

As a consequence, the degeneracy between σ and π-masses is split. From the terms $\pi^2/2$ and $\sigma'^2/2$ in $\mathcal{L}$ one finds

$$m_\pi^2 = \mu_0^2 - \lambda \sigma_0^2 \tag{8.8}$$

$$m_\sigma^2 = \mu_0^2 - 3\,\lambda \sigma_0^2 \tag{8.9}$$

and the σ-nucleon interaction gives rise to a nucleon mass term $-m \bar{\Psi} \Psi$ with

$$m = g\,\sigma_0. \tag{8.10}$$

In the absence of the symmetry breaking, the nucleons would be massless. The constant c is found to be

$$c = -\frac{\mu_0^2}{2}\,\sigma_0^2 + \frac{\lambda}{4}\,\sigma_0^4 + f_\pi \mu^2 \sigma_0 = \frac{\sigma_0^2}{8}\,(m_\sigma^2 - 5 m_\pi^2) + f_\pi \mu^2 \sigma_0. \tag{8.11}$$

Combining (8.7) and (8.8) and requiring $m_\pi^2 = \mu^2$, we determine the potential minimum

$$\sigma_0 = f_\pi. \tag{8.12}$$

Quantization of the Lagrangian will yield σ_0 as the vacuum expectation value of the field σ. It is therefore convenient to introduce a new field

$$\sigma' \equiv \sigma - \sigma_0 \tag{8.13}$$

which oscillates around zero.
The most important coupling constants are found by looking at the corresponding vertices[38])

$$g_{\pi NN} = -g_{\sigma NN} = g = \frac{m}{f_\pi} \qquad (\mathcal{L}_{\pi NN} \equiv g_{\pi NN} \bar{\Psi} i \gamma_5 \boldsymbol{\tau} \Psi \boldsymbol{\pi}) \tag{8.14}$$

$$g_{\sigma\pi\pi} = -\frac{m_\sigma}{f_\pi}\left(1 - \frac{\mu^2}{m_\sigma^2}\right) \qquad \left(\mathcal{L}_{\sigma\pi\pi} \equiv g_{\sigma\pi\pi} \frac{m_\sigma}{2}\,\sigma \boldsymbol{\pi}^2\right) \tag{8.15}$$

$$g_{\sigma\sigma\sigma} = -\frac{m_\sigma}{f_\pi}\left(1 - \frac{\mu^2}{m_\sigma^2}\right) \qquad \left(\mathcal{L}_{\sigma\sigma\sigma} \equiv g_{\sigma\sigma\sigma} \frac{m_\sigma}{2}\,\sigma'^3\right). \tag{8.16}$$

Numerically, the first relation[39])

$$g_{\pi NN}\ (= 13.5) = (-g_{\sigma NN} \approx 15.)$$

---

[38]) Recall that $g_{\pi NN} = m/f_\pi$ is the model's version of the Goldberger Treiman relation $g_{\pi NN} = m g_A / f_\pi$.

[39]) We shall choose the sign of $g_{\sigma\pi\pi}$ to be negative, as in the linear σ-model. Then equ. (3.4) determines $g_{\sigma NN} \approx -15$.

is borne out by the analyses of πN backward-scattering [11]. The $g_{\sigma\pi\pi}$ coupling of the model is

$$g_{\sigma\pi\pi} \approx -7.4$$

which is too large by a factor of about $\sqrt{2}$.
We can introduce dilatations in the model by means of the current (7.1) *choosing* the dimensions of $\Psi(x)$, $\sigma(x)$, $\pi(x)$ to be $^3/_2$, 1, 1. With this choice, the divergence $\partial\mathscr{D}$ becomes

$$\partial\mathscr{D}(x) = \mu_0^2(\sigma^2 + \pi^2) - 3 f_\pi \mu^2 \sigma + 4c. \tag{8.17}$$

This agrees with our general theorem (6.11), since due to (8.2) the terms $\mu_0^2/2\,(\sigma^2 + \pi^2)$, $-f_\pi\mu^2\sigma$, and c are scalar symmetry breakers of dimensions 2, 1 and zero, respectively. In terms of σ' we find

$$\partial\mathscr{D}(x) = \mu_0^2(\sigma'^2 + \pi^2) - f_\pi m_\sigma^2 \sigma'. \tag{8.18}$$

We can now easily calculate any matrix elements of $\partial\mathscr{D}$ in the tree graph approximation. For example:

$$\langle N(p')\,|\partial\mathscr{D}|\,N(p)\rangle = -f_\pi m_\sigma^2 \frac{g_{\sigma\mathrm{NN}}}{m_\sigma^2 - q^2} = m\,\frac{m_\sigma^2}{m_\sigma^2 - q^2} \tag{8.19}$$

$$\langle \pi(p')\,|\,\partial\mathscr{D}\,|\,\pi(p)\rangle = 2\mu_0^2 - f_\pi m_\sigma^2 \frac{g_{\sigma\pi\pi} m_\sigma}{m_\sigma^2 - q^2} = 2\mu^2 - f_\pi m_\sigma g_{\sigma\pi\pi} \frac{q^2}{m_\sigma^2 - q^2} \tag{8.20}$$

$$\langle \sigma(p')\,|\,\partial\mathscr{D}\,|\,\sigma(p)\rangle = 2\mu_0^2 - f_\pi m_\sigma^2 \frac{3 g_{\sigma\sigma\sigma} m_\sigma}{m_\sigma^2 - q^2} = 2m_\sigma^2 - f_\pi m_\sigma g_{\sigma\sigma\sigma} \frac{3q^2}{m_\sigma^2 - q^2}. \tag{8.21}$$

These matrix elements satisfy at $q^2 = 0$ the fundamental low energy theorem (2.11).
The $SU(2) \times SU(2)$ breaking term $-\Sigma = f_\pi \mu^2 \sigma$ and the divergence of the axial current have the dimension one. Therefore our equation (4.32) should be true. Indeed, the term linear in σ' in (8.18) gives us

$$\frac{m_\sigma^3}{\gamma} = -f_\pi m_\sigma^2 \text{ or } g = -\frac{m_\sigma}{\gamma_\pi} \tag{8.22}$$

such that (4.32) for $d = 1$

$$g_{\sigma\pi\pi} = \gamma\left(1 - \frac{\mu^2}{m_\sigma^2}\right) \tag{8.23}$$

leads to the correct $\sigma\pi\pi$ coupling.
Further, the matrix element (8.20) is once substracted taking at $q^2 = \infty$ the value

$$\langle \pi(p')\,|\,\partial\mathscr{D}\,|\,\pi(p)\rangle_{q^2\to\infty} = 2\mu_0^2 = 3\mu^2 - m_\sigma^2 \tag{8.24}$$

which agrees with (4.33) for $d = 1$.
Notice that in the model we have exactly the situation which was the basis of our assumptions of Section V. While $\partial\mathscr{D}$ is once subtracted between pions, the symmetry breaker Σ is *always* σ-pole dominated. For this reason our theorems

(5.1)—(5.3) about the size of the Σ-term are necessarily correct. The $\partial\mathscr{D}$NN vertex (8.19) is unsubtracted and therefore the coupling $g_{\pi NN} = -m/f_\pi$ agrees with the general result (3.13) if one uses the model's value for γ (8.22).
From (8.21) we suspect that the $\partial\mathscr{D}\ \partial\mathscr{D}\ \partial\mathscr{D}$ vertex will be subjected to a similar theorem as (4.20). This can indeed be verified by means of Ward identities. Since this coupling is most academical, though, we shall not consider it any further.
Notice that, due to appearance of the operator $(\mu_0^2/2)(\sigma^2 + \pi^2)$ of dimension two, the symmetry limit $\Sigma \to 0$ causes only the pion to have zero mass (becoming the Goldstone boson of spontaneous breakdown of chiral symmetry). The σ-mass is still finite. If we want both the chiral and scale symmetry breaking to be caused by the same term Σ[40]), we have to set $\mu_0^2 = 0$ and find

$$m_\sigma^2 = 3 m_\pi^2 \tag{8.25}$$

i.e. the σ-mass drops to about the size of the pion mass. This is clearly incompatible with experiment.
The linear σ-model can easily be modified to include the case of the divergence of the axial current having an arbitrary dimension d. For this we simply take as a symmetry breaker

$$\mathscr{L}_{SB} \equiv -\Sigma = f_\pi \sigma_0^{1-d} \mu^2 \sigma (\sigma^2 + \boldsymbol{\pi}^2)^{(d-1)/2} \tag{8.26}$$

such that

$$\partial A = \delta_\pi \sigma_0^{d-1} \mu^2 \boldsymbol{\pi} (\sigma^2 + \boldsymbol{\pi}^2)^{(d-1)/2} = f_\pi \mu^2 \pi + \cdots \tag{8.27}$$

The value σ_0 is now determined from

$$\mu_0^2 \sigma_0 - \lambda \sigma_0^3 = f_\pi \mu^2 d \tag{8.28}$$

while the meson mass formulas become

$$m_\pi^2 = \mu_0^2 - \lambda \sigma_0^2 - \frac{f_\pi}{\sigma_0} \mu^2 (d-1) \tag{8.29}$$

$$m_\sigma^2 = \mu_0^2 - 3\lambda \sigma_0^2 - \frac{f_\pi}{\sigma_0} \mu^2 (d-1)\, d. \tag{8.30}$$

Inserting (8.29) in (8.28) gives

$$m_\pi^2 = \frac{f_\pi}{\sigma_0} \mu^2 \tag{8.31}$$

If we want $m_\pi^2 = \mu^2$ it follows that again $\sigma_0 = f_\pi$.
It is obvious that the couplings πNN and σNN are just the same as before. For $\sigma\pi\pi$ and $\sigma\sigma\sigma$ we now obtain additional contribution from Σ:

$$\begin{aligned} -\Sigma &= f_\pi \mu^2 \sigma_0 \left(1 + \frac{\sigma'}{\sigma_0}\right) \left(1 + \frac{2\sigma_0\sigma' + \sigma'^2 + \boldsymbol{\pi}^2}{\sigma_0^2}\right)^{d-1/2} = \\ &= f_\pi \mu^2 \sigma_0 \left\{1 + d\frac{\sigma'}{\sigma_0} + (d-1)\frac{d}{2}\frac{\sigma'^2}{\sigma_0^2} + \frac{d-1}{2}\frac{\boldsymbol{\pi}^2}{\sigma_0^2} + \right. \\ &\quad \left. + (d-1)(d-2)\frac{\sigma'\boldsymbol{\pi}^2}{2\sigma_0^3} + (d-1)\left[1 + \frac{(d+1)(d-3)}{3}\right]\frac{\sigma'^3}{2\sigma_0^3} + \cdots\right\} \end{aligned} \tag{8.32}$$

[40]) Up to the trivial c-number term.

giving:

$$\mathcal{L}_{\sigma'\pi\pi} = \left(\lambda\sigma_0 + f_\pi\mu^2\sigma_0(d-1)\,(d-2)\,\frac{1}{2\sigma_0^3}\right)\sigma'\boldsymbol{\pi}^2. \tag{8.33}$$

Combining this with (8.29) and (8.30) we find

$$\mathcal{L}_{\sigma'\pi\pi} = -\frac{m_\sigma^2}{2f_\pi}\left(1 + (d-2)\,\frac{m_\pi^2}{m_\sigma^2}\right)\sigma'\boldsymbol{\pi}^2 \tag{8.34}$$

and therefore

$$g_{\sigma\pi\pi} = -\frac{m_\sigma}{f_\pi}\left(1 + (d-2)\,\frac{m_\pi^2}{m_\sigma^2}\right). \tag{8.35}$$

This verifies our general result (4.33). The direct coupling of $\partial\mathcal{D}$ to σ' is obviously also in this case

$$\gamma = -\frac{m_\sigma}{f_\pi}. \tag{8.36}$$

This can be seen as well by expanding

$$\begin{aligned}\partial\mathcal{D} &= \mu_0^2(\sigma^2 + \boldsymbol{\pi}^2) + (4-d)\,\Sigma + 4c = \\ &= (2\mu_0^2 - (4-d)\,d\mu^2)\,f_\pi\sigma' + \mu_0^2(\sigma'^2 + \boldsymbol{\pi}^2) + \\ &\quad + (d-4)\,(d-1)\,\frac{\mu^2}{2}\,(d\sigma'^2 + \pi^2).\end{aligned} \tag{8.37}$$

The factor of σ' is indeed equal to m_σ^3/γ as it should be. In this form we can also see the subtraction constant appearing in the $\partial\mathcal{D}\boldsymbol{\pi}\boldsymbol{\pi}$ vertex. The term $\propto\boldsymbol{\pi}^2$ shows

$$\begin{aligned}\langle\pi(p')\,|\,\partial\mathcal{D}\,|\,\pi(p)\rangle|_{q^3\to\infty} &= 2\mu_0^2 + (d-4)\,(d-1)\,\mu^2 \\ &= -m_\sigma^2 + (4-d)\,\mu^2.\end{aligned} \tag{8.38}$$

Also here, the term Σ is dominated by σ' with the normalization $\Sigma = -d\mu^2 f_\pi\sigma'$ which agrees with the general result (5.1) if we insert there γ from (8.36).
Notice that we can combine eqns. (8.29)—(8.31) and bring the mass formula for m_σ^2 to the form

$$m_\sigma^2 = (4-d)\,d\mu^2 - 2\mu_0^2. \tag{8.39}$$

If $\mu_0^2 = 0$, our Lagrangian has only one c-number δ-term, apart from $-\Sigma$, breaking scale symmetry, and our formula (6.37) should hold. Indeed, if we insert $\gamma = -m_\sigma/f_\pi$ and $d_\delta = 0$, (6.37) coincides exactly with (8.39).
If, in addition to $\mu_0^2 = 0$, also $d = 4$, the σ-particle becomes massless. Since the pion mass is still $\mu \neq 0$, the σ-particle is apparently just the Goldstone particle of a spontaneously broken scale symmetry. To see this consider the energy momentum tensor $\Theta_{\mu\nu}$ of (7.15):

$$\Theta_{\mu\nu} = \Theta^B_{\mu\nu} + \frac{1}{6}\,(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)\,(\sigma^2 + \boldsymbol{\pi}^2). \tag{8.40}$$

Between pions, only the terms

$$\Theta_{\mu\nu} = \partial_\mu \boldsymbol{\pi}\, \partial_\nu \boldsymbol{\pi} - g_{\mu\nu} \left(\frac{1}{2} (\partial_\mu \boldsymbol{\pi})^2 - \frac{\mu^2}{2} \boldsymbol{\pi}^2 \right)$$

$$+ \frac{1}{6} (\Box g_{\mu\nu} - \partial_\mu \partial_\nu) (\boldsymbol{\pi}^2 + 2 \sigma_0 \sigma') + \cdots \tag{8.41}$$

contribute. Then we obtain for the form factors $F_1(q^2)$, $F_2(q^2)$, defined in (7.25),

$$F_1(q^2) \equiv 1$$

$$F_2(q^2) \equiv \frac{1}{6} + \frac{f_\pi}{3} \frac{g_{\sigma\pi\pi} m_\sigma}{q^2}. \tag{8.42}$$

But

$$g_{\sigma\pi\pi} m_\sigma = - \frac{m_\sigma^2}{f_\pi} \left[1 + (d-2) \frac{\mu^2}{m_\sigma^2} \right] \tag{8.43}$$

becomes for $d = 4$, $m_\sigma^2 = 0$

$$g_{\sigma\pi\pi} m_\sigma = - \frac{2\mu^2}{f_\pi} \tag{8.44}$$

such that

$$F_2(q^2) = \frac{1}{6} - \frac{2}{3} \frac{\mu^2}{q^2} \tag{8.45}$$

just as is necessary to ensure the tracelessness (see (7.29)).

It is interesting to see what happens to the matrix elements of $\partial \mathscr{D}$ if a small scale breaker $\mu_0^2 = -\varepsilon^2/_2$ is present in the Lagrangian. Then (8.39) gives $m_\sigma^2 = \varepsilon^2$ and the matrix element of $\partial \mathscr{D}$ between pions becomes

$$\langle \pi(\boldsymbol{p}') \mid \partial \mathscr{D} \mid \pi(\boldsymbol{p}) \rangle = 2\mu^2 + 2\mu^2 \frac{q^2}{\varepsilon^2 - q^2}.$$

The result is zero for any finite q^2 except for $q^2 = 0$ where $\langle \pi(\boldsymbol{p}) \mid \partial \mathscr{D} \mid \pi(\boldsymbol{p}) \rangle = 2\mu^2$. Thus, close to the scale invariant limit, there is a strong singularity at small q^2 making $\partial \mathscr{D}$ run extremely fast from $2\mu^2$ to zero.

We can use this model also to illustrate the defects of some of the "proofs" claiming to show the impossibility of a Goldstone symmetry.[41]) One considers the commutator

$$[D(0), H] = -iH \mid 0\rangle \tag{8.46}$$

to demonstrate that $D(0) \mid 0\rangle$ has zero energy

$$HD(0) \mid 0 \rangle = 0 \tag{8.47}$$

such that either $D(0) \mid 0\rangle = 0$ (i.e. exact symmetry) or the vacuum is degenerate. In the latter case one uses the commutator of H with the conformal transformation K_0:

$$[K_0(0), H] = -2iD(0) \tag{8.48}$$

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[41]) See the first two of Refs. [*39*].

to obtain

$$\langle 0 | [K_0(0)H] \, D(0) | \, 0\rangle = -2i \, \langle 0 \, | D(0) \; D(0) | \; 0\rangle = 0. \tag{8.49}$$

This shows that the state $D(0)|0\rangle$ has zero norm from which people conclude $D(0)|0\rangle = 0$, i.e. exact symmetry.

However, this type of state of norm zero is nothing bad in a field theory containing a zero-mass particle. Consider our model for $d = 4$, $\mu_0^2 = 0$. Then $D(0)|0\rangle$ is just a state of the Goldstone boson σ of momentum zero:

$$D(0)|0\rangle = i\,\frac{f_\pi}{2}\, a_\sigma^+(0) |\, 0\rangle. \tag{8.50}$$

This is quite a necessary state of affairs in order to make the field $\sigma'(0)$ transform according to

$$i\Gamma D(0), \sigma'(0)] = \sigma'(0) + \sigma_0 = \sigma'(0) + f_\pi \tag{8.51}$$

or

$$i\,\langle 0 | [D(0), \sigma'(0)] |\, 0\rangle = f_\pi. \tag{8.52}$$

Even though the state $a_\sigma^+(0) \,|\, 0\rangle$ has zero norm,[42]) the product with a field as singular as

$$\sigma'(x) = \int \frac{d^3 q}{2 q_0 (2\pi)^3} \, (e^{-iqx} \, a_\sigma(\boldsymbol{q}) + \text{u. c.}) \tag{8.53}$$

gives

$$\langle 0 \,| \sigma'(\mathrm{p}) \, D(0) |\, 0\rangle = \frac{f_\pi}{2} \int \frac{d^3 q}{2 q_0 (2\pi)^3} \, \langle 0 \,| a_0(q) \, a_\sigma^+(0) |\, 0\rangle = \frac{f_\pi}{2}. \tag{8.54}$$

Since $a_\sigma^+(0) \,|\, 0\rangle$ is an isolated state of norm zero there is no problem with the axioms of quantum mechanics: Any wave packet formed of $a_\sigma^+(\boldsymbol{q}) |\, 0\rangle$ will have a non-vanishing norm.

Finally, we would like to use this σ-model to demonstrate the danger of deriving conclusions concerning the size of the Σ-term as long as we do not have specific ideas about PCAC breaking (see p. 16).

Suppose we want to include the effect of the higher singularities in the mass dispersion relation for ∂A correcting the Goldberger Treiman relation (4.14). This can be done most simple by adding to the Lagrangian a very small PCAC breaking term

$$\mathscr{L}_{\text{PCAC break}} = -m_0 \, \overline{\Psi}(x) \, \Psi(x). \tag{8.55}$$

This term will enter into ∂A as [43])

$$\partial \boldsymbol{A}(x) = \frac{\overline{\delta} \mathscr{L}}{\delta' \boldsymbol{\alpha}} = f_\pi \mu^2 \boldsymbol{\pi}(x) + m_0 \, \overline{\Psi} i \gamma_5 \boldsymbol{\tau} \, \Psi \tag{8.56}$$

such that between nucleons

$$\langle N(p') \,| \partial \boldsymbol{A}(0) |\, N(p)\rangle = \left[f_\pi \mu^2 \, \frac{g_{\pi\mathrm{NN}}}{\mu^2 - q^2} + m_0 \right] \overline{\Psi} i \gamma_5 \boldsymbol{\tau} \, \Psi \tag{8.57}$$

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[42]) Remember, our boson normalization amounts to $[a_\sigma(\boldsymbol{q}'), a_\sigma^+(\boldsymbol{q})] = 2 q_0 (2\pi)^3 \, \delta^3(q' - q)$.

[43]) Since $(\overline{\Psi}\Psi, -\overline{\Psi} i \gamma_5 \boldsymbol{\tau} \Psi)$ transforms in the same way as $(\sigma, \boldsymbol{\pi})$ (see (8.2) and (8.3)).

and the Goldberger Treiman relation becomes

$$m g_A = f_\pi g_{\pi NN} + m_0. \tag{8.58}$$

From the experiment numbers (see (4.14)) we know that m_0 should be chosen ≈ -120 MeV. This gives only a 10% contribution to the huge pion pole term at $q^2 = 0$.

For the Σ-term, on the other hand, the situation is quite different. The total symmetry breaker is

$$\Sigma = -f_\pi \mu^2 \, d\sigma' + m_0 \, \overline{\Psi}(x) \, \Psi(x) \tag{8.59}$$

yielding between nucleons

$$\langle N(p') \,|\Sigma(0)|\, N(p)\rangle = -f_\pi \mu^2 d \, \frac{g_{\sigma NN}}{m_\sigma^2 - q^2} + m_0. \tag{8.60}$$

Now even though $g_{\sigma NN}$ is quite large (≈ -15) and there is the possibility that the factor d could be 3, the large denominator m_σ^2 at $q^2 = 0$ makes the first, unsubstracted, contribution never outweigh the term m_0 (the largest estimate is $f_\pi \, \mu^2/m_\sigma \approx 3 g_{\sigma NN} \approx -160$ MeV).

Thus we see that in spite of PCAC breaking effects being small in matrix elements with a close lying pion pole, it might well become dominant when only the high σ-pole is present.

It is curious to note that in this model the on-shell value considered by Cheng and Dashen as the Σ-term

$$\langle N(p) \,|\Sigma|\, N(p)\rangle_{CD} \equiv f_\pi^2 \left[A(0, 2\,\mu^2, \mu^2, \mu^2) - \frac{g^2_{\pi NN}}{m} \right] \tag{8.61}$$

is, in fact, given by the unsubstracted part of (8.59) only.

If we abstract what are possibly the model independent features of this picture of PCAC breaking we conclude that the on-shell determination of Σ by means of (8.61) should follow the general formulas (5.2)—(5.4) while the true off-shell value (5.5) as appearing in the low-energy theorem and evaluated via the Fubini-Furlan method could quite possibly be *smaller* by substraction terms of the order of 100 MeV. As we see, this conclusion is roughly born out by experiment.

Certainly one can introduce also PCAC breaking by using a second chiral pair of fields $\bar{\sigma}$ and $\bar{\pi}$ of higher masses. Due to the additional parameters one would then be able to fit the Goldberger Treiman relation without determining the substraction term in (8.60) to be equal to $m_0 = -120$ MeV. All we wanted to demonstrate with this model is the general expectation that PCAC breaking should have dramatic effects on such tiny expressions as $\langle N \,|\Sigma|\, N\rangle \approx 25$ MeV. As always, people should be careful in not overstretching simple approximative ideas into regions where common sense casts strong doubts on their validity.

IX. Scale Properties of the Non-linear σ-Model

While linear chiral Lagrangian models allow to for an adjustment of the dimensional properties by using factors of the chiral invariant $(\sigma^2 + \pi^2)$ of dimension two, the conventional way of doing the same thing in nonlinear Lagrangians proceeds via the introduction of the so-called dilation fiel χ.

Consider the simplest non-linear Lagrangian obtained by eliminating σ in the linear σ-model [*44*]

$$\mathscr{L} = \frac{1}{2}\left[\partial_\mu \boldsymbol{\pi}\, \partial^\mu \boldsymbol{\pi} + \partial_\mu \sqrt{f_\pi^2 - \boldsymbol{\pi}^2}\, \partial^\mu \sqrt{f_\pi^2 - \boldsymbol{\pi}^2}\right] + f_\pi \mu \sqrt{f_\pi^2 - \boldsymbol{\pi}^2} - f_\pi^2 \mu^2$$

$$= \frac{1}{2} D_\mu \boldsymbol{\pi}\, D^\mu \boldsymbol{\pi} + f_\pi \mu^2 \sqrt{f_\pi^2 - \boldsymbol{\pi}^2} - f_\pi^2 \mu^2. \tag{9.1}$$

The covariant derivatiIe D_μ is defined as

$$D_\mu \boldsymbol{\pi}_a = d_{ab}(\pi).\partial_\mu \pi_b$$

with

$$d_{ab}(\boldsymbol{\pi}) = \delta_{ab} + \frac{1}{f_\pi}\left[\frac{1}{\sqrt{f_\pi^2 - \boldsymbol{\pi}^2}} - \frac{1}{\sqrt{f_\pi^2 - \boldsymbol{\pi}^2} + f_\pi}\right] \pi_a \pi_b. \tag{9.2}$$

Since the axial charge transforms $\boldsymbol{\pi}$ as

$$[Q_a^5(x_0), \pi_b(x)] = i \sqrt{f_\pi^2 - \boldsymbol{\pi}^2(x)}\, \delta_{ab} \tag{9.3}$$

we see that if we want the axial charge to have a definite dimension [44]), we have to assign the dimension zero to the pion field[45]). [*40, 5*]. In that case we see that $\mathscr{L}$ consists of terms of dimension two and zero only.

If we introduce the dilation field χ with a non-vanishing vacuum expectation value, say $\langle 0 | \chi | 0 \rangle = \chi_0$, we can construct Lagrangians like

$$\mathscr{L} = \frac{1}{2}(D_\mu \pi)^2 \chi^2/\chi_0^2 + f_\pi \mu^2 \sqrt{f_\pi^2 - \boldsymbol{\pi}^2}\, \chi^d/\chi_0^d +$$

$$+ \frac{1}{2b^2}(\partial_\mu \chi)^2/\chi_0^2 - \frac{\mu_0^2}{2}\chi^2 + \frac{\lambda}{4}\chi^4 - c \tag{9.4}$$

which have the property that the chiral symmetry breaker and therefore also

$$\partial \boldsymbol{A}(x) = f_\pi \mu^2 \boldsymbol{\pi}(x)\, \chi^d/\chi_0^d$$

have dimension d.

The value of χ_0 is, as usual, determined by the minimum of the potential

$$V_\chi = \frac{\mu_0^2}{2}\chi^2 - \frac{\lambda}{4}\chi^4 - f_\pi \mu^2 \sqrt{f_\pi^2 - \boldsymbol{\pi}^2}\, \chi^d/\chi_0^d + c \tag{9.5}$$

which gives the relation

$$\mu_0^2 \chi_0^2 - \lambda \chi_0^4 = \mu^2 f_\pi^2 d. \tag{9.6}$$

---

[44]) Which is then necessarily zero from the charge $SU(2) \times SU(2)$ algebra.

[45]) Notice that any bona fide axiomatic field of dimension zero is necessarily a constant. For our phenomenological fields introduced to enforce Ward identities we don't habe to worry about such theorems.

If one introduces the field χ' defined by $\chi = \chi_0(1 + b\chi')$ oscillating around zero, V_χ becomes

$$V_\chi = \left[\frac{\mu_0^2}{2}\chi_0^2 - \frac{\lambda}{4}\chi_0^4 - f_\pi^2\mu^2 + c\right] + \\ + \frac{1}{2}[\mu_0^2\chi_0^3 - 3\lambda\chi_0^4 - \mu^2 f_\pi^2 d(d-1)]b^2\chi'^2 + \frac{\mu^2}{2}\boldsymbol{\pi}^2. \tag{9.7}$$

The vanishing of $\langle 0 | \mathscr{L} | 0\rangle$ enforces

$$c = -\frac{\mu_0^2}{2} + \frac{\lambda}{4}\chi_0^4 + f_\pi^2\mu^2. \tag{9.8}$$

From the quadratic terms in (9.7) we find the mass of the χ particle

$$m_\chi^2 = [\mu_0^2\chi_0^2 - 3\lambda\chi_0^4 - \mu^2 f^2 d(d-1)]\, b^2 \tag{9.9}$$

which becomes with (9.6)

$$m_\chi^2 = -2\mu_0^2\chi_0^2 b^2 + (4-d)\, d b^2\mu^2 f_\pi^2. \tag{9.10}$$

The vertex of χ with two pions is given by the Lagrangian

$$\mathscr{L}_{\chi\pi\pi} = b\left(\partial_\mu\boldsymbol{\pi}\,\partial^\mu\boldsymbol{\pi} - \frac{\mu^2}{2}d\boldsymbol{\pi}^2\right)\chi'. \tag{9.11}$$

This amounts to an on-shell coupling constant of [*40*]

$$g_{\chi\pi\pi} = -bm_\chi\left(1 + (d-2)\frac{\mu^2}{m_\chi^2}\right) \tag{9.12}$$

in agreement with our theorem (4.31) if we identify

$$\gamma = -bm_\chi. \tag{9.13}$$

This identification can again be checked directly by looking at the term in

$$\partial\mathscr{D} = \mu_0^2\chi^2 - (4-d)\, f_\pi\mu^2\sqrt{f_\pi^2 - \mu^2}\,\chi^d/\chi_0^d + 4c \tag{9.14}$$

linear in χ'. Expanding $\partial\mathscr{D}$ we find

$$\begin{aligned}\partial\mathscr{D} &= \mu_0^2\chi_0^2(1 + 2b\chi' + \chi'^2) - (4-d)\, f_\pi^2\mu^2\Big(1 + db\chi' + \\ &\quad + \frac{1}{2}d(d-1)\, b^2\chi'^2 + (4-d)\frac{\mu^2}{2}\boldsymbol{\pi}^2 + \cdots + 4c = \\ &= [2\mu_0^2\chi_0^2 - (4-d)\, f_\pi^2\mu^2 d]\, b\chi' - (4-d)\, f_\pi^2\mu^2\frac{d(d-1)}{2}b^2\chi'^2 + \\ &\quad + (4-d)\frac{\mu^2}{2}\boldsymbol{\pi}^2 = \\ &= -\frac{m_\chi^2}{b}\chi' + (4-d)\, f_\pi^2\mu^2\frac{d(d-1)}{2}b^2\chi'^2 + (4-d)\frac{\mu^2}{2}\boldsymbol{\pi}^2 + \cdots\end{aligned} \tag{9.15}$$

The subtraction in the $\partial\mathscr{D}\pi\pi$ vertex comes about by a mechanism slightly different from the linear σ-model. From (9.15) we first find a contribution $(4 - d)\mu^2$. But the effective vertex (9.11) has in momentum space the form

$$V_{\chi\pi\pi} = -b(q^2 + (d - 2)\mu^2) \tag{9.16}$$

such that the complete form factor reads

$$\langle \pi(p') \mid \partial\mathscr{D} \mid \pi(p)\rangle = \left(q^2 + (d - 2)\,\mu^2\right) \frac{m_\chi^2}{m_\chi^2 - q^2} + (4 - d)\,\mu^2 \tag{9.17}$$

which gives us indeed the value (4.32) for $q^2 \to \infty$.
In this model we can also check our spectral function sum rule (6.36). If we choose $\mu_0^2 = 0$ there is only one operatorial scale breaker of dimension d and equation (6.36) should hold with $d = 0$.
From (9.10) we find for $\mu_0^2 = 0$:

$$m_\chi^2 = (4 - d)\,d b^2 \mu^2 f_\pi^2, \tag{9.18}$$

which indeed agrees with (6.36) using (9.13).
Notice that if in addition to $\mu_0^2 = 0$ also $d = 4$, equs (9.6) and (9.8) show that also $c = 0$ and the Lagrangian becomes scale invariant with $m_\pi^2 \neq 0$ and $m_\chi^2 = 0$. This situation identifies χ as the Goldstone particle of scale invariance.
Obviously, the essential difference between this model and the linear σ-model of arbitrary dimension d is that the direct $\partial\mathscr{D}\sigma$ coupling $b = -\chi/m_\chi$ is a free parameter. In the linear model it is the property of σ being the chiral partner of $\boldsymbol{\pi}$ that fixes this value to be $1/f_\pi$. Here the dilaton is an $SU(2) \times SU(2)$ singlet which is coupled completely independently of the pion. On the one hand, this is quite a pleasant situation since, as we noted before, $-m_\sigma/f_\pi$ appearently overestimates $g_{\sigma\pi\pi}$. Here we can correct this defect by *setting* $\gamma \approx -5$. Unfortunately, however, this adjustment makes the σNN coupling come out too small.
To see this let us include the baryons in this Lagrangian in the same way as in the linear σ-model only that $\sqrt{f_\pi^2 - \boldsymbol{\pi}^2}$ is used instead of σ in the interaction term, and a factor χ/χ_0 or $e^{b\sigma}$ has to be introduced to raise its dimension up to 4. Then every statement concerning the baryons remains the same, except that the free parameter γ appears instead of $-m_\sigma/f_\pi$. Therefore we obtain for $g_{\sigma\mathrm{NN}}$ the expression (3.13) which leads with $\gamma \approx -5.$ to

$$g_{\sigma\mathrm{NN}} \approx -5. \times \frac{m}{m_\sigma} \approx -7.$$

The linear σ-model just managed to keep $g_{\sigma\mathrm{NN}} \approx g_{\pi\mathrm{NN}} = 13.5$ at the cost of the too large value of $\gamma = -m_\sigma/f_\pi \approx -7$.
If the ratio $g_{\sigma\mathrm{NN}}/g_{\sigma\pi\pi}$ keeps staying up at around 3, also the unsubstractedness of the $\partial\mathscr{D}$NN, which is common to both models, will have to be abandoned. At present, the data on the σ-couplings are sufficiently uncertain to relieve us from such a conclusion.

X. Scale Properties of a Non-linear $SU(3) \times SU(3)$ σ-Model

Let us now see wether it is possible to reduce the breakdown of scale symmetry to the same source as the breakdown of chiral $SU(3) \times SU(3)$ symmetry. The standard Lagrangian for broken chiral $SU(3) \times SU(3)$ symmetry consists of an octet of baryon fields B_i belonging to an $(8, 1) \pm (1, 8)$ and of scalar and pseudoscalar meson fields belonging $(3, \bar{3}) \pm (\bar{3},3)$ representation of $SU(3) \times SU(3)$. If F_i are the octet generators of $SU(3)$ and φ and $\bar{\varphi}$ denote the rotation angles of vector and axial transformations, the baryon fields transform according to

$$U^{-1} B U = e^{-i(\varphi_i F_i + \bar{\varphi} F_i \gamma_5)} B \tag{10.1}$$

such that the kinetic term of the baryons

$$\mathscr{L} = \bar{B} \gamma^\mu \partial_\mu B \tag{10.2}$$

is $SU(3) \times SU(3)$ invariant.
It is convenient to decompose B into its chiral parts

$$B = \frac{1}{2} [(1 + \gamma_5) B + (1 - \gamma_5) B] \equiv \frac{1}{2} [B_+ + B_-] \tag{10.3}$$

ransforming irreducibly under the commuting $SU(3)$ groups generated by

$$F^+ \equiv F \left(\frac{1 + \gamma_5}{2}\right); \; F^- \equiv F \left(\frac{1 - \gamma_5}{2}\right) \tag{10.4}$$

in the form

$$U^{-1} B^\pm U = e^{-i\left\{\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right\}} B^\pm, \; \begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} = \varphi \pm \bar{\varphi}. \tag{10.5}$$

This form offers going from the octet indices to the 3×3 matrices by means of:

$$\mathsf{B}_a{}^{\pm b} = \sum_{i=1} \frac{1}{\sqrt{2}} \lambda_a{}^{ib} B_i{}^\pm \equiv \frac{1}{\sqrt{2}} (\lambda \boldsymbol{B})_a{}^b. \tag{10.6}$$

Obviously the matrices $\mathsf{B}_a{}^{\pm b}$ transform according to

$$U^{-1} \mathsf{B}^\pm U = e^{-i\left\{\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right\}\lambda/2} \mathsf{B}^\pm e^{i\left\{\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right\}\lambda/2}. \tag{10.7}$$

With this transformation law, $\mathsf{B}^\pm$ can easily be combined with any meson fields matrices of the $(3,\bar{3}) + (\bar{3},3)$ representation.
Such a matrix behaves under $SU(3) \times SU(3)$ as

$$U^{-1} \mathsf{M} U = e^{-i\alpha \lambda/2} \mathsf{M} e^{i\beta \lambda/2} \tag{10.8}$$

$$U^{-1} \mathsf{M}^\dagger U = e^{-i\beta \lambda/2} \mathsf{M}^+ e^{i\alpha \lambda/2}.$$

Therefore

$$\mathscr{L}_{BM} = \frac{g}{4} [\mathrm{tr}\, (\bar{\mathsf{B}}_+ \mathsf{M} \mathsf{B}_- \mathsf{M}^\dagger) + \mathrm{tr}(\bar{\mathsf{B}}_- \mathsf{M}^\dagger \mathsf{B}_+ \mathsf{M})] \tag{10.9}$$

is obviously a chiral invariant. The connection of M with fields u_a, v_a transforming in the way used in Sect. VI, equ. (6.7) is obtained by decomposing M into its hermitian and antihermitian parts:

$$\mathsf{M} = \sum_{a=0}^{8} (u_a + i v_a) \frac{\lambda^a}{\sqrt{2}}; \quad \lambda^0 = \sqrt{\frac{2}{3}}\, 1. \tag{10.10}$$

We shall construct such fields u_a and v_a in the standard non-linear fashion[46]) [*41*] out of a pseudoscalar octet

$$P_i = (\boldsymbol{\pi}, K, \bar{K}, \eta) \qquad\qquad i = 1, \ldots, 8$$

and a scalar quartet

$$S_r = (\varkappa, \bar{\varkappa}) \qquad\qquad r = 4, 5, 6, 7.$$

If F_i, F_i^5 denotes an arbitrary representation of $SU(3) \times SU(3)$[47]) (for example the octet representation used in (10.1)), then the nonlinear transformation among the fields P_i and S_r is defined uniquely by the requirement:

$$g\, e^{i\sum_1^8 P_i F_i^5}\, e^{i\sum_4^7 S_r F_r} = e^{i\sum_1^8 P_i' F_i^5}\, e^{i\sum_4^7 S_r' F_r} \times e^{i\sum_1^3 W_i' F_i + W_8' F^8} \tag{10.11}$$

where

$$g = \mathrm{e}^{-i(\varphi_i F_i + \bar{\varphi}_i F_i^5)} \tag{10.12}$$

is an arbitrary $SU(3) \times SU(3)$ transformation. Let now Λ_a, $\bar{\Lambda}_a$ be a linear $(3,\bar{3}) \pm (\bar{3},3)$ representation in the space of the matrices F_i, F_i^5 [47]). Then the fields u_a, v_a can be defined by

$$\sum_0^8 (u_a \Lambda_a + v_a \bar{\Lambda}_a) = \zeta e^{i\sum_7^8 P_i F_i^5}\, e^{i\sum_4^7 S_r F_r} (\Lambda_0 + c' \Lambda_a)\, e^{-i\sum_4^7 S_r F_r}\, e^{-i\sum_i^8 P_i F_i^5} \tag{10.13}$$

and are non-linear functions of P_i, S_r:

$$u_a = \xi \Big[\delta_{a0} + c' \delta_{a8} - c' S_r f_{r8a} + \frac{1}{2} c' S_r S_t f_{ria} f_{t8i} - \frac{1}{2} P_i P_j (d_{j0b} + c' d_{j8b})\, d_{iba} +$$

$$+ \frac{1}{2} c' S_r P_i P_j f_{r8k} d_{iba} d_{jkb} - \frac{1}{2} c' S_r S_s S_t f_{r8i} f_{sja} f_{tij} + O(\text{field}^4) \Big] \tag{10.14}$$

$$v_a = \zeta [P_b (d_{b0a} + c' d_{b8a}) - c' P_b S_r f_{r8i} d_{bia} + O\,(\text{field}^3)]. \tag{10.15}$$

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[46]) The author thanks J. Honerkamp for discussions of his paper Ref. [*41*].

[47]) For example

$$F_i = 1 \times \frac{\lambda_i}{2}, \quad F_i^5 = \gamma_5 \times \frac{\lambda_i}{2}$$

$$\Lambda_i = \gamma_0 \times \frac{\lambda_i}{2}, \quad \Lambda_i = i\gamma_5\gamma_0 \times \frac{\lambda_i}{2}$$

Note that the parameter c' denotes the breakdown of $SU(3)$ symmetry in the vacuum, since

$$\langle 0 | u_0 | 0 \rangle = \zeta$$
$$\langle 0 | u_8 | 0 \rangle = \zeta c'. \tag{10.16}$$

If $c' = 0$, the vacuum is $SU(3)$ invariant while $c' = -\sqrt{2}$ amounts to a chiral $SU(2) \times SU(2)$ invariance of the vacuum. The standard assumption of most models is $c' = 0$ [*30*].

We shall not go into the details of the purely mesonic parts of the non-linear Lagrangian. They have been dealt with extensively in the literature [*41*]. We only write down the relevant terms needed for our discussion.

The chirally invariant meson Lagrangian leads to massless mesons S_r, P_i described by

$$\mathcal{L}_M = \frac{1}{2}\left[\sum_1^8 f_{P_i}^2\, \partial_\mu P_i\, \partial^\mu P_i + \sum_4^7 f_{S_r}^2\, \partial_\mu S_r\, \partial^\mu S_r\right]. \tag{10.17}$$

Then the $SU(3) \times SU(3)$ breaking

$$\mathcal{L}_{SB} = -(u_0 + c u_8) = -\zeta \Bigg[(1 + cc') - \frac{1}{2}\, cc' S_r S_t f_{r8i} f_{t8i} - \tag{10.18}$$

$$- \frac{1}{2}\, P_i P_j\, (d_{ia0} + c d_{ia8})\, (d_{j0a} + c' d_{i8a}) + O(\text{field}^3)\Bigg] \tag{10.19}$$

lifts these masses up to values given by the equations:

$$f_\pi{}^2 m_\pi{}^2 \equiv \pi = -\frac{\zeta}{3}\,(\sqrt{2} + c)\,(\sqrt{2} + c') \equiv -\zeta\, W_\pi(c)\, W_\pi(c')$$

$$f_\varkappa{}^2 m_\varkappa{}^2 \equiv K = -\frac{\zeta}{3}\left(\sqrt{2} - \frac{c}{2}\right)\left(\sqrt{2} - \frac{c'}{2}\right) \equiv -\zeta W_\varkappa(c)\, W_\varkappa(c')$$

$$f_\eta{}^2 m_\eta{}^2 \equiv \eta = -\frac{\zeta}{3}\,(\sqrt{2} - c)\,(\sqrt{2} - c') \equiv -\zeta W_\eta(c)\, W_\eta(c')$$

$$f_\varkappa{}^2 m_\varkappa{}^2 \equiv \varkappa = -\frac{\zeta}{3}\,\frac{9}{4}\, cc'. \tag{10.20}$$

These equations can be inverted, solving for the parameters ζ, $\varkappa$, c' in terms of the better known parameters π, K, c:

$$\zeta = -\frac{1}{6r}\,\frac{3r + 2}{r + 1}\,[(r + 1)\,\pi + 2rK] \tag{10.21}$$

$$\varkappa = \frac{1}{1 + r}\, K - \frac{1}{r}\,\pi \tag{10.22}$$

$$c' = -2\sqrt{2}\,[rK - (r + 1)\,\pi]/[(r + 1)\,\pi + 2rK] \tag{10.23}$$

with one constraint

$$4K + 4\varkappa - 3\eta - \pi = 0 \tag{10.24}$$

expressing the pure $(3, \bar{3}) + (\bar{3}, 3)$ character of the symmetry breakdown. In an $(8, 1) \pm (1, 8)$ breaking this term would be non-zero.
Notice that the chiral low-energy theorem (6.26) is necessarily satisfied in this model

$$\begin{aligned}\langle 0 \,|\, u_0 + c u_8 \,|\, 0\rangle &= -\langle 0 \,|\mathscr{L}_{SB}|\, 0\rangle = \zeta(1 + cc') \\ &= \zeta\left(1 - \varkappa \frac{4}{3\zeta}\right) = \zeta - \frac{4}{3}\,\varkappa.\end{aligned} \tag{10.25}$$

Due to the non-zero vacuum expectation values of u_0 and u_8, masses are generated in the *chirally invariant* meson-baryon interaction Lagrangian (10.9).
Notice that nothing in the world forces us to use, in this coupling, the fields u_a, v_a with the same parameter c'. One could take any other value and even use different values of c' for either one of the two meson fields appearing in (10.9).
With this freedom it is no problem to fit the baryon masses[48]), and the mass term can be written immediately as

$$\mathscr{L}_{BM} = \sum_1^8 m_i\, \bar{B}_i B_i + \cdots \tag{10.26}$$

Now we can proceed to impose simple scale properties upon this Lagrangian. We want to see wether all terms can have dimension four, except for $\mathscr{L}_{SB}$ which we want to have dimension d. This is achieved by writing [*19*]

$$\begin{aligned}\mathscr{L} &= \mathscr{L}_B + \mathscr{L}_M e^{2b\sigma} + \mathscr{L}_{BM} e^{b\sigma} + \mathscr{L}_{SB} e^{db\sigma} + \frac{1}{2b^2}\,(\partial_\mu e^{b\sigma})^2 - \\ &\quad - \langle 0 \,|\mathscr{L}_{SB}|\, 0)\,\frac{d}{4}\, e^{4b\sigma} + \frac{d-4}{4}\,\langle 0 \,|\mathscr{L}_{SB}|\, 0\rangle.\end{aligned} \tag{10.27}$$

As a first result we obtain the coupling strength of the dilation σ:

$$g_{\sigma B_i B_i} = -b m_{B_i} \delta_{ij} = \frac{\gamma}{m_\sigma}\, m_{B_i} \delta_{ij}\,. \tag{10.28}$$

The value γ can again be read off the coefficient of σ in $\partial\mathscr{D}$

$$\partial\mathscr{D} = -(4 - d)\,\mathscr{L}_{SB} e^{db\sigma} = (4 - d)\,(1 + cc')\,\zeta\, db\sigma + \cdots \tag{10.29}$$

which tells us that

$$\frac{m_\sigma{}^3}{\gamma} = (4 - d)\,(1 + cc')\,\zeta db. \tag{10.30}$$

---

[48]) Since there are two meson fields, the mass formula is a product of two Gell-Mann Okubo formulas.

Taking b to the left hand side as $-\gamma/m_\sigma$, we find

$$-\frac{m_\sigma^4}{\gamma^2} = (4-d)\, d\, \langle 0\,|u|\,0\rangle \tag{10.31}$$

which verifies our sum rule (6.20).

This model illustrates an important point concerning the discussion on the size of the Σ-term. Due to the $SU(3)$ dependence of $g_{\sigma Bi\,Bi}$ the matrix elements of u_0 and u_8 between baryon states do not allow for a perturbation type of treatment. In fact, the terms u_0 and u_8 in $\mathscr{L}_{SB} e^{db\sigma}$ contribute[49])

$$\langle B_i\,|u_0|\,B_i\rangle = \zeta\, d\, b\, g_{\sigma Bi\,Bi}\,\frac{1}{m_\sigma^2}$$

$$\langle B_i\,|u_8|\,B_i\rangle = c'\,\zeta\, d\, b\, g_{\sigma B_i B_i}\,\frac{1}{m_\sigma^2}. \tag{10.32}$$

Instead, the matrix elements of u_0 and u_8 are $SU(3)$ broken *both* in the same way as the masses themselves. Therefore those of u_0 are almost $SU(3)$ invariant in spite of its index 8. If the vacuum is $SU(3)$ invariant, ($c' = 0$), the mass splitting of the baryons is *entirely* due to the *chirally symmetric* part of the Lagrangian $\mathscr{L}_{BM}$ and not to u_8!

XI. Coupling of σ to Vector and Axial Vector Mesons

We may extend the set of Ward identities derived in Sect. IV. and include also three point functions

$$\langle 0\,|T(\mathscr{D}_\mu(y)\, A_\nu(x)\, A_\lambda(0))|\,0\rangle \tag{11.1}$$

and

$$\langle 0\,|T(\mathscr{D}_\mu(y)\, V_\nu(x)\, V_\lambda(0))|\,0\rangle \tag{11.2}$$

and their derivatives.

The parametrization of these vertices will involve, besides pions, also vector and axial vector mesons and the Ward identities will give relations between their couplings. For simplicity, we shall study these couplings directly in appropriate effective Lagrangians.[50])

We shall take the standard point of view that vector and axial vector mesons dominate the corresponding currents. It is well known how to construct Lagrangians of this type. We shall use the following model:[44])

$$\mathscr{L} = \frac{1}{2}\left[(\Delta_\mu\sigma)^2 + (\Delta_\mu\boldsymbol{\pi})^2\right] + \bar{f}\mu^2\sigma -$$

$$-\frac{1}{4}\,(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu} + \boldsymbol{G}_{\mu\nu}\boldsymbol{G}^{\mu\nu}) + \frac{m_0^2}{2}\,(\boldsymbol{v}_\mu^2 + \boldsymbol{a}_\mu^2) - c \tag{11.3}$$

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[49]) Notice that due to PCAC no direct $\overline{\mathrm{B}}\mathrm{B}$ terms are allowed in contrary to Ellis' Lagrangian [*42*]. See the footnote on p. 24.

[50]) For the treatment of these couplings via Ward identities we refer the reader to the original literature [*43*].

where σ, π are non-linear representations of $SU(2) \times SU(2)$ like those employed in Sect. IX. We shall assume σ to have the vacuum expectation value

$$\sigma = \sqrt{\sigma_0^2 - \pi^2} = \sigma_0 - \frac{1}{2\sigma_0^2} = \boldsymbol{\pi}^2 - \frac{1}{8\sigma_0^4}(\boldsymbol{\pi}^2)^2 - \cdots \tag{11.4}$$

and find it convenient to define, in analogy with the linear model, a field with vanishing vacuum expectation value

$$\sigma' \equiv \sigma - \sigma_0 = -\frac{1}{2\sigma_0^2}\boldsymbol{\pi}^2 - \frac{1}{8\sigma_0^4}(\boldsymbol{\pi}^2)^2 - \cdots \tag{11.5}$$

The expressions Δ_μ denote covariant derivatives and are defined as

$$\Delta_\mu \sigma \equiv \partial_\mu \sigma + \gamma_0 \boldsymbol{a}_\mu \cdot \boldsymbol{\pi} \tag{11.6}$$

$$\Delta_\mu \boldsymbol{\pi} \equiv \partial_\mu \boldsymbol{\pi} + \gamma_0 \boldsymbol{v}_\mu \times \boldsymbol{\pi} - \gamma_0 \sigma \boldsymbol{a}_\mu \equiv D_\mu \boldsymbol{\pi} - \gamma_0 \sigma' \boldsymbol{a}_\mu . \tag{11.7}$$

The fields $\boldsymbol{v}_\mu$, $\boldsymbol{a}_\mu$ are supposed to describe vector and axial vector mesons $\varrho_\mu A_{1\mu}$, and $\boldsymbol{F}_{\mu\nu}$, $\boldsymbol{G}_{\mu\nu}$ are their covariant curls

$$\boldsymbol{F}_{\mu\nu} \equiv \boldsymbol{f}_{\mu\nu} + \gamma_0 \boldsymbol{v}_\mu \times \boldsymbol{v}_\nu + \gamma_0 \boldsymbol{a}_\mu \times \boldsymbol{a}_\nu \tag{11.8}$$

$$\equiv \partial_\mu \boldsymbol{v}_\nu - \partial_\nu \boldsymbol{v}_\mu + \gamma_0 \boldsymbol{v}_\mu \times \boldsymbol{v}_\nu + \gamma_0 \boldsymbol{a}_\mu \times \boldsymbol{a}_\nu \tag{11.9}$$

$$\boldsymbol{G}_{\mu\nu} \equiv \partial_\mu \boldsymbol{a}_\nu - \partial_\nu \boldsymbol{a}_\mu + \gamma_0 \boldsymbol{v}_\mu \times \boldsymbol{a}_\nu + \gamma_0 \boldsymbol{a}_\mu \times \boldsymbol{v}_\nu .$$

Apart from the mass term, this Lagrangian is invariant under all space tim dependent $SU(3) \times SU(3)$ gauge transformations.

$$\delta\sigma = 0, \qquad \delta\boldsymbol{\pi}(x) = \boldsymbol{\alpha}(x) \times \boldsymbol{\pi}(x)$$

$$\delta\boldsymbol{v}(x)_\mu = \boldsymbol{\alpha}(x) \times \boldsymbol{x}_\mu(x) - \frac{1}{\gamma_0}\partial_\mu \alpha(x), \quad \delta\boldsymbol{a}_\mu(x) = \boldsymbol{\alpha}(x) \times \boldsymbol{a}_\mu(x) \tag{11.10}$$

and

$$\bar\delta\sigma = \boldsymbol{\alpha}(x)\,\boldsymbol{\pi}(x), \quad \bar\delta\boldsymbol{\pi}(x) = -\boldsymbol{\alpha}(x)\,\sigma(\partial)$$

$$\bar\delta v_\mu(x) = \boldsymbol{\alpha}(x) \times \boldsymbol{a}_\mu(x), \quad \bar\delta\boldsymbol{a}_\mu(x) = \boldsymbol{\alpha}(x) \times \boldsymbol{v}_\mu(x) - \frac{1}{\gamma_0}\partial_\mu \alpha(x). \tag{11.11}$$

As a consequence, the vector and axial vector currents $V^\mu(x)$ and $A^\mu(x)$ obtained by varying $\mathscr{L}$ with respect to $\partial^\mu \alpha(x)$ arise from the non-invariant mass term only:

$$\boldsymbol{V}^\mu(x) \equiv \frac{\delta\mathscr{L}}{\delta\,\partial^\mu \boldsymbol{\alpha}(x)} = -\frac{m_0^2}{\gamma_0}\boldsymbol{v}_\mu(x) \tag{11.12}$$

$$\boldsymbol{A}^\mu(x) = \frac{\bar\delta\mathscr{L}}{\delta\,\partial^\mu \boldsymbol{\alpha}(x)} = -\frac{m_0^2}{\gamma_0}\boldsymbol{a}_\mu(x).$$

These equations express the vector meson dominance of the model. The covariant derivative (11.7) introduces terms of the type $a_\mu \partial^\mu \pi$ into the Lagrangian mixing up π and a_μ fields. One removes such terms by going to the pure vector field

$$\boldsymbol{A}_\mu(x) \equiv \boldsymbol{a}_\mu(x) - \xi D_\mu \boldsymbol{\pi}(x) \tag{11.13}$$

bringing the Lagrangian to the form:

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \sigma' + \gamma_0 \boldsymbol{\pi} \boldsymbol{A}_\mu + \gamma_0 \xi \boldsymbol{\pi} D_\mu \boldsymbol{\pi})^2 + \right.$$
$$\left. + \left((1-\gamma_0\xi\sigma_0)\, D_\mu \boldsymbol{\pi} - \gamma_0 \xi \sigma' D_\mu \boldsymbol{\pi} - \gamma_0 \sigma_0 \boldsymbol{A}_\mu - \gamma_0 \sigma' \boldsymbol{A}_\mu\right)^2\right] + f_\mu{}^2 \sigma$$
$$- \frac{1}{4}(\boldsymbol{f}_{\mu\nu} + \gamma_0 \boldsymbol{A}_\mu \times \boldsymbol{A}_\nu + \gamma_0 \xi \boldsymbol{A}_\mu \times \boldsymbol{A}_\nu + \gamma_0 \xi D_\mu \boldsymbol{\pi} \times \boldsymbol{A}_\nu + \gamma_0 \xi^2 D_\mu \boldsymbol{\pi} \times D_\nu \boldsymbol{\pi})^2 -$$
$$- \frac{1}{4}(D_\mu \boldsymbol{A}_\nu - D_\nu \boldsymbol{A}_\mu + \xi \gamma_0 \boldsymbol{f}_{\mu\nu} \times \boldsymbol{\pi})^2 + \frac{m_0{}^2}{2}(\boldsymbol{v}_\mu{}^2 + \boldsymbol{a}_\mu{}^2) - c \tag{11.14}$$

where in analogy to $D_\mu \boldsymbol{\pi}$ we have written

$$D_\mu \boldsymbol{A}_\nu \equiv \partial_\mu \boldsymbol{A}_\nu + \gamma_0 \boldsymbol{v}_\mu \times \boldsymbol{A}_\nu .$$

The πA_μ mixing terms are eliminated by setting

$$m_0{}^2 \xi - \gamma_0 \sigma_0 (1 - \gamma_0 \xi \sigma_0) = 0 \tag{11.15}$$

or

$$\xi = \gamma_0 \sigma_0 / [m_0{}^2 + (\gamma_0 \sigma_0)^2]. \tag{11.16}$$

By this mixing procedure one produces additional kinetic terms of the pion field which now appear with a factor

$$\frac{m_0{}^2}{[m_0{}^2 + (\gamma_0\sigma_0)^2]} \frac{1}{2}(D_\mu \boldsymbol{\pi})^2 \equiv Z^{-1} \frac{1}{2}(D_\mu \boldsymbol{\pi})^2 \tag{11.17}$$

such that one has to introduce a properly renormalizes pion field by

$$\boldsymbol{\pi} = Z^{1/2} \boldsymbol{\pi}^r . \tag{11.18}$$

Due to this renormalization factor the mass of the pion becomes

$$m_\pi{}^2 = \frac{\tilde{f} Z}{\sigma_0} \mu^2 \tag{11.19}$$

such that

$$\tilde{f} = \sigma_0 Z . \tag{11.20}$$

In addition, the PCAC condition

$$\partial \boldsymbol{A} = \tilde{f} \mu^2 \boldsymbol{\pi} = \tilde{f} Z^{1/2} \mu^2 \boldsymbol{\pi}^r \tag{11.21}$$

teaches us that $\tilde{f}$ has to be chosen

$$\tilde{f} = Z^{-1/2} f_\pi . \tag{11.22}$$

The masses of vector and axial vector mesons ρ and A_1 satisfy

$$m_\rho{}^2 = m_0{}^2$$
$$m_{A_1}{}^2 = Z m_0{}^2 . \tag{11.23}$$

such that the experimental ratio determines $Z \approx 2$. From the definition (11.17) of Z, we recover the KSFR relation

$$\gamma_0^2 = \frac{m_\rho^2}{2 f_\pi^2}. \tag{11.24}$$

The Lagrangian (11.3) has still some problems in explaining the $A_1 \rho \pi$ width. Therefore one usually introduces in addition a so called δ-term

$$\mathscr{L}_\delta = -\delta \frac{\gamma_0}{m_0^2} \left[\frac{1}{2} \boldsymbol{F}_{\mu\nu} \Delta^\mu \boldsymbol{\pi} \times \Delta^\nu \boldsymbol{\pi} + \boldsymbol{G}_{\mu\nu} \Delta^\mu \boldsymbol{\pi}\, \Delta^\nu \sigma\right]. \tag{11.25}$$

With this term the $\rho\pi\pi$ and $A_1 \rho \pi$ couplings come out as[51]) [4]

$$g_{\rho\pi\pi} = \frac{m_\rho}{\sqrt{2}\, f_\pi} \left(\frac{3-\delta}{4}\right)$$

$$g_{A\rho\pi} = \frac{m_\rho}{4 f_\pi} (2 + \delta)$$

$$h_{A\rho\pi} = \frac{m_\rho}{2 f_\pi} \delta \tag{11.26}$$

By choosing $\delta \approx -1/2$ one can get

$$\Gamma_{\rho\pi\pi} \approx 110 \text{ MeV}, \quad \Gamma_{A\rho\pi} \approx 100 \text{ MeV}$$

roughly in agreement with the experimental widths. There was always a discrepancy of the longitudinal to transverse ratio.

$$\frac{g_L}{g_T} \approx \frac{\frac{2}{f_\pi}\left(\frac{3+\delta}{2}\right)}{-\frac{2}{f_\pi}(2+\delta)} \approx -\frac{3+\delta}{4+2\delta} \approx -.8 \tag{11.27}$$

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[51]) The standard definition of these couplings is

$$\mathscr{L} = g_{\rho\pi\pi}\, \varrho^\mu \boldsymbol{\pi} \times \partial_\mu \boldsymbol{\pi} + g_{A\rho\pi} m_\rho\, \boldsymbol{\varrho}_\mu \boldsymbol{A}^\mu \times \boldsymbol{\pi} + h_{A\varrho\pi}/m_\varrho\, \boldsymbol{\varrho}_\mu\, \partial^\mu \boldsymbol{A}^\nu \times \partial_\nu \boldsymbol{\pi}$$

such that the widths become

$$\Gamma_{\sigma\pi\pi} = \frac{g_{\rho\pi\pi}^2}{4\pi} \frac{1}{12} \frac{(m_\rho^2 - 4 m_\pi^2)^{3/2}}{m_\rho^2} \approx 140 \left(\frac{3-\delta}{4}\right)^2 \text{MeV}$$

$$\Gamma_{A\rho\pi} = \frac{1}{4\pi} \frac{p^5}{3 m_A^2} \left[g_L^2 \frac{m_A^2}{m_\rho^2} + 2 g_T^2\right] \approx .095 \left[100\, g_{A\rho\pi}^2 - 12\, g_{A\rho\pi} h_{A\rho\pi} + h_{A\rho\pi}^2\right] \text{MeV}$$

$$\approx 6.2 \left[5\, \delta^2 + 22\, \delta + 25\right] \text{MeV}$$

where the longitudinal and transverse couplings are defined by [45]

$$g_T \equiv -g_{A\rho\pi} \left(\frac{2 m_A}{m_A^2 - m_\rho^2}\right) m_\rho^2 \approx -g_{A\rho\pi} \frac{8}{m_\rho}$$

$$g_L \equiv -h_{A\rho\pi} \frac{1}{m_\rho} - g_T \frac{m_A^2 + m_\rho^2}{2 m_A^2} \approx (-h_{A\rho\pi} + 6 g_{A\rho\pi}) \frac{1}{m_\rho}.$$

with the measured value [46][52])

$$\left|\frac{g_L}{g_L}\right| \approx 1.4$$

originating in the models failure to produce odd $\pi\pi \to \pi\pi$, $\pi\rho \to \pi\rho$, $\pi A_1 \to \pi A_1$, scattering amplitudes which are asymptotically well behaved. For this reason the couplings do not satisfy the corresponding Adler Weisberger relations[53]). Therefore some more terms are really needed in the Lagrangian to inforce the correct high energy behaviour.

For the purpose of showing how to deal with scale invariance we shall be satisfied with discussing only $\mathscr{L} + \mathscr{L}_\delta$. Obviously, the Lagrangian (11.3) is made almost scale invariant by writing it in the form

$$\begin{aligned}\mathscr{L} = \frac{1}{2}\,[(\partial_\mu \sigma)^2 + (\varDelta_\mu \boldsymbol{\pi})^2]\, e^{2b\sigma} + \tilde{f}\mu^2 \sigma e^{db\sigma} - \\ - \frac{1}{4}\,(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu} + \boldsymbol{G}_{\mu\nu}\boldsymbol{G}^{\mu\nu}) + \frac{{m_0}^2}{2}\,({v_\mu}^2 + {a_\mu}^2)\, e^{2b\sigma} \\ - \tilde{f}\mu^2\sigma_0 \frac{d}{4}\, e^{4b\sigma} - \tilde{f}\mu^2\sigma_0\left(1 - \frac{d}{4}\right) + \mathscr{L}_\delta\end{aligned} \tag{11.28}$$

where the only breaking comes form the Σ term of dimension d and the c-number. By looking at the form (11.14) we see directly that the absence of $A_\mu \pi$ mixing terms due to (11.15) implies the absence of the $\sigma A_1 \pi$ couplings. For $\sigma\rho\rho$ and $\sigma A_1 A_1$ on the other hand, we obtain contributions

$$\mathscr{L} = b\,[{m_\rho}^2 {\boldsymbol{v}_\mu}^2 + m_{A^2}]\,\sigma \tag{11.29}$$

showing that $h_{\sigma\rho\rho} = h_{\sigma AA} = 0$ and[54])

$$\begin{aligned} g_{\sigma\rho\rho} &= 2b{m_\rho}^2 = -2\gamma\,\frac{{m_\rho}^2}{m_\sigma} \\ g_{\sigma AA} &= 2b{m_A}^2 = -2\gamma\,\frac{{m_A}^2}{m_\sigma}. \end{aligned} \tag{11.30}$$

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[52]) Ref. 46) measures

$$\left|\frac{g_1}{g_0}\right| \equiv \left|\frac{g_T}{-\sqrt{2}\,g_L}\right| = .48 \pm .2$$

Notice that this value is predicted on the basis of Adler Weisberger relations in Ref. [47].

[53]) Even though the low-energy theorems derived from current algebra are certainly fulfilled by construction of the chirally invariant Lagrangian containing PCAC.

[54]) With the definition:

$$\mathscr{L}_{\sigma VV} = g_{\rho VV}\, m_V \sigma V_\mu V^\mu + h_{\sigma VV}\,\frac{1}{m_\nu}\,\sigma_\nu\, \partial_\mu V_\nu \partial^\nu V_\mu.$$

XII. On the Coupling of σ to Two Photons

Let us now see what our simple Lagrangian model has to say about the $\sigma\gamma\gamma$ vertex.[55]) We make the assumption that the photon couples in the standard fashion to leptonic currents $I_\mu^e(x)$ and communicates with hadronic matter only via a gauge invariant direct interaction with the ρ^0 meson[56]). [48]
In order to be specific let us assume the matter Lagrangian to be of the type

$$\mathscr{L}_h = \mathscr{L} + \mathscr{L}_\delta$$

discussed in the last section. In this Lagrangian we take out explicitly the ϱ^0 field in the form:

$$\mathscr{L}_h(\varrho^0) = -\frac{1}{4} f^0_{\mu\nu} f^{0\mu\nu} + \frac{m_\rho^2}{2} \varrho^0{}_\mu \varrho^{0\mu} - \gamma \varrho^{0\mu} I_\mu^h(\varrho^0) + \mathscr{L} \quad \text{(other hadron fields)} \tag{12.1}$$

where $f^0_{\mu\nu}$ is the field tensor of the ϱ^0 field

$$f^0_{\mu\nu} = \partial_\mu \varrho_\nu{}^0 - \partial_\nu \varrho_\mu{}^0 \tag{12.2}$$

and $I_\mu^h(x)$ is the neutral hadronic current. Since our model does not contain any weak interactions this current is conserved.

$$\partial^\mu I_\mu^h(x) = 0.$$

The photon field A_μ interacts with the leptonic current in just the same way:

$$\mathscr{L}_e = -\frac{1}{4} A_{\mu\nu} A^{\mu\nu} - e A^\mu I_\mu^l(x) + \mathscr{L} \text{ (other lepton fields).} \tag{12.3}$$

Now according to the idea of vector meson dominance of electromagnetic interaction all the interaction between leptonic and hydronic worlds is coming from a gauge invariant modification of the mass term $(m_\rho^2/2)\, \varrho_\mu{}^{02}$ in (12.1):

$$\frac{m_\rho^2}{2} \varrho^0{}_\mu{}^2 \to \frac{m_\rho^2}{2} \frac{1}{\sqrt{e^2+\gamma^2}} (\gamma \varrho_\mu - e A_\mu)^2. \tag{12.4}$$

The resulting Lagrangian is obviously invariant under the joint gauge transformations:

$$e A_\mu \to e A_\mu - \partial_\mu \alpha(x) \tag{12.5}$$

$$\gamma \varrho_\mu{}^0 \to \gamma \varrho_\mu{}^0 - \partial_\mu \alpha(x). \tag{12.5}$$

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55) For the Ward Identity approach see Ref. [15].
56) We neglet ω and φ contributions for simplicity. They don't change the result much, quantitatively.

For a moment, the new Lagrangian seems to contain a mass term for the photon field. However, if one transforms ϱ^0 and A_μ to new, physical, fields

$$\tilde{\varrho}_\mu{}^0(x) = \frac{1}{\sqrt{e^2+\gamma^2}}(\gamma\varrho_\mu{}^0 - eA_\mu) \tag{12.6}$$

$$\tilde{A}_\mu(x) = \frac{1}{\sqrt{e^2+\gamma^2}}(e\varrho_\mu{}^0 + \gamma A_\mu)$$

then $\mathscr{L}$ goes over into

$$\begin{aligned}\mathscr{L}_{\text{tot}} = & -\frac{1}{4}\tilde{\varrho}_{\mu\nu}{}^2 - \frac{1}{4}\tilde{A}_{\mu\nu}{}^2 + \frac{m_\rho{}^2}{2}\tilde{\varrho}^0{}_\mu{}^2 - \\ & - \frac{\gamma}{\sqrt{e^2+\gamma^2}}(\gamma\tilde{\varrho}^{0\mu} + e\tilde{A}^\mu)\, I_\mu{}^h\left(\frac{\gamma\tilde{\varrho}+e\tilde{A}}{\sqrt{e^2+\gamma^2}}\right) - \\ & - \frac{e}{\sqrt{e^2+\gamma^2}}(-e\tilde{\varrho}^{0\mu} + \gamma\tilde{A}^\mu)\, I_\mu{}^l(x)\,.\end{aligned} \tag{12.7}$$

Let us now assume that the Lagrangian $\mathscr{L}_h$ has been made approximately scale invariant as described in the last section. Then the currents $I_\mu{}^h$ have all dimension 3. So does $I_\mu{}^e$ by its definition. The mass term appears with a factor $e^{2b\sigma}$. Obviously the only $\sigma\tilde{A}_\mu\tilde{A}^\mu$ coupling that could appear would have to come from the term

$$-\frac{e\gamma}{\sqrt{e^2+\gamma^2}}\tilde{A}^\mu I_\mu{}^h\left(\frac{\gamma\tilde{\varrho}+e\tilde{A}}{\sqrt{e^2+\gamma^2}}\right). \tag{12.8}$$

But inspection of $\mathscr{L}^h$ in (11.28) shows that there is no $\tilde{A}_\mu\sigma$ contribution to the hadronic current. Hence

$$g_{\sigma\gamma\gamma} = 0 \tag{12.9}$$

in agreement with our conclusion in Sect. III.
Notice that our Lagrangian allows us, in addition, to predict the σ particle's decoupling[57]) from photons and vector mesons

$$g_{\sigma\rho\gamma} = 0. \tag{12.10}$$

Both of these predictions are rather unfortunate from the experimental point of view. As we said in Sect. III the vanishing of $g_{\sigma\gamma\gamma}$ prevents us from producing σ in the Primakoff effect. If the model's result on $g_{\sigma\rho\gamma}$ is true, also the colliding beam experiment proposed by CREUTZ and EINHORN *[49]*

$$e^+e^- \to \sigma\gamma \to \pi^+\pi^-\gamma$$

will be searching in vain *[40]* for a σ-resonance in the $\pi^+\pi^-$ system when s is close to $m_\rho{}^2$.

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[57]) See the first of Ref. *[40]*.

XIII. Conclusion

By investigating the low-energy properties of some hadron vertices we have been able to obtain some evidence for an approximate scale invariance of the world. The scale properties of the Hamiltonian density become most simple if one assumes the symmetry breaking to be mainly due to an almost Goldstone boson σ.

Unfortunately, the mass of this boson is rather high. Therefore the assumption of σ dominance of the divergence of the dilatation current (PCDC) should be expected to be a rather crude approximation for vertices where the coupling of σ is not very large. Indeed, by using additional information from the chiral properties of the world, we can *show* a subtraction constant to occur, for example, in the $\partial\mathscr{D}\pi\pi$ vertex.

Thus the approximation of PCDC is certainly much less accurate than good old PCAC.

Since scale transformations form a group with a rather poor structure, there is not much information coming from commutation rules of the scale currents with fields. The main result is (see Sect. IV) that the dimension of a field tells us how fast a vertex varies when going off shell in this particular field. For the pion we showed the canonical dimension $d = 1$ to produce the smoothest off-mass shell continuation.

The reader will have noticed that we have left out unitarity completely in our discussion. Now one certainly may argue that the large width of the σ resonance requires unitarizing our vertices if one wants to have any better than 40% accuracy. However, we do not think that the intrinsic crudity of the approximation of PCDC warrants such an elaborate correction procedure. All the frame work of Ward identities and PCDC should be expected to give us is some rough ideas on the order of magnitude of the σ couplings. With this reservation in mind we think that the whole scheme provides some interesting addition to the framework of current algebra.

We should also like to mention here that there are, in general, problems associated with the covariantization of Ward identities as soon as we form derivatives with respect to more than one current (for example in the $T(\mathscr{D}_\mu A_\lambda A_2)$ vertex). In the absence of a definite model one does not have a definite prescription how to proceed. For this reason we have not dealt with such cases in these lectures. Instead, we have taken directly some effective Lagrangians in order to obtain predictions.

The reader should be aware of the model dependence of all such results. For a complete study of this problem via Ward identities we suggest a study of Ref. [*43*].

Finally, the whole idea that scale invariance is broken in some soft way may be wrong alltogether. If Regge trajectories really rise up to infinity this is certainly the case. Then the highly massive and energetic photon never sees pointlike partons in deep-inelastic scattering [*8*]. There is no contradiction with the phenomena of scaling if only all form factors drop off the same way.

As usual, we shall have to wait and see what will survive of all these hypotheses.

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Appendix A: Some Theorems

We give here a brief derivation[58] [*50*] of the commutator (4.16)

$$i[\partial\mathscr{D}(x)\,Q_5(x_0)] = (4-d)\,\partial A(x) \tag{A. 1}$$

needed in deriving the Ward identity (4.19). The assumptions are:
1) In the commutator of the densities

$$i[\partial\mathscr{D}(0,\boldsymbol{x})\,A_0(0,\boldsymbol{y})] = \alpha(y)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) + S^i(\gamma)\,\partial_i^x\delta^3(\boldsymbol{x}-\boldsymbol{y}) + + \sum_{n=2}^{N}\sigma^{l_1-b_n}(\boldsymbol{y})\,\partial_{k_1}^x\cdots\partial_{k_n}^x\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{A. 2}$$

the Schwinger terms $\sigma^{k_1\cdots k_n}(\boldsymbol{y})$ all vanish.
2) The dimension of A_0 is definite, such that durrent algebra enforces

$$i[D(x_0),A_0(x)] = (3+x\partial)\,A_0(x). \tag{A. 3}$$

3) The dimension of ∂A is d.
The proof proceeds in two steps.
First we integrate (A. 2) over d^3x determing

$$i\int d^3x\,[\partial\mathscr{D}(0,\boldsymbol{x})\,A_0(0)] = \alpha(0). \tag{A. 4}$$

Commuting (2.8)

$$i\int d^3x\,\partial\mathscr{D}(0,\boldsymbol{x}) = iH + [D(0)H] \tag{A. 5}$$

with $A_0(0)$ we obtain:

$$\begin{aligned}\alpha(0) &= \partial_0 A_0(0) + \big[[D(0),H],A_0(0)\big]\\ &= \partial_0 A_0(0) + i\big[H,i[D(0),A_0(x)]\big] - i\big[D(0)\,i[H,A_0(0)]\big]\\ &= 4\,\partial_0 A_0(0) - i[D(0),\partial^0 A_0(0)]\\ &= 4\,\partial_0 A_0(0) - d\,\partial A + i[D(0),\partial^i A_i(0)].\end{aligned} \tag{A. 6}$$

Now from the basic property (2.4) of $D(x_0)$, the local operator $A_i\,(x)$ fulfils

$$i[D(x_0),A_i(x)] = x\partial A_i(x) + A_i'(x). \tag{A.7}$$

As a consequence, $\partial^i A_i$ satisfies

$$i[D(x_0)\,\partial^i A_i(x)] = x\partial\,\partial^i A_i(x) + \partial^i A_i(x) + \partial^i A_i'(x). \tag{A.8}$$

[58]) Our derivation proceeds under somewhat weaker assumptions than that of Ref. [*50*]. We do not require $A_i(x)$ and $\partial^i A_i(x)$ to have a definite dimension.

Such that (A. 6) can be rewritten as

$$\alpha(0) = (4 - d)\, \partial A(0) - 4\, \partial^i A_i(0) + \partial^i A_i(0) + \partial^i A_i'(0). \qquad \text{(A. 9)}$$

As the second step we observe that the Schwinger term S^i is determined by integrating (A. 2)

$$i \int d^3x x^i\, [\partial D(0, \boldsymbol{x}), A_0(0)] = -S^i(0). \qquad \text{(A. 10)}$$

The left-hand side is evaluated in the following fashion: One uses the vector property of $\mathscr{D}_\mu$:

$$i[M_{0i}, \mathscr{D}_0(x)] = (x_0\, \partial_i - x_i\, \partial_0)\, \mathscr{D}_0(x) - \mathscr{D}_i(x) \qquad \text{(A.11)}$$

to integrate

$$i[M_{0i}, D(0)] = \int d^3x x_i\, \partial \mathscr{D}(0, \boldsymbol{x}). \qquad \text{(A. 12)}$$

Then one commutes this with $-iA_0(0)$ to get

$$\begin{aligned} S(0) &= \big([[M_{0i}, D(0)], A_0(0)\big] \\ &= -i\big[M_{0i}\, i[D(0)\, A_0]\big] + i\big[D(0), i[M_{0i} A_0]\big] \qquad \text{(A.13)} \\ &= -3A_i(0) + A_i'(0). \end{aligned}$$

Using the results (A. 6) and (A.13) and integrating (A. 2) over d^3y we finally obtain

$$i[\partial \mathscr{D}(0, \boldsymbol{x}), Q^5(0)] = \alpha(0) - \partial_i S^i(0) = (4 - d)\, \partial A(0), \qquad \text{(A. 14)}$$

completing the proof.
Further, we want to show here that the result of the Σ-commutator (4.17):

$$\Sigma(\partial) \equiv i[Q_5(x_0), \partial A(x)] \qquad \text{(A. 15)}$$

is always a scalar, if only the commutator

$$[\partial A(0, \boldsymbol{x}), \partial A(0, \boldsymbol{y})] = \sigma^i(\boldsymbol{y})\, \partial_i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) + \sigma^{ii}(\boldsymbol{y})\, \partial_i \partial_j \delta^{(3)}(x - y) + \cdots \qquad \text{(A.16)}$$

has no lowest Schwinger term $\sigma^i(\boldsymbol{y})$. For a proof we simply commute

$$\begin{aligned} [M_{0i}, \Sigma(0)] &= i\big[M_{0i}, [Q_5(0), \partial A(0)]\big] \\ &= -i\big[\partial A(0), [M_{0i}, Q_5(0)]\big] + i\big[Q(0), [M_{0i}, \partial A(0)]\big]. \end{aligned}$$

The second term vanishes since ∂A is a scalar. In the first term we can use a relation like (A. 12) to get

$$[M_{0i}\, \Sigma(0)] = -\int d^3y x_i [\partial A(0, \boldsymbol{x})\, \partial A(0)] = \sigma_i(0) = 0, \qquad \text{(A.17)}$$

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