Extending Bogoliubov's Boson Theory to Strong Couplings

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We extend the theory of Bose-Einstein condensation from Bogoliubov's weak-coupling regime to large s-wave scattering lengths a_s. Our solution satisfies the Nambu-Goldstone theorem, thus avoiding an old problem of many-boson theories.

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1. For ϕ^4-theory in $D < 4$ Euclidean dimensions with O(N)-symmetry, a powerful strong-coupling theory has been developed in 1998 [1]. It has been carried to 7th order in perturbation theory in $D = 3$ [2], and to 5th order in $D = 4 - \epsilon$ dimensions [3]. The theory is an extension of a variational approach to path integrals set up by R.P. Feynman and collaborator in 1989 [4]. The extension to high orders is called *Variational Perturbation Theory* (VPT), and is developed in detail in the textbook [5]. Originally, the theory was designed to convert only the divergent perturbation expansions of quantum mechanics into exponentially fast convergent exressions [6]. In the papers [1–3], it was extended from quantum mechanics to ϕ^4-theory with its anomalous dimensions and produced all critical exponents. This is called quantum-field-theoretic VPT. That theory is explained in the textbook [7] and a recent review [8].

Surprisingly, this successful theory has not yet been applied to the presently so popular phenomena of Bose-Einstein condensation. These have so far mainly been focused [9] on the semiclassical treatments using the good-old Gross-Pitaevskii equations, or to the weak-coupling theory proposed many years ago by Bogoliubov [10]. This is somewhat surprising since the subject is under intense study by many authors. So far, only the shift of the critical temperature has been calculated to high orders [11]. There are only a few exceptions. For instance, a simple extension of Bogoliubov's thepry to strong couplings was proposed in [12] and pursued further in [13]. But that had an unpleasant feature that it needed two different chemical potentials to maintain the long-wavelength properties of Nambu-Goldstone excitations required by the spontaneously broken U(1)-symmetry in the condenste. For this reason it remained widely unnoticed. Another notable exception is the theory in [14] which came closest to our approach, since it wa also based on a variational optimization of the energy. But by following Bogoliubov in identifying a_0 as $\sqrt{\rho_0}$ from the outset, they ran into the notorious problem of violating the Nambu-Goldstone theorem. Another approach that comes close to ours is found in the paper [15]. Here the main difference lies in the popular use the Hubbard-Stratonovic transformation (HST) to introduce a fluctuating *collective pair field* [16]. But, as pointed out in [17] and re-emphasized in [18], this makes it impossible to calculate higher-order corrections [18]. The rules for applying VPT to nonrelativistic quantum field theories in 3+1 dimensions have been specified some time ago [19]. In this note we want to show how derive from them, to lowest order, the properties of the Bose-Einstein condensation at arbitrarily strong couplings.

It must be mentioned that in the literature, there have been many attempts to treat the strong-coupling regime of various field theories for models with a large number of identical field components (the so-called large-N-models). This has first been done for the so-called spherical model [20], later the Gross-Neveu model [21], and O(N)-symmetric φ^2-models [22]. In all these applications, the leading large-N limit has been easily solved with the help of the HST trick of introducing a fluctuating field variable [23, 24] for some dominant collective phenomenon (Collective Quantum Field Theory [16]). This approach has, however, the above-discussed problems of going to higher orders [18], which are absent here.

2. The Hamiltonian of the boson gas has a free term $H_0 \equiv \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}}(\varepsilon_{\mathbf{p}} - \mu)a_{\mathbf{p}} = \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}}\xi_{\mathbf{p}}a_{\mathbf{p}}$, where $\varepsilon_{\mathbf{p}} \equiv \mathbf{p}^2/2M$ are the single-particle energies and $\xi_{\mathbf{p}} \equiv \varepsilon_{\mathbf{p}} - \mu$ the relevant energies in a grand-canonical ensemble. As usual, $a^\dagger_{\mathbf{p}}$ and $a_{\mathbf{p}}$ are creation and annihilation operators defined by the canonical equal-time commutators of the local fields $\psi(\mathbf{x}) = \sum_{\mathbf{p}} e^{i\mathbf{px}/\hbar}a_{\mathbf{p}}$. The local interaction is

$$H_{\rm int} = \frac{g}{2V}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} a^\dagger_{\mathbf{p}+\mathbf{q}}a^\dagger_{\mathbf{p}'-\mathbf{q}}a_{\mathbf{p}'}a_{\mathbf{p}}. \tag{1}$$

Instead of following Bogoliubov in treating the $\mathbf{p} = 0$ modes of the operators $a_{\mathbf{p}}$ classically and identifying with the square-root of the condensate density ρ_0, we introduce the field expectation $\langle\psi\rangle \equiv \sqrt{V\Sigma_0/g}$ as a *variational parameter*, and rewrite $H_{\rm int}$ as $H^0_{\rm int} = (V/2g)\Sigma_0^2$ plus $H'_{\rm int} = \frac{1}{2}\sum_{\mathbf{p}\neq\mathbf{0}}\Big[2\Sigma_0\Big(a^\dagger_{\mathbf{p}}a_{\mathbf{p}}+a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}}\Big)+\Sigma_0\Big(a^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}+\text{h.c.}\Big)\Big]$, plus a fluctuation Hamiltonian $H''_{\rm int}$, which looks like (1), except that the sum contains only nonzero-momenta. Now we proceed according to the rules of VPT [18]. We introduce dummy variational parameter $\Sigma_{\mathbf{p}}$ and $\Delta_{\mathbf{p}}$ via an auxiliary Hamiltonian

$$\bar{H}_{\rm trial}=\frac{1}{2}\sum_{\mathbf{p}\neq 0}\Big[\Sigma_{\mathbf{p}}\Big(a^\dagger_{\mathbf{p}}a_{\mathbf{p}}+a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}}\Big)+\Delta_{\mathbf{p}}a_{-\mathbf{p}}a_{\mathbf{p}}+\text{h.c.}\Big],\quad (2)$$

leading a *harmonic* Hamiltonian

$$\begin{aligned} H_0' &\equiv -V\frac{\mu}{g}\Sigma_0+\frac{V}{2g}\Sigma_{\mathbf{0}}^2+\sum_{\mathbf{p}\neq\mathbf{0}}(\varepsilon_{\mathbf{p}}-\mu+2\Sigma_0)a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\\ &+\frac{1}{2}\Sigma_0\sum_{\mathbf{p}\neq\mathbf{0}}\Big(a^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}+\text{h.c.}\Big)+\bar{H}_{\rm trial}. \end{aligned}\quad (3)$$

From this we calculate the energy order by order in perturation theory considering the subtracted interaction

$$H^{\rm var}_{\rm int}=H''_{\rm int}-\bar{H}_{\rm trial}.\quad (4)$$

as a perturbation. The zeroth-order variational energy is $W_0=\langle H_0'\rangle$, and the lowest-order correction comes from the expectation value $\Delta_1 W=\langle H^{\rm var}_{\rm int}\rangle$. If the energy could be calculated to *all* orders in $H^{\rm var}_{\rm int}$, the result would be independent of the variational parameters Σ_0, $\Sigma_{\mathbf{p}}$, and $\Delta_{\mathbf{p}}$. At any *finite* order, however, the energy *will* depend on these parameters. Their best values are found by optimization (usually extremization), and the results converge exponentially fast as a function of the order [5–8].

A Bogoliubov transformation with as yet undetermined coefficients $u_{\mathbf{p}}$, $v_{\mathbf{p}}$ constrained by the condition $u^2_{\mathbf{p}}-v^2_{\mathbf{p}}=1$, produces a ground state with vacuum expectation values $\langle a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\rangle=v^2_{\mathbf{p}}$ and $\langle a_{\mathbf{p}}a_{-\mathbf{p}}\rangle=u_{\mathbf{p}}v_{\mathbf{p}}$, so that

$$\begin{aligned} W_0 &= -V\frac{\mu}{g}\Sigma_0+\frac{V}{2g}\Sigma_{\mathbf{0}}^2\\ &+\sum_{\mathbf{p}\neq\mathbf{0}}\{[\varepsilon_{\mathbf{p}}-\mu+2\Sigma_0+\Sigma_{\mathbf{p}}]\,v^2_{\mathbf{p}}+(\Sigma_0+\Delta_{\mathbf{p}})u_{\mathbf{p}}v_{\mathbf{p}}\}. \end{aligned}\quad (5)$$

The first-order variational energy W_1 contains, in addition, the expectation value $\langle H^{\rm var}_{\rm int}\rangle$. Of this, the first part, $\Delta_{(1,0)}W=\langle H''_{\rm int}\rangle$, is found immediately with the help of the standard commutation rules as a sum of three pair terms $\langle a^\dagger_{\mathbf{p+q}}a^\dagger_{\mathbf{p'-q}}a_{\mathbf{p'}}a_{\mathbf{p}}\rangle=\langle a^\dagger_{\mathbf{p+q}}a^\dagger_{\mathbf{p'-q}}\rangle\langle a_{\mathbf{p'}}a_{\mathbf{p}}\rangle+\langle a^\dagger_{\mathbf{p+q}}a_{\mathbf{p}}\rangle\langle a^\dagger_{\mathbf{p'-q}}a_{\mathbf{p'}}\rangle+\langle a^\dagger_{\mathbf{p+q}}a_{\mathbf{p'}}\rangle\langle a^\dagger_{\mathbf{p'-q}}a_{\mathbf{p}}\rangle$, which yields

$$\Delta_{(1,0)}W=\langle H''_{\rm int}\rangle=\frac{g}{2V}\sum_{\mathbf{p},\mathbf{p}'\neq\mathbf{0}}\big(2v^2_{\mathbf{p}}v^2_{\mathbf{p}'}+u_{\mathbf{p}}v_{\mathbf{p}}u_{\mathbf{p}'}v_{\mathbf{p}'}\big).\quad (6)$$

The second part $\langle -\bar{H}_{\rm trial}\rangle$ adds to this:

$$\Delta_{(1,1)}W=-\sum_{\mathbf{p}\neq\mathbf{0}}\big(\Sigma_{\mathbf{p}}v^2_{\mathbf{p}}+\Delta_{\mathbf{p}}u_{\mathbf{p}}v_{\mathbf{p}}\big).\quad (7)$$

Let us now fix the total number of particles N on the average. We differentiate $W_1\equiv W_0+\Delta_{(1,0)}W+\Delta_{(1,1)}W$ with respect to $-\mu$ and set the result equal to N, leading for the density $\rho=N/V$ to the equation $\rho=\Sigma_0/g+V^{-1}\sum_{\mathbf{p}\neq 0}v^2_{\mathbf{p}}$. The momentum sum on the right is the density of particles *outside* the condensate, the *uncondensed density* $\rho_{\mathbf{u}}=\sum_{\mathbf{p}\neq 0}\langle a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\rangle=V^{-1}\sum_{\mathbf{p}}v^2_{\mathbf{p}}$. Hence $\Sigma_0/g=\rho-\rho_{\mathbf{u}}$ is the condensate density.

Now we extremize W_1 with respect to the variational parameter Σ_0 which yields the equation $(\mu-\Sigma_0)/g=\sum_{\mathbf{p}\neq\mathbf{0}}(2v^2_{\mathbf{p}}+u_{\mathbf{p}}v_{\mathbf{p}})=2\rho_{\mathbf{u}}+\sum_{\mathbf{p}\neq\mathbf{0}}u_{\mathbf{p}}v_{\mathbf{p}}=2\rho_{\mathbf{u}}+\delta$.

We are now able to determine the size of the Bogoliubov coefficients $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$. The original way of doing this is algebraic, based on the elimination of the off-diagonal elements in the transformed Hamiltonian operator. In the framework of our variational approach it is more natural to use the equivalent procedure of *extremizing* the energy expectation W_0 with respect to $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ under the constraint $u^2_{\mathbf{p}}-v^2_{\mathbf{p}}=1$, so that $\partial u_{\mathbf{p}}/\partial v_{\mathbf{p}}=v_{\mathbf{p}}/u_{\mathbf{p}}$. Varying W_0, we obtain for each nonzero momentum the equation

$$2\big(\varepsilon_{\mathbf{p}}-\mu+2\Sigma_0+\Sigma_{\mathbf{p}}\big)v_{\mathbf{p}}+\big(\Sigma_0+\Delta_{\mathbf{p}}\big)\big(u_{\mathbf{p}}+v^2_{\mathbf{p}}/u_{\mathbf{p}}\big)=0.\quad (8)$$

In order to solve this we introduce the parameters $\bar{\Sigma}_{\mathbf{p}}\equiv-\mu+2\Sigma_0+\Sigma_{\mathbf{p}}$, $\bar{\Delta}_{\mathbf{p}}\equiv\Sigma_0+\Delta_{\mathbf{p}}$, and rewrite (8) in the simple form

$$2\big(\varepsilon_{\mathbf{p}}+\bar{\Sigma}_{\mathbf{p}}\big)v_{\mathbf{p}}+\bar{\Delta}_{\mathbf{p}}\big(u_{\mathbf{p}}+v^2_{\mathbf{p}}/u_{\mathbf{p}}\big)=0.\quad (9)$$

This is solved for all $\mathbf{p}$ by the Bogoliubov transformation coefficients

$$u^2_{\mathbf{p}}=\tfrac{1}{2}\big[1+(\varepsilon_{\mathbf{p}}+\bar{\Sigma}_{\mathbf{p}})/\mathcal{E}_{\mathbf{p}}\big]\,,\ v^2_{\mathbf{p}}=-\tfrac{1}{2}\big[1-(\varepsilon_{\mathbf{p}}+\bar{\Sigma}_{\mathbf{p}})/\mathcal{E}_{\mathbf{p}}\big]\,,(10)$$

with $u_{\mathbf{p}}v_{\mathbf{p}}=-\bar{\Delta}_{\mathbf{p}}/2\mathcal{E}_{\mathbf{p}}$, where $\mathcal{E}_{\mathbf{p}}$ are the quasiparticle energies $\mathcal{E}_{\mathbf{p}}=[(\varepsilon_{\mathbf{p}}+\bar{\Sigma}_{\mathbf{p}})^2-\bar{\Delta}^2_{\mathbf{p}}]^{1/2}$.

Here we make use of the U(1)-symmetry of the Hamiltonian to impose the Nambu-Goldstone property of the quasiparticle energies by setting $\bar{\Delta}_{\mathbf{p}=\mathbf{0}}=\bar{\Sigma}_{\mathbf{p}=\mathbf{0}}=\bar{\Sigma}$. This ensures that the zero-momentum excitations have no energy, since they reduce to symmetry transformations.

Having determined the Bogoliubov coefficients, we can calculate the momentum sums contained in $\mu-\Sigma_0$. For simplicity we shall, in lowest approximation, seek for variational parameters that are indepencent of the momenta $\mathbf{p}$, i.e., $\bar{\Sigma}_{\mathbf{p}}\equiv\bar{\Delta}_{\mathbf{p}}\equiv\bar{\Sigma}$. In higher approximations we shall eventually generalize this to $\bar{\Sigma}_{\mathbf{p}}\equiv\bar{\Sigma}+\bar{\Sigma}'\varepsilon_{\mathbf{p}}$ and $\bar{\Delta}_{\mathbf{p}}\equiv\bar{\Sigma}+\bar{\Delta}'\varepsilon_{\mathbf{p}}$. The parameter $\bar{\Sigma}'$ causes a wave function renormalization corresponding to a new effective mass $M'=M/(1+\Sigma')$, the paramter $\bar{\Delta}'$ ntroduces a gradient term in the anomalous expectations. Both maintain the renormalizability of the variational procedure.

In the lowest approximation, the uncondensed particle density is obtained using (10) and abbreviating $\bar{d}p^3\equiv d^3p/(2\pi\hbar)^3$:

$$\rho_{\mathbf{u}}=V^{-1}\sum_{\mathbf{p}}v^2_{\mathbf{p}}=\frac{1}{2}\int\bar{d}^3p[(\varepsilon_{\mathbf{p}}+\bar{\Sigma})/\mathcal{E}_{\mathbf{p}}-1].\quad (11)$$

The integral is easily done if we set $|\mathbf{p}|\equiv\hbar k_{\bar{\Sigma}}\,\kappa$ with $k_{\bar{\Sigma}}=\sqrt{2M\bar{\Sigma}}/\hbar$, so that $\rho_{\mathbf{u}}=k^3_{\bar{\Sigma}}I_{\rho_{\mathbf{u}}}/4\pi^2$, where $I_{\rho_{\mathbf{u}}}\equiv$

$\int_0^\infty d\kappa\left(\frac{\kappa^2(\kappa^2+1)}{\sqrt{(\kappa^2+1)^2-1}}-1\right)=\sqrt{2}/3$. The other important momentum sum in $\mu-\Sigma_0$ becomes, after inserting (10),

$$\delta \equiv \sum_{\mathbf{p}\neq 0}\langle a_{\mathbf{p}}a_{-\mathbf{p}}\rangle = V^{-1}\sum_{\mathbf{p}\neq 0}u_{\mathbf{p}}v_{\mathbf{p}} = -\frac{\bar\Sigma}{2}\int d\!\!{}^-{}^3p\frac{1}{\mathcal{E}_{\mathbf{p}}}. \quad (12)$$

In contrast to (11), this is a divergent quantity. However, as a consequence of the renormalizability of the theory with the harmonic bare Hamiltonian (3) and the interaction $H_{\rm int}^{\rm var}$, the divergence can be removed by renormalizing the parameters of the initial Hamiltonian. We may use dimensional regularization and apply Veltman's rule [7] to set $\int d^Dp\,p^\alpha=0$, in particular $\int d^3p\,1/\varepsilon_{\mathbf{p}}=0$. This amounts to introducing the finite renormalized quantity $1/g_R \equiv 1/g-\int d\!\!{}^-{}^3p/2\varepsilon_{\mathbf{p}}$. The constant g_R is *measurable* in two-body scattering as an s-wave *scattering length*: $g_R=4\pi\hbar^2a_s/M$. Similarly we define the subtracted finite momentum sum $\delta_R=V^{-1}\sum_{\mathbf{p}}u_{\mathbf{p}}v_{\mathbf{p}}=-(\bar\Sigma/2)\int d\!\!{}^-{}^3p\,(1/\mathcal{E}_{\mathbf{p}}-1/\varepsilon_{\mathbf{p}})$. Thus the above equation for $\mu-\Sigma_0$ becomes $(\mu-\Sigma_0)/g=2\rho_{\mathbf{u}}+\delta_R$. Inserting $\Sigma_0/g=\rho-\rho_{\mathbf{u}}$, this implies $\mu/g=\rho+\rho_{\mathbf{u}}+\delta_R$.

If we evaluate the momentum sum for δ_R in the same way as (11), it yields $\delta_R=k_{\bar\Sigma}^3I_\delta/4\pi^2$, where I_δ is given by the integral $I_\delta\equiv-\int_0^\infty d\kappa\left(\frac{\kappa^2}{\sqrt{(\kappa^2+1)^2-1}}-\frac{1}{\kappa^2}\right)=\sqrt{2}=3I_{\rho_{\mathbf{u}}}$. In terms of this, we have the relation $\bar\Sigma/g_R=\rho-3\rho_{\mathbf{u}}-\delta_R+\Sigma/g$.

Finally, we calculate the total variational energy W_1. Inserting the Bogoliubov coefficients (10) into W_0 of Eq. (5), and adding the interaction energies $\Delta_1W\equiv\Delta_{(1,0)}W+\Delta_{(1,1)}W$ of (6) and (7), we obtain $W_1=-\frac{V\mu}{g}\Sigma_0+\frac{V}{2g}\Sigma_0^2+w_0(\bar\Sigma)+\Delta_1W$, where $w_0(\bar\Sigma)$ is the momentum sum $w_0(\bar\Sigma)\equiv\frac{1}{2}\sum_{\mathbf{p}\neq\mathbf{0}}\left\{\left[\mathcal{E}_{\mathbf{p}}-\varepsilon_{\mathbf{p}}-\bar\Sigma+\bar\Sigma^2/2\varepsilon_{\mathbf{p}}\right]\right\}$, made convergent after a minimal subtraction. It is evaluated as in (11) to $w_0(\bar\Sigma)=V\bar\Sigma k_{\bar\Sigma}^3I_E/4\pi^2$, where $I_E\equiv\int_0^\infty d\kappa\kappa^2\left[\sqrt{(\kappa^2+1)^2-1}-\kappa^2-1+1/2\kappa^2\right]=8\sqrt{2}/15$. If we rename all $I/4\pi^2$'s to $\bar I$'s, the energy W_1 becomes

$$\begin{aligned}W_1=&-\frac{V}{g}\mu\Sigma_0+\frac{V}{2g}\Sigma_0^2+V\bar\Sigma k_{\bar\Sigma}^3\bar I_E \\ &+\frac{Vg}{2}k_{\bar\Sigma}^6(2\bar I_{\rho_{\mathbf{u}}}^2+\bar I_\delta{}^2)-Vk_{\bar\Sigma}^3(\Sigma\bar I_{\rho_{\mathbf{u}}}+\Delta\bar I_\delta).\end{aligned} \quad (13)$$

The expression is finite by dimensional regularization. It can be extremized with respect to $\bar\Sigma$. We insert $\Sigma/g\equiv\rho-3\rho_{\mathbf{u}}-\delta_R-\bar\Sigma/g$ and $\Delta/g\equiv\rho-\rho_{\mathbf{u}}-\bar\Sigma/g$, and vary W_1 in $\delta\bar\Sigma$ to find

$$\frac{\bar\Sigma}{g}=\rho-\rho_{\mathbf{u}}\frac{4\rho_{\mathbf{u}}'+2\delta'}{\rho_{\mathbf{u}}'+\delta'}-\delta\frac{2\rho_{\mathbf{u}}'-\delta'}{\rho_{\mathbf{u}}'+\delta'}. \quad (14)$$

Here a prime denotes the derivative with respect to $\bar\Sigma$, and we have used the relation $\partial_{\bar\Sigma}\bar\Sigma k_{\bar\Sigma}^3=5k_{\bar\Sigma}^3/2$ to equate $\partial_{\bar\Sigma}\bar\Sigma k_{\bar\Sigma}^3I_E=k_{\bar\Sigma}^3(I_{\rho_{\mathbf{u}}}+I_\delta)=\rho_{\mathbf{u}}+\delta$. Inserting $\delta_R=3\rho_{\mathbf{u}}$, Eq. (14) becomes $\bar\Sigma/g=\rho-7\rho_{\mathbf{u}}/4$.

3. To extract experimental consequences it is useful to re-express all equations in a dimensionless form by introducing the reduced variables $s\equiv\bar\Sigma/\varepsilon_a$, where $\varepsilon_a\equiv\hbar^2/2Ma^2$ is the natural energy scale of the system. We also introduce the reduced s-wave scattering length $\hat a_s\equiv 8\pi a_s/a$, in terms of which the renormalized coupling constant is $g_R=4\pi\hbar^2a_s/M=8\pi\varepsilon_aa^2a_s=\varepsilon_aa^3\hat a_s$, while $k_{\bar\Sigma}=\sqrt{s}/a$, $\bar\Sigma/g_R=s/8\pi a^2a_s=s/a^3\hat a_s$, and the second-sound velocity reads $c=\sqrt{s/2}\,v_a$, $v_a\equiv p_a/M\equiv\hbar/aM$. Finally we define the reduced quantities $\hat\rho_{\mathbf{u}}\equiv\rho_{\mathbf{u}}/\rho=s^{3/2}\bar I_{\rho_{\mathbf{u}}}$ and $\hat\delta\equiv\delta_R/\rho=s^{3/2}\bar I_\delta$, so that the relation between s and $\hat a_s$ is $s/\hat a_s=1-7s^{3/2}\bar I_{\rho_{\mathbf{u}}}$. In the strong-coupling limit $\hat a_s\to\infty$, this yields a maximal s-value

$$s^{\rm sc}=\left(\frac{1152\pi^4}{49}\right)^{1/3}\approx 13.18\ldots, \quad (15)$$

where the maximal depletion is

$$\rho_{\mathbf{u}}^{\rm sc}=4/7\approx 0.571\,. \quad (16)$$

For arbitrary $\hat a_s$, we find

$$s=\hat a_s-\frac{19}{24\sqrt{2}\pi^2}\hat a_s^{5/2}+\frac{361}{786\pi^4}\hat a_s^4+\ldots\,, \quad (17)$$

leading to

$$\frac{\rho_{\mathbf{u}}}{\rho}=1-\frac{1}{6\sqrt{2}\pi^2}\hat a_s^{3/2}+\frac{19}{192\pi^4}\hat a_s^3+\ldots\,. \quad (18)$$

These functions are plotted in Fig. 18.

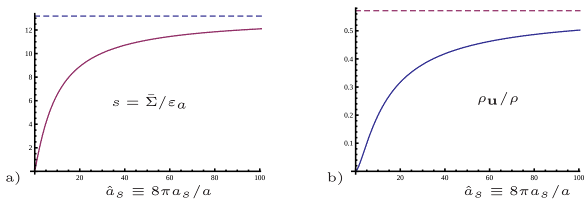

Figure 1: Reduced gap $s\equiv\bar\Sigma/\varepsilon_a$ as a function of reduced s-wave scattering length $\hat a_s=8\pi a_s/a=8\pi a_s\rho^{1/3}$. The maximal depletion value is $\rho_{\mathbf{u}}/\rho=4/7\approx 0.571$, reached in the strong-coupling limit $\hat a_s\to\infty$.

Finally we calulate the reduced variational energy $w_1\equiv W_1/N\varepsilon_a$ from Eq. (13)

$$\begin{aligned}w_1=&-\hat a_s(1+\hat\rho_{\rm u}+\hat\delta)(1-\hat\rho_{\rm u})+\frac{\hat a_s}{2}(1-\hat\rho_{\rm u})^2+\frac{s^{5/2}}{2}\bar I_E \\ &+\frac{\hat a_s}{2}(2\hat\rho_{\rm u}^2+\hat\delta^2)-\hat a_s(\sigma_\Sigma\hat\rho_{\rm u}+\sigma_\Delta\hat\delta),\end{aligned} \quad (19)$$

with $\sigma_\Sigma\equiv 1-3\hat\rho_{\mathbf{u}}-\hat\delta-s/\hat a_s$ and $\sigma_\Delta\equiv 1-\hat\rho_{\mathbf{u}}-s/\hat a_s$. Inserting $\hat\rho_{\mathbf{u}}=s^{3/2}\bar I_{\rho_{\mathbf{u}}}$ and $\hat\delta=3s^{3/2}\bar I_{\rho_{\mathbf{u}}}$, and going from the grand-canonical to the true proper energies by adding μN to W_1 forming $W^{\rm e}=W_1+\mu V\rho$, we obtain the reduced energy $w_1^{\rm e}=\frac{\hat a_s}{2}+\frac{\sqrt{2}}{3\pi^2}\hat a_ss^{3/2}-\frac{\sqrt{2}}{5\pi^2}s^{3/2}+\frac{\hat a_s}{72\pi^4}s^3$.

Inserting here (17), we find $w_1^{\rm e} = \frac{\hat a_s}{2} + \frac{\sqrt{2}}{15\pi^2}\hat a_s^{5/2} - \frac{1}{1152\sqrt{2}\pi^6}\hat a_s^{11/2} + \mathcal{O}(\hat a_s^7)$.

In Fig. 2, we compare the reduced total energy $w_1^{\rm e}$ with Bogoliubov's result $w_1^{\rm Bog} = \frac{\hat a_s}{2} + \frac{\sqrt{2}}{15\pi^2}\hat a_s^{5/2}$.

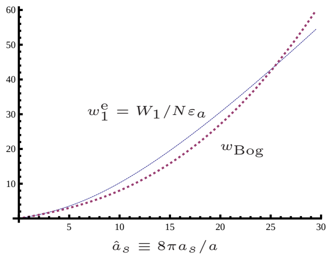

Figure 2: Reduced energy per particle $w_1^{\rm e} = W_1/N\varepsilon_a$ as a function of the reduced s-wave scattering length $w_{\rm Bog} = \hat a_s = 8\pi a_s/a = 8\pi a_s\rho^{1/3}$, compared with Bogoliubov's weak-coupling result.

4. In principle, the accuracy of our results can be increased to any desired level, with an exponentially fast convergence, as was demonstrated by the calculation of critical exponents in all Euclidean φ^4 theories with N components in D dimensions [7]. The fact that the theory is renormalizable, so that all divergencies can be removed by Veltman's rule, is an essential advantage of the present theory over any previous strong-coupling scheme.[1]

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