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Fractional quantum field theory, path integral, and stochastic differential equation for strongly interacting many-particle systems

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Abstract – While free and *weakly* interacting particles are described by a second-quantized nonlinear Schrödinger field, or relativistic versions of it, the fields of *strongly* interacting particles are governed by *effective actions*, whose quadratic terms are extremized by fractional wave equations. Their particle orbits perform universal Lévy walks rather than Gaussian random walks with perturbations.

editor's choice

Quantum-mechanical physics is explained with high accuracy by Schrödinger theory. The wave equation for many particles can conveniently be reformulated as a second-quantized *field theory*, with an action that is the sum of a quadratic and an interacting term:

$$\mathcal{A} = \mathcal{A}_2 + \mathcal{A}_{\text{int}}, \tag{1}$$

where the term $\mathcal{A}_2$ has typically the form

$$\mathcal{A}_2 = \int \mathrm{d}^D x \mathrm{d}t \psi^*(\mathbf{x},t)[i\partial_t + \hbar^2\nabla^2/2m - V(\mathbf{x})]\psi(\mathbf{x},t), \tag{2}$$

with D being the space dimension, m the mass, and $V(\mathbf{x})$ some external potential. The interaction term $\mathcal{A}_{\text{int}}$ may be approximated in molecular systems by a fourth-order term in the field

$$\frac{1}{2}\int \mathrm{d}^D x \mathrm{d}^D x' \mathrm{d}t \psi^*(\mathbf{x}',t)\psi^*(\mathbf{x},t)V_{12}(\mathbf{x},\mathbf{x}')\psi(\mathbf{x},t)\psi(\mathbf{x}',t), \tag{3}$$

where $V_{12}(\mathbf{x},\mathbf{x}')$ is some two-body potential.

If relativistic velocities are present, the field is generalized to a scalar Klein-Gordon field, or a quantized Dirac field. In molecular physics, the fourth-order term is due to the exchange of a minimally coupled quantized photon field and is proportional to e^2, where e is the electric charge. The field equations may be studied with any standard method of quantum field theory, and corrections can be derived using perturbation theory in powers of $\alpha \equiv e^2/\hbar \approx 1/137$. Since α is very small, this approach is quite successful.

If time is continued analytically to imaginary values $t = it$, one is faced with the so-called Euclidean version of quantum field theory. Then perturbation theory may be understood as developing a theory of particle physics from an expansion around Gaussian random walks. Indeed, the relativistic scalar free-particle propagator of mass m in $(D+1)$-dimensional Euclidean energy-momentum space $p^\mu = (\mathbf{p}, p_4)$, has the form

$$G(p) = \frac{1}{\mathbf{p}^2 + p_4^2 + m^2} = \int_0^\infty \mathrm{d}s\, e^{-sm^2} e^{-s(\mathbf{p}^2 + p_4^2)}, \tag{4}$$

where the energy has been continued analytically to $p_4 = -iE$. The Fourier transform of $e^{-s(\mathbf{p}^2+p_4^2)}$ is the distribution of Gaussian random walks of length s in $D+1$ Euclidean dimensions:

$$P(\mathbf{x}, x_4) = (4\pi s)^{-(D+1)/2} e^{-(|\mathbf{x}|^2 + x_4^2)/4s}, \tag{5}$$

which makes the propagator (4) a superposition of such walks with lenghts distributed like e^{-sm^2} [1–3]. This propagator is the relativistic version of the free-field propagator of the action (2). The second-quantized field

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theory described by (1) accounts for grand-canonical ensembles of orbits with their two-body interactions [4].

Gaussian random walks are a natural and rather universal starting point for many stochastic processes. For instance, they form the basis of the most important tool in the theory of financial markets, the Black-Scholes option price theory [5] (Nobel Prize 1997), by which a portfolio of assets is hoped to remain steadily growing through hedging. In fact, the famous *central-limit theorem* permits us to prove that many independent random movements of finite variance always pile up to display a Gaussian distribution [3,6].

However, since the last stock market crash and the still ongoing financial crisis it has become clear that realistic distributions belong to a more general universality class, the so-called Lévy stable distribution. They are the universal results of a pile-up of random movements of infinite variance[1]. They account for the fact that rare events, which initiate crashes, are much more frequent than in Gaussian distributions. These are events in the so-called Lévy tails $\propto 1/|x|^{1+\lambda}$ of the distributions, whose description requires a Hamiltonian [3]

$$H = \text{const}\,(p^2)^{\lambda/2}. \tag{6}$$

Such tail events are present in the self-similar distribution of matter in the universe [7–9], in velocity distributions of many-body sytems with long-range forces [10], and in the distributions of windgusts [11] and earthquakes [12], with often catastrophic consequences. They are a consequence of rather general maximal entropy assumptions [13]. In the limit $\lambda \to 2$, the Lévy distributions reduce to Gaussian distributions.

The purpose of this note is to point out, that such distributions occur quite naturally also in many-particle systems, provided the interactions are very strong [14]. They have been observed in numerous experiments at second-order phase transitions. The most accurate measurement of this type was done in a satellite (the so-called Infrared Atronomical Satellite, IRAS) by studying the singularity of the specific heat of superfluid ^{4}He near the critical temperature [15]. The observation agreed extremely well with the theoretical strong-coupling prediction [16,17].

The field of a strongly interacting N-body system is usually a multivalued function. Singularities perforate the space via vortex lines (for instance in type-II superconductors or in superfluid ^{4}He), or via line-like defects in the displacement field of a world-crystal formulation of Einstein(-Cartan) gravity [18]. If the positions of two particles are exchanged, one obtains a factor $+1$ for bosons or -1 for electrons. In two dimensions, one may even obtain a general phase $e^{i\phi}$ (anyons) [19].

A strongly interacting field system has a conformally invariant Green function [16,19,20]

$$G(\mathbf{p}, p_4) = [p_4^{1-\gamma}\phi(\mathbf{p}^2/p_4^z)]^{-1}. \tag{7}$$

If the dimension D differs only by a very small amount ϵ from the critical dimension D_c, where the theory is scale-invariant, *i.e.*, $D = D_c + \epsilon$, then γ is of order ϵ and z differs from unity by a similar amount. Such a power behavior is assured near D_c if the Gell-Mann-Low function [21] has an infrared-stable fixed point in the renormalization flow of the coupling constant. Very close to the critical dimension, a lowest approximation to $G(\mathbf{p}, p_4)$ is

$$G(\mathbf{p}, p_4) = \{p_4^{1-\gamma}[1 + D_\lambda(\mathbf{p}^2/p_4^z)^{\lambda/2}]\}^{-1}, \tag{8}$$

where λ is close to 2, and D_λ is a generalization of the diffusion constant in a Fokker-Planck equation.

Time-independent propagators involve the limit $p_4 \to 0$, where the correlation function behaves like

$$G(\mathbf{p}, 0) \propto |\mathbf{p}|^{-2+\eta}. \tag{9}$$

The index η is the *anomalous dimension* of the field, which is also of order ϵ. The existence of this limit in (8) fixes the scaling relation

$$\lambda = 2(1-\gamma)/z = 2 - \eta. \tag{10}$$

See appendix for the calculation of the exponents to order ϵ. The Green function (8) determines the particle probability distribution after a time t via the *double fractional Fokker-Planck equation*

$$[\hat{p}_4^{1-\gamma} + D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}]P(\mathbf{x}, t) = \delta(t)\delta^{(D)}(\mathbf{x}), \tag{11}$$

where $\hat{p}_4 \equiv \partial_t$, $\hat{\mathbf{p}} \equiv i\partial_{\mathbf{x}} \equiv i\boldsymbol{\nabla}$. A convenient definition of the fractional derivatives uses the same formula as in the dimensional continuation of Feynman diagrams $(-\boldsymbol{\nabla}^2)^{\lambda/2} = \Gamma[\lambda/2]^{-1}\int d\sigma\sigma^{-\lambda/2-1}e^{\lambda\boldsymbol{\nabla}^2/2}$ (see [2] and ref. [22]). The solution of (11) is given in the literature [23] and reads

$$\frac{t^{-\gamma}}{\pi^{D/2}|\mathbf{x}|^{D/2}} H^{2,1}_{2,3}\left(\frac{|\mathbf{x}|^\alpha}{2^\lambda D_\lambda t^{1-\gamma}}\middle|\begin{matrix}(1,1);(1-\gamma,1-\gamma)\\ \\(1,1),(D/2,\lambda/2);(1,\lambda/2)\end{matrix}\right), \tag{12}$$

where $H^{2,1}_{2,3}$ is a *Fox H-function* [24]. In the limits $\gamma = 0$ and $\alpha = \lambda$, this reduces to the standard quantum-mechanical Gaussian expression $(4\pi D_\lambda t)^{-D/2}e^{-|\mathbf{x}|^2/4D_\lambda t}$. For $\gamma = 0$, $\lambda = 1$, the result is

$$P(\mathbf{x}, t) = \frac{D_\lambda t}{\pi^{(D+1)/2}|\mathbf{x}|^{D+1}} H^{1,1}_{1,1}\left(\frac{D_\lambda^2 t^2}{|\mathbf{x}|^2}\middle|\begin{matrix}(1/2-D/2,1)\\ \\(0,1)\end{matrix}\right), \tag{13}$$

[1] A travelling pedestrian salesman is a Gaussian random walker, as a jetsetter he becomes a Lévy random walker.

[2] The relevant functional matrix is $\langle\mathbf{x}|(-\boldsymbol{\nabla}^2)^{\lambda/2}|\mathbf{x}'\rangle = \Gamma[-\lambda/2]^{-1}\int d\sigma\,\sigma^{-\lambda/2-1}(4\pi\sigma)^{-D/2}e^{R^2/4\sigma} = {}^{D}c_\lambda R^{-\lambda-D}$, where ${}^{D}c_\lambda = 2^\lambda\Gamma((D+\lambda)/2)/\pi^{D/2}\Gamma(-\lambda/2)$, and $R \equiv |\mathbf{x}-\mathbf{x}'|$. If λ is close to an even integer, it needs a small positive shift $\lambda \to \lambda_+ \equiv \lambda + \epsilon$ and we can replace $\epsilon R^{\epsilon-1}/2$ by $\delta(R) = S_D R^{D-1}\delta^{(D)}(\mathbf{R})$. For $A > 0$ we have $|\mathbf{x}'|^{-A} = {}^{D}c^{-1}_{\lambda_A}\langle\mathbf{x}'|(-\boldsymbol{\nabla}^2)^{\lambda_A/2}|\mathbf{0}\rangle$ with $\lambda_A \equiv A - D$, so that we find $\int d^D x'\langle\mathbf{x}|(-\boldsymbol{\nabla}^2)^{\lambda/2}|\mathbf{x}'\rangle|\mathbf{x}'|^{-A} = {}^{D}c^{-1}_{\lambda_A}\langle\mathbf{x}|(-\boldsymbol{\nabla}^2)^{(\lambda+A-D)/2}|\mathbf{0}\rangle = {}^{D}c_{\lambda+A-D}\,{}^{D}c^{-1}_{\lambda_A}|\mathbf{x}|^{-A-\lambda}$.

which is simply the Cauchy-Lorentz distribution function

$$[\Gamma(D/2+1/2)/\pi^{(D+1)/2}]D_\lambda t/[(D_\lambda t)^2+|\mathbf{x}|^2]^{D/2+1/2}.$$

The probability (11) may be calculated from the *doubly fractional canonical path integral* over fluctuating orbits $t(s), \mathbf{x}(s)$ $p_4(s), \mathbf{p}(s)$ viewed as functions of some pseudotime s (see [3])

$$\{\mathbf{x}_b t_b s_b|\mathbf{x}_a t_s s_a\} = \int \mathcal{D}\mathbf{x}\mathcal{D}t\mathcal{D}\mathbf{p}\mathcal{D}p_4 e^{-\mathcal{A}}, \qquad (14)$$

where $\mathcal{A}$ is the Euclidean action of the paths $t(s), \mathbf{x}(s)$:

$$\mathcal{A} = \int \mathrm{d}s[i(\mathbf{p}\mathbf{x}' - ip_4 t') - \mathcal{H}(\mathbf{p}, p_4)]. \qquad (15)$$

Here $t'(s) \equiv \mathrm{d}t(s)/\mathrm{d}s$, $\mathbf{x}'(s) \equiv \mathrm{d}\mathbf{x}(s)/\mathrm{d}s$, and $\mathcal{H}(\mathbf{p}, p_4) = p_4^{1-\gamma} + D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}$. At each s, the integrals over the components of $\mathbf{p}(s)$ and $p_4(s)$ run from $-\infty$ to ∞, whereas those over $p_4(s)$ run from $-i\infty$ to $i\infty$. At the end we obtain $P(\mathbf{x}, t)$ from the integral $\int_0^\infty \mathrm{d}s\{\mathbf{x}\,t\,s|\mathbf{0}\,0\,0\}$.

If $\gamma = 0$, the path integral over $p_4(s)$ yields the functional $\delta[t'(s) - 1]$, which brings (14) to the canonical path integral

$$(\mathbf{x}_b t_b|\mathbf{x}_a t_a) = \int \mathcal{D}\mathbf{x}\mathcal{D}\mathbf{p} e^{-\mathcal{A}'}, \qquad (16)$$

with

$$\mathcal{A}' = \int \mathrm{d}t[i\mathbf{p}\dot{\mathbf{x}} - D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}]. \qquad (17)$$

Now $P(\mathbf{x}, t) = (\mathbf{x}t|\mathbf{0}\,0)$ satisfies the ordinary fractional Fokker-Planck equation

$$[\hat{p}_4 + D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}]P(t, \mathbf{x}) = \delta(t)\delta^{(D)}(\mathbf{x}). \qquad (18)$$

This has been discussed at length in recent literature [25].

At this point it is worth mentioning that the probability can be written as a superposition $\int_0^\infty(\mathrm{d}\sigma/\sigma)f_\lambda(\sigma t^{-2/\lambda})P_\mathrm{G}(\sigma, \mathbf{x})$ of Gaussian distributions $P_\mathrm{G}(\sigma, \mathbf{x}) = (4\pi\sigma)^{-D/2}e^{-\mathbf{x}^2/4\sigma}$ with weight

$$f_\lambda(\sigma) = S_D \sum_{n=1}^{\infty} \frac{(-1)^n \sigma^{-n\lambda/2}}{(n+1)!\Gamma(D-1-n\lambda/2)} D_\lambda^{n/\lambda}, \qquad (19)$$

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface of a sphere in D dimensions.

If $\gamma \neq 0$, the above functional δ-function is softened, and the relation between the pseudotime s and the physical time becomes stochastic. It is governed by the probability distribution that solves the path integral

$$\{t_b s_b|t_a s_a\} = \int \mathcal{D}t\mathcal{D}p_4 \exp\left\{\int \mathrm{d}s\,[p_4 t' - p_4^{1-\gamma})]\right\}. \quad (20)$$

For imaginary $p_4 = -iE$, we define a *noise Hamiltonian* $\tilde{H}(\eta)$ which has the property that[3,4]

$$e^{-p_4^{1-\gamma}} = \int_{-\infty}^{\infty} \mathrm{d}\eta e^{-p_4\eta - \tilde{H}(\eta)}. \qquad (21)$$

The inverse of the Fourier integral yields the *noise probability* $P(\eta) = \int_{-i\infty}^{i\infty} \mathrm{d}p_4 e^{p_4\eta - p_4^{1-\gamma}}$, and a probability functional[5]

$$P[\eta] \equiv e^{-\int \mathrm{d}s\tilde{H}(\eta)} = \int \mathcal{D}p_4 \exp\left[\int \mathrm{d}s\,(p_4\eta - p_4^{1-\gamma})\right]. \qquad (22)$$

Using this we may solve the stochastic differential equation of the Langevin type

$$t'(s) = \eta(s), \qquad (23)$$

in which the noise $\eta(s)$ has a zero expectation value for each s, and the correlation functions, for $n = 2, 4, 6, \ldots$,

$$\langle\eta(s_1)\ldots\eta(s_{2n})\rangle \equiv \int \mathcal{D}\eta\,\eta(s_1)\ldots\eta(s_{2n})P[\eta]. \qquad (24)$$

If $\gamma = 0$, the solution of (22) is $P[\eta] = \delta[\eta(s) - 1]$, implying that $\eta(s)$ ceases to fluctuate, and (23) becomes $t'(s) \equiv 1$, so that $t \equiv s$.

In the past, many nontrivial Schrödinger equations (for instance that of the $\mathbf{1}/r$-potential) have been solved with path integral methods by re-formulating them on the pseudotime axis s, that is related to the time t via a *space-dependent differential equation* $t'(s) = f(x(t))$. This method, invented by Duru and Kleinert [26] to solve the path integral of the hydrogen atom, has recently been applied successfully to various Fokker-Planck equations [27,28]. The stochastic differential equation (23) may be seen as a stochastic version of the Duru-Kleinert transformation that promises to be a useful tool to study non-Markovian systems.

Certainly, the solutions of eq. (18) can also be obtained from a stochastic differential equation

$$\dot{\mathbf{x}} = \boldsymbol{\eta}, \qquad (25)$$

whose noise is distributed with a fractional probability

$$P[\boldsymbol{\eta}] = \int \mathcal{D}^D x e^{\int \mathrm{d}t(i\mathbf{p}\cdot\boldsymbol{\eta} - D_\lambda(\mathbf{p}^2)^{\lambda/2})}. \qquad (26)$$

Experimentally, a system within the strong-coupling limit can be produced by forming a Bose-Einstein condensate (BEC) in a magnetic field whose strength is tuned to a Feshbach resonance [29] of the two-particle interaction. In a BEC, the four-field term in the interaction (3) is local and parametrized by $V_{12}(\mathbf{x}, \mathbf{x}) \propto g\delta(\mathbf{x} - \mathbf{x}')$. At the Feshbach resonance, the bare coupling strength

[3]This technique is explained in Chapts. 12 and 19 of ref. [3]. The pseudotime s resembles the so-called Schwinger proper time used in relativistic physics.

[4]There should be no danger of confusing the fluctuating noise variable η in this equation with the constant critical exponent η in (9).

[5]See eq. (29.165) in ref. [3].

g goes to infinity [30], and the renormalized coupling times $12\mu^{-\epsilon}/(4\pi)^2$ converges to a fixed point $g^* \approx 0.503$. (See fig. 17.1 in ref. [16].).

The theoretical tool to describe the physics in this regime is the effective action $\Gamma[\Psi,\Psi^*]$. This a functional of the expectation values of the quantum fields $\Psi(t,\mathbf{x}) \equiv \langle\psi(t,\mathbf{x})\rangle$, a classical expression that contains all information of the full quantum theory [16,31]. It is the Legendre transform of the generating functional $Z[\eta,\eta^*] = \int \mathcal{D}\psi\mathcal{D}\psi^* e^{-\mathcal{A}-\eta^*\psi-\eta\psi^*}$ of the full quantum theory, and is extremal on the phasical field expections. All its vertex functions can be found from the functional derivatives of $\Gamma[\Psi,\Psi^*]$. In the strong-coupling limit, the effective interaction changes the interaction (3) to an anomalous power law $\Gamma^{\rm int}[\Psi,\Psi^*] = (g_c/2)\int \mathrm{d}t\mathrm{d}^D x\,|\Psi(t,\mathbf{x})|^{\delta+1}$, where $g_c = (2g^*)^{(\delta-1)/2}(4\pi)^2/24$. The power δ is a critical exponent that is measured experimentally by the relation $B = |\Psi|^\delta$. Its value is determined by η via the so-called hyperscaling relation[6] $\delta = (D+2-\eta)/(D-2+\eta)$. As a possible application we may study the behavior of a triangular lattice of vortices which form in a rotating Bose-Einstein condensate [32], and letting the magnetic field approach a Feshbach resonance.

The results may then be compared with a calculation based on a new field equation that generalizes the famous Gross-Pitaevskii equation [33]

$$\left[\hat{E} - \frac{1}{2m}\hat{\mathbf{p}}^2 + g|\Psi(t,\mathbf{x})|^2\right]\Psi(t,\mathbf{x}) = 0. \qquad (27)$$

The new equation is obtained by extremizing the effective action $\Gamma[\Psi,\Psi^*] = \Gamma_0[\Psi,\Psi^*] + \Gamma^{\rm int}[\Psi,\Psi^*]$, where

$$\Gamma_0 \equiv \int \mathrm{d}t\mathrm{d}^D x\,\Psi^\dagger(t,\mathbf{x})[\hat{E}^{1-\gamma} - D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}]\Psi(t,\mathbf{x}). \qquad (28)$$

By forming $\delta\mathcal{A}^{\rm eff}/\delta\Psi^\dagger(t,\mathbf{x})$, we obtain what may be called the *fractional Gross-Pitaevskii equation*:

$$\left[\hat{E}^{1-\gamma} - D_\lambda(\hat{\mathbf{p}}^2)^{1-\eta/2} - \frac{\delta+1}{4}g_c|\Psi(t,\mathbf{x})|^{\delta-1}\right]\Psi(t,\mathbf{x}) = 0. \qquad (29)$$

The fractional Schrödinger equation has many problems, such as the nonvalidity of the quantum superposition law, the violation of unitarity of the time evolution, and the violation of probability conservation which can produce nonsensical probabilities >1. However, these problems exist only if we restrict ourselves only to the free effective action (28), and this is meaningless, since the entire theory is only defined by the effective action in the strong-coupling limit —and this contains necessarily additional nonquadratic terms. Hence it does not possess free quasiparticles as in the time-honored Landau theory of Fermi liquids [34]. There is always an interaction that invalidates the standard discussion of Schrödinger equations. In fact, the theory of high-T_c superconductivity must probably be built as a true strong-coupling theory of this type with electrons being non-Fermi liquids.

[6]See eq. (1.35) in ref. [16].

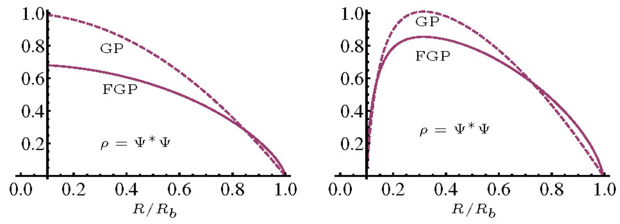

Fig. 1: (Colour on-line) Condensate density from Gross-Pitaevskii equation (27) (GP, dashed line) and its fractional version (29) (FGP), both in Thomas-Fermi approximation where the gradients are ignored. The FGP-curve shows a marked depletion of the condensate. On the right hand, a vortex is included. The zeros at $R/R_b \approx 1$ will be smoothened by the gradient terms in (32).

The relativistic version of the entire discussion is simpler since it is based on the Euclidean Green function (9) in which $\mathbf{p}$ denotes the $(D-1)$-dimensional vectors $(\mathbf{p}, p_4)$. The Fourier transform is the distribution fulfilling the Fokker-Planck equation

$$[\partial_s + (\hat{\mathbf{p}}^2)^{1-\eta/2}]P(s,\hat{\mathbf{x}}) = \delta(s)\delta^{(D+1)}(\mathbf{x}) \qquad (30)$$

and possessing the path integral representation

$$P(s,\hat{\mathbf{x}}) = \int \mathcal{D}\mathbf{x}\mathcal{D}\mathbf{p}\,e^{\int \mathrm{d}s[i\mathbf{p}\dot{\mathbf{x}} - (\hat{\mathbf{p}}^2)^{1-\eta/2}]}. \qquad (31)$$

The ϵ-expansion is now around $D_c = 4$ in powers of $\epsilon = -(D-D_c)$. The critical exponent η is small of order ϵ^2: $\eta = \epsilon^2/50 + \cdots \approx 0.04$. It can be ignored for $\epsilon = 1$. The power δ in the interaction is $3 + \epsilon + 23\epsilon^2/50 + \cdots \approx 4.76$ (see [7]).

The time-independent *fractional Gross-Pitaevskii equation* reads now

$$\left[(\hat{\mathbf{p}}^2)^{1-\eta/2} + \frac{\delta+1}{4}g_c|\Psi(\mathbf{x})|^{\delta-1}\right]\Psi(\mathbf{x}) = 0, \qquad (32)$$

with $g_c \approx 6.7$. For a $d = (D-1)$-dimensional vortex in $D = 3$ dimensions, it is solved by $\Psi(\mathbf{x}) = a|\mathbf{x}_\perp|^{-A}$ with $A = (2-\eta)/(\delta-1) = D/2 - 1 + \eta/2 \approx 1/2$ and $(\delta+1)a^{\delta-1}/2g_c = -{}^d c_{\lambda+A-d}\,{}^d c_{A-d}^{-1} \approx 0.2$, $\lambda = 2-\eta$ (see footnote [2]).

To compare our theory with experimental data, we must study the BEC in the scale-invariant strong-coupling limit. This is reached either by going to the temperature T_c of the second-order phase transition, or by raising the magnetic field B towards the field strength B_c of a Feshbach resonance. Then the coherence length ξ grows like $\xi \propto |T_c - T|^{-\nu}$, where $\nu \approx 2/3$ [16,35], or like $\xi \propto (B - B_c)^{-\nu}$ [29]). If the BEC is enclosed in a weak harmonic trap, this adds in the brackets of (27) a term $\propto |\mathbf{x}|^2 = R^2$. This is normally observed by the condensate density going

[7]The decimal numbers are from seven-loop calculation in $D = 3$ dimensions in table 20.2 of ref. [16].

to zero linearly like $R_b^2 - R^2$ near the border R_b (in the Thomas-Fermi approximation) [36]. For B near B_c (or T near T_c), however, the anomalous power δ will lead to the steeper approach to zero $(R_b^2 - R^2)^{2\nu(2-\eta)/(\delta-1)} \approx (R_b^2 - R^2)^{0.7}$. In addition, the central region is depleted (see fig. 1).

Moreover, the resonance frequency of a forced collective oscillation will depend on the field strength B near the Feshbach resonance [37].

A finite mass will enter the brackets of eq. (32) in the form $(\hat{m}^2)^{\nu(2-\eta)} f(|\Psi|^2/(\hat{m}^2)^{2\beta})$, with a Taylor expansion of $f(x)$, where μ is a mass scale and $\hat{m}^2$ a reduced mass (in a trap $\hat{m}^2 \propto R_b^2 - R^2$). For small $\hat{m}^2$, the Taylor expansion can be resummed to a Widom-type expression $[(\delta+1)g_c/4]|\Psi|^{\delta-1} w(\hat{m}^2/|\Psi|^{1/\beta})$ [17]. This explains the above-stated steeper density profiles in fig. 1. The function w can be expanded in powers of $(\hat{m}^2)^{\omega/2\nu} \propto \xi^{-\omega}$ which contain the Wegner critical exponent $\omega \approx 0.8$ governing the *approach to scaling*[8]. Thereby the kinetic term $(\hat{\mathbf{p}}^2)^{1-\eta/2}$ in (32) (and of course (29)) is modified to $(\hat{\mathbf{p}}^2)^{1-\eta/2}[1 + \text{const} \times \xi^{-\omega}(\hat{\mathbf{p}}^2)^{-\omega/2} + \ldots]$, and the interaction term $|\Psi|^{\delta-1}$ to $|\Psi|^{\delta-1}(1 + \text{const} \times \xi^{-\omega}|\Psi|^{-2\omega/(D-2+\eta)})$ (see [9]).

Summarizing we have seen that a many-body system with strong couplings between the constituents satisfies a more general form of the Schrödinger equation, in which the momentum and the energy appear with a power different from $\alpha = 2$ and $\gamma = 0$, respectively. The associated Green function can be written as a path integral over fluctuating time and space orbits that are functions of some pseudotime s. This is a Markovian object, but non-Markovian in the physical time t that is related to s by a stochastic differential equation of the Langevin type. The particle distributions can also be obtained by solving a Langevin type of equation in which the noise has correlation functions whose probability distribution is specified.

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Appendix

The lowest-order critical exponents can be extracted directly from the one-loop-corrected inverse Green function $G^{-1}(E, \mathbf{p})$ in $D = 2 + \epsilon$ dimensions after a minimal subtraction of the $(1/\epsilon)$-pole at [38]:

$$E - \mathbf{p}^2 + a\left(\tfrac{1}{3}\mathbf{p}^2 - E\right)^{D-1} \qquad \text{(A.1)}$$

For $\mathbf{p} = 0$, this has a power $-(-E)^{1-a\epsilon}$, so that $\gamma = a\epsilon$. For $E = 0$, on the other hand, we obtain $(-\mathbf{p}^2)^{1-a\epsilon/3}$, so that $(1-\gamma)/z - 1 \approx \gamma/3$.

[8] See sect. 10.8 in ref. [16], *viz.* eq. (10.151). Also compare (1.28) and expand $f(r/\xi) = 1 + c(r/\xi)^{\omega} + \ldots$.

[9] See eq. (10.191) in ref. [16] and expand $f(t/M^{1/\beta}) \sim \tilde{f}(\xi\Phi^{2/(D-2+\eta)})$ like $\tilde{f}(x) = 1 + cx^{-\omega} + \ldots$.

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