ALGEBRAIZATION OF REGGE COUPLINGS

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In any finite saturation scheme of the SU(2)×SU(2) algebra at infinite momentum the mass^2 operator is known to consist of an SU(2)×SU(2) singlet and a fourth component of a $(\frac{11}{22})$ representation (m_3^2, m_4^2). In this note we show that m_4^2 and $[m_a^2, m_b^2]$ represent directly the Regge couplings of f and ρ trajectories in the scattering amplitudes $\pi_a \alpha \to \pi_b \beta$ at $t = 0$.

According to the ideas of the current algebra, PCAC, and the narrow resonance approximation, the collinear pionic matrix elements of hadron resonances at $q^2 = (p'-p)^2 = 0$ *

$$\langle p'\lambda\beta | j_a(0) | p\lambda\alpha \rangle \equiv \frac{i}{F_\pi}(m_\beta^2 - m_\alpha^2)[\chi_a(\lambda)]_{\beta\alpha} \quad (1)$$

satisfy, together with the isospin matrices T_c, the algebra of SU(2)×SU(2) [1,2]

$$[X_b(\lambda), X_a(\lambda)] = i\epsilon_{bac} T_c . \quad (2)$$

The standard prood proceeds by inserting the low-energy theorem for the isospin odd amplitude of the process $\beta\pi_b \to \alpha\pi_a$ continued to zero-mass pions **

$$\frac{1}{\nu} T^{(-)ba}_{\beta\alpha} \Bigg|_{\substack{t = 0 \\ \nu = (m_\alpha^2 - m_\beta^2)/2}} = -\frac{i}{F_\pi^2} \epsilon^{bac}[T_c]_{\beta\alpha} \quad (3)$$

is an unsubtracted dispersion relation at $t = 0$ with the imaginary part

$$\text{Im } T^{(\pm)ba}_{\beta\alpha} = -\pi \sum_\gamma [\delta(\nu - \nu_\gamma^{\beta\alpha}) \mp \delta(\nu + \nu_\gamma^{\beta\alpha})] \rho^{(\pm)ba}_{\substack{\beta\alpha \\ \gamma}}(\lambda) \quad (4)$$

where from the definition (1)

$$\rho^{(\pm)ba}_{\substack{\beta\alpha \\ \gamma}}(\lambda) \equiv \quad (5)$$

$$-\frac{1}{2F_\pi^2}(m_\beta^2 - m_\gamma^2)(m_\gamma^2 - m_\alpha^2)[X_{b\beta\gamma}(\lambda), X_{a\gamma\alpha}(\lambda)]_\pm .$$

* Conventions: $\langle p' | p \rangle = 2p_0(2\pi)^3 \delta^3(p'-p)$,
$s = 1 - i(2\pi)^4 \delta^4(p_f - p_i) T$, $\nu \equiv (s - u)/2$,
$\nu_{\gamma}^{\beta\alpha} = m_\gamma^2 - (m_\beta^2 + m_\alpha^2)/2$, $F_\pi \approx 0.095$ GeV.
** By using an interpolating field $\pi(x) \equiv \partial A(x)/F_\pi \mu_\pi^2$.

The unsubtractedness is suggested by the ρ trajectory governing the high energy behaviour of $(1/\nu)T^{(-)}$ as $\nu^{\alpha\rho(0)-1} \sim \nu^{-0.5}$.

In solving eq. (2) one usually profits from the fast convergence of the dispersion integrals. Therefore one truncates the sum at some mass^2 $< N$ and solves for a finite number of resonances only. This procedure is called single particle saturation. It is clear that the resulting couplings are the better the more both initial and final particles lie below N.

Given such a saturation scheme it is possible to find a mass formula by making use of the superconvergence of the exotic $I_t = 2$ part of the $T^{(+)}$ amplitude. One finds that the mass operator can be written as the sum of an SU(2)×SU(2) invariant m_0^2 and a fourth component m_4^2 of a $(\frac{11}{22})$ representation (m_4^2, m_a^2) defined by [2]

$$m_a^2(\lambda) \equiv -i[X_a(\lambda), m^2] \quad (6)$$

$$m_4^2(\lambda)\delta_{ba} \equiv i[X_b(\lambda), m_a^2(\lambda)] = [X_b(\lambda), [X_a(\lambda), m^2]]. \quad (7)$$

It was recognized in the beginning [2] that there is a connection between m_4^2 and Regge exchange in the $T^{(+)}$ amplitude. It is the purpose of this note to point out that m_4^2 determines directly the contribution of Reggeized f exchange in a finite energy sum rule for $\beta\pi_b \leftarrow \alpha\pi_a$ ‡. With the same techniques we show, in addition, that the commutator $[m_a^2, m_b^2]$ describes the couplings of the ρ trajectory in $T^{(-)}$.

‡ In accordance with the standard ideas of duality we shall talk only about the non-diffractive part of $T^{(+)}$ which is dominated by resonances in the low energy region while the f trajectory governs its asymptotic behaviour.

Let $T^{(\pm)}(\nu,\, t=0)$ be decomposed into $t^{(\pm)}(\nu) \pm t^{(\pm)}(-\nu)$ where $t^{(\pm)}(\nu)$ has only the right-hand cut. Suppose that in the region where our saturation scheme is truncated, $t^{(+)}(\nu)$ can well be approximated by a Regge formula

$$t^{(+)}(\nu) \approx -c(0)\, \frac{\exp\{-i\pi\alpha_f(0)\}}{\sin\pi\alpha_f(0)} \left(\frac{\nu}{M^2}\right)^{\alpha_f(0)} . \qquad (8)$$

Then we insert the slight modification of $t^{(+)}(\nu)$,

$$\tilde{t}^{(+)ba}_{\beta\alpha}(\nu) \equiv \frac{2\nu}{\nu_1^2 - \nu^2}\, t^{(+)ba}_{\beta\alpha}(\nu); \quad \nu_1 \equiv \frac{m_\beta^2 - m_\alpha^2}{2}, \qquad (9)$$

in the Cauchy formula and obtain

$$0 = \frac{1}{2\pi i}\oint d\nu\, \tilde{t}^{(+)ba}_{\beta\alpha}(\nu) \approx T^{(+)ba}_{\beta\alpha}(\nu_1) - \qquad (10)$$

$$-\sum_\gamma \frac{2\nu_\gamma^{\beta\alpha}}{\nu_1^2 - \nu_\gamma^{\beta\alpha 2}}\, \rho^{(+)ba}_{\beta\alpha}(\lambda) + \frac{2}{\pi} c^{ba}_{\beta\alpha}(0)\, \frac{1}{\alpha_f(0)}\left(\frac{N}{M^2}\right)^{\alpha_f(0)}$$

where up to $\nu = N$ we have used the saturation scheme and above N the Regge formula. The first term vanishes in the limit of chiral symmetry[*]. From the other two parts we find, using eq. (5), immediately the desired result:

$$\frac{1}{F_\pi^2}[m_4^2]_{\beta\alpha}\,\delta^{ba} \approx \frac{2}{\pi} c^{ba}_{\beta\alpha}(0)\, \frac{1}{\alpha_f(0)}\left(\frac{N}{M^2}\right)^{\alpha_f(0)}. \qquad (11)$$

Experimentally, $\alpha_f(0)$ lies between $\frac{1}{2}$ and 1. Therefore m_4^2 increases indefinitely if one lets the truncation energy of the saturation scheme go to infinity[**]. Notice that if the f trajectory has an intercept $\alpha_f(0) < 0$, m_4^2 would vanish[***] and the masses would be $SU(2) \times SU(2)$ invariant.

In just the same way we can insert the isospin odd amplitude $t^{(-)ba}_{\beta\alpha}(\nu)$ in Cauchy's formula and obtain

$$-\frac{1}{2F_\pi^2}[m_b^2,\, m_a^2]_{\beta\alpha} \approx$$

$$\approx \frac{M^2}{\pi} c^{(-)ba}_{\beta\alpha}(0)\, \frac{1}{\alpha_\rho(0)+1}\left(\frac{N}{M^2}\right)^{\alpha_\rho(0)+1} . \qquad (12)$$

Let us calculate the predictions of the simplest saturation scheme containing π, A_1, σ and ρ [2]. In this scheme we find at $t = 0$

$$[m_4^2]_{\pi\beta\pi\alpha} = -m_\rho^2\,\delta_{\beta\alpha};$$

$$[m_b^2,\, m_a^2]_{\pi_p\pi\alpha} = i\epsilon_{bac}(-i\epsilon_{c\beta\alpha})\, m_\rho^4. \qquad (13)$$

[*] We shall disregard, for convenience, the small breaking of chiral symmetry due to the Σ term $\Sigma(x) \equiv i[Q_a(x_0),\, \partial A_a(x)] \neq 0$.
[**] Certainly also m_0^2 diverges, since $m^2 = m_0^2 + m_4^2$ gives the finite masses of the particles.
[***] If chiral symmetry is not exact, one finds, in this case, $[m_4^2(\lambda)]_{\beta\alpha} = \langle p\lambda\beta \mid \Sigma \mid p\lambda\alpha\rangle \approx 0$.

Hence, eq. (11) gives

$$-\frac{m_\rho^2}{F_\pi^2} \approx \frac{2}{\pi}\, c^{(+)}(0)\, \frac{1}{\alpha_f(0)}\left(\frac{N}{M^2}\right)^{\alpha_f(0)} \qquad (14)$$

while eq. (12) becomes

$$-\frac{m_\rho^4}{2F_\pi^2} \approx \frac{M^2}{\pi}\, c^{(-)}(0)\, \frac{1}{\alpha_\rho(0)+1}\left(\frac{N}{M^2}\right)^{\alpha_\rho(0)+1} . \qquad (15)$$

As in standard finite energy sum rules, the value to be taken for N should lie somewhat above $m_{A_1}^2 \approx 2m_\rho^2$, say $3m_\rho^2$. The sensitivity of the results on the exact value of N will certainly decrease for larger saturation schemes.

It is interesting to compare our predicted values of $c^{(\pm)}$ with those of the Veneziano model[†]. There we find

$$T^{(+)ba}_{\beta\alpha} = \delta_{\beta\alpha}\delta^{ba}[\tfrac{1}{2}(A(s,t)+A(t,u)) +$$

$$-\tfrac{1}{6}A(s,u)] + I_t = 2 \text{ part} \qquad (16)$$

with

$$\operatorname{Im} t^{(+)ba}_{\beta\alpha} \approx \delta_{\beta\alpha}\delta^{ba}\left[-\frac{m_\rho^2}{F_\pi^2}\,\frac{1}{\Gamma(\alpha(t))}\left(\frac{\nu}{2m_\rho^2}\right)^{\alpha(t)}\right] \qquad (17)$$

such that

$$c^{(+)}_{\text{Ven}}(0) = -\frac{m_\rho^2}{F_\pi^2}\,\frac{1}{\sqrt{\pi}} \approx -0.56\,\frac{m_\rho^2}{F_\pi^2}. \qquad (18)$$

Our first relation (14), on the other hand, amounts to

$$c^{(+)}_{\text{Alg}}(0) \approx -\frac{\pi}{4}\,(\tfrac{2}{3})^{1/2}\,\frac{m_\rho^2}{F_\pi^2} \approx -0.64\,\frac{m_\rho^2}{F_\pi^2}. \qquad (19)$$

For $T^{(-)}$ the Veneziano model gives

$$T^{(-)ba}_{\beta\alpha} = (i\epsilon_{bac})(-i\epsilon_{\beta\alpha})\,\tfrac{1}{2}[A(s,t)-A(u,t)] \qquad (20)$$

with

$$\operatorname{Im} t^{(-)ba}_{\beta\alpha} \approx$$

$$(i\epsilon_{bac})(-i\epsilon_{\beta\alpha})\left[-\frac{m_\rho^2}{F_\pi^2}\,\frac{1}{\Gamma(\alpha(t))}\left(\frac{\nu}{2m_\rho^2}\right)^{\alpha(t)}\right] \qquad (21)$$

such that we obtain for $c^{(-)}_{\text{Ven}}$ the same value as for $c^{(+)}_{\text{Ven}}$:

[†] With the standard form

$$A(s,t) = \frac{1}{\pi}\,\frac{2m_\rho^2}{F_\pi^2}\,\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))};$$

$$\alpha(s) = \tfrac{1}{2} + s/2m_\rho^2.$$

$$c_{Ven}^{(-)}(0) = -\frac{m_\rho^2}{F_\pi^2}\frac{1}{\sqrt{\pi}} = c_{Ven}^{(+)}. \tag{22}$$

This is to be compared with our prediction given by eq. (15) :

$$c_{Alg}^{(-)}(0) \approx -\frac{3\pi}{8}(\tfrac{2}{3})3/2\frac{m_\rho^2}{F_\pi^2} = c_{Alg}^{(+)}. \tag{23}$$

We see that both results are in reasonable agreement with the Regge parameters of the Veneziano model. Notice that as a consequence of the absence of exotics in s and t channel our algebraic scheme automatically exhibits exchange degeneracy.

Certainly, using larger saturation schemes will lead to better results both by yielding more reliable pionic couplings $[X_a(\lambda)]_{\beta\alpha}$ by improving on the quality of the Regge approximation of the high energy behaviour of the amplitudes.

The strength of the present scheme is the great *predictive power* for Regge couplings of large sets of resonances. In particular, if there is really some connection between f and the Pomeron trajectory [3]*, one may obtain an answer to the intriguing question of which processes will show diffraction and which will not [4]. In a forthcoming work we shall use the largest available representations [5] of eqs. (2) for meson and baryon resonances to calculate these predictions.

As in the case of usual finite energy sum rules one can consider also higher moments of the amplitude $f^{(\pm)}$ leading to further algebraic rules for Regge couplings. However, the reliability of such rules will decrease rapidly, with increasing order, for any given finite saturation scheme.

Further, we would like to mention that for helicity flip amplitudes the Regge exchange can be related to commutators like

$$[X_b, [X_a, mI_+]] .$$

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* It may, in fact, be that m_0^2 describes directly the coupling of the Pomeron trajectory. Since $m^2 = m_0^2 + m_4^2$ is finite for $N \to \infty$ this would imply $\alpha_f(0) = 1$.

This, in turn, can serve to predict the helicity properties of f (and possibly Pomeron) and ρ trajectories which have been the subject of much recent discussions [6].

Finally, we want to remark that any other good-bad, etc., commutators allow for a similar evaluation in the infinite momentum frame as long as a corresponding Regge term is added to the naive sum rule.

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