Conformal Gravity with Fluctuation-Induced Einstein Behavior at Long Distances

Hagen Kleinert*

Institut für Theoretische Physik, Freie Universität Berlin, 14195 Berlin, Germany
ICRANeT Piazzale della Repubblica, 10 -65122, Pescara, Italy

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Abstract: Conformal Gravity is renormalizable and has strong fluctuations capable of generating spontaneously an Einstein term in the action, as a kind of "dimensionally transmuted coupling constant". We show that this may produce the correct long-range behavior of gravitational forces.

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Conformal Gravity has recently become a fashionable object of study since it appears to be a possible alternative to standard Einstein gravity [1]. It is a pure metric theory that possesses general coordinate invariance and satisfies the equivalence principle of standard gravity, while augmenting it with the additional symmetry of invariance under local conformal transformations on the metric $g_{\mu\nu}(x) \to e^{2\alpha(x)} g_{\mu\nu}(x)$, where $\alpha(x)$ is an arbitrary local function. The action reads (see e.g. [2])

$$\mathcal{A}_{\rm conf} = \frac{1}{8\alpha} \int d^4x \, (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \tag{1}$$

where $C_{\lambda\mu\nu\kappa}$ is the conformal *Weyl tensor*:

$$\begin{aligned} C_{\lambda\mu\nu\kappa} \; &= \; R_{\lambda\mu\nu\kappa} - \frac{1}{2} \left(g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right) \\ &+ \; \frac{1}{6} R \left(g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right), \end{aligned} \tag{2}$$

with $R_{\lambda\mu\nu\kappa}$ being the *Riemann curvature tensor*, $R_{\lambda\nu} = R_{\lambda\mu\nu\mu}$ the *Ricci tensor*, and $R \equiv R_\mu{}^\mu$ the *scalar curvature* [3]. Equivalenty we have

$$\mathcal{A}_{\rm conf} = \frac{1}{4\alpha} \int d^4x \, (-g)^{1/2} \left[R_{\mu\kappa} R^{\mu\kappa} - \frac{1}{3} R^2 \right]. \tag{3}$$

* h.k@fu-berlin.de

The quantum theory is defined by a generating functional

$$Z = \oint \mathcal{D} g_{\mu\nu} e^{i\mathcal{A}_{\rm conf}}, \tag{4}$$

where the integral symbol $\oint$ includes a Fadeev-Popov determinant that cancels the superfluous integrations over gauge degrees of freedom.

It has recently been shown [4], that such an action arises in the spirit of Sakharov [5] from the fluctuations of the conformal factor in the partition function involving a sum of matter field actions of spin s in a D-dimensional Riemann spacetime with metric $g_{\mu\nu}$. The coupling parameter was calculated to be $-1/4\alpha = 1/[8\pi^2(4-D)] \times [(1+N_0)/120 + N_{1/2}/40 + N_1/10 - 233N_{3/2}/720 + 53N_2/45]$, where N_s is the number of fields of spin s. If we write $R_{\mu\nu\lambda}{}^{\kappa}$ as a covariant curl of the 4×4 matrix formed from the matrix of Christoffel symbols $\{\Gamma_\nu\}_\lambda{}^\kappa \equiv \Gamma_{\nu\lambda}{}^\kappa$:

$$R_{\mu\nu\lambda}{}^{\kappa} = \{\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu - [\Gamma_\mu, \Gamma_\nu]\}_\lambda{}^\kappa, \tag{5}$$

the action is seen to have a form that is completely analogous to the SU(3)-invariant non-abelian gauge theory QCD of strong interactions. This similarity is part of the esthetical appeal of the actions (1), (3).

As in QCD, the coupling constant α is dimensionless, and this makes the theory renormalizable, thus becoming an attractive candidate for a quantum theory of gravitation.

Adding to (3) a source term $-\int d^4x\,(-g)^{1/2}\delta g_{\mu\nu}T^{\mu\nu}$, variation with respect to the metric yields the field equation [1]

$$\frac{1}{2\alpha} B^{\mu\nu} = T^{\mu\nu}, \tag{6}$$

where $B^{\mu\nu}$ is defined by $2(-g)^{-1/2}\delta\mathcal{A}_{\rm conf}/\delta g_{\mu\nu} \equiv B_{\mu\nu}/2\alpha$. Functional differentiation of (1) yields the covariantly conserved and traceless *Bach tensor*:

$$B^{\mu\nu} \equiv 2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa}R_{\lambda\kappa}, \tag{7}$$

whose explicit form is

$$\begin{aligned} B^{\mu\nu} =& \frac{1}{2} g^{\mu\nu} R^{;\kappa}{}_{;\kappa} + R^{\mu\nu;\kappa}{}_{;\kappa} - R^{\mu\kappa;\nu}{}_{;\kappa} - R^{\nu\kappa;\mu}{}_{;\kappa} - 2R^{\mu\kappa}R^{\nu}{}_{\kappa} \\ &+ \frac{1}{2} g^{\mu\nu} R_{\lambda\kappa} R^{\lambda\kappa} - \frac{2}{3} g^{\mu\nu} R^{;\kappa}{}_{;\kappa} + \frac{2}{3} R^{;\mu;\nu} + \frac{2}{3} R R^{\mu\nu} - \frac{1}{6} g^{\mu\nu} R^2. \end{aligned} \tag{8}$$

The purpose of this note is to point out a phenomenon that has been observed a long time ago in the context of biomembranes [6], later in string theories with extrinsic curvature [7, 8], and after that in a gravity-like theory [9], that the large number of derivative in the free graviton propagator make fluctuations so violent that the theory creates spontaneously a new mass term. In the case of biomembranes and stiff strings this was a tension, here it is an Einstein-Hilbert action

$$\mathcal{A}_{\rm EH} = \frac{1}{2\kappa} \int d^4x\,(-g)^{1/2} R\,. \tag{9}$$

In this, the constant κ is proportional to *Newton's gravitational constant*

$$G_{\rm N} = 6.672 \times 10^{-8}{\rm cm}^3 g^{-1}{\rm sec}^{-2}, \tag{10}$$

in the combination

$$\kappa = \frac{8\pi G_{\rm N}}{c^3}. \tag{11}$$

It specifies the attractive force between two planets of masses M and M' at a distance r:

$$F = -G_{\rm N}MM'/r^2.$$

Instead of κ one may also use the so-called *Planck mass*

$$M_P \equiv \sqrt{\hbar c/G_{\rm N}} = 2.1737 \times 10^{-5}g = 1.22 \times 10^{19}{\rm GeV}, \tag{12}$$

or the *Planck length*

$$l_P \equiv \hbar/M_P c = 1.616 \times 10^{-33}{\rm cm}, \tag{13}$$

to express the prefactor $1/2\kappa$ as

$$\frac{1}{2\kappa} = \frac{M_P^2}{16\pi}\frac{c^2}{\hbar} = \frac{\hbar}{16\pi l_P^2}.$$

Let us see how the spontaneous generation of the Einstein action comes about. First we observe that the second term in the action (3) is the scale-invariant expression

$$\mathcal{A}_{\rm si} = -\frac{1}{4\alpha}\int d^4x\,(-g)^{1/2}\frac{1}{3}(R^\alpha{}_\alpha)^2. \tag{14}$$

Now we observe that we can set up an alternative scale-invariant action with the help of an auxiliary field $\lambda(x)$ as

$$\mathcal{A}_{\rm si'} = \int d^4x\,(-g)^{1/2}\left(-\frac{1}{2\kappa}\lambda R + 3\frac{\alpha\lambda^2}{4\kappa^2}\right). \tag{15}$$

In fact, integrating out λ in the generating functional $Z_{\rm si'} = \oint \mathcal{D}g_{\mu\nu}\mathcal{D}\lambda e^{i\mathcal{A}_{\rm si'}}$ leads back to the initial scale-invariant theory $Z_{\rm si} = \oint \mathcal{D}g_{\mu\nu}e^{i\mathcal{A}_{\rm si}}$. The field can be separated into an average background field $\bar\lambda$ plus fluctuations $\delta\lambda$ which have only nonzero momenta. The fluctuations are necessary to make the theory with $Z_{\rm si'}$ completely equivalent to that with $Z_{\rm si}$. But a useful approximation to be applied later will be based by neglecting all terms involving $\delta\lambda$, and taking the saddle-point approximation to the remaining integral over $\bar\lambda$.

Next we introduce an *arbitrary* mixing angle θ and with it a third version of the same action $\mathcal{A}_{\rm si''} \equiv C^2\mathcal{A}_{\rm si} - S^2\mathcal{A}_{\rm si'}$ where $C \equiv \cosh\theta$, $S \equiv \sinh\theta$. After replacing λ by λ/C^2, we may write

$$\begin{aligned}\mathcal{A}_{\rm si''} = &-\frac{C^2}{12\alpha}\int d^4x\,(-g)^{1/2}R^2 - \frac{1}{2\kappa}\int d^4x\,(-g)^{1/2}\lambda R\\ &- \frac{3\alpha}{4S^2\kappa^2}\int d^4x\,(-g)^{1/2}\lambda^2.\end{aligned} \tag{16}$$

Hence the second term in the action (3) can be replaced by the completely equivalent action (16), for any choice of the mixing angle θ. This will change the field equation (6) to

$$\frac{1}{2\alpha}B^{\mu\nu}+\frac{\lambda}{\kappa}G^{\mu\nu}+\frac{3\alpha}{4S^2\kappa^2}\lambda^2 g^{\mu\nu}=T^{\mu\nu}. \tag{17}$$

where the Bach tensor has the form (8), except that the last four terms containing factor $1/3$ are replaced by $C^2/3$ [10].

Let us now see that the fluctuations of the metric can make the average $\bar{\lambda}$ a finite quantity, which by a suitable choice of the parameters can be made equal to unity. As a result, the large-distance forces of gravity become Einsteinian (i.e. Newtonian).

Our starting point is the weak-field expansion of the action arising from an expansion of the metric $g_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}$. Setting $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, we may write the curvature tensor as

$$R_{\mu\nu\lambda\kappa}\approx\frac{1}{2}\left[\partial_\mu\partial^\lambda h_{\nu\kappa}-\partial_\nu\partial_\kappa h_\mu{}^\lambda-(\mu\leftrightarrow\nu)\right]+\dots\ , \tag{18}$$

implying for the *Ricci tensor*

$$R_\mu{}^\kappa\equiv g^{\nu\lambda}R_{\nu\mu\lambda}{}^\kappa \tag{19}$$

the small-$h_{\mu\nu}$ expansion

$$R_{\mu\kappa}\approx-\frac{1}{2}(\partial_\mu\partial_\lambda h_{\lambda\kappa}+\partial_\kappa\partial_\lambda h_{\lambda\mu}-\partial_\mu\partial_\kappa h-\partial^2 h_{\mu\kappa})+\dots\ ,$$

and for the *scalar curvature*

$$R\equiv R_\mu{}^\mu\approx(\partial^2 h-\partial_\mu\partial_\nu h^{\mu\nu})+\dots\ ,$$

where $h\equiv\eta^{\mu\nu}h_{\mu\nu}$. The Bach tensor becomes $B^{\mu\nu}=\frac{1}{2}\partial^2K^{\mu\nu}$, where $K^{\mu\nu}=h^{\mu\nu}-\frac{1}{4}\eta^{\mu\nu}h$.

For the linearized conformal gravity, the action (3) reads [1]

$$\mathcal{A}_{\rm conf}\approx-\frac{1}{16\alpha}\int d^4x\,\partial^2K_{\mu\nu}\partial^2K^{\mu\nu}. \tag{20}$$

The quadratic part of the Einstein-Hilbert action (9) in linearized approximation comes from the Γ^2-terms in (5) and reads

$$\mathcal{A}_{\rm EH}\approx\frac{1}{2\kappa}\int d^4x\,g^{\mu\nu}\left(\Gamma_{\mu\lambda}{}^\kappa\,\Gamma_{\nu\kappa}{}^\lambda-\Gamma_{\mu\nu}{}^\lambda\,\Gamma_{\lambda\kappa}{}^\kappa\right). \tag{21}$$

It can be rewritten as

$$\mathcal{A}_{\rm EH}=-\frac{1}{8\kappa}\int d^4x\,h_{\mu\nu}\,\epsilon^{\lambda\mu\kappa\sigma}\epsilon_\lambda{}^{\nu\tau\delta}\partial_\kappa\partial_\tau h_{\sigma\delta}. \tag{22}$$

The most compact way of writing the action is in terms of the *Einstein tensor*

$$G^{\mu\nu}\equiv R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R \tag{23}$$

whose linear approximation is

$$G^{\mu\kappa} = \frac{1}{2}(\partial^2 h^{\mu\kappa} + \partial^\mu \partial^\kappa h - \partial^\mu \partial_\lambda h^{\lambda\kappa} - \partial^\kappa \partial_\lambda h^{\lambda\mu}) - \frac{1}{2}\eta^{\mu\kappa}(\partial^2 h - \partial_\nu \partial_\lambda h^{\nu\lambda}), \tag{24}$$

and has as trace $G \equiv G^\mu{}_\mu = -R = -(\partial^2 h - \partial_\mu \partial_\nu h^{\mu\nu})$. This may be written as a four-dimensional double curl:

$$G^{\mu\nu} \approx \frac{1}{4}\epsilon^{\lambda\mu\kappa\sigma}\epsilon_\lambda{}^{\nu\tau\delta}\partial_\kappa \partial_\tau h_{\sigma\delta}, \tag{25}$$

from which we see that the Einstein-Hilbert action (22) becomes simply

$$\mathcal{A}_{\rm EH} = \int d^4x\, \mathcal{L}(x) = \frac{1}{2\kappa}\int d^4x\, h_{\mu\nu}G^{\mu\nu}, \tag{26}$$

With these approximations, the action (16) reads

$$\mathcal{A}_{\rm si''} = -\frac{C^2}{12\alpha}\int d^4x\, R^2 + \frac{1}{2\kappa}\int d^4x\, \lambda h_{\mu\nu}\partial^2\phi^{\mu\nu} + \frac{3\alpha}{4S^2\kappa^2}\int d^4x\, \lambda^2\eta^{\mu\nu}(\eta_{\mu\nu} + h_{\mu\nu}), \tag{27}$$

where $\phi_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu}h/2$. This action can now be used to replace the second term in (3) without changing the conformal field theory in the weak-field limit.

For the new action $\mathcal{A}_{\rm si''}$, the linear approximation to the field equation (17) turns into the simple differential equation

$$-\frac{1}{2\alpha}\partial^2\partial^2\phi^{\mu\nu} + \frac{\lambda}{2\kappa}\partial^2\phi^{\mu\nu} + \frac{3\alpha}{4S^2\kappa^2}\lambda^2\eta^{\mu\nu} = T^{\mu\nu}. \tag{28}$$

The correct long-range behavior of the gravitational field is ensured if the average value of the λ-field is $\bar{\lambda} = 1$. Moreover, the dimensionless constant α can be chosen such as to reproduce the experimentally observed cosmological constant.

Note that the full theory described by the action (27) is independent of the mixing angle θ. However, if we calculate the effective energy only up to a finite loop order in the fluctuating $h_{\mu\nu}$-field, the result *will* depend on θ. The optimal result is obtained from that θ-value at which the energy has the weakest dependence on θ. At the one-loop level, this is the place where the derivative of the energy with respect to θ vanishes [11, 12]. This criterion has been used successfully in the calculation of critical exponents in all O(N)-symmetric Φ^4-theories at large coupling constants [13, 14].

The one-loop Euclidean effective action is obtained by functionally integrating out the fields $h_{\mu\nu}$ in the exponential $e^{i\mathcal{A}_{\rm si''}}$ in which λ is approximated by its average value $\bar{\lambda}$. The result is $e^{-iV_4\Gamma}$ where V_4 is the total four-volume of the universe, and Γ is the effective Euclidean Lagrangian

$$\Gamma = \int' \frac{d^Dk}{(2\pi)^D}\log\left(k^4 + \frac{\bar{\lambda}\alpha}{\kappa C^2}k^2\right) - \frac{3\alpha}{4S^2\kappa^2}\bar{\lambda}^2. \tag{29}$$

The prime indicates a trivial subtraction of the leading divergence. After this, the integral can be done and yields in $D = 4 - \epsilon$ dimensions

$$\Gamma = \frac{\alpha^2\lambda^2}{32\pi^2 C^4\kappa^2}\left[\log\frac{4\alpha\bar{\lambda}}{C^2\kappa\mu^2\pi e^\gamma} - 1\right] - \frac{3\alpha}{4S^2\kappa^2}\bar{\lambda}^2, \tag{30}$$

where μ is an arbitrary renormalization scale, and we have not written down the pole term $\propto 1/\epsilon$ since this can eventually be discarded in a renormalization procedure by minimal subtraction[14].

The saddle point in $\bar{\lambda}$ is now determined by the vanishing of $\Gamma_{\bar{\lambda}} \equiv \partial\Gamma/\partial_{\bar{\lambda}}$, where

$$\frac{\Gamma_{\bar{\lambda}}}{\bar{\lambda}} = \frac{\alpha^2}{16\pi^2 C^4\kappa^2}\left[\log\frac{4\alpha\bar{\lambda}}{C^2\kappa\mu^2\pi e^\gamma} - \frac{1}{2}\right] - \frac{3\alpha}{2S^2\kappa^2} = 0, \tag{31}$$

and we may ignore again a pole term $\propto 1/\epsilon$ in minimal subtraction.

Finally, the optimal value of θ is determined from the vanishing of the derivative of Γ with respect to C^2, $\Gamma_{C^2} \equiv \partial\Gamma/\partial C^2$, i.e., of

$$\frac{C^6\Gamma_{C^2}}{2\bar{\lambda}^2} = -\frac{\alpha^2}{16\pi^2\kappa^2}\left[\log\frac{4\alpha\bar{\lambda}}{C^2\kappa\mu^2\pi e^\gamma} - \frac{1}{2}\right] - \frac{3C^6\alpha}{8S^4\kappa^2} = 0. \tag{32}$$

and we ignore once more the pole term $\propto 1/\epsilon$. In the combination $\lambda\Gamma_\lambda + C^2\Gamma_{C^2} = 0$ the pole term is absent and from the zero we determine the optimal C^2 to have the value 2, so that $S^2 = 1$.

From the vanishing of the $\bar{\lambda}$-derivative (31) we find that the extremal $\bar{\lambda}$ is given by

$$\bar{\lambda} = \frac{C^2\kappa\mu^2\pi e^{\gamma+1/2}}{4\alpha} e^{-6\pi^2 C^6/S^4\alpha}. \tag{33}$$

For any value of the dimensionless coupling strength α, we can choose the renormalization mass scale μ, in such a way that $\bar{\lambda}$ has the value 1, that will guarantee the correct gravitational forces at long distances. It is the *dimensionally transmuted coupling constant* of the massless theory. Its role here is completely analogous to the role of the dimensionally transmuted coupling constant in the Coleman-Weinberg treatment of the scale-invariant quantum electrodynamics of a scalar ϕ^4-theory [15], and to the famous QCD mass-scale $\Lambda_{\rm QCD} \approx 217 \pm 25$ MeV in the massless quantum chromodynamics of quarks and gluons.

The question may arise whether the field λ could have a gradient term and thus have a particle associated with it. However, such a term would destroy the renormalizability of the theory so that we have to keep λ purely auxiliary. Its spacetime fluctuations are important for the short-distances behavior of the gravitational forces but not for the evolution of the cosmos, except for the baby stage.

From the outset, conformal gravity has many problems, such as states with negative norm. These problems can be shifted to such large masses, for instance a Planck mass or a multiple of it, so that they will not contradict experiments at present and in the foreseeable future. Thus they should not worry us, in particular, if we recall that similar problems exist within the best quantum field theory in use, quantum electrodynamics (QED). This also possesses unphysical states known as *Landau poles* which fortunately lie at such large masses that other physical phenomena intervene at much lower energies, long before the diseases of QED show up. In conformal gravity, there have also been other proposals for dealing with such states: One is based on the choice of suitable boundary conditions at large infinity [16], the other sees hope in different quantization procedures [17]. The spontaneous generation of an Einstein action proposed in this letter seems to be a much more satisfactory way towards a physically acceptable quantum gravity.

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