Composite Fermions and their Pair States in a Strongly-Coupled Fermi Liquid

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Our goal is to understand the phenomena arising in a Fermi liquid at low temperature in an external magnetic field. Varying the field, the attraction between any two fermions can be made arbitrarily strong until it reaches the unitarity limit, where composite bosons form via so-called Feshbach resonances. By setting up strong-coupling equations for fermions, we find that in spatial dimension $d > 2$ and the unitarity limit that they couple to a gas of bosons which dress up the fermions and lead to new massive composite fermions. At low enough temperature, these form tightly bound pair states which are new bosonic quasi-particles producing a condensate of Bose-Einstein type. The mass of the new bosonic quasi-particles is much larger and the new condensate happens at a much higher temperature. This may be the origin of high-T_c superconductivity and a similar form of composite superfluidity.

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Introduction. The attraction between any two fermions can be tuned, as a function of an external magnetic field, and be made so strong that the coupling constant reaches the unitarity limit of infinite s-wave scattering length via a Feshbach resonance. At that point, the Cooper pairs, which form in the weak-coupling limit at low temperature and make the system a superconductor, become so strongly bound that they behave like elementary bosonic quasi-particles with a pseudogap at high temperature, and form a new type of Bose-Einstein condensate (BEC). The problem of understanding a strong-coupled Fermi gas has been investigated with reasonable success, see review by Randeria and Taylor [1, 2]. In this letter we shall argue that results should be corrected.

In addressing strong-coupling fermions at finite temperature we incorporate the relevant s-wave scattering physics via a "ℓ_0-range" contact potential in the Hamiltonian for spinor wave functions $\psi_{\uparrow,\downarrow}(x)$,

$$\beta\mathcal{H} = \int_0^\beta d\tau \int d\mathbf{x}\Big[\mathcal{H}_0 - g\psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x)\Big], \quad (1)$$

where $x = (\mathbf{x}, \tau)$, and $\beta = 1/T$. The kinetic energy of fermions with mass m and chemical potential μ is collected in

$$\mathcal{H}_0 = \sum_{\sigma=\uparrow,\downarrow} \psi^\dagger_\sigma(x)[\partial_\tau - \nabla^2/2m - \mu]\psi_\sigma(x). \quad (2)$$

The attraction between the up- and down-spins is characterized by a coupling constant $g(\Lambda) > 0$, where $\Lambda = \pi\ell_0^{-1}$, and the range ℓ_0 is usually much smaller than the lattice spacing ℓ of the mean separation between two atoms. The "bare" $g(\Lambda)$ is related to a "renormalized" coupling described by the s-wave scattering length a via the two-particle Schrödinger equation at finite temperature T,

$$\frac{m}{4\pi a} = -\frac{1}{g(\Lambda)} + \frac{T}{V}\sum_{\omega_n,|\mathbf{k}|<\Lambda} \frac{1}{\omega_n^2 + \epsilon_\mathbf{k}^2}$$
$$= -\frac{1}{g(\Lambda)} + \frac{1}{V}\sum_{|\mathbf{k}|<\Lambda} \frac{1}{2\epsilon_\mathbf{k}} \tanh\frac{\epsilon_\mathbf{k}}{2T}. \quad (3)$$

Here $\epsilon_\mathbf{k}^2 = |\mathbf{k}|^2/2m$ denotes the energy of the free fermions and $\sum_{\omega_n,|\mathbf{k}|<\Lambda}$ contains the phase-space integral, and the sum over the *Matsubara frequencies* $\omega_n = 2\pi Tn$ for $n = 0, \pm1, \pm2, \cdots$, as derived in Eq. (3.61) of textbook [3]. The second sum can be written in d dimensions as

$$\frac{4\pi a}{mg(\Lambda)} = \frac{ak_F}{4\pi b\hbar^3}\mathcal{S}_d(T) - 1. \quad (4)$$

In $d = 3$ dimensions, at half-filling electron density $n \approx 1/\ell^3 \approx k_F^3/3\pi^2\hbar^3$, Fermi momentum $k_F \approx (3\pi^2\hbar^3)^{1/3}/\ell$, and Fermi energy $\epsilon_F = k_F^2/2m$, we introduce the dimensionless length parameter $b = 2^{-1}(3\pi^2\hbar^3)^{1/3}\ell_0/\ell \ll 1$, and find $\mathcal{S}_3(T) \equiv \int_0^1 dt \tanh\left[\frac{\epsilon_F}{T}\frac{\pi^2 t^2}{8b^2}\right]$ and $S_3(0) = 1$. For low temperatures and small a, Eq. (3) reduces properly to its well-known BCS version [4, 5].

As the attractive coupling g or the inverse scattering strength $1/ak_F$ increases, the Cooper pairs become tightly composite bosons. They form a normal Bose liquid, provided the temperature T is less than the crossover temperature T^* of Cooper-pair formation.Otherwise, the Cooper pairs dissociate into two fermions and form a normal Fermi liquid of unpaired fermions.These composite bosons undergo the BEC and become superfluid, as T decreases below the transition temperature T_c. The T^* diverges away from the T_c as the $1/ak_F$ increases. These results are summarized in the phase diagram of T/ϵ_F vs $1/ak_F$[1, 4]. We study quasi-particle spectra for the phase $1/ak_F \geq 0$, and discuss the ultra-violate (UV) scaling domain in the unitarity limit $1/ak_F \to 0^\pm$.

Strong-coupling limit and expansion. Inspired by strong-coupling quantum field theories [7, 8], we calculate the two-point Green functions of composite boson and fermion fields to effectively diagonalize the Hamiltonian into the bilinear form of these composite fields, and find the composite-particle spectra in the phase $1/ak_F \geq 0$.

The lattice representation of the Hamiltonian (1), for one electron per cubic lattice site (half filling), reads

$$\beta\mathcal{H} = \beta \sum_{i,\sigma=\uparrow,\downarrow} (\ell^d)\psi^\dagger_\sigma(i)\Big[-\nabla^2/(2m\ell^2) - \mu\Big]\psi_\sigma(i) - g\beta\sum_i(\ell^d)\psi^\dagger_\uparrow(i)\psi^\dagger_\downarrow(i)\psi_\downarrow(i)\psi_\uparrow(i), \quad (5)$$

where each fermion field is defined at a lattice site "i" as $\psi_\sigma(i) = \psi_\sigma(x)$ and the index i runs over all lattice sites. The fermion field ψ_σ has a length dimension $[\ell^{-d/2}]$, and the coupling g has a dimension $[\ell^{d-1}]$. The Laplace operator ∇^2 is defined as

$$\nabla^2\psi_\sigma(i) \equiv \sum_{\hat\ell}\Big[\psi_\sigma(i+\hat\ell) + \psi_\sigma(i-\hat\ell)\Big] - 2\psi_\sigma(i) \Rightarrow 2\Big[\sum_{\hat\ell}\cos(k\hat\ell) - 1\Big]\psi_\sigma(k) \approx -k^2\ell^2\psi_\sigma(k), \quad (6)$$

where $\hat\ell$ for $l = 1, \dots, d$ indicate the orientated lattice space vectors to the nearest neighbors, and the $\psi_\sigma(k)$ is the Fourier component of $\psi_\sigma(i)$ in momentum k-space. In the last line we assume that $k^2\ell^2 \ll 1$.

To calculate the expansion in strong-coupling limit, here we relabel $\beta\ell^d \to \beta$ and $2m\ell^2 \to 2m$, so the lattice spacing ℓ is set equal to unity; this rescales $\psi_\sigma(i) \to (\beta g)^{1/4}\psi_\sigma(i)$ and $\psi^\dagger_\sigma(i) \to (\beta g)^{1/4}\psi^\dagger_\sigma(i)$, so that the Hamiltonian (5) can be written as $\beta\mathcal{H} = \sum_i [h\mathcal{H}_0(i) + \mathcal{H}_{\rm int}(i)]$, where the hopping parameter $h \equiv \beta/(\beta g)^{1/2}$ and

$$\mathcal{H}_0(i) = \sum_{\sigma=\uparrow,\downarrow}\psi^\dagger_\sigma(i)(-\nabla^2/2m - \mu)\psi_\sigma(i), \quad (7)$$

$$\mathcal{H}_{\rm int}(i) \equiv -\psi^\dagger_\uparrow(i)\psi^\dagger_\downarrow(i)\psi_\downarrow(i)\psi_\uparrow(i), \quad (8)$$

and the partition function with expectation values

$$\mathcal{Z} = \Pi_{i,\sigma}\int d\psi_\sigma(i)d\psi^\dagger_\sigma(i)\exp(-\beta\mathcal{H}), \quad (9)$$

$$\langle\cdots\rangle = \mathcal{Z}^{-1}\Pi_{i,\sigma}\int d\psi_\sigma(i)d\psi^\dagger_\sigma(i)(\cdots)\exp(-\beta\mathcal{H}). \quad (10)$$

Fermion fields $\psi_\uparrow$ and $\psi_\downarrow$ are one-component Grassman variables with $\psi_\sigma(i)\psi_{\sigma'}(j) = -\psi_{\sigma'}(j)\psi_\sigma(i)$ and integrals $\int d\psi_\sigma(i)\psi_{\sigma'}(j) = \delta_{\sigma,\sigma'}\delta_{ij}$, $\int d\psi^\dagger_\sigma(i)\psi^\dagger_{\sigma'}(j) = \delta_{\sigma,\sigma'}\delta_{ij}$, and all others vanishing.

In the strong-coupling limit $h \to 0$ for $g \to \infty$ and finite T, the kinetic terms (7) are neglected, the partition function (9) becomes the one-site integral

$$\Pi_i\int_{i\downarrow}\int_{i\uparrow}\exp(-\mathcal{H}_{\rm int}) = -\Pi_i\int_{i\downarrow}\psi_\downarrow(i)^\dagger\psi_\downarrow(i) = (1)^{\mathcal{N}}, \quad (11)$$

where $\mathcal{N}$ is the total number of lattice sites, $\int_{i\uparrow} \equiv \int[d\psi^\dagger_\uparrow(i)d\psi_\uparrow(i)]$ and $\int_{i\downarrow} \equiv \int[d\psi^\dagger_\downarrow(i)d\psi_\downarrow(i)]$. The lattice hopping expansion can now be performed at strong couplings in powers of the hopping parameter h.

Composite bosons. We first consider a composite bosonic pair field $\mathcal{C}(x) = \psi_\downarrow(x)\psi_\uparrow(x)$ and study its two-point function,

$$G(x) = \langle\psi_\downarrow(0)\psi_\uparrow((0), \psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x)\rangle = \langle\mathcal{C}(0), \mathcal{C}^\dagger(x)\rangle. \quad (12)$$

Here the fermion fields are not re-scaled by $(\beta g)^{1/4}$. The leading strong-coupling approximation to (12) is $G(x) = \delta^{(d)}(x)/\beta g$. The first correction is obtained by using the one-site partition function $Z(i)$ and the integral

$$\langle\psi_\uparrow\psi_\downarrow\rangle \equiv \frac{1}{Z(i)}\int_{i\downarrow}\int_{i\uparrow}\psi_\uparrow(i)\psi_\downarrow(i)e^{-h\mathcal{H}_0(i)-\beta\mathcal{H}_{\rm int}(i)} = h^2\sum_{\hat\ell}^{\rm ave}\psi_\uparrow(i;\hat\ell)\sum_{\hat\ell'}^{\rm ave}\psi_\downarrow(i;\hat\ell') \approx h^2\sum_{\hat\ell}^{\rm ave}\psi_\uparrow(i;\hat\ell)\psi_\downarrow(i;\hat\ell), \quad (13)$$

where the non-trivial result needs $\psi^\dagger_{\uparrow,\downarrow}(i)$ fields in the hopping expansion of $e^{-h\mathcal{H}_0(i)}$, and $\sum^{\rm ave}_{\hat\ell}\psi_\sigma(i;\hat\ell) \equiv \sum_{\hat\ell}\Big[\psi_\sigma(i+\hat\ell) + \psi_\sigma(i-\hat\ell)\Big]$. In Eq. (12), integrating over fields $\psi_{\uparrow,\downarrow}(i)$ at the site "i", the first corrected version reads:

$$G(x) = \frac{\delta^{(d)}(x)}{\beta g} + \frac{1}{\beta g}\left(\frac{\beta}{2m}\right)^2\sum_{\hat\ell}^{\rm ave}G^{\rm nb}(x;\hat\ell), \quad (14)$$

where $\delta^{(d)}(x)$ is a spatial δ-function and $G^{\rm nb}(x \pm \hat\ell)$ is the Green function (12) without integration over fields ψ_σ at the neighbor site x. Note that the nontrivial contributions come only from kinetic hopping terms $\propto (h/2m)^2 = (1/\beta g)(\beta/2m)^2$ dimensionless parameter. The chemical potential term $\mu\psi^\dagger_\sigma(i)\psi_\sigma(i)$ in the Hamiltonian (7) does not contribute to the hopping.

Replacing $G^{\rm nb}(x \pm \hat\ell)$ by $G(x \pm \hat\ell)$ converts Eq. (14) into a recursion relation for $G(x)$, which actually takes into account of all high-hopping corrections in a strong-coupling expansion. Going to momentum space we obtain $G(q) = \frac{1}{\beta g} + \frac{2}{\beta g}\left(\frac{\beta}{2m}\right)^2 G(q)\sum_{\hat\ell}\cos(q\hat\ell)$, which is solved by

$$G(q) = \frac{[2m/(\beta\ell)]^2}{4\ell^{-2}\sum_{\hat\ell}\sin^2(q\hat\ell/2) + M_B^2}. \quad (15)$$

Here we have resumed the original lattice spacing ℓ by settting back $\beta \to \beta\ell^3$ and $2m \to 2m\ell^2$.

We find that in the strong-coupling effective Hamiltonian, $\mathcal{C} = \psi_\downarrow\psi_\uparrow$ represents a massive composite boson with propagator

$$gG(q) = \frac{gR_B^2/(2M_B)}{(q^2/2M_B) + M_B/2} \Rightarrow \frac{gR_B^2}{q^2 + M_B^2}, \ (q\ell \ll 1), \quad (16)$$

with pole of mass M_B and residue of form factor gR_B^2:

$$M_B^2 = \left[g(2m)^2(\ell/\beta) - 2d\right]\ell^{-2} > 0, \ R_B^2 = (2m/\beta\ell)^2. \quad (17)$$

From Eq. (16), the effective Hamiltonian of the composite boson field $\mathcal{C}$ can be written as,

$$\mathcal{H}^B_{\rm eff} = \sum_i (\ell^3) Z_B^{-1}\mathcal{C}^\dagger(i)\Big[-\nabla^2/(2M_B\ell^2) - \mu_B\Big]\mathcal{C}(i). \quad (18)$$

The chemical potential is $\mu_B = -M_B/2$ and the wave function renremalization is $Z_B = gR_B^2/2M_B$. As long as Z_B is finite, we renormalize the elementary fermion field and the composite boson field as

$$\psi \to (gR_B^2)^{-1/4}\psi, \quad \text{and} \quad \mathcal{C} \to (2M_B)^{1/2}\mathcal{C}, \qquad (19)$$

so that the composite boson $\mathcal{C}$ behaves like a quasi particle in Eq. (18). Contrary to the loosely-bound state of two electrons in a Cooper pair in the weak-coupling region $(k_F a)^{-1} \ll 0$, this is a pair in a tightly bound Feshbach resonance.

The bound states are composed of two constituent fermions $\psi_\downarrow(k_1)$ and $\psi_\uparrow(k_2)$ around the Fermi surface, $k_1 \approx k_2 \approx k_F$ and $k_2 - k_1 = q \ll k_F$. The wave-function renomalization $Z_B \propto gT^2$ (16) relates to the bound-state size $\xi_{\rm boson}$. As $gT^2 \to 0$, Z_B decreases and the pair field $\mathcal{C}(x)$ descibes loose Cooper pair. The vanishing form factor represents that bosonic bound state (pole) dissolves into two fermionic constituents (cut) [9]. At this dissociation scale, i.e., crossover temperature T^* at $1/ak_F$, the phase transition to a normal Fermi liquid of unpaired fermions takes place. Limited by the validity of strong-coupling expansion, we are not able to quantitatively obtain the dissociation scale T^* as it results from $1/ak_F$. We can estimate at the unitarity limit $1/ak_F = 0$ the crossover temperature $T^* \approx \epsilon_B/\log(\epsilon_B/\epsilon_F)^{3/2}$ [4]. The binding energy $\epsilon_B/\epsilon_F = 2f_-/f_+, f_\pm = \sqrt{1+\hat\mu^2} \pm \hat\mu$ from Eq. (3.285) of textbook [3]. Here we insert for the crossover parameter $\hat\mu = \mu_B/M_B = -1/2$, using M_B as the mass gap at $1/ak_F = 0$ and find $\epsilon_B/\epsilon_F \approx 5.24$ and $T^*/\epsilon_F \approx 4.86$.

On the other hand, the mass term $M_B^2\mathcal{C}\mathcal{C}^\dagger$ changes its sign from $M_B^2 > 0$ to $M_B^2 < 0$ and the pole M_B becomes imaginary, implying the second-order phase transition from the symmetric phase to the condensate phase [8]. $M_B^2 = 0$ gives rise to the critical line: $m^2 g_c T_c = d/(2\ell)$. Substituting g_c from Eq. (4), we find for $1/ak_F \geq 0$

$$T_c/\epsilon_F = (T_c^u/\epsilon_F)\left[1 - \frac{4\pi\hbar^3 b}{\mathcal{S}_d(T_c)}\frac{1}{ak_F}\right], \qquad (20)$$

where T_c^u is the critical temperature at $1/ak_F = 0$,

$$T_c^u/\epsilon_F = (3\pi^2)^{-1/d} d\mathcal{S}_d(T_c^u)/[(4\pi)^2\hbar^3 b]. \qquad (21)$$

In the superfluid phase ($T < T_c$) tightly composite bosons $\mathcal{C}(x)$ develop a nonzero expectation value $\langle\mathcal{C}(x)\rangle$ and undergo BEC.

We numerically calculate in $d = 3$ for the parameters $b = 0.02, 0.03$ corresponding to the ratios $\ell_0/\ell = 0.013, 0.02$, find $T_c^u/\epsilon_F \approx 0.31, 0.2$. Equation (20) is plotted in Fig. 1. In contrast to the practically horizontal phase boundary in Figure 3 in Ref. [1], we obtain a decreasing critical temperature T_c as a function of $1/ak_F \geq 0$. At an "infinite" coupling strength $g_c \to \infty$ we find at constant $T_c g_c$ that $T_c = 0$ for $1/ak_F \to (1/ak_F)_{qc} \equiv \mathcal{S}_d(0)/4\pi b\hbar^3 = 1/4\pi b\hbar^3$. This is a quantum critical point, in which all thermal fluctuations are absent. What about the phase $(1/ak_F) > (1/ak_F)_{qc}$, it seems inconsistently $g < 0$ from Eq. (4). In fact, this phase should be related to the formation and condensate of more complex composite quasi-particles, e.g., spin triplet $\mathcal{C}^{\rm tri} \equiv (\psi_\uparrow\psi_\uparrow, \psi_\uparrow\psi_\downarrow, \psi_\downarrow\psi_\downarrow)$ with mass gap $M_B^{\rm tri}(T)$ and phase factor $\mathcal{S}_d^{\rm tri}(T)$, which are *not* given by Eqs. (3) and (17), and the bosonic triplet $\mathcal{C}^{\rm tri}$ dresses up an elementary fermion to form a three-fermion state discussed below. The critical line $T^{\rm tri}(1/ak_F)$ from $M_B^{\rm tri}(T) = 0$ separates the quasi-particle $\mathcal{C}^{\rm tri}$ formation from the condensate phases. Starting from the quantum critical point it increases as T and $(1/ak_F)$ increase (g decreases) for $(1/ak_F) > (1/ak_F)_{qc}$. Viewing the four-fermion interaction as an attractive potential, this "infinite" coupling point indicates the most tightly bound state locating at the lowest energy level of the potential, with a scattering length $a = 2\pi\ell_0$. If the attraction comes from a δ -function, the length parameters a, and b vanish, while $(1/k_F a)_{qc} \to \infty$, recovering the nearly horizontal critical line presented in Figure 3 in Ref. [1].

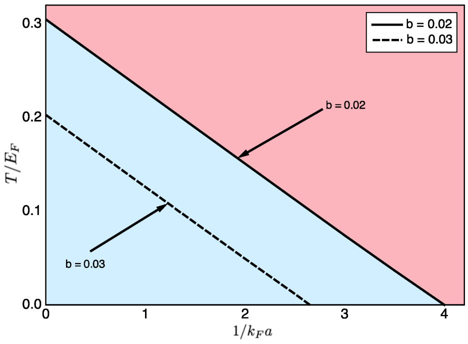

FIG. 1: Qualitative phase diagram in the unitarity limit. Transition temperature T_c/ϵ_F is plotted as a function of $1/k_F a \geq 0$ for the selected parameters $b = 0.02, 0.03$. The "infinite"-coupling points are shown to lie at one of the corresponding zeros $(1/k_F a)_{T_c=0} = 4.0, 2.7$. That is a point of quantum phase transition, Above the critical line is a normal liquid consisting of massive composite bosons and fermions. Below the critical line lies a superfluid phase with a new type of BEC involving composite massive fermions.

Analogously, we consider the composite field of electron and hole, i.e., the plasmon field $\mathcal{P}(x) = \psi_\downarrow^\dagger(x)\psi_\uparrow(x)$, the same calculations are applied for the two-point Green function $G_\mathcal{P}(x) = \langle\mathcal{P}(0), \mathcal{P}^\dagger(x)\rangle$. In the lowest nontrivial order of strong-coupling expansion, we obtain the same result as (16) and (17), indicating a tightly bound state of plasmon field, whose Hamiltonian is (18) with $\mathcal{C}(i) \to \mathcal{P}(i)$. This is not surprised since the Cooper $\mathcal{C}(x)$ and plasmon $\mathcal{P}(x)$ fields are symmetric in the strong-interacting Hamiltonian (5). However, the charged pair field $\mathcal{C}(x)$ and neutral plasmon $\mathcal{P}(x)$ field can be different up to a relative phase of field $\theta(x)$. We select the relative phase field as such that $\langle|\mathcal{P}(x)|\rangle = \langle|\mathcal{C}(x)|\rangle$. We also obtain [10] the identically vanishing two-point Green function $\langle \mathcal{P}(0), \mathcal{C}^\dagger(x)\rangle$ of Cooper $\mathcal{C}(x)$ with plasmon $\mathcal{P}(x)$ fields, as one is charged the other is neutral.

Composite Fermions. To exhibit the presence of composite fermions in the strong-coupling Hamiltonian (7) and (8), following [8] and using the Cooper field $\mathcal{C}(x)$ we calculate the two-point Green functions:

$$S_{LL}(x) \equiv \langle \psi_\uparrow(0), \psi^\dagger_\uparrow(x)\rangle, \tag{22}$$
$$S_{ML}(x) \equiv \langle \psi_\uparrow(0), \mathcal{C}^\dagger(x)\psi_\downarrow(x)\rangle, \tag{23}$$
$$S^\dagger_{ML}(x) \equiv \langle \psi^\dagger_\downarrow(0)\mathcal{C}(0), \psi^\dagger_\uparrow(x)\rangle \tag{24}$$
$$S_{MM}(x) \equiv \langle \psi^\dagger_\downarrow(0)\mathcal{C}(0), \mathcal{C}^\dagger(x)\psi_\downarrow(x)\rangle \tag{25}$$

By the analogy to (12)–(15), we obtain three recursion relations [10]

$$S_{LL}(p) = \frac{1}{\beta g}\left(\frac{\beta}{2m}\right)^3 \Big[2\sum_{\hat\ell}\cos(p\hat\ell)\Big] S_{ML}(p), \tag{26}$$
$$S_{ML}(p) = \frac{1}{\beta g} + \frac{1}{\beta g}\left(\frac{\beta}{2m}\right)\Big[2\sum_{\hat\ell}\cos(p\hat\ell)\Big] S_{LL}(p), \tag{27}$$
$$S_{MM}(p) = \frac{1}{\beta g}\left(\frac{\beta}{2m}\right)\Big[2\sum_{\hat\ell}\cos(p\hat\ell)\Big] S^\dagger_{ML}(p). \tag{28}$$

We solve these recursion relations and obtain

$$S_{ML}(p) = \frac{(1/\beta g)}{1-(1/\beta g)^2(\beta/2m)^4\Big[2\sum_{\hat\ell}\cos(p\hat\ell)\Big]^2}, \tag{29}$$

$S_{LL}(p)$ and $S_{MM}(p)$. Defining for the propagator of the composite fermion $S_{\rm Fermion}(p)$ the quantity $gS(p)$,

$$\begin{aligned} S(p) &= R_B^{-1}S_{LL}(p) + 2R_B^{-2}S_{ML}(p) + R_B^{-3}S_{MM}(p) \\ &= \frac{2}{4\ell^{-2}\sum_{\hat\ell}\sin^2(p\hat\ell/2)+M_F^2} \Rightarrow \frac{2}{p^2+M_F^2}, \end{aligned} \tag{30}$$

where R_B and $M_F^2 = M_B^2$ follow Eq. (17) in lowest order calculation. $S_{\rm Fermion}(p)$ represents a composite fermion composed of the elementary fermion $\psi_\uparrow$ and the three-fermion state $\mathcal{C}(x)\psi^\dagger_\downarrow(x)$ [8],

$$\begin{aligned} \Psi_\uparrow(x) &= R_B^{-1/2}\psi_\uparrow(x) + R_B^{-3/2}\mathcal{C}(x)\psi^\dagger_\downarrow(x) \\ &\Rightarrow g^{1/4}\psi_\uparrow(x) + g^{3/4}\mathcal{C}(x)\psi^\dagger_\downarrow(x), \end{aligned} \tag{31}$$

where the three-fermion state $\mathcal{C}(x)\psi^\dagger_\downarrow(x)$ is made of a hole $\psi^\dagger_\downarrow(x)$ "dressed" by a cloud of Cooper pairs. The associated two-point Green function reads

$$\begin{aligned} \langle \Psi_\uparrow(0), \Psi^\dagger_\uparrow(x)\rangle &= \langle \psi_\uparrow(0), \psi^\dagger_\uparrow(x)\rangle + \langle \psi_\uparrow(0), \mathcal{C}^\dagger(x)\psi_\downarrow(x)\rangle \\ &+\langle \mathcal{C}(0)\psi^\dagger_\downarrow(0), \psi^\dagger_\uparrow(x)\rangle + \langle \mathcal{C}(0)\psi^\dagger_\downarrow(0), \mathcal{C}^\dagger(x)\psi_\downarrow(x)\rangle, \end{aligned} \tag{32}$$

whose momentum transformation satisfies (30). A similar result holds for the spin-down composite fermion $\Psi_\downarrow(x) = R_B^{-1/2}\psi_\downarrow(x) + R_B^{-3/2}\mathcal{C}(x)\psi^\dagger_\uparrow(x)$. They can be represented in the effective Hamiltonian

$$\mathcal{H}^F_{\rm eff} = \sum_{i,\sigma=\uparrow\downarrow}(\ell^3)Z_F^{-1}\Psi^\dagger_\sigma(i)\Big[-\nabla^2/(2M_F\ell^2) - \mu_F\Big]\Psi_\sigma(i). \tag{33}$$

Here $\mu_F = -M_F/2$ is the chemical potential and $Z_F = g/M_F$ the wave-function renormalization. Following the renormalization (19) of elementary fermion fields, we renormalize composite fermion field $\Psi_{\uparrow,\downarrow} \Rightarrow (Z_F)^{-1/2}\Psi_{\uparrow,\downarrow}$, which behaves as a quasi-particle in Eq. (33), analogously to the composite boson (18). The negatively charged (e) three-fermion state is a negatively charged ($2e$) Cooper field $\mathcal{C}(x) = \psi_\downarrow(x)\psi_\uparrow(x)$ of two electrons combining with a hole $\psi^\dagger_\downarrow(x)$. These negatively charged (e) composite fermions $\Psi_{\uparrow\downarrow}(x)$ are composed of three-fermion states $\mathcal{C}\psi^\dagger_\uparrow$ or $\mathcal{C}\psi^\dagger_\downarrow$ and an electron $\psi_\uparrow$ or $\psi_\downarrow$. Similarly, positively charged ($-e$) composite fermions $\Psi^\dagger_\uparrow(x)$ or $\Psi^\dagger_\downarrow(x)$ are composed by three-fermion states $\mathcal{C}^\dagger\psi_\uparrow$ or $\mathcal{C}^\dagger\psi_\downarrow$ combined with a hole state $\psi^\dagger_\uparrow$ or $\psi^\dagger_\downarrow$. Suppose that two constituent electrons $\psi_\downarrow(k_1)$ and $\psi_\uparrow(k_2)$ of the Cooper field, one constituent hole $\psi^\dagger_\uparrow(k_3)$ are around the Fermi surface, $k_1 \approx k_2 \approx k_3 \approx k_F$, then the Cooper field $q = k_2 - k_1 \ll k_F$ and three-fermion bound state $p = k_1 - k_2 + k_3 \approx k_3 \approx k_F$ is around the Fermi surface. As a result, the composite fermions $\Psi_{\uparrow\downarrow}$ live around the Fermi surface as well.

The same results (30)–(33) are obtained for the plasmon field $\mathcal{P}(x) = \psi^\dagger_\downarrow(x)\psi_\uparrow(x)$ combined with another electron or hole, and the associated composite fermion

$$\begin{aligned} \Psi^{\mathcal{P}}_\uparrow(x) &= R_B^{-1/2}\psi_\uparrow(x) + R_B^{-3/2}\mathcal{P}(x)\psi_\downarrow(x) \\ &\Rightarrow g^{1/4}\psi_\uparrow(x) + g^{3/4}\mathcal{P}(x)\psi_\downarrow(x), \end{aligned} \tag{34}$$

whose two-point Green function,

$$\begin{aligned} &\langle \Psi^{\mathcal{P}}_\uparrow(0), \Psi^{\mathcal{P}\dagger}_\uparrow(x)\rangle = \langle \psi_\uparrow(0), \psi^\dagger_\uparrow(x)\rangle + \langle \psi_\uparrow(0), \mathcal{P}^\dagger\psi^\dagger_\downarrow(x)\rangle \\ &+\langle \mathcal{P}\psi_\downarrow(0), \psi^\dagger_\uparrow(x)\rangle + \langle \mathcal{P}\psi_\downarrow(0), \mathcal{P}^\dagger\psi^\dagger_\downarrow(x)\rangle. \end{aligned} \tag{35}$$

The same is for $\Psi^{\mathcal{P}}_\downarrow(x) = R_B^{-1/2}\psi_\downarrow(x) + R_B^{-3/2}\mathcal{P}(x)\psi_\uparrow(x)$ the spin-down field. They can be represented in the effective Hamiltonian (33) with $\Psi_\sigma(i) \to \Psi^{\mathcal{P}}_\sigma(i)$, following the renormalization (19) of elementary fermion fields, and renormalization $\Psi^{\mathcal{P}}_{\uparrow,\downarrow} \Rightarrow (Z_F)^{-1/2}\Psi^{\mathcal{P}}_{\uparrow,\downarrow}$. The charged three-fermion states $\mathcal{P}\psi_{\uparrow\downarrow}$ or $\mathcal{P}^\dagger\psi^\dagger_{\uparrow\downarrow}$ are composed of one electron or one hole combined with a neutral plasmon field $\mathcal{P}(x) = \psi^\dagger_\downarrow(x)\psi_\uparrow(x)$ or $\mathcal{P}^\dagger(x) = \psi^\dagger_\uparrow(x)\psi_\downarrow(x)$ of an electron and a hole. The composite fermions $\Psi^{\mathcal{P}}_{\uparrow,\downarrow}(x)$ are composed of a three-fermion states $\mathcal{P}\psi_{\uparrow,\downarrow}$ in combination with a further elementary fermion $\psi_\uparrow$ or $\psi_\downarrow$. The same thing is true for its charge-conjugate state. Suppose that constituent electron $\psi_\downarrow(k_1)$ and hole $\psi^\dagger_\downarrow(k_2)$, another constituent electron $\psi_\uparrow(k_3)$ are all around the Fermi surface, $k_1 \approx k_2 \approx k_3 \approx k_F$, and the plasmon field $q = k_2 - k_1 \ll k_F$ and composite fermion $p = k_1 - k_2 + k_3 \approx k_3 \approx k_F$ is around the Fermi surface as well. The three-fermion states in Eqs. (31) and (34) are related, $\mathcal{C}(x)\psi^\dagger_\downarrow(x) = -\mathcal{P}(x)\psi_\downarrow(x)$. This implies that the three-fermion states $\mathcal{C}(x)\psi^\dagger_\downarrow(x)$ and $\mathcal{P}(x)\psi_\downarrow(x)$ are the same up to a definite phase factor $e^{i\pi}$. Thus the composite fermions $\Psi_\sigma(x)$ (31) and $\Psi^{\mathcal{P}}_\sigma(x)$ (34) are indistinguishable up to a definite phase factor.

All composite fermions are of Dirac type, due to the interaction (1). For stronger couplings $1/ak_F > (1/ak_F)_{qc}$, Eq. (23) should be extended by more complex composite fermions $S_{ML}(x) \equiv \langle\psi_\uparrow(0), \mathcal{C}^\dagger(x)\psi_\downarrow(x)\rangle + \langle\psi_\uparrow(0), \mathcal{S}_\Uparrow(x)\psi^\dagger_\uparrow(x)\rangle$, where $\mathcal{S}_\Uparrow(x) = \psi^\dagger_\uparrow(x)\psi_\uparrow(x)$ brings in a spin-vector field.

Conclusion and Remarks. We present some discussions of the effective Hamiltonian (18) and (33) of composite boson and fermion for $1/ak_F \geq 0$ and different values of temperature T. (i) In the regime $T^* > T > T_c$, there is a mixed liquid of composite bosons and fermions with the pseudogap $M_{F,B}(T)$, which is expected to dissolve to normal unpaired Fermi gas at the crossover temperature T^*. These composite quasi particles are either charged or neutral. They behave as superfluids up to a relatively high crossover temperature T^*. (ii) In the regime $T < T_c$, the superfluid phase of composite bosons undergoes BEC and one finds in the ground state the coexistence of BEC and semi-degenerate fermions $\Psi_\uparrow(x)$ and $\Psi_\downarrow(x)$, the latter couple to the BEC background to form massive quasi-particles of fermion type, moreover they form tightly bound states $\Psi_\uparrow\Psi_\downarrow$ or $\Psi^\dagger_\uparrow\Psi_\downarrow$ which are new bosonic quasi-particles producing a new condensate of Bose-Einstein type. In both cases, if the Coulomb repulsion between electrons could be compensated by "phonons" in an analogous way to either composite bosons via a Feshbach resonance, or new bosonic quasi-particles via a composite-fermion pair state, this would result in superconductivity and superfluity at high temperature $T_c \propto \mathcal{O}(\epsilon_F)$. The scale of that is the result of a large coherent mass gap $M_{F,B}(T)$, being much larger than the BCS gap. The coherent supercurrents consist of composite fermions and bosons. These features though discussed in $1/ak_F \geq 0$ are expected to be also true in $1/ak_F \ll 0$ with much smaller scale $M_{F,B}(T)$. Due to the presence of composite fermions in addition to composite bosons, we expect a further suppression of the low-energy spectral weight for single-particle excitations and the material following harder equation of state. Its observable consequences include a further T-dependent suppression of heat capacity and gap-like dispersion in the density-of-states and spin susceptibility. Moreover, we discuss the quantum critical point and speculate the phase of complex quasi-particles.

It is known that the limit $1/ak_F \ll 0$ produces an IR-stable fixed point, and its scaling domain is described by an effective Hamiltonian of BCS physics with the gap scale $\Delta_0 = \Delta(T_c)$ in $T \sim T_c \lesssim T^*$. This is analogous to the IR-stable fixed point and scaling domain of an effective Lagrangian of Standard Model (SM) with the electroweak scale in elementary particle physics [11, 12].

The unitarity limit $1/ak_F \to 0^\pm$ representing a scale invariant point [13] was formulated in a renormalization group framework [14], implying an UV stable fixed point of large coupling. The couplings $g > g_{\rm UV}$ and $g < g_{\rm UV}$ approach $g_{\rm UV}$, as running energy scale becomes larger. In the scaling domain of this UV fixed point $1/ak_F \to 0^\pm$ and $T \to T^u_c$, an effective Hamiltonian of composite bosons and fermions is realized with characteristic scale

$$M_{B,F}(T) = \left(\frac{T - T^u_c}{T^u_c}\right)^{\nu/2} \frac{(2d)^{1/2}}{(3\pi^2)^{1/d}} k_F, \quad T \gtrsim T^u_c \quad (36)$$

where $\nu = 1$ is the critical exponent derived from the β-function which determines the scaling laws. Equation (36) shows that the relevant cutoff is the Fermi momentum k_F and the physical correlation length $\xi \propto M^{-1}_{B,F}$, which characterizes the size of composite particles via their form factor $Z_{B,F} \propto M^{-1}_{B,F}$ (18) and (33). This domain should be better explored experimentally. The analogy was discussed in elementary particle physics with anticipations of the UV scaling domain at TeV scales and effective Lagrangian of composite particles made by SM elementary fermions including Majorana type [15].

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