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Regge Couplings From $SU(2) \times SU(2)$ Saturation Schemes*)

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Abstract

For any saturation scheme of the chiral $SU(2) \times SU(2)$ charge algebra we develop a simple algebraic method of calculating the couplings of ϱ and f trajectories to all particles involved. The information on these coupling is shown to be directly contained in the chiral mass splittings among the different isospin multiplets.

I. Introduction

In these lectures*) we shall present a method of predicting numerous Regge couplings by means of simple algebraic calculations. The starting point is any reasonable saturation scheme of the chiral $SU(2) \times SU(2)$ charge algebra. In recent years, rather complete schemes of this type have been developed accommodating most of the known baryon and meson resonances. Via the assumption of PCAC, these saturation schemes provide us with the couplings of all these resonances to pions. They can therefore be efficiently used as an input to dispersion relations for any process

$$\alpha\pi_a \to \beta\pi_b$$

if one is content with a narrow-resonance approximation. Since these couplings satisfy simple commutation rules, any calculation can easily be performed by purely algebraic means.

II. Brief Review of SU(2) × SU(2) Charge Algebra

Recall that the $SU(2) \times SU(2)$ current commutation rules[1]) allow us to derive a Ward identity, relating the axial vector amplitude between two arbitrary hadron states β and α:

$$\tau^{ba}_{\mu\nu\ \beta\alpha} = -i \int e^{iqx} \langle \beta p' |\, T\big(A_\mu{}^b(x)\, A_\nu{}^a(0)\big)\, | \alpha p \rangle\, dx \qquad (2.1)$$

*) Lectures presented at the Discussion Meeting on High-Energy Scattering of Elementary Particles in the Black Forest, May 23—27, 1972.

[1]) Conventions: $\langle \mathbf{p}' | \mathbf{p} \rangle = 2p_0 (2\pi^3 \int \delta^3 (\mathbf{p}' - \mathbf{p})$, $S = 1 - i\,(2\pi)^4 \delta^4 (p_f - p_i)\,\mathrm{T}$

to the amplitude of the divergences

$$\tau^{ba}_{\beta\alpha} = -i \int e^{iqx} \langle \beta p' | \, T\big(\partial A^b(x) \, \partial A^a(0)\big) \, |\alpha p\rangle \, dx \tag{2.2}$$

via

$$q'^\mu q^\nu \tau^{ba}_{\mu\nu\beta\alpha} = \tau^{ba}_{\beta\alpha} + i f^{bac} \frac{(q' + q)^\mu}{2} \langle \beta p' | \, V_\mu{}^c \, |\alpha p\rangle \tag{2.3}$$

$$- \frac{i}{2} \big[\langle \beta p' | \, [Q^b_5(0), \partial A^a(0)] \, |\alpha p\rangle + (ba)\big].$$

The $\alpha\pi_a \to \beta\pi_b$ scattering amplitude can be obtained from $\tau^{ba}_{\beta\alpha}$ by picking up the residues of $\tau^{ba}_{\beta\alpha}$ at the pion poles in q'^2 and q^2:

$$T^{ba}_{\beta\alpha}(\nu, t) = \lim_{\substack{q'^2 \to \mu^2 \\ q^2 \to \mu^2}} T^{ba}_{\beta\alpha}(\nu, t, q'^2, q^2) \equiv \lim_{\substack{q'^2 \to \mu^2 \\ q^2 \to \mu^2}} \frac{(q'^2 - \mu^2)(q^2 - \mu^2)}{f_\pi{}^2 \mu^4} \tau^{ba}_{\beta\alpha}. \tag{2.4}$$

If q'^2, q^2 leave their mass shells, T *defines* an off-shell continuation of the $\alpha\pi_a \to \beta\pi_b$ amplitude. For this off-shell continuation we can derive a low-energy theorem by setting $q'^2 = q^2 = 0$ and going to the threshold point in the Ward identity (2.3). If one, instead of s, introduces the crossing symmetric variable

$$\nu \equiv \frac{s - u}{2} = \frac{1}{2}(p' + p)(q' + q) \tag{2.5}$$

one finds that at the threshold point $t = 0$, $\nu = \nu_{\text{th}} \equiv (m_\alpha{}^2 - m_\beta{}^2)2$ the isospin odd and even parts of $T^{ab}_{\beta\alpha}$

$$T^{(\mp)ba}_{\beta\alpha}(\nu, t) \equiv \frac{1}{2} \left(T^{ba}_{\beta\alpha} \mp T^{ab}_{\beta\alpha}\right)(\nu, t) \tag{2.6}$$

satisfy the low-energy theorems

$$\frac{1}{\nu} T^{(-)ba}_{\beta\alpha}(\nu, t) \Big|_{\substack{t=0 \\ \nu=\nu_{\text{ta}}}} = -i\varepsilon^{bac} \frac{1}{f\pi^2} [T_c]_{\beta\alpha} \tag{2.7}$$

$$T^{(+)ba}_{\beta\alpha}(\nu, t) \Big|_{\substack{t=0 \\ \nu=\nu_{\text{ta}}}} = \frac{1}{f_\pi{}^2} [\Sigma^{ba}]_{\beta\alpha}. \tag{2.8}$$

Here T_c is simply the (necessarily diagonal) isospin matrix between the particles β and α and Σ^{ba} denotes the commutator

$$\Sigma^{ba} \equiv (x) \frac{i}{2} \big([Q^b_5(x_0) \, \partial A^a(x)] + (ab)\big) \tag{2.9}$$

which has been the subject of much recent discussion.[2])

[2]) For a detailed discussion see H. Kleinert, Fortschr. Phys. **21**, 1 (1973).

Now one invokes the Regge pole model to argue that $T^{(-)}$ at high energy should be governed by the exchange of a ϱ trajectory of intercept $\alpha_\varrho(0)\approx 1/2$. This implies an asymptotic behaviour

$$\frac{1}{\nu}\,T^{(-)ba}_{\beta\alpha}(\nu)\underset{\nu\to\infty}{\sim}\nu^{\alpha_\varrho(0)-1}\approx\nu^{-0.5} \tag{2.10}$$

such that $1/\nu\,T^{(-)}$ obeys an unsubtracted dispersion in ν^2:

$$\frac{1}{\nu}\,T^{(-)}_{(\nu)}=\frac{1}{\pi}\int\limits_0^\infty\frac{\operatorname{Im}\frac{1}{\nu}\,T^{(-)}(\nu')}{\nu'^2-\nu^2}\,d\nu'^2. \tag{2.11}$$

If one assumes that the dispersion integral can be approximated by a set of narrow resonances only, the amplitude can be expressed completely in terms of the collinear couplings of the divergence of the axial currents to resonances of helicity λ [3]:

$$[X_a(\lambda)]_{\gamma\alpha}\equiv-\frac{i}{m_\gamma^2-m_\gamma^2}\,\langle\gamma p_\gamma\lambda|\,\partial A_a\,|\alpha p_\alpha\lambda\rangle\Big|_{(p_\gamma-p_\alpha)^2=0}. \tag{2.12}$$

Using this definition, the imaginary part of $T^{(\mp)ba}_{\beta\alpha}$ at the resonance $s=m_\gamma^2$ becomes explicitly[3])

$$\operatorname{Im}T^{(\mp)ba}_{\beta\alpha}=-\pi\,[\delta(\nu-\nu_\gamma{}^{\beta\alpha})\pm\delta(\nu+\nu_\gamma{}^{\beta\alpha})]\,\varrho^{(\mp)ba}(\lambda)_\gamma{}^{\beta\alpha} \tag{2.13}$$

with[4])

$$\begin{aligned}\varrho^{(\pm)ba}_{\beta\alpha}(\lambda)&=\frac{1}{2f_\pi^2}\,\{\langle\beta p'\lambda|\partial A^b|\gamma p_\gamma\lambda\rangle\,\langle\gamma p_\gamma\lambda|\partial A^a|\alpha p\lambda\rangle\mp(ab)\}\\&=-\frac{1}{2f_\pi^2}\,(m_\beta^2-m_\gamma^2)\,(m_\gamma^2-m_\alpha^2)\,[X_{b_{\beta\gamma}}(\lambda),X_{a_{\beta\alpha}}(\lambda)]_\mp\end{aligned} \tag{2.14}$$

such that equ. (2.11) yields

$$\begin{aligned}\frac{1}{\nu}\,T^{(-)ba}_{\beta\alpha}(\nu)&=\frac{1}{\nu}\sum_\gamma\left\{\frac{1}{\nu-\nu_\gamma{}^{\beta\alpha}}+\frac{1}{\nu+\nu_\gamma{}^{\beta\alpha}}\right\}\varrho^{(-)ba}_{\beta\gamma\alpha}(\lambda)\\&=-\frac{1}{f_\pi^2}\sum_\gamma\frac{(m_\beta^2-m_\gamma^2)\,(m_\gamma^2-m_\alpha^2)}{\nu^2-\nu_\gamma{}^{\beta\alpha 2}}\,[X_b(\lambda),X_a(\lambda)]_{\beta\alpha}.\end{aligned} \tag{2.15}$$

At $\nu=\nu_{\text{th}}$ the right-hand side reduces simply to the commutator of the matrices

$$\frac{1}{\nu}\,T^{(-)ba}(\nu)_{\beta\alpha}\Big|_{\nu=\nu_{\text{th}}}=-\frac{1}{f_\pi^2}\,[X_b(\lambda)\,X_a(\lambda)]_{\beta\alpha}. \tag{2.16}$$

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[3]) The pole positions of $T^{(F)ba}_{(\nu)\beta\alpha}$ are determined by $s-m_\gamma^2=\nu-\nu_\gamma^{\beta\alpha}$ with

$$\nu_\gamma^{\beta\alpha}\equiv m_\gamma^2-\frac{m_\alpha^2+m_\beta^2}{2}.$$

[4]) We use PCAC as $\partial A_0=f_\pi\mu^2\pi_a$ such that $f_\pi\approx .095$ GEV.

Invoking now the low-energy theorem of equ. (2.7), we find out that the matrices $X_a(\lambda)$ have to form, together with the isospin operator T_c, the algebra of $SU(2) \times SU(2)$:

$$[X_b(\lambda), X_a(\lambda)] = i\varepsilon_{bac}T_c. \tag{2.17}$$

Historically, these commutation rules for the coupling matrices $X_a(\lambda)$ to resonances were first derived by saturating the $SU(2) \times SU(2)$ charge algebra [1, 2]

$$[Q_b{}^5(x_0), Q_a{}^5(x_0)] = i\varepsilon_{bac}Q_c \tag{2.18}$$

with resonances in the infinite-momentum frame. Obviously, this procedure is completely equivalent to the use of a low-energy theorem, an unsubtracted dispersion relation and the narrow-resonance approximation for the imaginary part of the amplitude [3].

Given any representation of this algebra, the couplings $X_a(\lambda)$ can directly be compared with experiment if one makes use of the assumption of PCA C. According to this assumption the matrix element of

$$(\mu^2 - q^2) \langle \gamma p_\gamma \lambda | \, \partial A \, |\alpha p_\alpha \lambda\rangle \tag{2.19}$$

does not vary much when $q^2 = (p_\gamma - p_\alpha)^2$ is continued from $q^2 = 0$ to the physical mass^2-of the pion. Therefore we can write the on shell coupling constant

$$\langle \gamma p_\gamma \lambda \, | j_a{}^\pi | \, \alpha p_\alpha \lambda \rangle \Big|_{q^2 = m_\pi{}^2} \approx \frac{i}{f_\pi} (m_\beta{}^2 - m_\alpha{}^2) \, [X_a(\lambda)]_{\beta\alpha} \tag{2.20}$$

and we can express the pionic decay width $\Gamma_{\gamma\,\beta\pi_a}$ for the process $\gamma \to \alpha\pi_a$ directly in terms of $X_a(\lambda)$

$$\Gamma_{\gamma\to\alpha\pi_a} = \frac{(m_\gamma{}^2 - m_\alpha{}^2)^3}{16\pi f_\pi{}^2 \, m_{\pi\gamma}{}^3} \frac{f_l}{2J_\gamma + 1} \sum_\lambda |[X_a(\lambda)]_{\gamma\alpha}|^2 .^{5)} \tag{2.21}$$

The isospin even part of the amplitude $T^{(+)}(\nu)$ cannot as readily be turned over into a sum rule. The reason is that $T^{(+)}(\nu)$ will, in general, allow for the exchange of Regge poles with intercept $\alpha(0) > 0$, such that no unsubtracted dispersionrelation can be written down. Only if one considers the projection into the t-channel isospin $I_t = 2$ does $T^{(+)}$ presumably tend to zero for large ν since no Regge trajecsory is known with this exotic quantum number. In going through the same procedure for the $I_t = 2$ projection as for $1/\nu T^{(-)}$ we obtain then the sum rule

$$\begin{aligned} T^{(+)ba}_{\beta\alpha}(\nu) \Big|_{\substack{I_t=2 \\ \nu=\nu_{\text{th}}}} &= -\frac{1}{2f_\pi{}^2} \sum_\gamma (2m_\gamma{}^2 - m_\alpha{}^2 - m_\beta{}^2) \, [X_b(\lambda)_{\beta\gamma} \, X_a(\lambda)_{\gamma\alpha}] + \\ &= -\frac{1}{2f_\pi{}^2} \{[X_b, [m^2, X_a]] + (ab)\} \Big|_{I_t=2} \end{aligned} \tag{2.22}$$

which has to be equal to the $I_t = 2$ projection of the $\sum$ term.

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5) The factor f_l corrects for the finite pion mass in the phase space of the final state, which depends on the threshold behaviour of the amplitude

$$f_l \approx \left(1 - 2\,\frac{m_\pi{}^2(m_\gamma{}^2 + m_\alpha{}^2)}{(m_\gamma{}^2 - m_\alpha{}^2)^2}\right)^{l+1/2}.$$

In addition, one has to divide by a factor 2 if particle α and π_a are identical.

About the detailed form of $\sum$ term not much is really known. However, our present ideas concerning the mechanism that generates PCAC (via spontaneous symmetry breakdown of $SU(2)\times SU(2)$ with only little symmetry breaking in the Hamiltonian) tells us that it is small, proportional to the mass² of the pion. If we neglect at least the $I_t = 2$ part of $\sum^{ba}$, we find

$$[X_b, [m^2 X_a]]|_{I_t=2} = 0. \tag{2.23}$$

This equation is equivalent to saying that the mass² when seen as a (diagonal) matrix between the different particles, can be written as

$$[m^2]_{\beta\alpha} = m_\alpha^2\, \delta_{\beta\alpha} = [m_0^2]_{\beta\alpha} + [m_4^2]_{\beta\alpha} \tag{2.24}$$

where m_0^2 is $SU(2)\times SU(2)$ symmetric while m_4^2 transforms as the isosinglet component of a representation (1/2, 1/2) under $SU(2)\times SU(2)$ [3]. Thus for any solution of the $SU(2)\times SU(2)$ algebra, not only the pionic decay widths are determined but also the masses can be calculated in terms of a few reduced matrix elements only.

Also this result can be obtained by saturating directly the commutator

$$[Q_b^5(x_0), \int d^3x\, \partial A_a(x)]\,|_{I_t=2} = [Q_b(x_0), \dot{Q}_a(x_0)]|_{I_t=2} = 0 \tag{2.25}$$

with resonances in the infinite-momentum frame.

For the purpose of calculations it is convenient to introduce (hermitian) reduced matrix elements of X_a. Between isospin o and 1 we can write

$$[X_b(\lambda)]_{\gamma 0,\alpha a} \equiv \delta_{ba}\, G^{01}_{\gamma\alpha}(\lambda)\,;\ [X_b(\lambda)]_{\gamma c,\alpha 0} \equiv \delta_{cb} G^{10}_{(\lambda)\,\gamma\alpha} \tag{2.26}$$

and between isospin 1 and 1:

$$[X_b(\lambda)]_{\gamma c,\alpha a} \equiv i\varepsilon_{cba}\, G^{11}_{\gamma\alpha}(\lambda). \tag{2.27}$$

Then the commutation rules (2.17) are equivalent to the Adler Weisberger relations

$$\sum_\gamma G^{01}_{\beta\gamma}(\lambda)\, G^{11}_{\gamma\alpha}(0) = 0$$

$$\sum_\gamma G^{10}_{\beta\gamma}(\lambda)\, G^{01}_{\gamma\alpha}(\lambda) + G^{11}_{\beta\gamma}(\lambda)\, G^{11}_{\gamma\alpha}(\lambda) = 1_{\beta\alpha} \tag{2.28}$$

and the width formula becomes[6])

$$\Gamma_{\gamma\to\alpha\pi} = \frac{(m_\gamma^2 - m_\alpha^2)^3}{16\pi f_\pi^2\, m_\gamma^3}\, \frac{f_l}{(2J_\gamma + 1)} \sum_\lambda \begin{Bmatrix} |G^{10}_{\gamma\alpha}(\lambda)|^2 \\ 3|G^{01}_{\gamma\alpha}(\lambda)|^2 \\ 2|G^{11}_{\gamma\alpha}(\lambda)|^2 \end{Bmatrix} \text{ for } \begin{Bmatrix} I_\gamma = 1,\ I_\alpha = 0 \\ I_\gamma = 0,\ I_\alpha = 1 \\ I_\gamma = 1,\ I_\alpha = 1 \end{Bmatrix} \tag{2.29}$$

If a saturation scheme involves half integer isotopic spins 1/2 and 3/2, one intro-

[6]) Again with an additional Bose fator 1/2 if $\alpha \equiv \pi$.

duces the reduced coupling matrixes G by[7]) [4]

$$[X_b(\lambda)]^{11}_{\gamma\alpha} = \chi^{+'} \frac{\tau_b}{2} \chi\, G^{11}_{\gamma\alpha}(\lambda);\; [X_b(\lambda)]^{31}_{\gamma\alpha} = \frac{\sqrt{3}}{2} \chi_b{}^{+'} \chi G^{31}_{\gamma\alpha}$$

$$[X_b(\lambda)]^{33}_{\gamma\alpha} = \frac{3}{2}\, i\varepsilon_{cba}\, \chi_c{}^{+'}\, \chi_a\, G^{33}_{\gamma\alpha}(\lambda) \tag{2.30}$$

In this case the commutation rules (2.17) are equivalent to the Adler Weisberger relations

$$\sum_\gamma \left(G^{11}_{\beta\gamma}(\lambda)\, G^{11}_{\gamma\alpha}(\lambda) - G^{13}_{\beta\gamma}(\lambda)\, G^{31}_{\gamma\alpha}(\lambda)\right) = 1_{\beta\alpha}$$

$$\sum_\gamma \left(-G^{31}_{\beta\gamma}(\lambda)\, G^{11}_{\gamma\alpha}(\lambda) + 5G^{33}_{\beta\gamma}(\lambda)\, G^{31}_{\gamma\alpha}(\lambda)\right) = 0$$

$$\sum_\gamma \left(\frac{1}{2}\, G^{31}_{\beta\gamma}(\lambda)\, G^{13}_{\gamma\alpha}(\lambda) + G^{33}_{\beta\gamma}(\lambda)\, G^{33}_{\gamma\alpha}(\lambda)\right) = 1_{\beta\alpha} \tag{2.31}$$

and the width formula (2.21) becomes

$$\Gamma_{\Upsilon\to\alpha\pi} = \frac{(m_\gamma{}^2 - m_\alpha{}^2)^3}{16\pi\, f_\pi{}^2 m_\gamma{}^3}\, \frac{2f_l}{2J_\gamma + 1} \sum_{\lambda>0} \frac{3}{4} \begin{Bmatrix} |G^{11}_{\gamma\alpha}(\lambda)|^2 \\ |G^{31}_{\gamma\alpha}(\lambda)|^2 \\ 2|G^{13}_{\gamma\alpha}(\lambda)|^2 \\ 5|G^{33}_{\gamma\alpha}(\lambda)|^2 \end{Bmatrix} \text{ for } \begin{Bmatrix} I_\gamma = \frac{1}{2},\; I_\alpha = \frac{1}{2} \\ I_\gamma = \frac{3}{2},\; I_\alpha = \frac{1}{2} \\ I_\gamma = \frac{1}{2},\; I_\alpha = \frac{3}{2} \\ I_\gamma = \frac{3}{2},\; I_\alpha = \frac{3}{2} \end{Bmatrix}. \tag{2.32}$$

III. The Use of Saturation Schemes in Finite Energy Sum Rules

A few years ago it was observed that scattering amplitudes approach their asymptotic Regge limes at relatively low energies as soon as they have passed a few prominent low-energy resonances. If the scattering amplitude $T^{(\pm)}(\nu)$ at $t = 0$ is decomposed into pieces $t^{(\pm)}(\nu)$ and $t^{(\pm)}(-\nu)$ containing only the left and the right-hand cut, respectively,

$$T^{(\pm)}(\nu) = t^{(\pm)}(\nu) \pm t^{(\pm)}(-\nu) \tag{3.1}$$

then Regge theory says that for $\nu \to \infty$

$$t(\nu) \simeq -c(0)\frac{e^{-i\pi\alpha(0)}}{\sin \pi\alpha(0)} \left(\frac{\nu}{M^2}\right)^{\alpha(0)} \tag{3.2}$$

with

$$\operatorname{Im} t(\nu) \simeq c(0) \left(\frac{\nu}{M^2}\right)^{\alpha(0)} \tag{3.3}$$

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[7]) χ and χ_a are the isospinors of isospin 1/2 and 3/2, respectively. Since χ_a is pure $T = 3/2$, it satisfies $\tau^a \chi_a = 0$ and.

where $\alpha(0)$ is the intercept of the leading Regge trajectory and M^2 is some mass parameter (usually $\approx 1\ GeV^2$).
Since $t(\nu)$ has only a right hand cut, we can insert it into a Cauchy integral

$$0 = \frac{1}{2\pi i}\oint t(\nu)\, d\nu \tag{3.4}$$

where the contour passes below and above the cut in t and is closed by a circle of radius N. If we choose the energy N just in the region, where the resonance structure starts smoothening out to the Regge formulas (3.2), (3.3), we can ingegrate from zero to N by inserting the imaginary part coming from resonances only, and integrate over the circle by using the Regge approximation (3.2). In this way we obtain [5]

$$\frac{1}{\pi}\int_0^N \operatorname{Im} t(\nu)\, d\nu \simeq \frac{M^2}{\pi}\, c(0)\, \frac{1}{\alpha(0)+1}\left(\frac{\nu}{M^2}\right)^{\alpha(0)+1}. \tag{3.5}$$

This formula is called finite-energy sum rule (FESR). It provides a powerful tool relating low-energy resonance parameters to Regge couplings $c(0)$.
Suppose now we have calculated the coupling constants of pions in a saturation scheme of the $SU(2)\times SU(2)$ algebra up to an energy N, where Regge behaviour is setting in. These coupling constants can then be used to evaluate the left hand side of the FESR for any process $\alpha\pi_a \to \beta\pi_b$. Take for example $t^{(-)ba}(\nu)_{\beta\alpha}$ which is equal to the first term in equ. (2.15).

$$\begin{aligned} t^{(-)ba}_{\beta\alpha}(\nu) &= \sum_\gamma \frac{1}{\nu-\nu_\gamma^{\beta\alpha}}\, \varrho^{(-)ba}_{\beta\alpha\,\gamma}(\lambda) \\ &= -\frac{1}{2f_\pi^2}\sum_\gamma \frac{1}{\nu-\nu_\gamma^{\beta\alpha}}\,(m_\beta^2-m_\gamma^2)\,(m_\beta^2-m_\alpha^2)\,[X_{b_{\beta\gamma}}(\lambda),\ X_{a_{\gamma\alpha}}(\lambda)]_- \\ &= -\frac{1}{2f_\pi^2}\sum_\gamma \frac{1}{\nu-\nu_\gamma^{\beta\alpha}}\left[[m^2, X_b(\lambda)]_{\beta\gamma}, [m^2, X_a(\lambda)]_{\gamma\alpha}\right]_-. \end{aligned} \tag{3.6}$$

Inserting it into the left-hand side gives

$$\frac{1}{\pi}\int_0^N \operatorname{Im} t^{(-)ba}_{\beta\alpha}(\nu)\, d\nu = \frac{1}{2f_\pi^2}\sum_\gamma \left[[m^2, X_b(\lambda)]_{\beta\gamma}, [m^2, X_a(\lambda)]_{\gamma\alpha}\right]_-. \tag{3.7}$$

Let us call the commutators

$$i[m^2, X_a] \equiv m_a^2. \tag{3.8}$$

If we insert the chiral decomposition (2.24) of the mass matrix we find

$$m_a^2 = -i[X_a, m_4^2]. \tag{3.9}$$

Using this matrix the resonance contribution in the finite energy sum rule for $t^{(-)ba}_{\beta\alpha}$ can be replaced by the commutator $[m_b^2 m_a^2]$ giving [6]

$$-\frac{[m_b^2, m_a^2]_{\beta\alpha}}{2f_\pi^2} = \frac{M^2}{\pi}\, c^{(-)ba}_{\beta\alpha}(0)\, \frac{1}{\alpha_\varrho(0)+1}\left(\frac{N}{M^2}\right)^{\alpha_\varrho(0)+1}. \tag{3.10}$$

We have right away written $\alpha(0) = \alpha_\varrho(0)$ since the ϱ trajectory with $\alpha_\varrho \approx .5$ is governing the high energy behaviour. For any saturation scheme, the matrices $m_a{}^2$ can easily be calculated and equ. (3.10) provides us with a set of coupling constants for the ϱ trajectory at $t = 0$.

A similar treatment can be given to the $t^{(+)}$ amplitude. In order to obtain the amplitude with the weakest divergence at large energy we make use of the low-energy theorem at $t = 0$, $\nu = \nu_{\text{th}}$:

$$T^{(+)ba}_{\beta\alpha}(\nu)|_{\nu=\nu_{\text{th}}} = \frac{1}{f_\pi{}^2} \Sigma^{ba}_{\beta\alpha}. \tag{3.11}$$

Let us define the auxiliary amplitude

$$\tilde{t}^{(+)ba}_{\beta\alpha}(\nu) \equiv \frac{2\nu}{\nu^2_{\text{th}} - \nu^2}\, t^{(+)ba}_{\beta\alpha}(\nu). \tag{3.12}$$

Suppose above the energy N, $t^{(+)}$ behaves approximately like

$$t^{(+)ba}_{\beta\alpha}(\nu) \simeq -c^{(+)ba}_{\beta\alpha}(0)\, \frac{e^{-i\pi\alpha(0)}}{\sin \pi\alpha(0)} \left(\frac{\nu}{M^2}\right)^{\alpha(0)}. \tag{3.13}$$

Then the Cauchy integral gives

$$T^{(+)ba}_{\beta\alpha}(\nu_{\text{th}}) - \sum \frac{2\nu^{\beta\alpha}_\gamma}{\nu^2_{\text{th}} - \nu^{\beta\alpha 2}_\gamma}\, \varrho^{(+)ba}_{\beta\alpha} \simeq -\frac{2}{\pi}\, c^{(+)}_{\beta\alpha}(0)\, \frac{1}{\alpha(0)} \left(\frac{N}{M^2}\right)^{\alpha(0)} \times \left(1 - \varepsilon\left(\frac{m_\beta{}^2}{N}, \frac{m_\alpha{}^2}{N}\right)\right) \tag{3.14}$$

where $1 - \varepsilon(m_\beta{}^2/N, m_\alpha{}^2/N)$ denotes the following infinite series

$$1 - \varepsilon\left(\frac{m_\beta{}^2}{N}, \frac{m_\alpha{}^2}{N}\right) \equiv \frac{1}{2} \sum_{n=0}^{\infty} \frac{\alpha(0)}{\alpha(0) - n} \left[\left(\frac{m_\beta{}^2}{N}\right)^n + \left(\frac{m_\alpha{}^2}{N}\right)^n\right]. \tag{3.15}$$

This term has its origin in the denominator $1/(\nu^2_{\text{th}} - \nu^2)$ of the amplitude $\tilde{t}^{(+)ba}_{\beta\alpha}$. Notice that $\varepsilon(m_\beta{}^2/N, m_\alpha{}^2/N)$ vanishes as $m_\beta{}^2/N \to 0$ and $m_\alpha{}^2/N \to 0$ and can therefore be safely neglected if $m_\beta{}^2$ and $m_\alpha{}^2$ lie sufficiently low in the saturation scheme. If initial and final masses are taken to be large, however, the factor $1 - \varepsilon$ will become small or even negative and equation (3.14) does not serve any more for a reliable estimate of the Regge coupling.

The sum over resonances can be rewritten explicitly as

$$\begin{aligned} &\frac{1}{2f_\pi{}^2} \sum_\gamma (2m_\gamma{}^2 - m_\alpha{}^2 - m_\beta{}^2)\, [X_b(\lambda)_{\beta\gamma}\, X_a(\lambda)_{\gamma\alpha}]_+ \\ &= \frac{1}{2f_\pi{}^2} \left([X_b\, [m^2 X_a]_{\beta\alpha} + (ba)\right) = -\frac{[m_4{}^2]_{\beta\alpha}}{f_\pi{}^2}\, \delta_{ba}. \end{aligned} \tag{3.16}$$

Together with the low energy theorem (3.11) we finally find [*6*]:

$$\frac{[m_4{}^2]_{\beta\alpha}}{f_\pi{}^2}\, \delta_{ba} = \frac{\Sigma^{ba}_{\beta\alpha}}{f_\pi{}^2} + \frac{2}{\pi}\, c^{(+)ba}_{\beta\alpha}(0)\, \frac{1}{\alpha(0)} \left(\frac{N}{M^2}\right)^{\alpha(0)} (1 - \varepsilon). \tag{3.17}$$

Since the $\sum$ term vanishes in the chiral limit and is therefore expected to be small compared with the Regge term, we shall neglect it alltogether.
What value do we have to choose for the intercept $\alpha(0)$? The phenomenological discussion of FESR for $T^{(+)}$, mainly in the case of πN scattering, has shown that resonances in the direct channel are apparently not able to build up the asymptotic behaviour due to the diffraction part of the amplitude [7]. The non-diffractive part appears to be dominated by the exchange of an f trajectory. It's intercept is usually taken to be 1/2 (if one prefers a universal slope of all trajektories). For this value the series (3.15) can be summed explicitly giving

$$\varepsilon\left(\frac{m_\beta{}^2}{N}, \frac{m_\alpha{}^2}{N}\right) = \frac{1}{4}\sqrt{\frac{m_\beta{}^2}{N}}\ln\left[\frac{1+\sqrt{\frac{m_\beta{}^2}{N}}}{1-\sqrt{\frac{m_\alpha{}^2}{N}}}\right] + (m_\alpha{}^2 \leftrightarrow m_\beta{}^2). \tag{3.18}$$

Notice that as $m_\beta{}^2$ and $m_\alpha{}^2$ approach the upper region of the saturation scheme, ε moves close to one and aquires a strong dependence on N thus spoiling the sensitivity of formula (3.17) to $c_{\beta\alpha}^{(+)ba}$.

IV. Calculation of $[m_b{}^2, m_a{}^2]$

Suppose we are given a saturation sheme of the $SU(2) \times SU(2)$ algebra in formof the reduced coupling matrices

$$G^{01}_{\beta\alpha},\ G^{01}_{\beta\alpha} = [G^{01}]^+_{\beta\alpha},\ G^{11}_{\beta\alpha}. \tag{4.1}$$

A matrix $m_4{}^2$ can be constructed in terms of a few free parameters using its property of being the forth component of a (1/2, 1/2) representation. Let us introduce reduced matrix elements for $m_4{}^2$ between isospin zero and isospin one states, respectively:

$$[m_4{}^2]_{\beta 0,\alpha 0} \equiv M^{00}_{4\ \alpha\beta} \tag{4.2}$$

$$[m_4{}^2]_{\beta b,\alpha a} \equiv M^{11}_{4\ \beta\alpha}\,\delta_{ba}. \tag{4.3}$$

Then M_4 decribes directly the couplings of the f trajectory via equ. (3.16).
In order to obtain ϱ couplings we construct first $m_a{}^2 = -i[X_a(\lambda), m_4{}^2]$. If one introduces for $m_a{}^2$ reduced matrix elements M_V in a completely analogous fashion as for X_a, the commutator tells us:

$$M_V{}^{01} = -i(G^{01}M_4{}^{11} - M_4{}^{00}G^{01}), \qquad M_V{}^{10} = [M_V{}^{01}]^+$$
$$M_V{}^{11} = -i[G^{11}, M_4{}^{11}] = -iG^{11}M_4{}^{11} + \text{h.c.}. \tag{4.4}$$

Now consider the commutator $[m_b{}^2, m_a{}^2]$. Due to its antisymmetry in b, a, it is necessarily an object of isospin one. Therefore it vanishes between two isospin zero states. Between isospin zero and one we find:

$$[m_b{}^2, m_a{}^2]^{01}_{\beta 0,\alpha a'} = i\varepsilon_{baa'} \quad 2M_V{}^{01}M_V{}^{11}$$
$$[m_b{}^2, m_a{}^2]^{10}_{\beta b',\alpha 0} = i\varepsilon_{b'ba} \quad 2M_V{}^{11}M_V{}^{10} \tag{4.5}$$

and between two isospin one states

$$[m_b{}^2, m_a{}^2]^{11}_{\beta b?,\alpha a'} = i\varepsilon_{bac}\, i\varepsilon_{b'ca'}\, [M_V{}^{10} M_V{}^{01} + M_V{}^{11}\, M_V{}^{11}].$$

For saturation schemes involving the isotopic spins 1/2 and 3/2, we introduce the reduced matrix elements of $m_4{}^2$.

$$[m_4{}^2]^{11} = \chi^{+'}\chi\, M_4{}^{11} \tag{4.6}$$

$$[m_4{}^2]^{33} = \chi_a{}^{+'}\chi_a\, M_4{}^{33} \tag{4.7}$$

The reduced matrix M_V corresponding to $m_a{}^2$ is defined in analogy with equs (2.30). Using equ. (3.8) we calculate

$$\begin{aligned} M_V{}^{11} &= -i[G^{11}, M_4{}^{11}] = -iG^{11}M_4{}^{11} + \text{h.c.} \\ M_V{}^{31} &= -i(G^{31}M_4{}^{11} - M_4{}^{33}G^{31}) \\ M_V{}^{33} &= -i[G^{33}, M_4{}^{33}] = -iG^{33}M_4{}^{33} + \text{h.c.} \end{aligned} \tag{4.8}$$

From these expressions we can derive the commutators $[m_b{}^2, m_a{}^2]$ as

$$\begin{aligned} [m_b{}^2, m_a{}^2]^{11} &= i\varepsilon_{bac}\, \chi^{+'} \frac{\tau_c}{2}\, \chi\, \{M_V{}^{11}M_V{}^{11} - M_V{}^{13}M_V{}^{31}\} \\ [m_b{}^2, m_a{}^2]^{32} &= i\varepsilon_{bac}\, \frac{\sqrt{3}}{2}\, \chi_c{}^{+'}\, \chi \left\{-\frac{1}{2}\, M_V{}^{31}M_V{}^{11} + \frac{5}{2}\, M_V{}^{33}M_V{}^{31}\right\} \\ [m_b{}^2, m_a{}^2]^{33} &= i\varepsilon_{bac}\, \frac{3}{2}\, i\varepsilon_{b'ca'}\chi_{b'}^{+'}\chi_{a'} \left\{\frac{1}{2}\, M_V{}^{31}M_V{}^{13} + M_V{}^{33}M_V{}^{33}\right\} \end{aligned} \tag{4.9}$$

V. A Simple Saturation Scheme Involving $\pi\varrho A_1\sigma$ Mesons

The construction of saturation models proceeds by assigning certain irreducible $SU(2) \times SU(2)$ contents to every particle. Take for example the irreducible representation

$$v \equiv \left(\frac{1}{2}, \frac{1}{2}\right) \tag{5.1}$$

consisting of an isosinglet[8]) v_4 and an isovector v_a. The matrix elements X between these states are given by[9])

$$\begin{aligned} [X_b]_{v4,va} &= \delta_{ba} \\ [X_b]_{v4,v4} &= [X_b]_{vb,va} = 0\,. \end{aligned} \tag{5.2}$$

[8]) In our notation we make use of the isomorphism of T, X with the fourdimensional rotation group L, M. A four-vector transforms like (1/2 1/2) under $SU(2) \times SU(2)$.
[9]) Check that x_a satisfies the commutation rules Equ. (2.17)!

Another irreducible representation consists of an antisymmetric tensor[8])

$$t_{AB} = \begin{pmatrix} t_{ab} & -t_{4a} \\ t_{4a} & 0 \end{pmatrix} \equiv \begin{pmatrix} \varepsilon_{abc}\bar{t}_c & -t_{4a} \\ t_{4a} & 0 \end{pmatrix}; \quad \begin{matrix} \bar{t}_a \equiv [(1,0)_a + (0,1)_a]/\sqrt{2} \\ t_{4a} \equiv [(1,0)_a - (0,1)_a]/\sqrt{2} \end{matrix} \tag{5.3}$$

with the matrix elements[9])

$$\begin{aligned} [X_b]_{t_{4c},\bar{t}_a} &= i\varepsilon_{cba} \\ [X_b]_{t_{4c},t_{4a}} &= [X_b]_{\bar{t}_c,\bar{t}_a} = 0. \end{aligned} \tag{5.4}$$

Both representation can be mixed to form a reducible representation. We shall assign the most general mixture conserving isospin and G parity[10])to the mesons $\pi A_1 \varrho$ and σ at helicity $\lambda = 0$: [3]

$$\begin{aligned} \pi_a &= \cos\Psi\, t_{4a} + \sin\Psi\, v_a \\ A_{1a} &= -\sin\Psi\, t_{4a} + \cos\Psi\, v_a \\ \varrho_a &= \bar{t}_a; \qquad \sigma = v_4 \end{aligned} \tag{5.5}$$

with an arbitrary mixing angle Ψ. For the reduced matrix elements this amounts to

$$\begin{aligned} G^{00} &= 0 \\ G^{01} &= \sigma \begin{pmatrix} \pi & A_1 & \varrho \\ \sin\Psi & \cos\Psi & 0 \end{pmatrix} \\ G^{11} &= \begin{matrix} \pi \\ A_1 \\ \varrho \end{matrix} \begin{pmatrix} 0 & 0 & \cos\Psi \\ 0 & 0 & -\sin\Psi \\ \cos\Psi & -\sin\Psi & 0 \end{pmatrix} \end{aligned} \tag{5.6}$$

which easily can be checked to satisfy the Adler Weisberger relations (2.28). From the width formulas we find

$$\begin{aligned} \Gamma_{\rho\to\pi\pi} &= \frac{m_\varrho^3 f_1}{48\,\pi\, f_\pi^2} \cos^2\Psi \approx .135\,(2\cos^2\Psi)\ \text{GeV (for } m_\varrho \approx .76) \\ \Gamma_{\sigma\to\pi\pi} &= \frac{9}{2}\,\frac{m_\sigma^3 f_0}{48\,\pi\, f_\pi^2} \sin^2\Psi \approx .600\ (2\sin^2\Psi)\ \text{GeV (for } m_\sigma \approx .76) \\ \Gamma_{A_1\to\sigma\pi} &= \frac{(m_{A_1}^2 - m_\sigma^2)^3 f_1}{48\pi\, f_\pi^2\, m_{A_1}^3} \cos^2\Psi \approx .050\ (2\cos^2\Psi)\ \text{GeV (for } m_{A_1}^2 \approx 2m_\varrho^2). \end{aligned} \tag{5.7}$$

The experimental ϱ width tells us that we have to choose Ψ somewhere around 45°. The prediction for σ is about of the correct size. One presently prefers values for $\Gamma_{\sigma\to\pi\pi}$ of about 400–500 MeV. For $A_1 \to \sigma\pi$ no experimental results are available. Let us now see what restrictions are implied on the masses of these mesons by the superconvergence of the $I_t = 2$ amplitude. The most general matrix elements of

[10]) Since the G parity of X is negative, v_4 and v_a; t_{4a} and t_a have opposite G parities.

m_4^2 between v, t_{4a} and $\bar{t}_a$ are given by

$$[m_4^2]_{vv} = [m_4^2]_{t_{4b},t_{4a}} = [m_4^2]_{\bar{t}_b,\bar{t}_a} = 0$$
$$[m_4^2]_{v_b,t_{4a}} \equiv M_4 \delta_{ba}. \tag{5.8}$$

Between the particle states (5.5) this leads to a reduced matrix M_4:

$$M_4^{00} = 0$$
$$M_4^{11} = \begin{pmatrix} \sin 2\Psi & \cos 2\Psi & 0 \\ \cos 2\Psi & -\sin 2\Psi & 0 \\ 0 & 0 & 0 \end{pmatrix} M_4. \tag{5.9}$$

Together with the $SU(2) \times SU(2)$ singlet the total masses become

$$\begin{aligned} m^2_{\pi\pi} &= \cos^2 \Psi m_t^2 + \sin^2 \Psi m_0^2 + \sin 2\Psi M_4 \\ m^2_{A_1A_1} &= \sin^2 \Psi m_t^2 + \cos^2 \Psi m_1^2 - \sin 2\Psi M_4 \\ m^2_{\rho\rho} &= m_t^2 \\ m^2_{\sigma\sigma} &= m_v^2 \\ m^2_{\pi A_1} &= -\frac{1}{2} \sin 2\Psi (m_t^2 - m_v^2) + \cos 2\Psi M_4. \end{aligned} \tag{5.10}$$

Since m^2 between π and A_1 is zero, we find

$$\begin{aligned} m^2_{\pi\pi} &= \frac{1}{\cos 2\Psi} [\cos^2 \Psi m_\varrho^2 - \sin^2 \Psi m_\sigma^2] \\ m^2_{A_1A_1} &= \frac{1}{\cos 2\Psi} [-\sin^2 \Psi m_\varrho^2 + \cos^2 \Psi m_\sigma^2]. \end{aligned} \tag{5.11}$$

This implies one prediction independent of the mixing angle:

$$m_\pi^2 + m^2_{A_1} = m_\rho^2 + m_\sigma^2. \tag{5.12}$$

The mixing angle can be determined from the masses:

$$\tan^2 \Psi = \frac{m_\rho^2 - m_\pi^2}{m^2_{A_1} - m_\rho^2}. \tag{5.13}$$

Experimentally, $m_\sigma^2 \approx m_\rho^2$, $m_\pi^2 \approx 0$ and $m^2_{A_1} \approx 2m_\rho^2$, such that (5.12) is satisfied and equ. (5.13) yields $\tan \Psi \approx 1$ in good agreement with the value giving the correct ϱ width.

Let us calculate the Regge couplings of f and ϱ trajectories explicity in this saturation scheme. As we can read off equs. (5.9) the reduced matrix M_4 is in this case

$$M_4^{00} = 0$$
$$M_4^{11} = \begin{pmatrix} \sin 2\Psi & \cos 2\Psi & 0 \\ \cos 2\Psi & -\sin 2\Psi & 0 \\ 0 & 0 & 0 \end{pmatrix} M_4 \tag{5.14}$$

Thus we obtain the predictions that the f trajectory does not couple to $\sigma\sigma$ and $\rho\rho$ The transition from π to A_1 and the elastic A_1A_1 case cannot be calculated reliably since the correction terms ε are not small in these cases. The zero of M_4^{11} in the πA_1 transition ($\cos 2\Psi \approx 0$) and the negative value for A_1A_1 ($-\sin 2\Psi \approx -1$) are properties of the factor $1 - \varepsilon$ rather than the Regge coupling $c^{(+)}$. The elastic vertex $f\pi\pi$ comes out as follows: Since M_4 is equal to

$$M_4 = \frac{1}{2} \tan 2\Psi(m_t^2 - m_v^2) = \frac{1}{2} \sin 2\Psi(m_\pi^2 - m_{A_1}^2) \tag{5.15}$$

we insert equs (5.12) and (5.13) to find

$$M_4 = -\tan \Psi(m_{A_1}^2 - m_\rho^2) \approx -m_\rho^2. \tag{5.16}$$

Therefore our predicitions for the coupling strength of the f trajectory to π and A_1 are

$$\mp \frac{m_\varrho^2}{f_\pi^2} = \frac{2}{\pi} c^{(+)}_{\substack{\pi\pi \\ A_1A_1}} \frac{1}{\alpha_f(0)} \left(\frac{N}{M^2}\right)^{\alpha_f(0)}. \tag{5.17}$$

Let us now calculate the ϱ couplings. From (4.4) we find

$$M_V^{01} = -i(\sin\Psi, \cos\Psi, 0) \begin{pmatrix} \sin 2\Psi & \cos 2\Psi & 0 \\ \cos 2\Psi & -\sin 2\Psi & 0 \\ 0 & 0 & 0 \end{pmatrix} M_4$$

$$= -i(\cos\Psi, -\sin\Psi, 0)\, M_4$$

$$M_V^{10} = i \begin{pmatrix} \cos\Psi \\ -\sin\Psi \\ 0 \end{pmatrix} M_4$$

$$M_V^{11} = -i \begin{pmatrix} 0 & 0 & \cos\Psi \\ 0 & 0 & -\sin\Psi \\ \cos\Psi & -\sin\Psi & 0 \end{pmatrix} \begin{pmatrix} \sin 2\Psi & \cos 2\Psi & 0 \\ \cos 2\Psi & -\sin 2\Psi & 0 \\ 0 & 0 & 0 \end{pmatrix} M_4 + \text{h.c.}$$

$$= -i \begin{pmatrix} 0 & 0 & -\sin\Psi \\ 0 & 0 & -\cos\Psi \\ \sin\Psi & \cos\Psi & 0 \end{pmatrix} M_4. \tag{5.18}$$

Therefore the commutators become

$$[m_b^2, m_a^2]^{01} = [m_b^2, m_a^2]^{10} = 0$$

$$[m_b^2, m_a^2]^{11}_{\beta b', \alpha a'} = i\varepsilon_{bac}\, i\varepsilon_{b'ca'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_4^2. \tag{5.19}$$

Thus the ϱ trajectory does not connect with π, A_1 and ρ. The first two transitions are forbidden by G parity. The last one, however, is a dynamical result. It is

consistent with the prediction of scale invariant effective Lagrangians which states that the coupling of $\sigma\varrho\gamma$ should vanish[11]).
A second non-trivial prediction is that the ϱ trajectory at $t = 0$ fails to excite the pion to A_1. To my kn wledge there exists, as yet, no reliable analysis confirming this results.
Third, we predict that the ϱ trajectory couples to π, A_1 and ϱ, *all with equal strength* [*8*] $M_4 \approx m_\rho^4$. Explicitly, equ. (5.19) tells us that

$$-\frac{m_\varrho^4}{2f_\pi^2} = \frac{M^2}{\pi}\; c^{(-)}_{\substack{\pi\pi\\ \mathrm{A,A,}\\ \rho\rho}}\; \frac{1}{\alpha_\varrho(0)+1}\left(\frac{N}{M^2}\right)^{\alpha_\varrho(0)+1} \tag{5.20}$$

where, for brevity, we have removed the isospin factor $i\varepsilon_{bac}\, i\varepsilon_{b'ca'}$ from $C^{(-)ba}_{\beta\alpha}$ and the left hand side of the equation.

VI. Comparison with Veneziano Model of $\pi\pi$ Scattering

Since not much experimental information is available on the couplings of f and ϱ mesons, let us compare our predictions with another model, which incorporates some of the essential features of our scheme: the Veneziano model of $\pi\pi$ scattering [*9*—*11*]. This model satisfies the correct low-energy theorems of $T^{(\pm)}$, is asymptotically dominated by reggeized f and ϱ exchange in the t-channel, and builds up the Regge behaviour by a string of s-channel resonances. If β, α denote directly the isospin indices of target and final pion, the Veneziano amplitude is given by

$$\begin{aligned} T^{ba}_{\beta\alpha} &= T^{I_t=0}\,\frac{1}{3}\,\delta^{ba}\delta_{\beta\alpha} + T^{I_t=1}\,\frac{1}{2}\,(\delta_\alpha{}^a\delta_\beta{}^b - \delta_\beta{}^a\delta_\alpha{}^b) \\ &\quad + T^{I_t=2}\left[\frac{1}{2}(\delta_\alpha{}^a\delta_\beta{}^b + \delta_\beta{}^a\delta_\alpha{}^b) - \frac{1}{3}\,\delta^{ba}\delta_{\beta\alpha}\right] \end{aligned} \tag{6.1}$$

with

$$\begin{aligned} \frac{1}{3}\,T^{I_t=0} &= \frac{1}{2}\,\big(A(s,t) + A(t,u)\big) - \frac{1}{6}\,A(s,u) \\ \frac{1}{2}\,T^{I_t=1} &= \frac{1}{2}\,\big(A(s,t) - A(u,t)\big) \\ T^{I_t=2} &= A(s,u) \end{aligned} \tag{6.2}$$

where

$$A(s,t) = \frac{2m_\varrho^2}{f_\pi^2}\,\frac{1}{\pi}\,\frac{\Gamma\big(1-\alpha(s)\big)\,\Gamma\big(1-\alpha(t)\big)}{\Gamma\big(1-\alpha(s)-\alpha(t)\big)};\ \alpha(s) = \frac{1}{2} + \frac{s}{2m_\varrho^2}. \tag{6.3}$$

The coupling strengths of the s-channel resonances ϱ and γ are not too much different from those of our saturation scheme. They are somewhat smaller,

$$G^{11}_{\pi\varrho} = \frac{1}{\sqrt{\pi}},\qquad G^{10}_{\pi\sigma} = \frac{1}{\sqrt{\pi}}, \tag{6.4}$$

(instead of $\cos \approx 1/\sqrt{2}$, $\sin \nu \approx 1/\sqrt{2}$).

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[11]) Assuming that γ is emitted via a ρ meson continued to zero mass and that this in turn can be related to the coupling of the ρ trajectory at $t = 0$. See Sections XI and XII of the Reference given in the footnote on p. 378.

since more resonances are present filling up the Adler Weisberger relation (2.28). However, all those other resonances couple with considerable less strength (The next are f, ϱ' with $G^{01}_{f\pi} = G^{11}_{\varrho'\pi} = 1/\sqrt{6\pi}$). At high energies, Im $A(s, t)$ approaches

$$\operatorname{Im} A(s,t) \approx -\frac{2m_\varrho^2}{f_\pi^2} \frac{1}{\Gamma(\alpha(t))} \left(\frac{\nu}{2m_\varrho^2}\right)^{\alpha(t)} \tag{6.5}$$

which becomes at $t = 0$

$$\operatorname{Im} A(s,0) \approx -\frac{2m_\varrho^2}{f_\pi^2} \frac{1}{\sqrt{\pi}} \left(\frac{\nu}{2m_\varrho^2}\right)^{\alpha(0)} \tag{6.6}$$

As a consequence, Im $t^{(\pm)}$ behave asymptotically as

$$\operatorname{Im} t^{(+)ba}_{\beta\alpha} \approx \delta^{ba}\delta_{\beta\alpha} \left[-\frac{m_\varrho^2}{f_\pi^2} \frac{1}{\sqrt{\pi}} \left(\frac{\nu}{2m_\varrho^2}\right)^{1/2}\right] \tag{6.7}$$

$$\operatorname{Im} t^{(-)ba}_{\beta\alpha} \approx i\varepsilon_{bac}\, i\varepsilon_{\beta c\alpha} \left[-\frac{m_\varrho}{f_\pi^2} \frac{1}{\sqrt{\pi}} \left(\frac{\nu}{2m_\varrho^2}\right)^{1/2}\right]. \tag{6.8}$$

Therefore $c^{(+)}_{\text{Van}}$ and $c^{(-}_{\text{Van}}$ are both equal to

$$c^{(+)}_{\text{Ven}} = c^{(-)}_{\text{Ven}} = -\frac{m_\varrho^2}{f_\pi^2} \frac{1}{\sqrt{\pi}} \approx -.56 \frac{m_\varrho^2}{f_\pi^2}. \tag{6.9}$$

This property of being equal is called exchange degeneracy of ϱ and f trajectory. It is shared by all amplitudes satisfying FESR in which the exotic $I_t = 2$ amplitude vanishes at high energies and in which the exotic s-channel $I_s = 2$ is free of resonance poles.

Now consider our predictions (5.17) and (5.19). In order to compare the results we have to fix $M^2 = 2m_\varrho^2 \approx 1$ GeV^2. Therefore we find

$$c^{(+)} \simeq -\frac{\pi}{2} \left(\frac{2m_\varrho^2}{N}\right)^{1/2} \frac{m_\varrho^2}{f_\pi^2}$$

$$c^{(-)} \simeq -\frac{3\pi}{4} \left(\frac{2m_\varrho^2}{N}\right)^{3/2} \frac{m_\varrho^2}{f_\pi^2}.$$

We now have to choose the energy N. Since our saturation scheme is rather small, the results will be somewhat sensitive to the exact value of N to be taken. Considering that our scheme contains all important resonances up to the f meson at $3m_\varrho^2$, we choose simply

$$N = 3m_\varrho^2.$$

For this value, $c^{(+)}$ and $c^{(-)}$ take both the value[6]

$$c^{(\pm)} \simeq -\frac{\pi}{2} \left(\frac{2}{3}\right)^{1/2} \frac{m_\varrho^2}{f_\pi^2} \approx -.64 \frac{m_\varrho^2}{f_\pi^2}.$$

The agreement with the couplings predicted by the Veneziano model is excellent. Notice also, that our predictions will always show exchange degeneracy (at least approximately). The reason is simply that we have enforced $T^{(+)}|_{I_t=2} \xrightarrow{\nu\to\infty} 0$ by taking $m^2 = m_0^2 + m_4^2$ and that the s-channel contains no exotic resonances by construction. Therefore if one considers the exact choice of N to be dubious one may simply prefer the value exhibiting exact exchange degeneracy.

VII. A Simple Saturation Scheme involving N, Δ (1236), N*

In the case of baryons the representation (1, 1/2) contains the most important particles N and Δ and the couplings are the same as those obtained from $SU(4)$ symmetry[12])

$$G^{11} = \frac{5}{3}, \qquad G^{31} = \frac{4}{3}, \qquad G^{33} = \frac{1}{3}.$$

The simplest saturation scheme going beyond this is constructed by admixing a (1/2, 0) representation which amounts to introducing one more $T = 1/2$ resonance N^*[3]. Defining the physical states as

$$N = \cos\Psi \left(1, \frac{1}{2}\right)_1 + \sin\Psi \left(\frac{1}{2}, 0\right)$$

$$N^* = -\sin\Psi \left(1, \frac{1}{2}\right)_1 + \cos\Psi \left(\frac{1}{2}, 0\right)$$

$$\Delta = \left(1, \frac{1}{2}\right)_3 \tag{7.1}$$

we find

$$G^{11} = \frac{1}{3}\begin{pmatrix} 4 + \cos 2\Psi & -\sin 2\Psi \\ -\sin 2\Psi & 4 - \cos 2\Psi \end{pmatrix} \tag{7.2}$$

$$G^{31} = \frac{4}{3}\,(\cos\Psi, -\sin\Psi)$$

$$G^{33} = \frac{1}{3}.$$

Experimentally, one knows the couplings

$$g_A \equiv G^{11}_{NN} = 1 + \frac{2}{3}\cos^2\Psi \approx 1.23. \tag{7.3}$$

and

$$G^{31}_{\Delta N} = \frac{4}{3}\cos\Psi \approx 1 \tag{7.4}$$

from the width $\Gamma_{\Delta\to N\pi} \approx 120$ MeV.
The scheme has the basic defect that no choice of Ψ can really satisfy both. The mixing angle Ψ may vary from 54^0, where g_A is correct but $G^{31}_{\Delta N} \approx .78$ too small, to

[12]) They obviously fulfill the Adler Weisberger relations (2.31).

41^0, where $G^{31}_{\Delta N}$ is correct but g_A is too large (≈ 1.37).
If we enforce $m^2 = m_0{}^2 + m_4{}^2$ we find

$$\tan^2 \Psi = \frac{m_\Delta{}^2 - m_N{}^2}{m^2_{N^*} - m_\Delta{}^2} \tag{7.5}$$

which puts the resonance N* anywhere from 1.38 to 1.55 GeV. Therefore we shall call it tentatively the Roper resonance and denote it by R. Its coupling is predicted as $G^{11}_{RN} \approx -.33$ in the whole range of Ψ. Experimentally the Roper resonance decays with $\Gamma_{R\to N\pi} \approx 100$ to 240 MeV giving $\|G^{11}_{RN}\| \approx .32$ to .5 compatible with the predicted number.
Since m_4 can connect only (1, 1/2) with (1/2, 0) the mass splitting matrix $M_4{}^{2T_\beta, 2T_\alpha}$ has the form

$$M_4{}^{11} = \begin{pmatrix} \sin 2\Psi & \cos 2\Psi \\ \cos 2\Psi & -\sin 2\Psi \end{pmatrix} M_4$$

$$M_4{}^{33} = 0 \tag{7.6}$$

where

$$M_4 \equiv \left\langle \left(1, \frac{1}{2}\right) \middle| m_4{}^2 \middle| \left(\frac{1}{2}, 0\right) \right\rangle = \frac{1}{2} \sin 2\Psi \,(m_N{}^2 - m_R{}^2)$$

$$\approx -\begin{Bmatrix} .5 \\ .76 \end{Bmatrix} \mathrm{GeV}^2 \text{ for } \Psi \approx \begin{Bmatrix} 54^\circ \\ 41^\circ \end{Bmatrix} \tag{7.7}$$

Thus we obtain for the coupling of f to NN:

$$(c_f)^{-\pi-\pi}_{\rho\rho} \approx -\frac{\pi}{4f_\pi{}^2} \left(\frac{M^2}{N}\right)^{1/2} \begin{Bmatrix} .5 \\ .76 \end{Bmatrix} \mathrm{GeV}^2$$

$$\approx -\begin{Bmatrix} 30 \\ 46 \end{Bmatrix} \text{ for } \Psi \approx \begin{Bmatrix} 54^0 \\ 41^0 \end{Bmatrix}, \tag{7.8}$$

while experimentally one finds[13] the value

$$(c_f)^{\pi-\pi-}_{PP} \equiv c^{(+)\pi\pi}_{NN} = -2m\beta^{(+)} = -2m(27.2 \pm 1)\ \mathrm{GeV}^{-1} \approx -53.6 \pm 2. \quad (7.9).$$

In order to calculate the coupling of the ϱ trajectory we insert equ. (7.6) in equs (4.9).
If we denote the reduced matrix elements contained in the curly brackets of (4.9) by R, we calculate

$$R^{11} = \frac{4}{9} \begin{pmatrix} 1 - 4\sin^2 \Psi & -2 \sin 2\Psi \\ -2\sin 2\Psi & 1 - 4\cos^2 \Psi \end{pmatrix} M_4{}^2$$

$$R^{31} = \frac{4}{9} (-\cos \Psi, \sin \Psi)\, M_4{}^2$$

$$R^{33} = \frac{8}{9} M_4{}^2 . \tag{7.10}$$

[13]) According to the review of C. Michael, Springer Tracts of Modern Physics **55**, 182 (1971) one has for πN scattering $\mathrm{Im}\,(A + \nu B) \approx \beta(\nu/M^2)^\alpha$ with $\beta^{(+)} = (27.2 \pm 1)\ \mathrm{GeV}^{-1}$ and $\beta^{(-)} = (14.7 \pm .8)/3\ \mathrm{GeV}^{-1}$ $(A + \nu B \equiv -T/2m)$.

For the angles $\Psi = \{^{54^0}_{41^0}\}$ this gives numerically

$$R^{11} \approx -\begin{pmatrix} .18 & \begin{Bmatrix} .22 \\ .51 \end{Bmatrix} \\ \begin{Bmatrix} .22 \\ .51 \end{Bmatrix} & \begin{Bmatrix} .00 \\ .33 \end{Bmatrix} \end{pmatrix} \text{GEV}^4$$

$$R^{31} \approx \left(-\begin{Bmatrix} .07 \\ .20 \end{Bmatrix}, \begin{Bmatrix} .90 \\ .17 \end{Bmatrix} \right) \text{GeV}^4$$

$$R^{33} \approx \begin{Bmatrix} .22 \\ .52 \end{Bmatrix} \text{GeV}^4. \tag{7.11}$$

Notice that several numbers show an unpleasant strong dependence on the mixing angle Ψ. It is gratifying to note, that the most interesting matrix element R^{11}_{NN} is rather stable in Ψ. Unfortunately it leads to a completely wrong ρNN Regge coupling. From equ. (3.10) we find

$$(c_\varrho)^{\pi^-\pi^-}_{\text{PP}} \approx \frac{3}{4}\frac{\pi}{M^2}\left(\frac{M^2}{N}\right)^{3/2} R^{11}_{\text{NN}}/2 \approx 6 \tag{7.12}$$

while experimentally [14]) this value is found to be $-2m(14.7 \pm .8)/3\ \text{GeV}^{-1} \approx -10$. The reason for this serious defect of the saturation scheme is easily discovered: If we write R^{11}_{NN} explicitly as a sum of intermediate states of isospin 1/2 and 3/2 we find from (4.9),

$$R^{11}_{\text{NN}} = \sum_i (m_{\text{N}}{}^2 - m^2_{\text{N}_i})^2 (G^{11}_{\text{N}_i\text{N}})^2 - (m_{\text{N}}{}^2 - m^2_{\Delta_i})^2 (G^{31}_{\Delta_i\text{N}})^2.$$

The experimental sign is due to the fact that this sum is dominated by large masses of isospin 1/2. This interpretation is confirmed by a look at the standard figure [*5*] showing the validity of finite-energy sum rules for πN scattering in the $T^{(-)}$ amplitude. The Regge behaviour builds up in the region where isospin 1/2 resonances are dominant. In our scheme, however, we have explicitely

$$R^{11}_{\text{NN}} = \Big(\underset{N}{\frac{4}{9}} - \frac{16}{9}\sin^2\Psi\underset{\Delta}{\Big)} M_4{}^2 \tag{7.13}$$

and the Δ resonance makes R^{11}_{NN} negative. The only way to remedy this defect is by going to a more realistic scheme containing high-mass $T = 1/2$ resonances. This will be done in future work.

VIII. Outlook

Encouraged by these first results based on an extremely simple saturation scheme we suggest investigations on the following subjects:

1. Properties of f and ρ couplings in larger meson as well as baryon saturation schemes [*4*].

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[14]) See footnote on p. 393

2. Helicity flip properties of f and ρ couplings. For this one simply has to apply our procedure to helicity flip amplitudes.
3. Identification of $i[m_a^2 m_4^2]$ as the A_1 Regge couplings. Here one has to algebraize appropriate finite-energy sum rules of amplitudes involving three pions.
4. Possibility that the operators T_a, X_a, m_a^2, m_4^2 form a larger Lie algebra. The observation that $[m_a^2, m_b^2]$ comes out proportional to T_c (ϱ universality) is certainly an indication that this could happen.
5. Links with the algebra of Regge residues of Cabbibo, Horwitz and Ne'eman [12].
6. Possible connections with the infinitely rising meson schemes of Brout et al [13].
7. Extension of the method of chiral $SU(3) \times SU(3)$ saturation schemes.

Once these questions will be answered we may obtain a joint understanding of particle *and* Regge couplings within *one* algebraic scheme.

References

[1] S. Adler and R. Dashen, *Current Algebras*, W. A. Benjamin, Inc., N.Y., 1968.
B. Renner, *Current Algebras and their Applications*, Pergamon Press, Oxford 1968.
[2] R. Dashen and M. Gell-Mann, Coral Gables Conference on Symmetry Principles at High Energy, Vol. 3 (1966);
R. Gatto, L. Maiani and G. Preparata, Physics **3**, 1 (1967); Phys. Rev. Letters **16**, 377 (1966);
I. Gerstein and B. W. Lee, Phys. Rev. Letters **16**, 1060 (1966); Nuovo Cimento 44 A, 1279 (1966);
H. Harari, Phys. Rev. Letters **16**, 964 (1966);
N. Cabibbo and H. Ruegg, Phys. Letters **22**, 85 (1966);
F. Buccella et. al., Nuclear Phys. **6B**, 430 (1968); Nuclear Phys. **8B**, 521 (1968); Nuovo Cimento Letters **1**, 556 (1969); **2**, 57 (1969); Nuclear Phys. **22B**, 651 (1970).
[3] S. Weinberg, Phys. Rev. **177**, 2604 (1969).
[4] F. Buccella, H. Kleinert, C. A. Savoy et al., Nuovo Cimento **69A**, 133 (1970).
[5] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).
[6] H. Kleinert, Phys. Letters **39B**, 511 (1972).
[7] H. Harari and Y. Zarmi, Phys. Rev. **187**, 2230 (1969).
[8] G. Costa and C. A. Savoy, Nuovo Cimento **56A**, 1168 (1968).
[9] G. Veneziano, Nuovo Cimento **62A**, 190 (1968);
[10] C. Lovelace, Phys. Letters **28A**, 265 (1968);
C. A. Savoy, Lettere Nuovo Cimento **1**, 263 (1969).
H. Osborn, ibid., p. 513 (1969).
[11] J. Shapiro and J. Yellin UCRL-18500 (1968);
J. Yellin, Notes on $\pi\pi$ Scattering UCRL Reports (1969).
[12] R. Brout, F. Englert, and C. Truffin, Phys. Letters **B29**, 590, 688 (1969);
F. Englert, R. Brout, and H. Stern, Nuovo Cimento **66A**, 845 (1970).
[13] N. Cabibbo, L. Horwitz, Y. Ne'eman, Phys. Letters **22**, 336 (1966).

Note added in proof:

While this paper was in print, most of the investigations proposed in the outlook have successfully been performed. The couplings of *f* and ϱ trajectories to larger meson schemes are given in H. Kleinert and L. R. Ram Mohan, Nuclear Phys. **52B**, 253 (1973). Points 3., 4. and 5 of the outlook are solved in H. Kleinert, Letters Nuovo Cimento **6**, 583

(1973). There it is shown that T_a, X_a, $-i\,[m_a^2, m_b^2]$, $-i\,[m_a^2, m_4^2]$, m_a^2, m_4^2 form the „superalgebra“ $SU(2) \times SU(2) \times O(5)$ of charges and Regge couplings. In addition, the connection of this superalgebra with the algebra of bilocal form factors of Fritzsch and Gell-Mann has been established in H. KLEINERT, Berlin preprint, March 1973, and a combined algebraic view of currents and Regge couplings has been obtained.