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Collective Quantum Fields

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Abstract

A general quantum field theory of collective excitations is presented for many-body systems. Lagrangians involving fundamental particles are transformed to collective fields exactly via Feynman path integrals. Graphically, this amounts to a complete resummation of the perturbation series without the danger of double counting. Plasma and pairing effects provide special examples corresponding to mutually complementary transformations. Bose and Fermi systems can be treated on equal footing. The methods are illustrated by giving the *exact* collective Lagrangian for the BCS model and approximate ones for type II superconductors and ^{3}He.

Contants

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I. Introduction

Under convenient circumstances, many-body systems can well be approximated by a gas of weakly interacting collective excitations. If this happens, it is desirable to replace the original action involving the fundamental fields (electrons, nucleons, ^{3}He, ^{4}He atoms, etc.) by another one in which all these excitations appear as explicit *independent* quantum fields. It will turn out that such replacements can be performed in many different ways without changing the physical content of the theory. Under certain circumstances there may be a choice of fields associated with *dominant* collective excitations displaying weak residual interactions which can be treated perturbatively. In such situations the collective field language greatly simplifies the description of the physical system.
Consider a system of fermions interacting via a two-body potential which may be generated by the exchange of some more fundamental particles such as photons or phonons. It is well-known that, depending on the band structure, two distinct collective modes will be important: plasma oscillations or pairing vibrations. The first effect is seen if the exchanged particle generating the potential couples strongly to virtual fermion-hole states. The second case is observed if the fermions are likely to form two-particle bound states. Examples are the plasmons in a degenerate electron gas, or the excitons in a semi- and the Cooper pairs in a super-conductor. Graphically, it is known that one of the two effects will be dominant depending on whether ring or ladder graphs provide the main contributions.
It is the purpose of these notes to discuss a simple technique via Feynman path integral formulas in which the transformation to collective fields amounts to mere changes of integration variables in functional integrals. After the transformation, the path formulation will again be discarded. The resulting field theory is quantized in the standard fashion and the fundamental quanta directly describe the collective excitations.
For systems showing plasma type of excitations, a real field depending on one space and time variable is most convenient to describe all physics. For the opposite situation in which dominant bound states are formed, a complex field depending on two space and one or two time coordinates will render the more economic description. Such fields will be called bilocal. If the potential becomes extremely short range, the bilocal field degenerates into a local field. In the latter case a classical approximation to the action of a superconducting electron system has been known for some time: the Ginzburg-Landau equation. The complete *bilocal* theory has been studied in elementary-particle physics where it plays a role in the transition from inobservable quark to observable hadron fields.
The change of integration variables in path integrals will be shown to correspond to an exact resummation of the perturbation series thereby accounting for phenomena which cannot be described perturbatively. The path formulation has the great advantage of translating all quantum effects among the fundamental particles completely into the field language of collective excitations. All radiative corrections may be computed

using only propagators and interaction vertices of the collective fields. The method presented here is particularly powerful when a system is in a region where several collective effects become simultaneously important. An example is the electron gas at lower density where ladder graphs gain increasing importance with respect to ring graphs thus mixing plasma and pair effects. In ^{3}He pair effects are dominant but plasma effects provide strong corrections.

We shall illustrate the functional approach by discussing first conventional systems such as electron gas, superconductor and ^{3}He. After this, we investigate soluble models in order to understand precisely the mechanism of the functional field transformations as well as the relation between the Hilbert spaces generated once from fundamental and once from collective quantum fields.

II. Path integrals

II.1. The Functional Formulas for Vacuum Amplitudes

Consider the general case of many-body system described by an action

$$\mathcal{A} = \int d^3x\, dt \psi^+(\boldsymbol{x}, t) \big(i\partial_t - \varepsilon(-i\nabla)\big) \psi(\boldsymbol{x}, t) - \frac{1}{2} \int d^3x\, dt\, d^3x'\, dt' \psi^+(\boldsymbol{x}', t')\, \psi^+(\boldsymbol{x}, t) \times V(\boldsymbol{x}, t; \boldsymbol{x}'t')\, \psi(\boldsymbol{x}, t)\, \psi(\boldsymbol{x}', t') \equiv \mathcal{A}_0 + \mathcal{A}_{\text{int}} \tag{2.1}$$

with a translationally invariant two-body potential

$$V(\boldsymbol{x}, t; \boldsymbol{x}', t') = V(\boldsymbol{x} - \boldsymbol{x}', t - t'). \tag{2.2}$$

In many physically important cases the potential is, in addition, instantaneous in time

$$V(\boldsymbol{x}, t; \boldsymbol{x}', t') = \delta(t - t')\, V(\boldsymbol{x} - \boldsymbol{x}')$$

allowing for great simplifications of later results. The fundamental field $\psi(x)$ may describe bosons or fermions. The full solution of the theory amounts to the determination of all Green's functions in the Heisenberg picture:

$$G(\boldsymbol{x}_1 t_1, \ldots, \boldsymbol{x}_n t_n; \boldsymbol{x}_{n'} t_{n'}, \ldots, \boldsymbol{x}_{1'} t_{1'}) = \langle 0|\, T\big(\psi_H(\boldsymbol{x}_1 t_1) \vdots \psi_H(\boldsymbol{x}_n t_n)\, \psi_H{}^+(x_{n'} t_{n'}) \vdots \psi_H(\boldsymbol{x}_{1'} t_{1'})\big) |\, 0\rangle. \tag{2.3}$$

It is convenient to view all these Green's functions as derivatives of the generating functional

$$Z[\eta^+, \eta] = \langle 0 \big(T \exp\big\{i \int d^3x\, dt \big(\psi^+(\boldsymbol{x}t)\, \eta(\boldsymbol{x}t) + \eta^+(\boldsymbol{x}t)\, \psi(\boldsymbol{x}t)\big)\big\} |0\rangle \tag{2.4}$$

namely

$$G(\boldsymbol{x}, t_1, \ldots, \boldsymbol{x}_n t_n; \boldsymbol{x}_{n'} t_{n'}, \ldots, x_{1'} t_{1'}) = (-i)^{n+n'} \left. \frac{\delta^{n+n'} Z[\eta^+, \eta]}{\delta\eta^+(\boldsymbol{x}_1 t_1) \vdots \delta\eta^+(\boldsymbol{x}_n t_n)\, \delta\eta(\boldsymbol{x}_{n'} t_{n'}) \cdots \delta\eta(\boldsymbol{x}_{1'} t_{1'})} \right|_{\eta=\eta^+\equiv 0}. \tag{2.5}$$

Physically, the generating functional describes the amplitude that the vacuum will remain a vacuum in spite of the presence of external sources.

The calculation of these Green's functions is usually performed in the interaction picture which can be summarized by the operator expression for Z:

$$Z[\eta^+, \eta] = N \langle 0|\, T \exp\big\{i\mathcal{A}_{\text{int}}[\psi^+, \psi] + i \int d^3x\, dt \big(\psi^+(\boldsymbol{x}t)\, \eta(\boldsymbol{x}t) + \text{h.c.}\big)\big\} |0\rangle. \tag{2.6}$$

Here the fields ψ possess free-field propagators and the normalization constant N is determined by the condition (which is trivially true for (2.4)):

$$Z[0, 0] = 1. \tag{2.7}$$

The standard perturbation theory is obtained by expanding exp $(i\mathscr{A}_{\text{int}})$ of (2.6) in a power series and bringing the resulting expressions to normal order via Wick's expansion technique. The perturbation expansion of (2.6) often serves conveniently to *define* an interacting theory. Every term can be pictured graphically and has a physical interpretation as a virtual process.

Unfortunately, the perturbation series up to a certain order in the coupling constant is unable to describe many important physical phenomena, for example bound states in the vacuum and collective excitations in many-body systems. Those require the summation of infinite subsets of diagrams to all orders. In many situations it is well-known which subsets have to be taken in order to account approximately for specific effects. What is not so clear is how such lowest approximations can be improved in a systematic manner. The point is that as soon as a selective summation is performed, the original coupling constant has lost its meaning as an organizer of the expansion and there is need for a new systematics of diagrams. Such a systematics will be presented in what follows.

As soon as bound states or collective excitations are formed, it is very suggestive to use *them* as new quantum fields rather than the original fundamental particles ψ. The goal would then be to rewrite the expression (2.6) for $Z[\eta^+, \eta]$ in terms of new fields whose unperturbed propagator has the free energy spectrum of the *bound states* or collective excitations and whose $\mathscr{A}_{\text{int}}$ describes *their* mutual interactions. In the operator form (2.6), however, such changes of fields are hard to conceive.

A much more flexible description of the quantum physics contained in the functional $Z[\eta^+, \eta]$ is offered by Feynman's path integral formulas [*1*, *2*]. There, changes of fields amount to changes of integration variables. Feynman's formula is based on the observation that the amplitudes of diffraction phenomena of light are obtained by summing over the individual amplitudes for all paths the light could possibly have taken, each of them being a pure phase depending only on the action of the light particle along the path. In the general field system (2.1), this principle leads to the alternative formula for the amplitude, $Z[\eta^+, \eta]$, that the vacuum goes over into the vacuum in the presence of external sources:

$$Z[\eta^+, \eta] = N \int D\psi^+(\boldsymbol{x}, t)\, D\psi(\boldsymbol{x}, t) \exp\left\{i\mathscr{A}[\psi^+, \psi] + \int d^3x\, dt\big(\psi^+(\boldsymbol{x}, t)\, \eta(\boldsymbol{x}, t) + \text{h.c.}\big)\right\}. \tag{2.8}$$

Notice that the field ψ in this formulation is a complex number and *not* an operator. All quantum fluctuations are accounted for by the fact that the path integral includes also the classically forbidden paths, i.e. all those which do not run through the valley of extremal action in the exponent.

The integral may conveniently be defined by grating the space-time into finer and finer cubic lattices of size δ with corners at $(x, y, z, t) = (i_1, i_2, i_3, i_4)\,\delta$, introducing fields at each such points,

$$\psi_{i_1i_2i_3i_4} \equiv \psi(x_{i_1}, y_{i_2}, z_{i_3}, t_{i_4})\sqrt{\delta}^{\,4}, \tag{2.9}$$

and performing the product of all the integrals at each lattice point, i.e.

$$\int D\psi^+(\boldsymbol{x}t)\, D\psi(\boldsymbol{x}t) \equiv \prod_{\substack{i_1i_2i_3i_4\\ i_1'i_2'i_3'i_4'}} \iint \frac{d\psi^\dagger_{i_1i_2i_3i_4}\, d\psi_{i_1'i_2'i_3'i_4'}}{\sqrt{2\pi i}\,\sqrt{2\pi i}} \tag{2.10}$$

where the double integral over complex variables $\int\int d\psi^+ \, d\psi$ stands symbolic for the real integrals $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} d\left(\frac{\psi+\psi^+}{\sqrt{2}}\right) d\left(\frac{\psi-\psi^+}{\sqrt{2}\,i}\right)$. This naive definition of path integration is straightforward for Bose fields. A slight complication arises in the case of Fermi fields. Here the fields must be taken as anticommuting c-numbers[1]).

All results to be derived later will make us of only one simple class of integrals which are the generalization [2] of the elementary Gaussian (or Fresnel) formula for $A > 0$:

$$\int\limits_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi i}} \exp\left(\frac{i}{2}\,\xi A \xi\right) = A^{-1/2}. \tag{2.11}$$

First, one considers a multidimensional real space $(\xi_1, \ldots, \xi_k, \ldots)$ in which clearly

$$\int\limits_{-\infty}^{\infty} \prod_k \frac{d\xi_k}{\sqrt{2\pi i}} \exp\left(\frac{i}{2} \sum_k \xi_k A_k \xi_k\right) = \left[\prod_k A_k\right]^{-1/2}. \tag{2.12}$$

Now if A_{kl} is an arbitrary symmetric positive matrix, and the exponent has the form $i/2 \sum\limits_{k,l} \xi_k A_{kl} \xi_l$, an orthogonal transformation can be used to bring A_{kl} to diagonal form without changing the maesure of integration. Thus an equation like (2.12) is still valid with the right-hand side denoting the product of eigenvalues of A_{kl}. This can also be written as

$$\int\limits_{-\infty}^{\infty} \prod_m \frac{d\xi_m}{\sqrt{2\pi i}} \exp\left(\frac{i}{2} \sum_{k,l} \xi_k A_{kl} \xi_l\right) = [\det A]^{-1/2}. \tag{2.13}$$

If, more generally, ξ is complex and A hermitian and positive, the result (2.13) follows separately for the real and for the imaginary part yielding

$$\int \prod_m \frac{d\xi_m{}^+ d\xi_m}{\sqrt{2\pi i}\,\sqrt{2\pi i}} \exp\left(i \sum_{k,l} \xi_k{}^+ A_{kl} \xi_l\right) = [\det A]^{-1}. \tag{2.14}$$

If the integrals are performed over anticommuting real or complex variables ξ or ξ^+, ξ, the right-hand sides of formulas (2.13) and (2.14) appear in inverse form, i.e. as $[\det A]^{1/2}$, $[\det A]^1$ respectively. This is immediately seen in the complex case. After bringing the matrix A_{kl} to diagonal form via a unitary transformation, the integral reads

$$\int \prod_m \frac{d\xi_m{}^+ d\xi_m}{\sqrt{2\pi i}\,\sqrt{2\pi i}} \exp\left(i \sum_n \xi_n{}^+ A_n \xi_n\right) = \prod_m \int \frac{d\xi_m{}^+ d\xi_m}{\sqrt{2\pi i}\,\sqrt{2\pi i}} \exp(i\xi_m{}^+ A_m \xi_m). \tag{2.15}$$

Expanding the exponentials into a power series leaves only the first two terms since $(\xi_m{}^+\xi_m)^2 = 0$. Thus the integral becomes

$$\prod_m \int \frac{d\xi_m{}^+ d\xi_m}{\sqrt{2\pi i}\,\sqrt{2\pi i}} \,(1 + i\xi_m{}^+ A_m \xi_m). \tag{2.16}$$

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[1]) Such objects form a Grassmann algebra G. If ξ, ξ' are real elements of G, then $\xi\xi' = -\xi'\xi$ such that $\xi^2 = 0$. If ξ is complex $\in G$, then $(\xi^+\xi)^2 = 0$.

But each of these integrals can immediately be performed using the very simple integration rules of Grassmann algebras[2])

$$\int \frac{d\xi}{\sqrt{2\pi i}} = 0; \quad \int \frac{d\xi}{\sqrt{2\pi i}}\,\xi = 1; \quad \int \frac{d\xi}{\sqrt{2\pi i}}\,\xi^n = 0, \quad n > 1 \tag{2.17}$$

for real ξ and

$$\int \frac{d\xi^+}{\sqrt{2\pi i}}\frac{d\xi}{\sqrt{2\pi i}} = 0, \quad \int \frac{d\xi^+}{\sqrt{2\pi i}}\frac{d\xi}{\sqrt{2\pi i}}\, i\xi^+\xi = 1, \quad \int \frac{d\xi^+}{\sqrt{2\pi i}}\frac{d\xi}{\sqrt{2\pi i}}\,(\xi^+\xi)^n = 0, \quad n > 1$$

for complex ξ^+, ξ. As a consequence, the right-hand side of (2.16) becomes the product of eigenvalues A_m (apart from an irrelevant factor)

$$\prod_m A_m = [\det A]^1$$

which is exactly the inverse of the boson result (2.14). The case of real Fermi fields is slightly more involved since now the hermitian matrix A_{kl} can no longer be diagonalized by unitary transformations (i.e. without changing the measure of integration $\prod_m d\xi_m/\sqrt{2\pi i}$). The integral *can* be done after observing that A_{kl} may always be assumed to be antisymmetric. For if there was any symmetric part, it would cancel in the quadratic form $\sum_{kl} \xi_k A_{kl} \xi_l$ due to the anticommutativity of the Grassmann variables. But an antisymmetric hermitian matrix can always be written as $A = iA_R$ where A_R is real antisymmetric. Such a matrix is a standard metric in symplectic spaces and can be brought to a canonical form $\mathbf{C}$ which is zero except for 2×2 matrices $C = i\sigma^2$ along the diagonal. Thus $iA = -T^T\mathbf{C}T$. The matrix $-\mathbf{C}$ has unit determinant such that $\det T = \det^{1/2}(iA)$. Let $\xi_k' \equiv T_{kl}\xi_l$, then[2]) $\prod_k d\xi_k = (\det T)\prod_k d\xi_k'$.

Hence, the integral can be evaluated using (2.17):

$$\int \prod_m \frac{d\xi_m}{\sqrt{2\pi i}} \exp\Big(i\sum_{k,l} \xi_k A_{kl}\xi_l\Big) = (\det T)\int \prod_m \frac{d\xi_m'}{\sqrt{2\pi i}} \exp\Big(-\sum_{kl} \xi_k' \mathbf{C}_{kl} \xi_l'\Big)$$

$$= (\det iA)^{1/2} \prod_n^{\infty} \int \frac{d\xi_{2n}'}{\sqrt{2\pi i}}\frac{d\xi_{2n+1}'}{\sqrt{2\pi i}} (1 + \xi_{2n+1}'\xi_{2n}') = (\det iA)^{1/2}.$$

Again, the result is the inverse of the boson case (2.13).

In order to apply these formulas to fields $\psi(\boldsymbol{x}, t)$ defined on continuous space-time both formulas have to be written in such a way that the limit of infinitely fine lattice grating $\delta \to 0$ can be performed with no problem. For this one remembers the useful matrix identity

$$[\det A]^{\mp 1} = \exp\,[i(\pm i\,\mathrm{tr}\log A)] \tag{2.18}$$

where $\log A$ may be expanded in the standard fashion as

$$\log A = \log\big(1 + (A-1)\big) = -\sum_{n=1}^{\infty} [-(A-1)]^n \frac{1}{n}. \tag{2.19}$$

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[2]) Notice that these rules make the integral sign an unusual way of denoting the operation of *differentiation*: For a real Grassmann element ξ, a function $F(\xi)$ has only two terms in an expansion: $F(\xi) = F_0 + F'\xi$. F' is defined as derivative $F' \equiv d/d\xi F(\xi)$. But: $\int \frac{d\xi}{\sqrt{2\pi i}} F(\xi) = F'$ from (2.17)! For this reason, changes in the integration variable do not transform with the Jacobian but with its inverse: $\int d\xi/\sqrt{2\pi i} = a \int d(a\xi)/\sqrt{2\pi i}$!

This formula reduces the calculation of the determinant to a series of matrix multiplications. But in each of these the limit $\delta \to 0$ is straight-forward. One simply replaces all sums over lattice indices by integrals over $d^3x\, dt$, for instance

$$\operatorname{tr} A^2 = \sum_{kl} A_{kl} A_{lk} \Rightarrow \int d^3x\, dt\, d^3x'\, dt'\, A(\boldsymbol{x}t, \boldsymbol{x}'t')\, A(\boldsymbol{x}'t', \boldsymbol{x}t)\,. \tag{2.20}$$

With this in mind, the field versions of (2.13) and (2.14) amount to the following functional formulas:

$$\int D\varphi(\boldsymbol{x}t) \exp\left[\frac{i}{2} \int d^3x\, dt\, d^3x'\, dt'\, \varphi(\boldsymbol{x}t)\, A(\boldsymbol{x}t, \boldsymbol{x}'t')\, \varphi(\boldsymbol{x}'t')\right] = \exp\left[i\left(\pm \frac{i}{2} \operatorname{tr} \log \begin{Bmatrix} 1 \\ i \end{Bmatrix} A\right)\right]$$

$$\int D\psi^+(\boldsymbol{x}t)\, D\psi(\boldsymbol{x}t) \exp\left[i \int d^3x\, dt\, d^3x'\, dt'\, \psi^+(\boldsymbol{x}t)\, A(\boldsymbol{x}t, \boldsymbol{x}', ')\, \psi(\boldsymbol{x}', t')\right] = \exp\left[i(\pm i \operatorname{tr} \log A)\right]. \tag{2.21}$$

Here φ, ψ are arbitrary real and complex fields, with the upper sign holding for bosons, the lower for fermions. Notice that the same result is true if φ, ψ have several, say spin, components and A is a matrix in the corresponding space.

Finally, we may include an external source for the fields φ, ψ into the integral and solve by quadratic completion. In the elementary forms, (2.12), (2.14) one has for bosons as well as fermions (dropping product and summation symbols):

$$\int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi i}} \exp\left(\frac{i}{2}\, \xi A \xi + i j \xi\right) = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi i}} \exp\left[\frac{i}{2}\, (\xi + jA^{-1})\, A(\xi + A^{-1}j) - \frac{i}{2}\, jA^{-1}j\right]$$

$$\int \frac{d\xi^+ d\xi}{\sqrt{2\pi i}\, \sqrt{2\pi i}} \exp\left(i\xi^+ A\xi + ij^+\xi + i\xi^+ j\right)$$

$$= \int \frac{d\xi^+ d\xi}{\sqrt{2\pi i}\, \sqrt{2\pi i}} \exp\left[i(\xi^+ + j^+ A^{-1})\, A(\xi + A^{-1}j) - ij^+ A^{-1} j\right]. \tag{2.22}$$

The shift in the integral $\xi \to \xi + A^{-1}\xi$ gives no change due to the infinite range of integration such that

$$\int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi i}} \exp\left(\frac{i}{2}\, \xi A \xi + ij\xi\right) = \begin{Bmatrix} 1 \\ i^{1/2} \end{Bmatrix} A^{\mp 1/2} \exp\left(-\frac{i}{2}\, jA^{-1}j\right)$$

$$\int_{-\infty}^{\infty} \frac{d\xi^+ d\xi}{\sqrt{2\pi i}\, \sqrt{2\pi i}} \exp\left(i\xi^+ A\xi + ij^+\xi + i\xi^+ j\right) = A^{\mp 1} \exp\left(-ij^+ A^{-1} j\right). \tag{2.23}$$

A corresponding operation on the functional formula (2.21) leads to the generalized form:

$$\int D\varphi(\boldsymbol{x}, t) \exp\left\{\frac{i}{2} \int d^3x\, dt\, d^3x'\, dt' \left[\varphi(\boldsymbol{x}t)\, A(\boldsymbol{x}t, \boldsymbol{x}'t')\, \varphi(\boldsymbol{x}'t') + 2j(\boldsymbol{x}t)\, \varphi(\boldsymbol{x}t)\, \delta^3(\boldsymbol{x} - \boldsymbol{x}'t)\, \delta(t - t')\right]\right\}$$

$$= \exp\left\{i\left(\pm \frac{i}{2} \operatorname{tr} \log \begin{Bmatrix} 1 \\ i \end{Bmatrix} A\right) - \frac{i}{2} \int d^3x\, dt\, d^3x'\, dt'\, j(\boldsymbol{x}t)\, A^{-1}(\boldsymbol{x}t, \boldsymbol{x}'t')\, j(\boldsymbol{x}'t')\right\} \tag{2.24a}$$

$$\int \mathrm{D}\psi^+(\boldsymbol{x}, t)\, \mathrm{D}\psi(\boldsymbol{x}, t)$$
$$\times \exp\left\{i \int d^3x\, dt\, d^3x'\, dt' [\psi^+(\boldsymbol{x}t) A(\boldsymbol{x}t, \boldsymbol{x}'t')\, \psi(\boldsymbol{x}'t') + (\eta^+(\boldsymbol{x}t)\, \psi(\boldsymbol{x}t)\, \delta^3(\boldsymbol{x}-\boldsymbol{x}')\, \delta(t-t') + \text{h.c.})]\right\}$$
$$= \exp [i(\pm i \operatorname{tr} \log A) - i \int d^3x\, dt\, d^3x'\, dt \eta^+(\boldsymbol{x}t)\, A^{-1}(\boldsymbol{x}t, \boldsymbol{x}'t')\, \eta(\boldsymbol{x}'t')]. \tag{2.24b}$$

This form collects all information about functional integration necessary for the understanding of the remainder of these lectures.

II.2. Equivalence of Functionals and Operators

As an exercise we shall apply (2.24) to present a simple proof of the equivalence of Feynman's path integral formula (2.8) and the operator version (2.6). First we notice that the interaction can be taken outside the integral or the vacuum expectation value in either formula as

$$Z[\eta^+, \eta] = \exp \left\{i\mathscr{A}_{\text{int}} \left[\frac{1}{i}\frac{\delta}{\delta\eta}, \frac{1}{i}\frac{\delta}{\delta\eta^+}\right]\right\} Z_0[\eta^+, \eta], \tag{2.25}$$

where Z_0 is the generating functional for the free fields. Thus in eqn. (2.8) there is only $\mathscr{A}_0$ of (2.1) in the exponent. Since

$$\mathscr{A}_0[\psi^+, \psi] = \int dx\, dt \psi^+(\boldsymbol{x}t)\, \big(i\partial_t - \varepsilon(-i\nabla)\big)\, \psi(\boldsymbol{x}t) \tag{2.26}$$

the functional integral is of the type (2.24) with a matrix

$$A(\boldsymbol{x}t, \boldsymbol{x}'t') = \big(i\partial_t - \varepsilon(-i\nabla)\big)\, \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\, \delta(t-t'). \tag{2.27}$$

This matrix is the inverse of the free propagator

$$A(\boldsymbol{x}t, \boldsymbol{x}'t') = iG_0^{-1}(\boldsymbol{x}t, \boldsymbol{x}'t') \tag{2.28}$$

where

$$G_0(\boldsymbol{x}t, \boldsymbol{x}'t') = \int \frac{dEd^3p}{(2\pi)^4} \exp\left[-i\big(E(t-t') - \boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')\big)\right] \frac{i}{E - \varepsilon(\boldsymbol{p}) + i\eta}. \tag{2.29}$$

Inserting this into (2.24b), we see

$$Z_0[\eta^+, \eta] = N \exp\left[i(\pm i \operatorname{tr} \log iG_0^{-1}) - \int d^3x\, dt\, d^3x'\, dt' \eta^+(\boldsymbol{x}t)\, G_0(\boldsymbol{x}t, \boldsymbol{x}'t')\, \eta(\boldsymbol{x}'t')\right]$$
$$= \exp\left[- \int d^3x\, dt\, d^3x'\, dt' \eta^+(\boldsymbol{x}t)\, G_0(\boldsymbol{x}t, \boldsymbol{x}'t')\, \eta(\boldsymbol{x}'t')\right] \tag{2.30}$$

where N has been chosen according to the normalization (2.7).
But this result coincides exactly with what one would obtain from the operator expression (2.6) for $Z_0[\eta^+, \eta]$ (i.e. without $\mathscr{A}_{\text{int}}$): According to Wick's theorem [2], any time ordered product can be expanded as a sum of normal products with all possible contractions taken via Feynman propagators. The formula for an arbitrary functional of free fields ψ, ψ^+ is

$$TF[\psi^+, \psi] = \exp\left[\int d^3x dt d^3x' dt' \frac{\delta}{\delta\psi(\boldsymbol{x}t)}\, G_0(\boldsymbol{x}t, \boldsymbol{x}'t')\, \frac{\delta}{\delta\psi^+(\boldsymbol{x}'t')}\right]$$
$$:F[\psi^+, \psi]:. \tag{2.31}$$

Applying this to

$$\langle 0|\, TF[\psi^+, \psi]\, |0\rangle = \langle 0|\, T \exp\left[i \int dx\, dt(\psi^+\eta + \eta^+\psi)\right] |0\rangle$$

one finds:

$$Z_0[\eta^+, \eta] = \exp\left[-\int dx\, dt\, dx'\, dt' \eta^+(\boldsymbol{x}t)\, G_0(\boldsymbol{x}t, \boldsymbol{x}'t')\, \eta(\boldsymbol{x}'t')\right]$$
$$\langle 0|\, :\exp\left[i \int dx\, dt (\psi^+\eta + \eta^+\psi)\right]:\, |0\rangle . \tag{2.32}$$

But the second factor equals one, proving the equality of this Z_0 with the path integral result (2.30) (which holds for the full $Z[\eta^+, \eta]$ because of (2.25)).

II.3. Grand Canonical Ensembles at $T = 0$

All these results are easily generalized from vacuum expectation values to thermodynamic averages at fixed temperature T and chemical potential μ. The change at $T = 0$ is trivial: The single particle energies in the action (2.1) have to be replaced by

$$\xi(-i\nabla) = \varepsilon(-i\nabla) - \mu \tag{2.33}$$

and new boundary conditions have to be imposed upon all Green's functions via an appropriate $i\varepsilon$ prescription in $G_0(\boldsymbol{x}t, \boldsymbol{x}'t')$ of (2.29):[3] [3]

$${}^{T=0}G_0(\boldsymbol{x}t, \boldsymbol{x}'t') = \int \frac{dE d^3p}{(2\pi)^4} \exp[-iE(t-t') + i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')] \frac{i}{E - \xi(\boldsymbol{p}) + i\eta \operatorname{sgn} \xi(\boldsymbol{p})} . \tag{2.34}$$

In order to simplify the notation we shall sometimes employ four-vectors $p = (p^0, \boldsymbol{p})$ and write the Green's function as

$${}^{T=0}G_0(\boldsymbol{x}t, \boldsymbol{x}'t') \equiv \sum_p \exp[-ip(x - x')] \frac{i}{p^0 - \xi(\boldsymbol{p}) + i\eta \operatorname{sgn} \xi(\boldsymbol{p})} .$$

The resulting formulas for ${}^{T=0}Z[\eta^+, \eta]$ can be brought to conventional form by performing a Wick rotation in the complex energy plane in all energy integrals (2.34) implied by the formulae (2.32) and (2.6). For this, one sets $E = -p^0 \equiv i\omega$ and replaces

$$\int_{-\infty}^{\infty} \frac{dE}{2\pi} \equiv \sum_{p^0} \to i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} . \tag{2.35}$$

Thus the Green's function becomes

$${}^{T=0}G_0(\boldsymbol{x}t, \boldsymbol{x}'t') = -\int \frac{d\omega d^3p}{(2\pi)^4} \exp[\omega(t-t') + i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')] \frac{1}{i\omega - \xi(\boldsymbol{p})} . \tag{2.36}$$

With this Green's function, formulas (2.30), (2.25) for ${}^{T=0}Z[\eta^+, \eta]$ coincide exactly with the grand-canonical partition function in the presence of sources [3].

$$\Omega(T = 0, \mu, V) = {}^{T=0}Z[\eta^+, \eta] \tag{2.37}$$

---

[3]) In this way, fermions with $\xi < 0$ inside the Fermi sea propagate backwards in time. Bosons, on the other hand, have in general $\xi > 0$ and, hence, always propagate forward in time.

II.4. General Grand Canonical Ensembles at $T \neq 0$

The generalization to arbitrary temperature is finally achieved by imposing the boundary condition on $G_0(\boldsymbol{x}t, \boldsymbol{x}'t')$, and thus, by virtue of (2.25), (2.30) on all Green's functions [3], to be periodic or anti-periodic with period $1/T$ after the analytic continuation to imaginary time $t \equiv -i\tau$ via the Wick rotation:

$$ {}^TG_0\left(\boldsymbol{x}, -i\left(\tau + \frac{1}{T}\right); \boldsymbol{x}', -i\tau'\right) \equiv \pm {}^TG_0(\boldsymbol{x}, -i\tau; \boldsymbol{x}', -i\tau'). \tag{2.38} $$

The plus sign holds for bosons, the minus sign for fermions. This property is enforced automatically by replacing the energy integrations $\int_{-\infty}^{\infty} d\omega/2\pi$ in (2.36) by a summation over the discrete Matsubara frequencies

$$ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \to T \sum_{\omega_n} \equiv {}^T\sum_{\omega} \tag{2.39} $$

which are even or odd multiples of πT[4])

$$ \omega_n = \pi T\, 2 \begin{Bmatrix} n \\ n + \frac{1}{2} \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} \text{bosons} \\ \text{fermions} \end{Bmatrix}. \tag{2.40} $$

Thus:

$$ {}^TG_0(\boldsymbol{x}\tau, \boldsymbol{x}'\tau') = -T \sum_{\omega_n} \int \frac{d^3p}{(2\pi)^3} \exp[-i\omega_n(\tau - \tau') + i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}')] \frac{1}{i\omega_n - \xi(\boldsymbol{p})}. \tag{2.41} $$

Also in the case of $T \neq 0$ ensembles, the notation is simplified by the use of four-vectors. One writes

$$ \begin{aligned} p &\equiv (i\omega, \boldsymbol{p}) \\ x &\equiv (-i\tau, \boldsymbol{x}) \\ px &\equiv \omega\tau - \boldsymbol{p}\boldsymbol{x} \end{aligned} \tag{2.42} $$

and collect integral and sum in Eqn. (2.41) under one four-summation symbol.

$$ {}^TG_0(x - x') \equiv -{}^T\sum_p \exp[-ip(x - x')] \frac{1}{i\omega - \xi(\boldsymbol{p})}. \tag{2.43} $$

It is quite straightforward to make the general $T \neq 0$ Green's function emerge from a path integral formulation analogous to (2.8). For this one considers fields $\psi(\boldsymbol{x}, \tau)$ with the periodicity or anti-periodicity

$$ \psi(\boldsymbol{x}, \tau) = \pm \psi\left(\boldsymbol{x}, \tau + \frac{1}{T}\right). \tag{2.44} $$

They can be Fourier decomposed as

$$ \psi(\boldsymbol{x}, \tau) = T \sum_{\omega_n} \sum_{\boldsymbol{p}} \exp(-i\omega_n\tau + i\boldsymbol{p}\boldsymbol{x})\, a(\omega_n, \boldsymbol{p}) = {}^T\sum_p \exp(-ipx)\, a(p) \tag{2.45} $$

---

[4]) Throughout these lectures we shall use natural units sucht that $k_B = 1$, $\hbar = 1$.

with a sum over even or odd Matsubara frequencies ω_n.
If now a free action is defined as

$$\mathscr{A}_0[\psi^+, \psi] = -i \int_{-1/2T}^{1/2T} d\tau \int d^3x \psi^+(\boldsymbol{x}, \tau) \left(-\partial_\tau - \xi(-i\nabla)\right) \psi(\boldsymbol{x}, \tau) \tag{2.47}$$

formula (2.24) renders [1, 4, 5]

$$^TZ_0[\eta^+, \eta] = \exp\left[\mp \operatorname{tr} \log A + \iint_{-1/2T}^{1/2T} d\tau\, d\tau' \int d^3x\, d^3x' \eta^+(\boldsymbol{x}\tau)\, A^{-1}(\boldsymbol{x}\tau, \boldsymbol{x}'\tau')\, \eta(\boldsymbol{x}'\tau')\right] \tag{2.48}$$

with

$$A(\boldsymbol{x}\tau, \boldsymbol{x}'\tau') = \left(\partial_\tau + \xi(-i\nabla)\right) \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\, \delta(\tau - \tau') \tag{2.49}$$

and henceforth A^{-1} equal to the propagator (2.41), the Matsubara frequencies arising due to the finite τ interval of Euclidean space together with the periodic boundary condition (2.44).
Again, interactions are taken care of by multiplying $^TZ_0[\eta^+\eta]$ with the factor (2.25). In terms of the fields $\psi(\boldsymbol{x}, \tau)$, the exponent has the form:

$$\mathscr{A}_{\text{int}} = \frac{1}{2} \iint_{-1/2T}^{1/2T} d\tau\, d\tau' \int d^3x\, d^3x' \psi^+(\boldsymbol{x}, \tau)\, \psi^+(\boldsymbol{x}', \tau')\, \psi(\boldsymbol{x}', \tau')\, \psi(\boldsymbol{x}, \tau)\, V(\boldsymbol{x}, \boldsymbol{x}', -i\tau, -i\tau'). \tag{2.50}$$

In the case of an instantaneous potential (2.2), the potential becomes instantaneous in τ:

$$V(\boldsymbol{x}, \boldsymbol{x}', -i\tau, -i\tau') = V(\boldsymbol{x} - \boldsymbol{x}')\, i\delta(\tau - \tau'). \tag{2.51}$$

In this case $\mathscr{A}_{\text{int}}$ can be written in terms of the interaction Hamiltonian as

$$\mathscr{A}_{\text{int}} = i \int_{-1/2T}^{1/2T} d\tau H_{\text{int}}(\tau). \tag{2.52}$$

Thus the grand canonical partition function in the presence of external sources may be calculated from the path integral [4, 5]:

$$\begin{aligned} \Omega(\mu, T, V)\, [\eta^+, \eta] &= {}^TZ[\eta^+, \eta] \\ &= \int \mathrm{D}\psi^+(\boldsymbol{x}, \tau)\, \mathrm{D}\psi(\boldsymbol{x}, \tau) \exp\left[i\, {}^T\mathscr{A} + \int_{-1/2T}^{1/2T} d\tau \int d^3x \left(\psi^+(\boldsymbol{x}\tau)\, \eta(\boldsymbol{x}\tau) + \text{h.c.}\right)\right] \end{aligned} \tag{2.53}$$

where the grand-canonical action is

$$\begin{aligned} i\, {}^T\mathscr{A}[\psi^+, \psi] = &-\int_{-1/2T}^{1/2T} d\tau \int d^3x \psi^+(\boldsymbol{x}\tau) \left(\partial\tau + \xi(-i\nabla)\right) \psi(\boldsymbol{x}\tau) + \frac{i}{2} \iint_{-1/2T}^{1/2T} d\tau\, d\tau' \\ &\times \int d^3x\, d^3x' \psi^+(\boldsymbol{x}\tau)\, \psi^+(\boldsymbol{x}'\tau')\, \psi(\boldsymbol{x}'\tau')\, \psi(\boldsymbol{x}\tau)\, V(\boldsymbol{x}, \boldsymbol{x}', -i\tau, -i\tau'). \end{aligned} \tag{2.54}$$

It is this formulation for Ω which offers the advantageous flexibility with respect to changes in the field variables for arbitrary grand-canonical ensembles.
Summarizing we have seen that the functional (2.53) defines the most general type of theory involving two-body forces. It contains all information on the physical system

in the vacuum as well as in thermodynamic ensembles. The vacuum theory is obtained by setting $T = 0$, $\mu = 0$, and continuing the result back from T to physical times. Conversely, the functional (2.8) in the vacuum can be generalized to ensembles in the straight-forward manner by first continuing the time t to imaginary values $-i\tau$ via a Wick rotation in all energy integrals and then going to periodic functions in τ.
There is a complete correspondence between either formulation (2.8) and (2.53). For this reason it will be sufficient to exhibit all techniques only in one version for which we shall choose (2.8). It should, however, be pointed out that due to the singular nature of the propagators (2.29) in real energy-momentum a *proper* definition of the theory in the vacuum via path integrals always has to take place in the thermodynamic formulation. This has to be kept in mind when performing the path integral manipulations in the following sections.

III. Plasmons

Let us give a first application of the functional method by transforming the grand partition function (2.41) to plasmon coordinates.
For this, we make the basic observation based on formula (2.24a) that an interaction (2.1) in the generating functional can always be rewritten in terms of a functional integral involving a new auxiliary field $\varphi(x)$ as[5]) [*6*]

$$\exp\left[-\frac{i}{2}\int dx dx' \psi^+(x)\,\psi^+(x')\,\psi(x')\,\psi(x)\,V(x,x')\right]$$
$$= \text{const.} \int \mathrm{D}\varphi \exp\left\{\frac{i}{2}\int dx\,dx'[\varphi(x)\,V^{-1}(x,x')\,\varphi(x') - 2\varphi(x)\,\psi^+(x)\,\psi(x)\,\delta(x-x')]\right\} \quad (3.1)$$

where the constant is simply const. $= [\det V]^{-1/2}$. Absorbing this constant in the normalization factor N, the grand-canonical partition function $\Omega \equiv Z$ becomes

$$Z[\eta^+, \eta] = \int \mathrm{D}\psi^+\mathrm{D}\psi\mathrm{D}\varphi \exp\left[i\mathscr{A} + i\int dx\big(\eta^+(x)\,\psi(x) + \psi^+(x)\,\eta(x)\big)\right] \quad (3.2)$$

where the new action is

$$\mathscr{A}[\psi^+\psi\varphi]$$
$$= \int dx\,dx' \left\{\psi^+(x)\left(i\partial_t - \xi(-i\nabla) - \varphi(x)\right)\delta(x-x')\,\psi(x') + \frac{1}{2}\,\varphi(x)\,V^{-1}(x,x')\,\varphi(x')\right\}. \quad (3.3)$$

Notice that the effect of using formula (3.1) in the generating functional amounts to the addition of the complete square in φ in the exponent:

$$\frac{1}{2}\int dx\,dx'\big(\varphi(x) - \int dy V(x,y)\,\psi^+(y)\,\psi(y)\big)\,V^{-1}(x,x')\,\big(\varphi(x') - \int dy' V(x',y')\,\psi^+(y')\,\psi(y')\big)$$

together with the additonal integration over $\mathrm{D}\varphi$. This procedure of going from (2.1) to (3.3) is probably simpler mnemonically than formula (3.1). The fact that the functional Z remains unchanged by this addition follows, as before, since the integral $\mathrm{D}\varphi$ produces only the irrelevant constant $[\det V]^{-1/2}$.

---

[5]) We shall write in future $x \equiv (\boldsymbol{x}, t)$, $dx \equiv d^3x\,dt$, $\delta(x) \equiv \delta^3(\boldsymbol{x})\,\delta(t)$. The symbol $V^{-1}(x,x')$ denotes the functional matrix inverse of the matrix $V(x,x')$, i.e. $\int dx' V^{-1}(x,x')\,V(x',x'') = \delta(x-x'')$.

The physical significance of the new field $\varphi(x)$ is easy to understand: $\varphi(x)$ is directly related to the particle density. At the classical level this is seen immediately by extremizing the action (3.3) with respect to variations $\delta\varphi(x)$:

$$\frac{\delta \mathscr{A}}{\delta\varphi(x)} = \varphi(x) - \int dy V(x, y)\, \psi^+(y)\, \psi(y) = 0. \tag{3.4}$$

Quantum mechanically, there will be fluctuations around the equalities (3.4) and (3.6), making the Green's functions of $\varphi(x)$ and of the composite operator $\int dy V(x, y) \times \psi^+(y)\, \psi(y)$ different. But due to the Gaussian nature of the $D\varphi$ integration, the fluctuations are quite simple. One can easily show that, for example, the propagators of either field differ only by the direct interaction, i.e.

$$\langle T(\varphi(x)\, \varphi(x'))\rangle = V(x - x') + \langle T\big(\int dy V(x, y)\, \psi^+(y)\, \psi(y)\big)\big(\int dy' V(x', y')\, \psi^+(y')\, \psi(y')\big)\rangle. \tag{3.5}$$

For the proof, the reader is referred to Appendix B.

Notice, that for a potential V which is dominantly caused by a single fundamental-particle exchange, the field $\varphi(x)$ coincides with the field of this particle:

If, for example, $V(x, y)$ represents the Coulomb interaction

$$V(x, x') = \frac{e^2}{|\boldsymbol{x} - \boldsymbol{x}'|}\, \delta(t - t')$$

equ. (3.4) amounts to

$$\varphi(\boldsymbol{x}, t) = -\frac{4\pi e^2}{V^2}\, \psi^+(\boldsymbol{x}, t)\, \psi(\boldsymbol{x}, t) \tag{3.6}$$

revealing the auxiliary field as the electric potential.

If the particles $\psi(x)$ have spin indices, the potential will, in this example, be thought of as spin conserving at every vertex and equ. (3.4) must be read as spin contracted: $\varphi(x) \equiv \int dy V(x, y)\, \psi^{+\alpha}(y)\, \psi_\alpha(y)$. This restriction is just for convenience and can easily be lifted later. Nothing in our procedure depends on this particular property of V and φ. In fact, V could arise from the exchange of many different fundamental particles and their multiparticle configurations (for example π, $\pi\pi$, σ, ϱ, ect. in nuclei) such that the spin dependence is the rule rather than the exception.

The important point is now that the auxiliary field $\varphi(x)$ can be made the *only* field of the theory by integrating out ψ^+, ψ in the formula (3.2), using formula (2.24). Thus one obtains

$$Z[\eta^+, \eta] \equiv \Omega[\eta^+, \eta] = N e^{i\mathscr{A}} \tag{3.7}$$

where the new action is

$$\mathscr{A}[\varphi] = \pm i\, \mathrm{tr} \log (iG_\varphi^{-1}) + \frac{1}{2} \int dx\, dx' \varphi(x)\, V^{-1}(x, x')\, \varphi(x')$$
$$+ i \int dx\, dx' \eta^+(x)\, G_\varphi(x, x')\, \eta(x') \tag{3.8}$$

with G_φ being the Green's function of the fundamental particles in an external classical field $\varphi(x)$:

$$\big(i\partial_t - \xi(-i\nabla) - \varphi(x)\big)\, G_\varphi(x, x') = i\delta(x - x'). \tag{3.9}$$

The field $\varphi(x)$ is called a plasmon field. The new plasmon action can easily be interpreted graphically. For this, one expands G_φ in powers of φ

$$G_\varphi(x, x') = G_0(x - x') - i \int dx_1 G_0(x - x_1)\, \varphi(x_1)\, G_0(x_1 - x') + \cdots \tag{3.10}$$

Hence the couplings to the external currents η^+, η in (3.8) amount to radiating one, two, etc. φ fields from every external line of fundamental particles (see Fig. Ia). An expansion of the tr log expression in φ gives

$$\pm i \operatorname{tr} \log (iG_\varphi^{-1}) = \pm i \operatorname{tr} \log (iG_0^{-1}) \pm i \operatorname{tr} \log (1 + iG_0\varphi)$$
$$= \pm i \operatorname{tr} \log (iG_0^{-1}) \mp i \operatorname{tr} \sum_{n=1}^{\infty} (-iG_0\varphi)^n \frac{1}{n}. \tag{3.11}$$

The first term leads to an irrelevant multiplicative factor in (3.7). The n^{th} term corresponds to a loop of the original fundamental particle emitting n φ lines (see Fig. Ib).

Fig. Ia

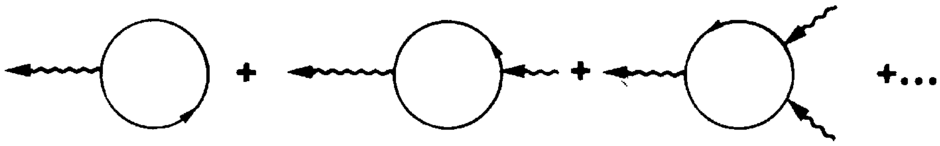


Fig. Ib

Fig. Ia). This diagram displays the last, pure current, piece of the collective action (3.8). The original fundamental particle (fat line) can enter and leave the diagrams only via external currents, emitting an arbitrary number of plasmons (wiggly lines) on its way

Fig. Ib). The non-polynomial self-interaction terms of the plasmons arising from the tr log in (3.8) are equal to the single loop diagrams emitting n plasmons

Let us now use the action (3.8) to construct a quantum field theory of plasmons. For this we may include the quadratic term

$$\pm i \operatorname{tr} (G_0\varphi)^2 \frac{1}{2} \tag{3.12}$$

into the free part of φ in (3.8) and treat the remainder perturbatively. The free propagator of the plasmon becomes

$$\{0|\, T\varphi(x)\, \varphi(x')\, |0\} \equiv G^{\text{pl}}(x - x') = \frac{i}{V^{-1} \pm iG_0 \wedge G_0} (x, x'). \tag{3.13}$$

Here $|0\}$ denotes the vacuum of the free plasmon field and $G_0 \wedge G_0$ is a matrix including the trace over $(2s + 1)$ spin indices:

$$(G_0 \wedge G_0)\ (x, x') \equiv (2s + 1)\, G_0(x, x')\, G_0(x', x).$$

This corresponds to an inclusion into the V propagator of all ring graphs (see Fig. II).

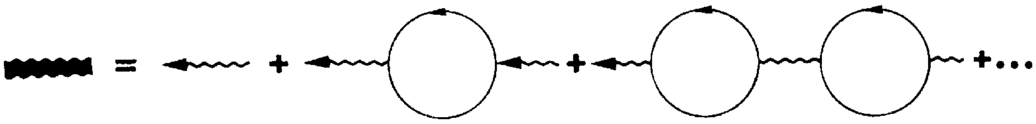


Fig. II. The free plasmon propagator contains an infinite sequence of single loop corrections ("bubblewise summation")

It is worth pointing out that the propagator in momentum space $G^{\text{pl}}(k)$ contains actually two important physical informations. From the derivation at fixed temperature it appears in the transformed action (3.8) as a function of discrete Euclidean frequencies

$\nu_n = \pi T 2n$ only. In this way it serves for the time independent fixed T description of the system. The calculation (3.13), however, renders it as a function in the whole complex energy plane. It is this function which determines the *time dependent* collective phenomena for *real* times.[6])
With the propagator (3.13) and the interactions given by (3.11), the original theory of fundamental fields ψ^+, ψ has been transformed into a theory of φ fields whose bare propagator accounts for the original potential which has absorbed ringwise an infinite sequence of fundamental loops.
This transformation is exact. Nothing in our procedure depends on the statistics of the fundamental particles nor on the shape of the potential. Such properties are important when it comes to *solving* the theory perturbatively. Only under appropriate physical circumstances will the field φ represent important collective excitations with weak residual interactions. It is then that the new formulation is of great use in understanding the dynamics of the system.
As an illustration consider a dilute fermion gas of very low temperature. Then the function $\xi(-i\nabla)$ is $\varepsilon(-i\nabla) - \mu$ with $\varepsilon(-i\nabla) = -\nabla^2/2m$.
Let the potential be translationally invariant and instantaneous

$$V(x, x') = \delta(t - t')\, V(\boldsymbol{x} - \boldsymbol{x}'). \tag{3.14}$$

Then plasmon propagator (3.13) reads in momentum space

$$G^{\mathrm{pl}}(\nu, \boldsymbol{k}) \doteq -\frac{1}{\big(V(\boldsymbol{k})\big)^{-1} - \pi(\nu, \boldsymbol{k})} \tag{3.15}$$

where the single electron loop is[7])

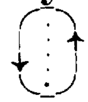
$$\pi(\nu, \boldsymbol{k}) = 2\ {}^{T}\!\sum_{p} \frac{1}{i\omega - \dfrac{\boldsymbol{p}^2}{2m} + \mu} \cdot \frac{1}{i(\omega + \nu) - \dfrac{(\boldsymbol{p} + \boldsymbol{k})^2}{2m} + \mu}. \tag{3.16}$$

The frequencies ω and ν are odd and even multiples of πT. The sum is calculated in the standard fashion by introducing a convergence factor $e^{i\omega\eta}$ and rewriting

$$\pi(\nu, \boldsymbol{k}) = 2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\xi(\boldsymbol{p} + \boldsymbol{k}) - \xi(\boldsymbol{p}) - i\nu}$$
$$\times\ T \sum_{\omega_n} e^{i\omega_n\eta} \left(\frac{1}{i(\omega_n + \nu) - \xi(\boldsymbol{p} + \boldsymbol{k})} - \frac{1}{i\omega_n - \xi(\boldsymbol{p})} \right) \tag{3.17}$$

via the basic formula[8])

$$T \sum_{\omega_n} e^{i\omega_n\eta} \frac{1}{i\omega_n - \xi(\boldsymbol{p})} = \frac{T}{2\pi i} \int dz \frac{e^{\eta z}}{e^{z/T} \pm 1} \frac{1}{z - x} = \mp \frac{1}{e^{\xi(\boldsymbol{p})} \mp 1} = \mp \frac{1}{e^{\varepsilon(\boldsymbol{p}) - \mu} \mp 1} = \mp n(\boldsymbol{p}) \tag{3.18}$$



[6]) See the discussion in Ch. 9 of the last of [3] and G. BAYM and N. D. MERMIN, J. Math. Phys. 2, 232 (1961).
[7]) The factor 2 stems from the trace over the electron spin.
[8]) This formula is written for bosons and fermions, only the lower sign applying in the present case. The contour encircles all Matsubara frequencies along the imaginary z axis.

as

$$\pi(\nu, \boldsymbol{k}) = 2 \int \frac{d^3p}{(2\pi)^3} \frac{n(\boldsymbol{p}+\boldsymbol{k}) - n(\boldsymbol{p})}{\varepsilon(\boldsymbol{p}+\boldsymbol{k}) - \varepsilon(\boldsymbol{p}) - i\nu}. \tag{3.19}$$

If one performs a long wavelength, small frequency expansion of one finds for $T \approx 0$:

$$\pi(\nu, \boldsymbol{k}) \approx -\frac{mp_F}{\pi^2}\,(1 - \varrho \cdot \operatorname{arc\,tan} \varrho^{-1}) \tag{3.20}$$

where p_F denotes the Fermi momentum and ϱ the ratio $\varrho \equiv m\nu/p_F|\boldsymbol{k}|$[9]).
The analytic continuation to physical energies $k_0 = i\nu$ yields, with $\tilde{\varrho} \equiv mk_0/p_F|\boldsymbol{k}| = i\varrho$:

$$\pi(k_0, \boldsymbol{k}) = -\frac{mp_F}{\pi^2}\left(1 - \frac{\tilde{\varrho}}{2}\log\left|\frac{\tilde{\varrho}+1}{\tilde{\varrho}-1}\right| - i\,\frac{\pi}{2}\,|\tilde{\varrho}|\,\theta(1-|\tilde{\varrho}|)\right). \tag{3.21}$$

The real poles of G^{pl} determine the elementary excitations. Suppose $\big(V(\boldsymbol{k})\big)^{-1}$ has a long-wavelength expansion

$$\big(V(\boldsymbol{k})\big)^{-1} = \big(V(0)\big)^{-1} + a\boldsymbol{k}^2 + \cdots \tag{3.22}$$

Then there are real poles at energies k_0 for which

$$\big(V(0)\big)^{-1} + a\boldsymbol{k}^2 + \cdots = -\frac{mp_F}{\pi^2}\left(1 - \frac{\tilde{\varrho}}{2}\log\left|\frac{\tilde{\varrho}+1}{\tilde{\varrho}-1}\right|\right) \tag{3.23}$$

as long as $\big(V(0)\big)^{-1}$ is finite and positive, i.e. for a well behaved overall repulsive potential $\big(V(0) = \int d^3x V(\boldsymbol{x}) > 0\big)$. The value $\tilde{\varrho}_0$ for which (3.23) is fulfilled at $\boldsymbol{k} = 0$ determines the zero-sound velocity c_0 according to

$$\tilde{\varrho}_0 = \frac{m}{p_F}\,\frac{k_0}{|\boldsymbol{k}|} = \frac{1}{v_F}\,c_0. \tag{3.24}$$

In the neighbourhood of the pole the propagator has the form

$$G^{\mathrm{pl}}(k_0, k) \approx \text{const.}\,\frac{|\boldsymbol{k}|}{k_0 - c_0|\boldsymbol{k}|}. \tag{3.25}$$

The case of an electron gas has to be discussed separately since the potential is not well behaved:

$$V(x, x') = \delta(t - t')\,\frac{e^2}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{3.26}$$

such that

$$\big(V(\boldsymbol{k})\big)^{-1} = \frac{\boldsymbol{k}^2}{4\pi e^2}. \tag{3.27}$$

Hence, (3.23) has to be solved for $\big(V(0)\big)^{-1} = 0$ and $a = 1/4\pi e^2$. Obviously, $\tilde{\varrho}$ has to go to infinity as $\boldsymbol{k} \to 0$. In this limit

$$\pi(k_0, \boldsymbol{k}) \to \frac{mp_F}{\pi^2}\,\frac{\tilde{\varrho}^{-2}}{3} = \frac{p_F{}^3}{3\pi^2 m}\,\frac{\boldsymbol{k}^2}{k_0{}^2} \tag{3.28}$$

[9]) For a discussion of this expression, see a standard textbook, for example Ref. [3].

and there is a pole at energy*)

$$k_0^2 = 4\pi e^2 \frac{p_F^3}{3\pi^2 m} = 4\pi e^2 \frac{n}{m}$$

which is the well-known plasmon frequency. Thus the long-range part of the propagator can be written as

$$G^{\text{pl}}(k_0, \boldsymbol{k}) \approx 4\pi e^2 \frac{k_0^2}{k_0^2 - 4\pi e^2 \frac{n}{m}} \frac{i}{\boldsymbol{k}^2}. \tag{3.29}$$

Using the plasmon propagator (3.15) and the multi-plasmon interactions from (3.11) one can develop a fully fledged quantum field theory of plasmons.
Great simplications arise if the system is investigated only with respect to its long-range behaviour in space and time. Then expressions like (3.25) and (3.29) become good approximations to the propagator. Moreover, the higher terms in the expansion (3.11) become more and more irrelevant due to their increasing field dimensionality. Such discussions are standard and will not be repeated her [7].

IV. Pairs

IV.1. General Formulation

There is a collective field complimentary to the plasmon field which describes dominant collective excitations in many systems such as type II superconductors, ^{3}He, excitonic insulators, etc. A pair field $\Delta(\boldsymbol{x}t, \boldsymbol{x}'t')$ with two space and two time indices, called bilocal, is introduced into the generating functional by rewriting the exponential of the interaction (2.8) different from (3.1) as[10] [8]

$$\exp\left[-\frac{i}{2}\int dx\,dx'\psi^+(x)\,\psi^+(x')\,\psi(x')\,\psi(x)\,V(x,x')\right]$$

$$= \text{const.}\int \mathrm{D}\Delta(x,x')\,\mathrm{D}\Delta^+(x,x')$$

$$\times \exp\left[\frac{i}{2}\int dx\,dx'\left\{|\Delta(x,x')|^2\frac{1}{V(x,x')} - \Delta^+(x,x')\,\psi(x)\,\psi(x') - \psi^+(x)\,\psi^+(x')\,\Delta(x,x')\right\}\right]. \tag{4.1}$$

Hence the grand-canonical potential becomes

$$Z[\eta,\eta^+] = \int \mathrm{D}\psi^+\mathrm{D}\psi\mathrm{D}\Delta^+\mathrm{D}\Delta \exp\left\{i\mathscr{A}[\psi^+,\psi,\Delta^+,\Delta] + i\int dx\big(\psi^+(x)\,\eta(x) + \text{h.c.}\big)\right\} \tag{4.2}$$

with an action

$$\mathscr{A}[\psi^+,\psi,\Delta^+,\Delta] = \int dx\,dx'\left\{\psi(x)\big(i\partial_t - \xi(-i\nabla)\big)\,\delta(x-x')\,\psi(x') - \frac{1}{2}\Delta^+(x,x')\,\psi(x)\,\psi(x')\right.$$
$$\left. - \frac{1}{2}\psi^+(x)\,\psi^+(x')\,\Delta(x,x') + \frac{1}{2}|\Delta(x,x')|^2\frac{1}{V(x,x')}\right\}. \tag{4.3}$$

---

*) n is the number density: $n = \dfrac{2\int d^3p_F}{(2\pi)^3} = p_F^3/3\pi^2$.

[10]) On the right hand side, $1/V(x,x')$ is understood as numeric division, no matrix inversion being implied. The hermitian conjugate of $\Delta(x,x')$ includes the transpose in the functional sense, i.e. $\Delta^+(x,x') \equiv [\Delta(x',x)]^+$.

Notice that this new action arises from the original one in (2.8) by adding the complete square $i/2 \int dx\, dx'\, |\Delta(x, x') - V(x', x)\, \psi(x')\, \psi(x)|^2\, 1/V(x', x)$ which is irrelevant upon functional integration over $\int \mathrm{D}\Delta^+ \mathrm{D}\Delta$ but has the virtue of removing the quartic interaction term.

At the classical level, the field $\Delta(x', x)$ is nothing but a convenient abbreviation for the composite field $V(x, x')\, \psi(x)\, \psi(x')$. This follows from the equation of motion obtained by extremizing the new action with respect to $\delta\Delta^+(x, x')$:

$$\frac{\delta \mathscr{A}}{\delta \Delta^+(x, x')} = \frac{1}{2V(x, x')} \left(\Delta(x', x) - V(x, x')\, \psi(x)\, \psi(x')\right) \equiv 0. \tag{4.4}$$

Quantum mechanically, there are again Gaussian fluctuations which are discussed in Appendix B.

The expression (4.3) is quadratic in the fundamental fields ψ and can be rewritten in matrix form as

$$\begin{aligned}&\frac{1}{2} f^+(x)\, A(x, x')\, f(x')\\ &= \frac{1}{2} f^+(x) \begin{pmatrix} \left(i\partial_t - \xi(-i\nabla)\right) \delta(x - x') & -\Delta(x, x') \\ -\Delta^+(x, x') & \mp\left(i\partial_t + \xi(i\nabla)\right) \delta(x - x') \end{pmatrix} f(x')\end{aligned} \tag{4.5}$$

where $f(x)$ denotes the fundamental field doublet $f(x) = \begin{pmatrix} \psi(x) \\ \psi^+(x) \end{pmatrix}$.

Now $f(x)$ is not independent of $f^+(x)$ such that f^+Af can also be written as

$$f^+Af = f^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Af.$$

Therefore, the rule (2.24a) applies in the real field version giving

$$Z[\eta, \eta^+] = N \int \mathrm{D}\Delta^+\, \mathrm{D}\Delta\, \exp\left(i\mathscr{A}[\Delta^+, \Delta] - \frac{1}{2} \int dx\, dx'\, j^+(x)\, \mathbf{G}_\Delta(x, x')\, j(x')\right\} \tag{4.6}$$

where $j(x) = \begin{pmatrix} \eta(x) \\ \eta^+(x) \end{pmatrix}$, with the collective action

$$\mathscr{A}[\Delta^+, \Delta] = \pm \frac{i}{2} \operatorname{tr} \log \left(i\mathbf{G}_\Delta^{-1}(x, x')\right) + \frac{1}{2} \int dx\, dx' |\Delta(x, x')|^2 \frac{1}{V(x, x')} \tag{4.7}$$

and $\mathbf{G}_\Delta$ denoting the propagator iA^{-1} satisfying

$$\int dx'' \begin{pmatrix} \left(i\partial_t - \xi(-i\nabla)\right) \delta(x - x'') & -\Delta(x, x'') \\ -\Delta^+(x, x'') & \mp\left(i\partial_t + \xi(i\nabla)\right) \delta(x - x'') \end{pmatrix} \mathbf{G}_\Delta(x'', x') = i\delta(x - x'). \tag{4.8}$$

Writing $\mathbf{G}_\Delta$ as a matrix $\begin{pmatrix} G & F \\ F^+ & \tilde{G} \end{pmatrix}$ we recognize a structural equality with the Gorkov equations in the theory of type II superconductors.[11])

Notice, however, that $Z[\eta^+, \eta]$ is the *full* partition function and unlike the Gorkov derivation no approximation is implied by (4.6), (4.7).

Let us now leave the functional formulation of $Z[\eta^+, \eta]$ and consider Δ as a collective *quantum* field. In order to develop the corresponding Feynman rules we shall assume

[11]) See, for example, p. 444 of [3].

that the Green's function $\mathbf{G}_\Delta$ can be expanded in powers of Δ as

$$\mathbf{G}_\Delta = \mathbf{G}_0 - i\mathbf{G}_0 \begin{pmatrix} 0 & \Delta \\ \Delta^+ & 0 \end{pmatrix} \mathbf{G}_0 - \cdots \tag{4.9}$$

where

$$\mathbf{G}_0(x, x') = \begin{pmatrix} \frac{i}{i\partial_t - \xi(-i\nabla)}\delta & 0 \\ 0 & \mp\frac{i}{i\partial_t + \xi(i\nabla)}\delta \end{pmatrix}(x - x').$$

We shall see later that this assumption is justified only in a very limited range of thermodynamic parameters, namely close to the critical temperature T_c. With such an expansion, the source term in (4.6) can be interpreted graphically by the absorption and emission of lines $\Delta(k)$ and $\Delta^+(k)$, respectively, from virtual zig-zag configurations of the underlying particles ψ, ψ^+ (see Fig. III)

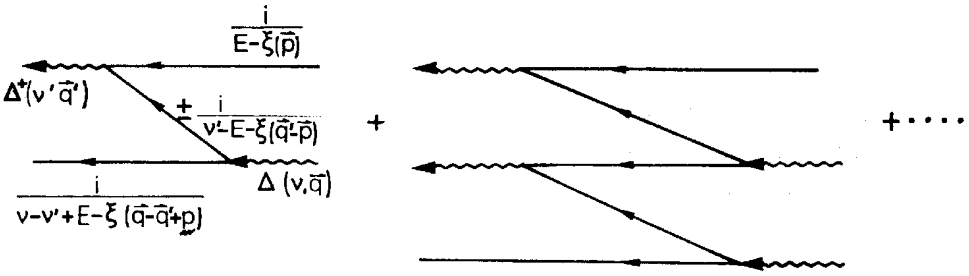


Fig. III. The fundamental particles (fat lines) may enter and any diagram only via the external currents in the last term of (4.6), absorbing n pairs from the right (the past) and emitting the same number of the left (the future).

Notice that the submatrices in $\mathbf{G}_0$ consist of

$$G_0(x, x') = \sum_p \frac{i}{p^0 - \xi(\boldsymbol{p})} \exp\left[-i(p^0 t - \boldsymbol{p}\boldsymbol{x})\right]$$

$$\tilde{G}_0(x, x') = \pm\sum_p \frac{i}{-p^0 - \xi(-\boldsymbol{p})} \exp\left[-i(p^0 t - \boldsymbol{p}\boldsymbol{x})\right]. \tag{4.10}$$

The first matrix coincides with the previous Green's function

$$G_0(x - x') = \langle 0|\, T\big(\psi(x)\, \psi^+(x')\big)\, |0\rangle. \tag{4.11}$$

The second one is obviously the opposite configuration

$$\tilde{G}_0(x - x') = \langle 0|\, \big(T\psi^+(x)\, \psi(x')\big)\, |0\rangle \tag{4.11a}$$
$$= \pm\langle 0|\, T\big(\psi(x')\, \psi^+(x)\big)\, |0\rangle = \pm G_0(x' - x) \equiv \pm G_0^T(x - x')$$

where T stands for the transposed of the functional matrix (i.e. $x \leftrightarrow x'$ exchanged). After the Wick rotation, the matrices read

$$G_0(\boldsymbol{x} - \boldsymbol{x}', \omega) = -\sum_{\boldsymbol{p}} \frac{1}{i\omega - \xi(\boldsymbol{p})} \exp\left[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}')\right]$$

$$\tilde{G}_0(\boldsymbol{x} - \boldsymbol{x}', \omega) = \mp\sum_{\boldsymbol{p}} \frac{1}{-i\omega - \xi(-\boldsymbol{p})} \exp\left[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}')\right] = \mp G_0(\boldsymbol{x}' - \boldsymbol{x}, -\omega). \tag{4.12}$$

The tr log term in eqn. (4.7) can be interpreted graphically just as easily by expanding according to (4.9):

$$\pm i \operatorname{tr} \log(i\mathbf{G}_0^{-1}) \mp \frac{i}{2} \operatorname{tr} \left[-i\mathbf{G}_0 \begin{pmatrix} 0 & \Delta \\ \Delta^+ & 0 \end{pmatrix} \right]^n \frac{1}{n}. \tag{4.13}$$

The first term only changes the irrelevant normalization N of Z. To the remaining sum only even powers can contribute such that we can rewrite

$$\begin{aligned}\mathscr{A}[\Delta^+, \Delta] &= \mp i \sum_{n=1}^{\infty} \frac{(-)^n}{2n} \operatorname{tr} \left[\left(\frac{i}{i\partial_t - \xi(-i\nabla)} \delta \right) \Delta \left(\frac{\mp i}{i\partial_t + \xi(i\nabla)} \delta \right) \Delta^+ \right]^n \\ &\quad + \frac{1}{2} \int dx\, dx'\, |\Delta(x, x')|^2 \frac{1}{V(x, x')} \\ &= \sum_{n=1}^{\infty} \mathscr{A}_n[\Delta^+, \Delta] + \frac{1}{2} \int dx\, dx' |\Delta(x, x')|^2 \frac{1}{V(x, x')}. \end{aligned} \tag{4.14}$$

This form of the action allows for an immediate quantization of the collective field Δ. The graphical rules are slightly more involved technically than in the plasmon case since the pair field is bilocal. Consider at first the *free* quanta which can be obtained from the quadratic part of the action:

$$\begin{aligned}\mathscr{A}_2[\Delta^+, \Delta] &= -\frac{i}{2} \operatorname{tr} \left[\left(\frac{i}{i\partial_t - \xi(-i\nabla)} \delta \right) \Delta \left(\frac{i}{i\partial_t + \xi(i\nabla)} \delta \right) \Delta^+ \right] \\ &\quad + \frac{1}{2} \int dx\, dx' |\Delta(x, x')|^2 \frac{1}{V(x, x')}. \end{aligned} \tag{4.15}$$

Variation with respect to Δ displays the equations of motion

$$\Delta(x, x') = iV(x, x') \left[\left(\frac{i}{i\partial_t - \xi(-i\nabla)} \delta \right) \Delta \left(\frac{i}{i\partial_t + \xi(i\nabla)} \delta \right) \right]. \tag{4.16}$$

This equation coincides exactly with the Bethe-Salpeter equation [9], in ladder approximation, for two-body bound state vertex functions usually denoted in momentum space by $\Gamma(p, p') = \int dx\, dx' \exp[i(px + p'x')]\, \Delta(x, x')$. Thus the free quanta of the field $\Delta(x, x')$ consist of bound pairs of the original fundamental particles. The field $\Delta(x, x')$ will consequently be called "pair field". If we introduce total and relative momenta $q = p + p'$ and $P = (p - p')/2$, then (4.16) can be written as[12])

$$\begin{aligned}\Gamma(P \mid q) &= -i \int \frac{dP'}{(2\pi)^4} V(P - P') \frac{i}{\frac{q_0}{2} + P_0' - \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}'\right) + i\eta \operatorname{sign} \xi} \\ &\quad \times \Gamma(P' \mid q) \frac{i}{\frac{q_0}{2} - P_0' - \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}'\right) + i\eta \operatorname{sign} \xi}. \end{aligned} \tag{4.17}$$

[12]) Here q stands short for $q_0 = E$ and $\boldsymbol{q}$.

Graphically this formula can be represented as follows:

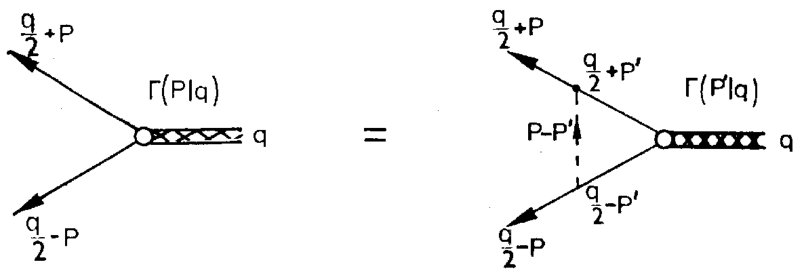


Fig. IV. The free pair field follows the Bethe-Salpeter equation pictured in this diagram

The Bethe-Salpeter *wave function* is related to the vertex $\Gamma(P \mid q)$ by

$$\Phi(P \mid q) = N \frac{i}{\frac{q_0}{2} + P_0 - \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) + i\eta \operatorname{sgn} \xi} \frac{i}{\frac{q_0}{2} - P_0 - \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right) + i\eta \operatorname{sgn} \xi} \times \Gamma(P \mid q). \tag{4.18}$$

It satisfies

$$G_0\left(\frac{q}{2} + P\right) G_0\left(\frac{q}{2} - P\right) \Phi(P \mid q) = -i \int \frac{dP'}{(2\pi)^4} V(P, P')\, \Phi(P' \mid q)$$

thus coinciding, up to a normalization, with the Fourier transform of the two-body state wave functions

$$\psi(\boldsymbol{x}t, \boldsymbol{x}'t') = \langle 0 |\, T\big(\psi(\boldsymbol{x}t)\, \psi(\boldsymbol{x}'t')\big)\, | B(q) \rangle . \tag{4.19}$$

If the potential is instantaneous, then (4.16) shows $\Delta(x, x')$ to be factorizable according to

$$\Delta(x, x') = \delta(t - t')\, \Delta(\boldsymbol{x}, \boldsymbol{x}'; t) \tag{4.20}$$

such that $\Gamma(P \mid q)$ becomes independent of P_0.

Consider now the system at $T = 0$ in the vacuum. Then $\mu = 0$ and $\xi(-i\nabla) = \varepsilon(-i\nabla) > 0$. One can perform the P_0 integral in (4.17) with the result

$$\Gamma(\boldsymbol{P} \mid q) = \int \frac{d^3P'}{(2\pi)^4} V(\boldsymbol{P} - \boldsymbol{P}') \frac{1}{q_0 - \varepsilon\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}'\right) - \varepsilon\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}'\right) + i\eta} \Gamma(\boldsymbol{P}' \mid q). \tag{4.21}$$

Now the equal-time Bethe-Salpeter wave function

$$\psi(\boldsymbol{x}, \boldsymbol{x}'; t) \equiv N \int \frac{d^3\boldsymbol{P}\, dq_0\, d^3\boldsymbol{q}}{(2\pi)^7} \exp\left[-i\left(q_0 t - \boldsymbol{q}\,\frac{\boldsymbol{x} + \boldsymbol{x}'}{2} - \boldsymbol{P}(\boldsymbol{x} - \boldsymbol{x}')\right)\right] \times \frac{1}{q_0 - \varepsilon\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) - \varepsilon\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right) + i\eta} \Gamma(\boldsymbol{P} \mid q) \tag{4.22}$$

satisfies

$$\big(i\partial_t - \varepsilon(-i\nabla) - \varepsilon(-i\nabla')\big)\, \psi(\boldsymbol{x}, \boldsymbol{x}'; t) = V(\boldsymbol{x} - \boldsymbol{x}')\, \psi(\boldsymbol{x}, \boldsymbol{x}'; t) \tag{4.23}$$

which is simply the Schrödinger equation. Thus, in the instantaneous case, the free collective excitations in $\Delta(x, x')$ are the bound states as they follow from the Schrödinger equation.

In a thermodynamic ensemble the energies in (4.17) have to be summed over Matsubara frequencies only. As a result, the Schrödinger equation is modified as ($q_0 \equiv i\nu$)

$$\Gamma(\boldsymbol{P} \mid q) = -\int \frac{d^3\boldsymbol{P}'}{(2\pi)^3} V(\boldsymbol{P} - \boldsymbol{P}')\, l(\boldsymbol{P}' \mid q)\, \Gamma(\boldsymbol{P}' \mid q) \tag{4.24}$$

with

$$\begin{aligned}
l(\boldsymbol{P} \mid q) &= -i \sum_{P_0} G_0\left(\frac{q}{2} + P\right) \tilde{G}_0\left(P - \frac{q}{2}\right) \\
&= -i \sum_{P_0} \frac{i}{\frac{q_0}{2} + P_0 - \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) + i\eta \operatorname{sgn} \xi} \; \frac{i}{\frac{q_0}{2} - P_0 - \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right) + i\eta \operatorname{sgn} \xi} \\
&\to -{}^{T}\sum_{\omega_n} \frac{1}{i\left(\omega_n + \frac{\nu}{2}\right) - \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right)} \; \frac{1}{i\left(\omega_n - \frac{\nu}{2}\right) + \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)} \\
&= {}^{T}\sum_{\omega_n} \frac{1}{i\nu - \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) - \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)} \\
&\quad \times \left[\frac{1}{i\left(\omega_n + \frac{\nu}{2}\right) - \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right)} - \frac{1}{i\left(\omega_n - \frac{\nu}{2}\right) + \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)}\right] \\
&= -\frac{1 \pm \left(n\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) + n\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)\right)}{i\nu - \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) - \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)}.
\end{aligned} \tag{4.25}$$

Here we have used the familiar sum (see (3.18))

$$^{T}\sum_{\omega_n} \frac{1}{(\omega_n - \xi(p))} = \mp \frac{1}{e^{\xi(\boldsymbol{p})/T} \mp 1} \equiv \mp n(\boldsymbol{p})$$

with $n(\boldsymbol{p})$ being the acupation number of the state of enerqy $\xi(\boldsymbol{p})$.

The expression in brackets is antisymmetric if both $\xi \to -\xi$ since under this substitution $n \to \mp 1 - n$. In fact, one can write it in the form $-N(\boldsymbol{P}, \boldsymbol{q})$ with

$$N(\boldsymbol{P} \mid q) \equiv 1 \pm \left(n\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) + n\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)\right) = \frac{1}{2}\left(\operatorname{th}^{\mp 1} \frac{\xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right)}{2T} + \operatorname{th}^{\mp 1} \frac{\xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)}{2T}\right)$$

such that

$$l(\boldsymbol{P} \mid q) = -\frac{N(\boldsymbol{P} \mid \boldsymbol{q})}{i\nu - \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) - \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)}.$$

Defining again a Schrödinger type wave function as in (4.22), the bound state problem can be brought to the form (4.21) but with a momentum dependent potential $V(\boldsymbol{P} - \boldsymbol{P}') \times N(\boldsymbol{P}' \mid \boldsymbol{q})$.

We are now ready to construct the propagator of the pair field $\Delta(x, x')$ for $T = 0$. In many cases, this is most simply done by considering equ. (4.17) with a potential $\lambda V(P, P')$ rather than V and asking for all eigenvalues λ_n at *fixed* q. Suppose this eigenvalue leads to a complete set of vertex functions $\Gamma_n(P \mid q)$. Then one can write the propagator as

$$\Delta(P \mid q)\, \Delta^+(P' \mid q') = -i \sum \frac{\Gamma_n(P \mid q)\, \Gamma_n^+(P' \mid q)}{\lambda - \lambda_n(q)} (2\pi)^4\, \delta^{(4)}(q - q')\bigg|_{\lambda=1}. \qquad (4.26)$$

Obviously the vertex functions have to be normalized in a specific way; this is discussed in Appendix A.
Expansion in powers of $(\lambda/\lambda_n(q))^n$ displays the propagator of Δ as a ladder sum of exchanges (see App. A)

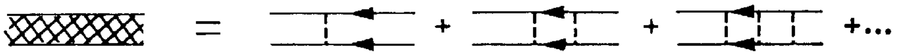


Fig. V. The free pair propagator amounts to summing a ladder of exchanges of the fundamental potential which is revealed explicitly in an expansion of (4.26) in powers $(\lambda/\lambda_n(q))$

In the instantaneous case either side is independent of P_0, P_0'. Then the propagator can be shown to coincide directly with the scattering matrix T of the Schrödinger equation (4.21), (4.23) (see equ. (A.13)).

$$\Delta\Delta^+ = iT \equiv iV + iV \frac{1}{E - H} V. \qquad (4.27)$$

Consider now the higher interactions $\mathscr{A}_n$, $n \geqq 3$ of equ. (4.14). They correspond to zig-zag loops

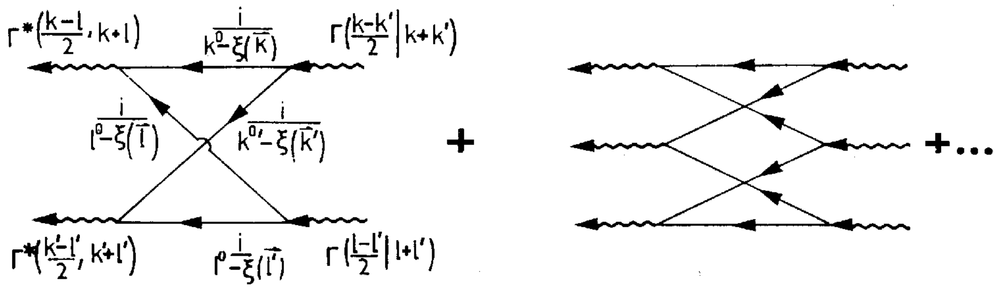


Fig. VI. The self-interaction terms of the non-polynomial pair Lagrangian amounts to the calculation of all single zig-zag loop diagrams absorbing and emitting n pair fields

which have to be calculated with every possible $\Gamma_n(P \mid q)$, $\Gamma_m^+(P \mid q)$ entering or leaving, respectively.
Due to the P dependence at every vertex, the loop integrals become very involved. A slight simplification arises for instantaneous potential in that at least the frequency sums can be performed immediately. Only in the special case of a completely local action the full P dependence disappears and the integrals can be calculated at least approximately. This will be done in the following section.

IV.2. Local Potential, Ginzburg-Landau Equations

As an illustration of the methods consider the latter case of a completely local potential. In order to be as general as possible let us allow for Bose and Fermi statistics. Then we are forced to carry along a spin index. Let us assume the fundamental interaction to be of the form

$$\mathscr{A}_{\text{int}} = \frac{g}{2} \sum_{\alpha,\beta} \int d^3x\, dt\, \psi_\alpha^+(\boldsymbol{x}, t)\, \psi_\beta^+(\boldsymbol{x}, t)\, \psi_\beta(\boldsymbol{x}, t)\, \psi_\alpha(\boldsymbol{x}, t). \qquad (4.28)$$

Following the general arguments leading to (4.1) we can rewrite the exponential of this interaction as[13])

$$\exp\left[\frac{i}{2}\, g \sum_{\alpha,\beta} \int d^3x\, dt \psi_\alpha^+ \psi_\beta^+ \psi_\beta \psi_\alpha(\boldsymbol{x}, t)\right]$$

$$= \text{const.} \int \mathrm{D}\Delta(\boldsymbol{x}t)\, \mathrm{D}\Delta^+(\boldsymbol{x}t) \exp\left[-\frac{i}{2}\int d^3x\, dt \sum_{\alpha\beta}\left(|\Delta_{\alpha\beta}(\boldsymbol{x}t)|^2 \frac{1}{g} - \Delta_{\alpha\beta}^+ \psi_\alpha \psi_\beta - \psi_\alpha^+ \psi_\beta^+ \Delta_{\alpha\beta}\right)\right] \tag{4.29}$$

where the new auxiliary field is a $(2s+1)\times(2s+1)$ non-hermitian matrix which satisfies the equation of constraint:

$$\Delta_{\alpha\beta}(\boldsymbol{x}, t) = g\psi_\alpha(\boldsymbol{x}, t)\, \psi_\beta(\boldsymbol{x}, t). \tag{4.30}$$

Thus the matrix A of (4.5) reads

$$A(\boldsymbol{x}, t; \boldsymbol{x}', t') = \begin{pmatrix} \big(i\partial_t - \xi(-i\nabla)\big)\, \delta^{(4)}(x - x')\, \delta_{\alpha\beta} & -\Delta_{\alpha\beta}(x)\, \delta^{(4)}(x - x') \\ -\Delta^+(x)_{\alpha\beta}\, \delta^{(4)}(x - x') & \mp\big(i\partial_t + \xi(i\nabla)\big)\, \delta^4(x - x')\, \delta_{\alpha\beta} \end{pmatrix} \tag{4.31}$$

and the action (4.14) becomes

$$\mathscr{A}[\Delta^+, \Delta] = \mp i \sum_{n=1}^{\infty} \frac{(-)^n}{2n} \operatorname{tr} \operatorname{tr}_{\text{spin}} \left[\left(\frac{i}{i\partial_t - \xi(-i\nabla)}\, \delta\right)(\Delta \cdot \delta)\left(\frac{\mp i}{i\partial_t + \xi(i\nabla)}\, \delta\right)(\Delta^+ \cdot \delta)\right]^n$$
$$- \frac{1}{2} \operatorname{tr}_{\text{spin}} \int d^4x\, \Delta^+(x)\, \Delta(x)\, \frac{1}{g} \tag{4.32}$$

where $\operatorname{tr}_{\text{spin}}$ is only over spin indices while tr is restricted to the functional matrix space.

Let us now confine our attention to fermions of spin 1/2 close to a critical region, i.e. $T \approx T_c$ in which long-range properties of the system dominate. As far as such questions are concerned, the expansion $\mathscr{A}[\Delta^+, \Delta] = \sum_2^\infty \mathscr{A}_n[\Delta^+, \Delta]$ may be truncated after the fourth term without much loss of information (the dimensions of the neglected terms are so high that they become invisible at long distances [7].

The free part of the action, $\mathscr{A}_2[\Delta^+, \Delta]$, is given by

$$\mathscr{A}_2[\Delta^+, \Delta] = \pm i \operatorname{tr} \operatorname{tr}_{\text{spin}}\left[\left(\frac{i}{i\partial_t - \xi(-i\nabla)}\, \delta\right)(\Delta\delta)\left(\frac{\mp i}{i\partial_t + \xi(i\nabla)}\, \delta\right)(\Delta^+\delta)\right]$$
$$- \frac{1}{2} \operatorname{tr}_{\text{spin}} \int dx\, \Delta^+(x)\, \Delta(x) \frac{1}{g}. \tag{4.33}$$

The spin traces can be performed by noticing that due to Fermi statistics and remembering the constraint equ. (4.4), (4.30), there is really only one independent pair field:

$$\Delta(x) \equiv \Delta_{\downarrow\uparrow}(x) = g\psi_\downarrow(x)\, \psi_\uparrow(x) = -\Delta_{\uparrow\downarrow}(x). \tag{4.34}$$

Thus $\mathscr{A}_2$ becomes:

$$\mathscr{A}_2[\Delta^+ \Delta] = -i \int dx\, dx' G_0(x, x')\, \tilde{G}_0(x', x)\, \Delta^+(x)\, \Delta(x') - \frac{1}{g} \int dx\, |\Delta(x)|^2. \tag{4.35}$$

---

[13]) In analogy to [10]), the hermitian adjoint $\Delta_{\alpha\beta}^+(x)$ comprises transposition in the spin indices, i.e. $\Delta_{\alpha\beta}^+(x) = [\Delta_{\beta\alpha}(x)]^+$.

In momentum space this can be rewritten as

$$\mathscr{A}_2[\Delta^+, \Delta] = \sum_k \Delta^+(k)\, L(k)\, \Delta(k) \tag{4.36}$$

where

$$L(k) = -i \sum_p \frac{i}{p^0 + k^0 - \xi(\boldsymbol{p}+\boldsymbol{k}) + i\eta \operatorname{sgn} \xi(\boldsymbol{p}+\boldsymbol{k})} \frac{i}{p^0 + \xi(\boldsymbol{p}) - i\eta \operatorname{sgn} \xi(\boldsymbol{p})} - \frac{1}{g}$$
$$= \sum_{\boldsymbol{p}} l(\boldsymbol{p} \mid k) - \frac{1}{g}$$

as pictured by the diagram in Fig .VII.

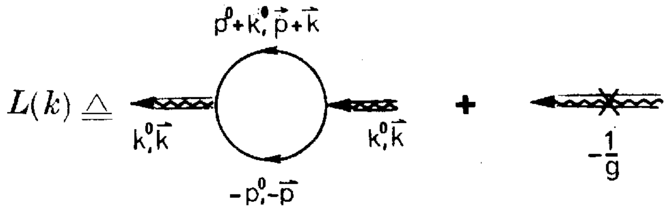


Fig. VII. The free part of the pair field Δ Lagrangian contains the direct term plus the one loop diagram. As a consequence, the free Δ propagator sums up an infinite sequence of such loops

The expression $l(\boldsymbol{p} \mid k)$ was discussed before in general and brought to the form (4.25). For the present case of Fermi statistics this leads to

$$L(\nu, \boldsymbol{k}) = \frac{1}{2} \sum_{\boldsymbol{p}} \frac{1}{\xi(\boldsymbol{p}+\boldsymbol{k}) + \xi(\boldsymbol{p}) - i\nu} \left(\operatorname{th} \xi(\boldsymbol{p}+\boldsymbol{k})/2T + \operatorname{th} \xi(\boldsymbol{p})/2T\right) - \frac{1}{g}. \tag{4.37}$$

At $k = 0$ one has[14])

$$L(0) = \frac{1}{2} \sum_{\boldsymbol{p}} \frac{\operatorname{th} \xi(\boldsymbol{p})/2T}{\xi(\boldsymbol{p})} - \frac{1}{g} \approx N(0) \int_0^{\omega_D} \frac{d\xi}{\xi} \operatorname{th} \xi/2T - \frac{1}{g} = N(0) \log\left(\frac{\omega_D}{T}\frac{2e^\gamma}{\pi}\right) - \frac{1}{g} \tag{4.38}$$

which vanishes at a critical temperature determined by

$$T_c \equiv \frac{2e^\gamma}{\pi}\, \omega_D \exp\left(-1/N(0)\, g\right). \tag{4.39}$$

In terms of T_c, $L(0)$ can be rewritten as

$$L(0) = N(0) \log \frac{T_c}{T} \approx N(0) \left(1 - \frac{T}{T_c}\right). \tag{4.40}$$

The constant $L(0)$ obviously plays the role of the chemical potential of the pair field. Its vanishing at $T = T_c$ implies that at that temperature the field propagates over long

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[14]) Here we have used the approximation $\sum_{\boldsymbol{p}} \equiv \int \frac{d^3p}{(2\pi)^3} = (2\pi)^{-3} \int p^2 \frac{dp}{d\xi}\, d\xi \approx N(0) \int d\xi$ with $N(0) = \frac{mp_F}{2\pi^2} = \frac{3}{4}\frac{n}{\mu}$, and the Debye frequency ω_D, which is typically of the order of a few hundred K, i.e. much larger than T_c which lies in the K range. Notice that the Fermi energy is usually an order of magnitude larger than ω_D which acts as a short-distance cutoff, due to the lattice system. The consant γ is Euler's: $\gamma = -\frac{\Gamma'(1)}{\Gamma(1)} \approx 0{,}577$, hence $2\frac{e^\gamma}{\pi} \approx 1{,}13$.

range (with a power law) in the system. Critical phenomena are observed [7]. For $T < T_c$ the chemical becomes positive indicating the appearence of a Bose condensate. If $\nu \neq 0$ but $\boldsymbol{k} = 0$ one can write (4.35) as in the subtracted form

$$L(\nu, \mathbf{0}) - L(0, \mathbf{0}) = \sum_{\boldsymbol{p}} \left(\frac{1}{2\xi(\boldsymbol{p}) - i\nu} - \frac{1}{2\xi(\boldsymbol{p})} \right) \operatorname{th} \frac{\xi(\boldsymbol{p})}{2T} \approx i\nu N(0) \int_{-\omega_D}^{\omega_D} \frac{d\xi}{2\xi - i\nu} \frac{\operatorname{th}\frac{\xi}{2T}}{2\xi}.$$

Since the integral converges fast it can be performed over the whole ξ axis with the small error $T/\omega_D \ll 1$. For $\nu < 0$, the contour may be closed above picking up poles exactly at the Matsubara frequencies $\xi = i(2n + 1)\pi T = i\omega_n$. Hence

$$L(\nu, \mathbf{0}) - L(0, \mathbf{0}) \approx \nu N(0)\, \pi T \sum_{\omega_n > 0} \frac{1}{\omega_n - \frac{\nu}{2}} \frac{1}{\omega_n}.$$

The sum can be expressed in terms of Digamma functions: For $|\nu| \ll T$ one expands

$$\sum_{\omega_n>0} \frac{1}{\omega_n^2} + \frac{\nu}{2} \sum_{\omega_n>0} \frac{1}{\omega_n^3} + \frac{\nu^2}{4} \sum_{\omega_n>0} \frac{1}{\omega_n^4} + \dots$$

and applies the formula

$$\sum_{\omega_n>0} \frac{1}{\omega_n^k} = \frac{1}{\pi^k T^k} (1 - 2^{-k})\, \zeta(k). \tag{4.41}$$

For example:

$$\sum_{\omega_n>0} \frac{1}{\omega_n^2} = \frac{1}{\pi^2 T^2} \frac{3}{4} \frac{\pi^2}{6} = \frac{1}{8T^2}$$

$$\sum_{\omega_n>0} \frac{1}{\omega_n^3} = \frac{1}{\pi^3 T^3} \frac{7}{8} \zeta(3)$$

$$\sum_{\omega_n>0} \frac{1}{\omega_n^4} = \frac{1}{\pi^4 T^4} \frac{15}{16} \frac{\pi^4}{90} = \frac{1}{96T^4}.$$

Using the power series for the Digamma function

$$\psi(1 - x) = -\gamma - \sum_{k=2}^{\infty} \zeta(k)\, x^{k-1}$$

the sum is

$$-\frac{2}{\nu\pi T} \left[\psi\left(1 - \frac{\nu}{2\pi T}\right) - \psi\left(1 - \frac{\nu}{4\pi T}\right) \Big/ 2 + \gamma/2 \right] = \frac{1}{\nu\pi T} \left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} - \frac{\nu}{4\pi T}\right) \right]$$

with the first terms

$$\frac{1}{8T^2} + \nu \frac{1}{2\pi^3 T^3} \frac{7}{8} \zeta(3) + \frac{\nu^2}{4 \cdot 96 T^4} + \cdots$$

For $\nu > 0$ the integration contour is closed below and the same result is obtained with $-\nu$. Thus one finds

$$L(\nu, 0) - L(0, 0) = N(0) \left(\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{|\nu|}{4\pi T}\right) \right)$$

$$\approx -N(0) \left\{ \frac{\pi}{8T} |\nu| - \nu^2 \frac{1}{2\pi^2 T^2} \frac{7}{8} \zeta(3) + \cdots \right\}. \tag{4.42}$$

The $\boldsymbol{k}$ dependence at $\nu = 0$ is obtained by expanding directly

$$L(0, \boldsymbol{k}) = \sum_{\omega,\boldsymbol{p}}^{T} \frac{1}{i\omega - \xi(\boldsymbol{p}+\boldsymbol{k})} \frac{1}{-i\omega - \xi(\boldsymbol{p})} - \frac{1}{g}$$

$$= \sum_{n=0}^{\infty} \sum_{\omega,\boldsymbol{p}}^{T} \frac{1}{\left(i\omega - \xi(\boldsymbol{p})\right)^{n+1}} \left(\frac{\boldsymbol{p}}{m}\,\boldsymbol{k} + \frac{\boldsymbol{k}^2}{2m}\right)^n \frac{1}{-i\omega - \xi(\boldsymbol{p})} - \frac{1}{g}. \tag{4.43}$$

The $\boldsymbol{p}$ integration may be split into radial and angular parts as

$$\int \frac{d^3p}{(2\pi)^3} \approx N(0) \int d\xi \int \frac{d\hat{p}}{4\pi}. \tag{4.44}$$

The denominators are strongly peaked at $\xi \approx 0$ such that only the narrow region $|\xi| \lesssim T$ contributes. Hence, the momentum p may be replaced by the Fermi momentum p_F with only a small error $O(T/\mu)$ ($\approx 10^{-3}$). Introducing now the Fermi velocity $v_F = p_F/m$, for convenience, and performing the ξ-integrals

$$\int d\xi \frac{1}{(i\omega - \xi)^{n+1}} \frac{1}{-i\omega - \xi} = (-i \operatorname{sgn} \omega)^n \, \pi/(2^n |\omega|^{n+1}) \tag{4.45}$$

one finds

$$L(0, \boldsymbol{k}) \approx 2N(0) \operatorname{Re} \sum_{n=0}^{\infty} \sum_{\omega>0} (-i)^n \frac{\pi}{2^n |\omega|^{n+1}} \int \frac{d\hat{p}}{4\pi} \left(v_F \hat{\boldsymbol{p}}\boldsymbol{k} + \frac{\boldsymbol{k}^2}{2m}\right)^n - \frac{1}{g}. \tag{4.46}$$

For $\boldsymbol{k} = 0$ we recover the logarithmically divergent sum

$$L(0, \boldsymbol{0}) = N(0) \sum_{\omega} \frac{\pi}{|\omega|} - \frac{1}{g}.$$

which was made finite before by the cutoff procedure (4.38). The higher powers can be summed via formula (4.41) with the result

$$L(0, \boldsymbol{k}) = L(0, \boldsymbol{0}) + 2N(0) \operatorname{Re} \sum_{n=1}^{\infty} \frac{(-i)^n}{2^n \pi^n T^n} (1 - 2^{-(n+1)})\, \zeta(n+1) \int \frac{d\hat{p}}{4\pi} \left(v_F \hat{\boldsymbol{p}}\boldsymbol{k} + \frac{\boldsymbol{k}^2}{2m}\right)^n$$

$$= L(0, \boldsymbol{0}) + N(0) \operatorname{Re} \int \frac{d\hat{p}}{4\pi} \left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} - i\left(v_F \hat{\boldsymbol{p}}\boldsymbol{k} + \frac{\boldsymbol{k}^2}{2m}\right)\Big/4\pi T\right)\right]. \tag{4.47}$$

Comparing this with equ. (4.42) one sees that the full $\boldsymbol{k}$ and ν dependence is obtained by adding $\frac{|\nu|}{4\pi T}$ to the arguments of the second Digamma function. This can also be checked by a direct calculation. In the long-wave length limit in which $kv_F/T \ll 1$ one has also $(k^2/2m)/T \approx \frac{k}{p_F}\frac{kv_F}{T} \ll \frac{kv_F}{T}$ and one may truncate the sum after the quadratic term as follows:

$$L(0, \boldsymbol{k}) = L(0, \boldsymbol{0}) + \sum A_{kl}(0)\, k_k k_l \tag{4.48}$$

where

$$A_{kl}(0) = -2N(0) \frac{1}{4\pi^2 T^2} \frac{7}{8} \zeta(3)\, v_F^2 \int \frac{d\hat{p}}{4\pi} \hat{p}_k \hat{p}_l. \tag{4.49}$$

The angular integration yields

$$\int \frac{d\hat{p}}{4\pi} \hat{p}_k \hat{p}_l = \frac{1}{3} \delta_{kl}. \tag{4.50}$$

Hence, the lowest terms in the expansion of $L(\nu, \boldsymbol{k})$ for $kv_F \ll T$ and $\nu \ll T$ are

$$L(\nu, \boldsymbol{k}) \approx L(0, 0) - N(0) \left[\frac{\pi}{8T} |\nu| + \frac{1}{6\pi^2 T^2} \frac{7}{8} \zeta(3) v_F^2 \boldsymbol{k}^2\right]. \tag{4.51}$$

The term (4.49) may also conveniently be calculated in x space, since there, for large x ($\gg 1/p_F$),

$$G_0(\boldsymbol{x}, \omega) \approx -\frac{m}{2\pi|\boldsymbol{x}|} \exp\left[ip_F|\boldsymbol{x}| \operatorname{sgn}\omega - \frac{|\omega|}{v_F} |\boldsymbol{x}|\right] \tag{4.52}$$

such that the second spatial derivative contributes to (4.32):

$$\int dx \left(\frac{1}{2} \int d^3x' \, T \sum_{\omega_n} G_0(\boldsymbol{x} - \boldsymbol{x}', \omega_n) \, G_0(\boldsymbol{x} - \boldsymbol{x}', -\omega_n) \, (x - x')_i \, (x - x')_j\right) \Delta^+(x) \, \nabla_i \nabla_j \Delta(x).$$

The parenthesis becomes

$$\frac{1}{2} \int d^3z T \sum_{\omega_n} \left(\frac{m}{2\pi|\boldsymbol{z}|}\right)^2 \exp\left(-2 \frac{|\omega_n|}{v_F} |\boldsymbol{z}|\right) z_i z_j$$
$$= \frac{1}{24} \delta_{ij} T \int d^3z \frac{1}{\operatorname{sh} 2\pi|\boldsymbol{z}|T/v_F} = \delta_{ij} \frac{7\zeta(3)}{48} N(0) \frac{v_F^2}{\pi^2 T^2} \tag{4.53}$$

making (4.51) coincide with (4.48).

For many formulas to come it is useful to introduce the characteristic length parameter (using $T_F \equiv \mu \equiv p_F^2/2m$)

$$\xi_0 \equiv \sqrt{\frac{7\zeta(3)}{48}} \frac{v_F}{\pi T_c} = \sqrt{\frac{7\zeta(3)}{48}} \frac{2T_F p_F^{-1}}{\pi T_c} \approx \frac{1}{4} \frac{T_F}{T_c} p_F^{-1} \tag{4.54}$$

which, in most superconductors, is of the order of 100 nm. Then, in the action (4.36), the low-frequency and long-wavelength result (4.51) corresponds to*)

$$\mathscr{A}_2[\Delta^+, \Delta] \approx -iN(0) \, T \sum_{\nu \ll T, \boldsymbol{k}} \Delta^+(\nu, \boldsymbol{k}) \left\{\left(1 - \frac{T}{T_c}\right) - \xi_0^2 \boldsymbol{k}^2 - \frac{\pi}{8T} |\nu|\right\} \Delta(\nu, \boldsymbol{k}). \tag{4.55}$$

For $T \gtrsim T_c$, the field Δ can now be quantized with a propagator

$$\overline{\Delta(\nu_n, \boldsymbol{k}) \, \Delta(\nu_m, \boldsymbol{k}')} = -(2\pi)^3 \, \delta^3(\boldsymbol{k} - \boldsymbol{k}') \frac{1}{T} \delta_{n,m} \frac{1}{N(0)} \left(-\frac{\pi}{8T} |\nu_n| + \left(1 - \frac{T}{T_c}\right) - \xi_0^2 \boldsymbol{k}^2\right)^{-1}. \tag{4.56}$$

The spectrum of collective excitations can be read off this expression by continuing the energy back to real values from the upper half of the complex plane:

$$k_0 = -i \frac{8}{\pi} (T - T_c) - i \frac{8T}{\pi} \xi_0^2 k^2.$$

These excitations are purely dissipative.

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*) Notice that only Matsubara frequency $\nu_0 = 0$ satisfies the condition $\nu \ll T$. The neighbourhood of $\nu_0 = 0$ with the linear behaviour $|\nu|$ becomes visible only after analytic continuation of (4.56) to the retarded Green's function which amounts to replacing $|\nu_n| \to -ik_0$.

If the system is close enough to the critical temperature all interaction terms except $\mathscr{A}_4[\Delta^+, \Delta]$ become irrelevant because of their high dimensions [7]. And in $\mathscr{A}_4$ only the momentum independent contribution is of interest, again because it has the lowest dimension.
Its calculation is standard:

$$\mathscr{A}_4[\Delta^+\Delta] = \frac{i}{4} \operatorname{tr} \operatorname{tr}_{\text{spin}} \left[\left(\frac{i}{i\partial_t - \xi(-i\nabla)}\,\delta\right) \Delta\delta \left(\frac{i}{i\partial_t + \xi(i\nabla)}\,\delta\right) \Delta^+\delta\right]^2$$

$$= -\frac{i}{2} \int dx_1\, dx_2\, dx_3\, dx_4 G_0(x_1x_2)\, \tilde{G}_0(x_2x_3)\, G_0(x_3x_4)\, \tilde{G}_0(x_4x_1)\, \Delta^+(x_1)\, \Delta(x_2)\, \Delta^+(x_3)\, \Delta(x_4)$$

$$\approx -\frac{1}{2} \int dx |\Delta(x)|^4 \int d^3x_2\, d^3x_3\, d^3x_4 T \sum_{\omega_n} [G_0(\boldsymbol{x} - \boldsymbol{x}_2, \omega_n)\, G_0(\boldsymbol{x}_3 - \boldsymbol{x}_2, -\omega_n)$$

$$\times\, G_0(\boldsymbol{x}_3 - \boldsymbol{x}_4, \omega_n)\, G_0(\boldsymbol{x} - \boldsymbol{x}_4, -\omega_n)] \equiv -\frac{\beta}{2} \int dx |\Delta(x)|^4. \tag{4.57}$$

The coefficient can be computed as usual

$$\beta = T \sum_{\omega_n \boldsymbol{p}} \frac{1}{\left(\omega_n^2 + \xi^2(\boldsymbol{p})\right)^2} \approx N(0)\, T \sum_{\omega_n} \int d\xi\, \frac{1}{(\omega_n^2 + \xi^2)^2}$$

$$= N(0)\, \frac{\pi}{2}\, T \sum_{\omega_n} \frac{1}{|\omega_n|^3} = N(0)\, \frac{7\zeta(3)}{8(\pi T_c)^2} = 6N(0)\, \frac{\xi_0^2}{v_F^2} \approx 2.6 \times 10^{-3}\, \frac{p_F^3}{T_F T_c^2}. \tag{4.58}$$

The time independent part of this action at the classical level has been derived a long time ago by Gorkov on the basis of Green's function techniques [*3*, *11*]. Certainly, his technical manipulations are exactly the same as presented here. The difference lies in the theoretical foundation [*4*, *5*, *6*, *7*] and the ensuing prescriptions on how to improve upon the approximations. Our action of (4.7) is the *exact* translation of the fundamental theory into pair fields. These fields are made quantum fields in the standard fashion by leaving the functional formalism and going to the operator language. The result is a perturbation theory of Δ quanta with (4.56) as a free propagator and $\mathscr{A}_n$, $n > 2$ treated as perturbations. The higher terms $\mathscr{A}_6, \mathscr{A}_8, \ldots$ are very weak residual interactions as far as long distance questions are concerned. In fact, for the calculation of the critical indices, $\mathscr{A}_2$ and $\mathscr{A}_4$ contain *all* information about the system.

IV.3. Inclusion of Electromagnetic Fields into the Pair Theory

The original action $\mathscr{A}$ of (2.1) can be made invariant under general spacetime dependent gauge transformations

$$\psi(\boldsymbol{x}, t) \to \exp\left[-i\Lambda(\boldsymbol{x}, t)\right] \psi(\boldsymbol{x}, t) \tag{4.59}$$

if an electromagnetic potential

$$A = (\varphi, \boldsymbol{A}) \tag{4.60}$$

is present, capable of absorbing the generated derivative terms via

$$\begin{aligned} \varphi &\to \varphi - \frac{1}{e}\, \partial_t \Lambda \\ A_i &\to A_i + \frac{c}{e}\, \nabla_i \Lambda. \end{aligned} \tag{4.61}$$

The complete action including electromagnetism in the Coulomb gauge, $\nabla A = 0$, becomes:

$$\mathscr{A}_{\text{compl.}} = \mathscr{A}[\psi^+, \psi]\left(i\partial_t \to i\partial_t + e\varphi,\ -i\nabla_i \to -i\nabla_i + \frac{c}{e} A_i\right)$$
$$+ \frac{1}{8\pi}\int dx \left(-\varphi\nabla^2\varphi + \frac{1}{c^2} \boldsymbol{A}^2 + \boldsymbol{A}\nabla^2\boldsymbol{A}\right) \tag{4.62}$$

where the arrows denote the gauge invariant substitutions in the action (2.1). Since the final pair action (4.14) is an exact translation of (2.1), it certainly has to possess the same invariance after inclusion of electromagnetism. But from the constraint equation (4.4) we see

$$\Delta(x, x') \to \exp\left[-i\big(\Lambda(x) + \Lambda(x')\big)\right] \Delta(x, x'). \tag{4.63}$$

For the local pair field appearing in (4.29) this gives

$$\Delta(x) \to \exp\left[-2i\Lambda(x)\right] \Delta(x). \tag{4.64}$$

Hence the final action (4.55) with $\mathscr{A}_4$ from (4.57) added is gauge invariant after replacing

$$\begin{aligned} i\partial_t &\to i\partial_t + 2e\varphi, \qquad -i\nabla_i \to -i\nabla_i + 2\,\frac{e}{c}\,A_i, \\ k_0 &\to k_0 + 2e\varphi, \qquad k_i \to k_i + 2\,\frac{e}{c}\,A_i. \end{aligned} \tag{4.65}$$

This leads to the full time dependent Lagrangian close to the critical point

$$\begin{aligned} \mathscr{L} = {} & \frac{N(0)\,\pi}{8T}\,\Delta^+(x)\,(-\partial_t + 2ie\varphi)\,\Delta(x) + N(0)\left(1 - \frac{T}{T_c}\right)\Delta^+\Delta \\ & - N(0)\,\xi_0^2\left(\nabla_i - 2i\,\frac{e}{c}\,A_i\right)\Delta^+(x)\left(\nabla_i + 2i\,\frac{e}{c}\,A_i\right)\Delta(x) \\ & - 3N(0)\,\frac{\xi_0^2}{v_F^2}\,|\Delta(x)|^4 + \frac{1}{8\pi}\left(-\varphi\nabla^2\varphi + \frac{1}{c^2}\,\dot{\boldsymbol{A}}^2 + \boldsymbol{A}\nabla^2\boldsymbol{A}\right). \end{aligned} \tag{4.66}$$

The discussion of this Lagrangian is standard. At the classical level there are, above T_c, doubly charged pair states of chemical potential

$$\mu_{\text{pair}} = L(0) = N(0)\left(1 - \frac{T}{T_c}\right) < 0; \quad T > T_c. \tag{4.67}$$

Below T_c the chemical potential becomes positive causing an instability which settles, due to the stabilizing quartic term, at a non-zero field value, the "gap":

$$\Delta_0(T) = \sqrt{\frac{\mu_{\text{pair}}}{\beta}} = \sqrt{\frac{8}{7\zeta(3)}}\,\pi T_c\left(1 - \frac{T}{T_c}\right)^{1/2} \approx 3.1 T_c\left(1 - \frac{T}{T_c}\right)^{1/2}. \tag{4.68}$$

The new vacuum obviously brakes gauge invariance spontaneously: the field Δ will now oscillate radially with a chemical potential

$$\mu_{\text{rad}} = -2N(0)\left(1 - \frac{T}{T_c}\right) < 0; \quad T < T_c. \tag{4.69}$$

Due to this, spatial changes of the field $|\Delta|$ can take place over a length scale, defined as coherence length [3, 11]

$$\xi_c(T) \equiv \sqrt{\frac{\text{coefficient of } |\nabla\Delta|^2}{|\mu_{\text{pair}}|}} = \xi_0 \left(1 - \frac{T}{T_c}\right)^{-1/2}. \tag{4.70}$$

The azimuthal oscillations experience a different fate in the absence of electromagnetism; they have a vanishing chemical potential due to the invariance of $\mathscr{L}$ under phase rotations. As an electromagnetic field is turned on, the new center of oscillations (4.68) is seen in (4.66) to generate a mass term $1/8\pi\ \mu_A{}^2 \boldsymbol{A}^2$ for the photon. The vector potential aquires a mass

$$\mu_A{}^2 = 8\pi \text{ coefficient of } \boldsymbol{A}^2 \text{ from } |\nabla\Delta|^2 = 8\pi \frac{4e^2}{c^2} N(0)\, \xi_0{}^2 \Delta_0{}^2. \tag{4.71}$$

This mass limits the penetration of magnetic field into a superconductor. The penetration depth is defined as [3, 11]

$$\lambda(T) \equiv \mu_A{}^{-1} = \sqrt{\frac{3}{\pi N(0)} \frac{c}{4ev_F}} \left(1 - \frac{T}{T_c}\right)^{-1/2} = \sqrt{\frac{3\pi}{8}} \sqrt{\frac{c}{v_F \alpha}}\, p_F{}^{-1} \left(1 - \frac{T}{T_c}\right)^{-1/2}. \tag{4.72}$$

The ratio

$$\varkappa(T) \equiv \frac{\lambda(T)}{\xi(T)} = \sqrt{\frac{9\pi^3}{14\zeta(3)}} \sqrt{\frac{c}{v_F\alpha}} \frac{T_c}{T_F} \approx 4.1 \sqrt{\frac{c}{v_F\alpha}} \frac{T_c}{T_F} \tag{4.73}$$

is the Ginzburg-Landau parameter deciding whether it is energetically preferable for the superconductor to have flux lines invading it or not ($\varkappa > 1/\sqrt{2}$ yes, type II superconductor, $\varkappa < 1/\sqrt{2}$ no, type I superconductor).

IV.4. Far below T_c

We have seen in the last section that for T smaller than T_c the chemical potential of the pair field becomes positive, causing oscillations around a new minimum which is the "gap" value Δ_0 given by (4.68). This formula is based on the expansion (4.9) of the pair action and can be valid only as long as $\Delta \ll T_c$, i.e. $T \approx T_c$. If T drops far below T_c, the gap value is expected to increase and the expansion (4.9) shows bad convergence. In this case it is appropriate to account for Δ_0 non-perturbatively by inserting it as an open parameter into $\mathbf{G}_\Delta$ of (4.8). In the general bilocal form one writes

$$\Delta(x, x') = \Delta_0(x - x') + \Delta'(x, x')$$

and expands $\mathbf{G}_\Delta$ around

$$\mathbf{G}_{\Delta_0}(x, x') = i \begin{pmatrix} (i\partial_t - \xi(-i\nabla))\,\delta & -\Delta_0 \\ -\Delta_0{}^\dagger & \mp i(\partial_t + \xi(i\nabla))\,\delta \end{pmatrix}^{-1} (x, x') \tag{4.74}$$

instead of (4.9). This leads to the replacement $\mathbf{G}_0 \to \mathbf{G}_{\Delta_0}$ in every term of (4.13). Observe that in the underlying theory of fields ψ^+, ψ the matrix $\mathbf{G}_{\Delta_0}$ collects the bare one-particle Green's functions:

$$\mathbf{G}_{\Delta_0}(x, x') = \begin{pmatrix} \overbrace{\psi(x)\, \psi^+(x')} & \overbrace{\psi(x)\, \psi(x')} \\ \overbrace{\psi^+(x)\, \psi^+(x')} & \overbrace{\psi^+(x)\, \psi(x')} \end{pmatrix}. \tag{4.75}$$

Contrary to (4.9) and (4.11) the off-diagonal Green's functions are nonvanishing, since at $T < T_c$ a condensate is present in the vacuum. The presence of Δ_0 causes a linear dependence of the action on $\Delta'(x, x')$

$$\mathscr{A}_1[\Delta'^+, \Delta'] = \pm \mathrm{tr}\left(\mathbf{G}_{\Delta_0}\begin{pmatrix} 0 & \Delta' \\ \Delta'^+ & 0 \end{pmatrix}\right) + \frac{1}{2}\int dx\, dx' \left(\Delta_0^+(x - x')\, \Delta'(x, x')\, \frac{1}{V(x, x')} + \text{h.c.}\right). \tag{4.76}$$

The gap function may now be determined optimally by minimizing the action with respect to $\delta\Delta'$ at $\Delta' = 0$ which amounts to the elimination of $\mathscr{A}_1[\Delta'^+, \Delta']$. Taking the functional derivative of (4.76) gives the "gap equation"

$$\Delta_0(x - x') = \pm V(x - x')\ \mathrm{tr}\left(\mathbf{G}_{\Delta_0}(x, x')\, \frac{\tau^-}{2}\right) \tag{4.77}$$

where $\tau^-/2$ is the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in the 2×2 dimensional matrix space of (4.8).
If the potential is instantaneous, the gap has a factor $\delta(t - t')$: $\Delta_0(x - x') \equiv \delta(t - t') \times \Delta_0(\boldsymbol{x} - \boldsymbol{x}')$ and the Fourier transform of the spatial part satisfies

$$\Delta_0(\boldsymbol{p}) = \pm \sum_{\omega, \boldsymbol{p}'}^{T} V(\boldsymbol{p} - \boldsymbol{p}')\ \mathrm{tr}\left(\mathbf{G}_{\Delta_0}(\omega, \boldsymbol{p}')\, \frac{\tau^-}{2}\right). \tag{4.78}$$

Inverting (4.74) renders the propagator:

$$\mathbf{G}_{\Delta_0}(\tau, x) = \mp \sum_{\omega, \boldsymbol{p}}^{T} \exp\left[-i(\omega\tau - \boldsymbol{p}\boldsymbol{x})\right] \frac{1}{\omega^2 + \xi^2(\boldsymbol{p}) \mp |\Delta_0(\boldsymbol{p})|^2} \begin{pmatrix} \mp(i\omega + \xi(\boldsymbol{p})) & \Delta_0(\boldsymbol{p}) \\ \Delta_0^+(\boldsymbol{p}) & i\omega - \xi(\boldsymbol{p}) \end{pmatrix} \tag{4.79}$$

such that the gap equation (4.78) takes the explicit form

$$\Delta_0(\boldsymbol{p}) = -\sum_{\omega, \boldsymbol{p}'}^{T} V(\boldsymbol{p} - \boldsymbol{p}')\, \frac{\Delta_0(\boldsymbol{p}')}{\omega^2 + \xi^2(\boldsymbol{p}') \mp |\Delta_0(\boldsymbol{p}')|^2}. \tag{4.80}$$

Performing the frequency sum yields

$$\Delta_0(\boldsymbol{p}) = -\sum_{\boldsymbol{p}'} V(\boldsymbol{p} - \boldsymbol{p}')\, \frac{\Delta_0(\boldsymbol{p}')}{2E(\boldsymbol{p}')}\, \mathrm{th}^{\mp 1} \frac{E(\boldsymbol{p}')}{2T} \tag{4.81}$$

where

$$E(\boldsymbol{p}) = \sqrt{\xi^2(\boldsymbol{p}) \mp |\Delta_0(\boldsymbol{p})|^2}.$$

For the case of the superconductor with an attractive local potential

$$V(x - x') = -g\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\, \delta(t - t')$$

this becomes

$$\Delta_0 = g \sum_{\omega, \boldsymbol{p}}^{T} \frac{\Delta_0}{\omega^2 + \xi^2(\boldsymbol{p}) + |\Delta_0|^2} = \left(g \sum_{\boldsymbol{p}} \frac{1}{2E(\boldsymbol{p})}\, \mathrm{th}\, \frac{E(\boldsymbol{p})}{2T}\right) \Delta_0. \tag{4.82}$$

The integral is evaluated in the standard approximation [3]. There is a non-zero gap if

$$g \sum_{\boldsymbol{p}} \frac{1}{2E(\boldsymbol{p})}\, \mathrm{th}\, \frac{E(\boldsymbol{p})}{2T} = 1. \tag{4.83}$$

Let $T = T_c$ denote the critical temperature at which the gap vanishes. There: $E(\boldsymbol{p}) = \xi(\boldsymbol{p})$ such that equ. (4.83) determines the same T_c as the previous equs. (4.38), (4.39) which where derived for $T \approx T_c$ in a different fashion. The result (4.83) holds for any temperature.

The full temperature dependence of the gap cannot be obtained in closed form from (4.83). For $T \approx T_c$ one may expand directly (4.82) in powers of Δ_0.

$$
\begin{aligned}
1 &= g \sum_{\omega,\boldsymbol{p}}^{T} \frac{1}{\omega^2 + \xi^2(\boldsymbol{p})} - \Delta_0^2 \sum_{\omega,\boldsymbol{p}}^{T} \frac{1}{\left(\omega^2 + \xi^2(\boldsymbol{p})\right)^2} + \cdots \\
&= gN(0)\left(\log \frac{\omega_D}{T}\, 2\frac{e^{\gamma}}{\pi} - \Delta_0^2 \frac{7\zeta(3)}{8\pi^2 T^2} + \cdots\right) \\
&= 1 + N(0)\left(\left(1 - \frac{T}{T_c}\right) - \Delta_0^2 \frac{7\zeta(3)}{8\pi^2 T^2} + \cdots\right)
\end{aligned}
$$

and finds

$$
\Delta_0^2(T) \approx \frac{8}{7\zeta(3)}\, \pi^2 T_c^2 \left(1 - \frac{T}{T_c}\right)
$$

in agreement with (4.68).

For very small temperatures, on the other hand, equ. (4.82) can be written as

$$
\begin{aligned}
1 &= gN(0) \int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta_0^2}} \left(1 - 2\exp\left(-\sqrt{\xi^2 + \Delta_0^2}/T\right) - \cdots\right) \\
&= gN(0)\left(\log \frac{2\omega_D}{\Delta_0} - 2K_0\left(\frac{\Delta_0}{T}\right)\right) + \cdots
\end{aligned}
\tag{4.84}
$$

For small T, K_0 vanishes as

$$
2K_0\left(\frac{\Delta_0}{T}\right) \to \frac{1}{\Delta_0} \sqrt{2\pi T \Delta_0}\; e^{-\Delta_0/T}.
$$

Hence one finds at $T = 0$ the gap

$$
\Delta_0(0) = 2\omega_D e^{-1/gN(0)}
$$

or, from (4.39),

$$
\Delta_0(0) = \pi e^{-\gamma} T_c \approx 1.76 T_c. \tag{4.85}
$$

This value is approached exponentially as $T \to 0$ since from (4.84)

$$
\log \frac{\Delta_0(T)}{\Delta_0(0)} \approx \frac{\Delta_0(T)}{\Delta_0(0)} - 1 \approx -\frac{1}{\Delta_0(0)} \sqrt{2\pi T\, \Delta_0(0)}\, \exp\left(-\Delta_0(0)/T\right). \tag{4.86}
$$

Let us now study the free pair quanta. The action quadratic in the pair fields Δ' reads

$$
\mathscr{A}_2[\Delta'^+, \Delta'] = \pm \frac{i}{4} \operatorname{tr}\left(\mathbf{G}_{\Delta_0} \begin{pmatrix} 0 & \Delta' \\ \Delta'^+ & 0 \end{pmatrix} \mathbf{G}_{\Delta_0} \begin{pmatrix} 0 & \Delta' \\ \Delta'^+ & 0 \end{pmatrix}\right) + \frac{1}{2} \int dx\, dx' |\Delta(x, x')|^2 \frac{1}{V(x, x')} \tag{4.87}
$$

with an equation of motion

$$
\begin{pmatrix} \Delta'(x, x') \\ \Delta'^+(x, x') \end{pmatrix} = \mp \frac{i}{2} V(x, x') \operatorname{tr}\left(\mathbf{G}_{\Delta_0} \begin{pmatrix} 0 & \Delta' \\ \Delta'^+ & 0 \end{pmatrix} \mathbf{G}_{\Delta_0} \begin{pmatrix} 0 & \Delta' \\ \Delta'^+ & 0 \end{pmatrix} \frac{\tau^{\pm}}{2}\right)(x, x') \tag{4.88}
$$

rather than (4.16). Inserting the momentum space representation (4.79) this amounts to the two equations

$$\Delta'(P \mid q) = -\sum_{p'}^{T} V(P - P')\,[l_{11}(P' \mid q)\,\Delta'(P' \mid q) + l_{12}(P' \mid q)\,\Delta'^{+}(P' \mid q)]$$

$$\Delta'^{+}(P \mid q) = -\sum_{p'}^{T} V(P - P')\,[l_{11}(P' \mid q)\,\Delta'^{+}(P' \mid q) + l_{12}(P' \mid q)\,\Delta'(P' \mid q)] \tag{4.89}$$

where ($P_0 \equiv i\omega$)

$$l_{11}(P \mid q) = \frac{\omega^2 - \frac{\nu^2}{4} + \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right)\xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)}{\left(\left(\omega + \frac{\nu}{2}\right)^2 + E^2\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right)\right)\left(\left(\omega - \frac{\nu}{2}\right)^2 + E^2\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)\right)}$$

$$l_{12}(P \mid q) = \pm\frac{\Delta_0^{2}\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right)}{\left(\left(\omega + \frac{\nu}{2}\right)^2 + E^2\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right)\right)\left(\left(\omega - \frac{\nu}{2}\right)^2 + E^2\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)\right)}\,. \tag{4.90}$$

Thus for $T \ll T_c$ the simple bound state problem (4.24) takes quite a different form due to the presence of the off-diagonal terms in the propagator (4.79).
Notice that the parenthesis on the right-hand side equs. (4.89) contain precisely the Bethe-Salpeter wave function of the bound state (compare (4.18), (4.19) in the gapless case)

$$\begin{aligned}\psi(P \mid q) &\equiv \pm\frac{i}{2}\,\mathrm{tr}\left(\mathbf{G}_{\Delta_0}\left(\frac{q}{2} + p\right)\begin{pmatrix}0 & \Delta'(P \mid q)\\ \Delta'^{+}(P \mid q) & 0\end{pmatrix}\mathbf{G}_{\Delta_0}\left(P - \frac{q}{2}\right)\frac{\tau^{+}}{2}\right)\\ &= l_{11}(P \mid q)\,\Delta'(P \mid q) + l_{12}(P' \mid q)\,\Delta'^{+}(P \mid q)\,.\end{aligned} \tag{4.91}$$

Not much is known on the general properties of solutions of equations (4.89). Even for the simple case of a $\delta^{(4)}(x - x')$ function potential, only the long wavelength spectrum has been studied. There is, however, one important solution which always occurs for $T < T_c$ due to symmetry considerations: The original action (2.3) is symmetric under phase transformations

$$\psi \to e^{i\alpha}\psi$$

guaranteeing the conservation of particle number. If the pair fields oscillate around a non-zero value $\Delta_0(x - x')$, this symmetry is spontaneously broken (since the complex c-number Δ_0 does not take part in such a phase transformation). As a consequence, there must now be an excitation of the system related to the infinitesimal symmetry transformation (Goldstone's Theorem). If the whole system is transformed at once thiscorresponds to $\boldsymbol{q} = 0$. The symmetry ensures that this corresponds to energy $q_0 = 0$. Indeed, suppose the gap equation does have a non-trivial solution $\Delta_0(P) \not\equiv 0$. Then we can easily see that

$$\Delta'(P \mid q = 0) \equiv i\,\Delta_0(P) \tag{4.92}$$

is a solution of the bound state equations (4.89) at $q = 0$. In order to prove this take

$$l_{11}(P \mid q = 0) = \frac{\omega^2 + \xi^2(\boldsymbol{P})}{\omega^2 + E^2(\boldsymbol{P})} \tag{4.93}$$

$$l_{12}(P \mid q = 0) = \pm \frac{\Delta_0{}^2(P)}{\omega^2 + E^2(\boldsymbol{P})} \tag{4.93}$$

and insert (4.92) into (4.89). One finds

$$\begin{aligned} \Delta_0(P) &= -\sum_{P'}^{T} V(P - P') \left[\frac{1}{\left(\omega'^2 + E^2(\boldsymbol{P}')\right)^2} \left(\omega^2 + \xi^2(\boldsymbol{P}') \mp |\Delta_0(P')|^2\right)\right] \\ &= -\sum_{P'}^{T} V(P - P') \frac{1}{\omega'^2 + E^2(\boldsymbol{P}')} \Delta_0(P') \end{aligned} \tag{4.94}$$

i.e. the bound state equation at $q = 0$ reduces to the gap equation. Moreover, due to (4.91) the expression

$$\psi_0(P \mid q = 0) \equiv \frac{1}{\omega^2 + E^2(\boldsymbol{P})} \Delta_0(P) \tag{4.95}$$

is the Bethe-Salpeter wave function of the $q = 0$ bound state.
If the potential is instantaneous, one can go to the equal-"time" amplitude $\psi_0(\boldsymbol{x} - \boldsymbol{x}', \tau) \equiv \psi(\boldsymbol{x}\tau, \boldsymbol{x}'\tau)$ by summing over ω with the result

$$\begin{aligned} \psi_0(x - x', \tau) &= \int \frac{d^3P}{(2\pi)^3} \exp\left[i\boldsymbol{P}(\boldsymbol{x} - \boldsymbol{x}')\right] \sum_{\omega}^{T} \psi_0(\boldsymbol{P} \mid q = 0) \\ &= \int \frac{d^3P}{(2\pi)^3} \exp\left[i\boldsymbol{P}(\boldsymbol{x} - \boldsymbol{x}')\right] \mathrm{th}^{\mp 1} \frac{E(\boldsymbol{P})}{2T} \frac{\Delta_0(\boldsymbol{P})}{2E(\boldsymbol{P})}. \end{aligned} \tag{4.96}$$

Actually, there is really no "time" dependence at all since the bound state has no energy. The $q = 0$ bound state described by $\psi_0(\boldsymbol{x} - \boldsymbol{x}')$ is called the Cooper pair.
Notice that in configuration space (4.94) amounts to a Schrödinger type of equation.

$$-2E(-i\nabla)\, \psi_0(\boldsymbol{x}) = V(\boldsymbol{x})\, \psi(\boldsymbol{x}). \tag{4.97}$$

This may be interpreted as the $q = 0$ bound state of two quasi-particles whose energies are

$$E(\boldsymbol{P}) = \sqrt{\xi^2(\boldsymbol{P}) \mp |\Delta_0(\boldsymbol{P})|^2}.$$

The equation (4.97) is, however, non-linear since $\Delta_0(\boldsymbol{P})$ in $E(\boldsymbol{P})$ depends itself on $\psi_0(x)$.
In order to establish contact with the standard discussion of pairing effects via canonical transformations (see Ref. [3]) a few comments may be useful. Let us restrict the discussion to instantaneous potentials. From equation (4.79) one sees that the propagator $\mathbf{G}_{\Delta_0}$ can be diagonalized by means of an ω-independent Bogoljubov transformation

$$B(\boldsymbol{p}) = \begin{pmatrix} u_{\boldsymbol{p}}{}^* & \mp v_{\boldsymbol{p}}{}^* \\ -v_{\boldsymbol{p}} & u_{\boldsymbol{p}} \end{pmatrix} \tag{4.98}$$

where

$$\begin{gathered} |u_{\boldsymbol{p}}|^2 = \frac{1}{2}\left(1 + \frac{\xi(\boldsymbol{p})}{E(\boldsymbol{p})}\right), \quad |v_{\boldsymbol{p}}|^2 = \mp \frac{1}{2}\left(1 - \frac{\xi(\boldsymbol{p})}{E(\boldsymbol{p})}\right) \\ 2u_{\boldsymbol{p}}v_{\boldsymbol{p}}{}^* = \frac{\Delta_0(\boldsymbol{p})}{E(\boldsymbol{p})}. \end{gathered} \tag{4.99}$$

Since

$$|u_{\boldsymbol{p}}|^2 \mp |v_{\boldsymbol{p}}|^2 = 1.$$

one finds

$$B^{-1}(\boldsymbol{p}) = \begin{pmatrix} u_{\boldsymbol{p}} & \pm v_{\boldsymbol{p}}^* \\ v_{\boldsymbol{p}} & u_{\boldsymbol{p}}^* \end{pmatrix} = \begin{Bmatrix} \sigma_3 B^+(\boldsymbol{p})\, \sigma_3 \\ B^+(\boldsymbol{p}) \end{Bmatrix}. \tag{4.100}$$

Thus $B(\boldsymbol{p})$ is a unitary spin rotation in the Fermi case whereas for bosons it is a non-unitary element of the non-compact group $SU(1, 1)$ [12].
The diagonalized propagator

$$\mathbf{G}^d_{\Delta_0}(\omega, \boldsymbol{p}) = B(\boldsymbol{p})\, \mathbf{G}^d_{\Delta_0}(\omega, \boldsymbol{p})\, B(\boldsymbol{p})^+ = -\begin{pmatrix} \dfrac{1}{i\omega - E(\boldsymbol{p})} & \\ & \pm\dfrac{1}{i\omega + E(\boldsymbol{p})} \end{pmatrix} \tag{4.101}$$

may be interpreted as describing free quasi-particles of energy

$$E(\boldsymbol{p}) = \sqrt{\xi^2(\boldsymbol{p}) \mp |\Delta_0(\boldsymbol{p})|^2}. \tag{4.102}$$

In fact, if one would introduce new creation and annihilation operators

$$\begin{pmatrix} \alpha(\boldsymbol{p}, \tau) \\ \beta^+(-\boldsymbol{p}, \tau) \end{pmatrix} = B(\boldsymbol{p}) \begin{pmatrix} a(\boldsymbol{p}, \tau) \\ a^+(-\boldsymbol{p}, \tau) \end{pmatrix} \tag{4.103}$$

their propagators would be

$$\mathbf{G}^d_{\Delta_0}(\tau - \tau', \boldsymbol{p}) \equiv \begin{pmatrix} \overbrace{\alpha(\boldsymbol{p}, \tau)\, \alpha^+(\boldsymbol{p}, \tau')} & \overbrace{\alpha(\boldsymbol{p}, \tau)\, \beta(-\boldsymbol{p}, \tau')} \\ \overbrace{\beta^+(-\boldsymbol{p}, \tau)\, \alpha^+(\boldsymbol{p}, \tau')} & \overbrace{\beta^+(-\boldsymbol{p}, \tau)\, \beta(-\boldsymbol{p}, \tau')} \end{pmatrix} = \sum_{\omega}^{T} e^{-i\omega(\tau-\tau')} \mathbf{G}^d_{\Delta_0}(\omega, \boldsymbol{p}). \tag{4.104}$$

At equal "times", $\tau' = \tau + \varepsilon$, the frequency sums may be performed with the result

$$\sum_{\omega} \mathbf{G}^d_{\Delta_0}(\omega, \boldsymbol{p}) = \begin{pmatrix} \pm n^{\mathrm{qu}}(\boldsymbol{p}) & 0 \\ 0 & \pm 1 + n^{\mathrm{qu}}(\boldsymbol{p}) \end{pmatrix} \tag{4.105}$$

where $n^{\mathrm{qu}}(\boldsymbol{p})$ are the usual Bose and Fermi occupation factors for the quasi-particle energy (4.102):

$$n^{\mathrm{qu}}(\boldsymbol{p}) = \frac{1}{e^{E(\boldsymbol{p})/T} \mp 1}.$$

The corresponding frequency sum for the original propagator becomes

$$\sum_{\omega} \mathbf{G}_{\Delta_0}(\omega, \boldsymbol{p}) = \sum_{\omega} B^{-1}(\boldsymbol{p})\, \mathbf{G}^d_{\Delta_0}(\omega, \boldsymbol{p})\, B^{-1}(\boldsymbol{p})^+$$

$$= \begin{pmatrix} \pm |v_{\boldsymbol{p}}|^2 \operatorname{th}^{\mp 1} \dfrac{E(\boldsymbol{p})}{2T} \pm n(\boldsymbol{p}) & u_{\boldsymbol{p}} v_{\boldsymbol{p}}^* \operatorname{th}^{\mp 1} \dfrac{E(\boldsymbol{p})}{2T} \\ u_{\boldsymbol{p}}^* v_{\boldsymbol{p}} \operatorname{th}^{\mp 1} \dfrac{E(\boldsymbol{p})}{2T} & \pm |u_{\boldsymbol{p}}|^2 \operatorname{th}^{\mp 1} \dfrac{E(\boldsymbol{p})}{2T} - n(\boldsymbol{p}) \end{pmatrix}. \tag{4.106}$$

The off-diagonal elements of $\mathbf{G}_{\Delta_0}$ describe, according to equ. (4.75), the vacuum expectation values of $\langle \overbrace{\psi(x)\, \psi(x')} \rangle_{\tau' = \tau + \varepsilon}$, i.e.

$$\langle \psi(\boldsymbol{x}, \tau)\, \psi(\boldsymbol{x}, \tau) \rangle = \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{p}\boldsymbol{x}} u_{\boldsymbol{p}} v_{\boldsymbol{p}}^* \operatorname{th}^{\mp 1} \frac{E(\boldsymbol{p})}{2T}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{p}\boldsymbol{x}} \operatorname{th}^{\mp 1} \frac{E(\boldsymbol{p})}{2T} \frac{\Delta_0(\boldsymbol{p})}{2E(\boldsymbol{p})}. \tag{4.107}$$

But from equ. (4.96) this coincides with the Schrödinger type of wave function of the bound state $\langle \psi(\boldsymbol{x}\tau)\,\psi(\boldsymbol{x}\tau) \mid B(q)\rangle$ at $q = 0$.
After this general discussion let us now return to the superconductor. The action quadratic in the pair fields Δ' reads (instead of (4.33))

$$\mathscr{A}_2[\Delta'^+, \Delta'] = -\frac{i}{2}\,\mathrm{tr}\left[\mathbf{G}_{\Delta_0}\begin{pmatrix}0 & \Delta'\\ \Delta'^+ & 0\end{pmatrix}\mathbf{G}_{\Delta_0}\begin{pmatrix}0 & \Delta'\\ \Delta'^+ & 0\end{pmatrix}\right] - \frac{1}{g}\int dx\,|\Delta'(x)|^2 \qquad (4.108)$$

where the spin traces have been taken. This action can be written in momentum space as

$$\mathscr{A}_2[\Delta'^+, \Delta'] = \frac{1}{2}\sum_k \big(\Delta'^+(k)\,L_{11}(k)\,\Delta'(k) + \Delta'(k)\,L_{22}(k)\,\Delta'^+(-k)$$
$$+ \Delta'^+(k)\,L_{12}(k)\,\Delta'^+(-k) + \Delta'(-k)\,L_{21}(k)\,\Delta'(k)\big). \qquad (4.109)$$

The Lagrangian matrix L_{ij} is obtained by inserting the Fermi form of (4.79) into (4.108) (compare (4.89), (4.90)). Setting $\nu = ik_0$ one has:

$$\mathscr{A}_2[\Delta'^+,\Delta'] = -\frac{1}{2}\sum_k \sum_{\omega,\boldsymbol{p}}^{T} \frac{1}{\left(\omega+\frac{\nu}{2}\right)^2 + E^2\left(\boldsymbol{p}+\frac{\boldsymbol{k}}{2}\right)}\;\frac{1}{\left(\omega-\frac{\nu}{2}\right)^2 + E^2\left(\boldsymbol{p}-\frac{\boldsymbol{k}}{2}\right)} \qquad (4.110)$$

$$\mathrm{tr}\left[\begin{pmatrix} i\left(\omega+\frac{\nu}{2}\right)+\xi\left(\boldsymbol{p}+\frac{\boldsymbol{k}}{2}\right) & \Delta_0\\ \Delta_0^+ & i\left(\omega+\frac{\nu}{2}\right)-\xi\left(\boldsymbol{p}+\frac{\boldsymbol{k}}{2}\right)\end{pmatrix}\begin{pmatrix}0 & \Delta'(k)\\ \Delta'^+(-k) & 0\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix} i\left(\omega-\frac{\nu}{2}\right)+\xi\left(\boldsymbol{p}-\frac{\boldsymbol{k}}{2}\right) & \Delta_0\\ \Delta_0^+ & i\left(\omega-\frac{\nu}{2}\right)-\xi\left(\boldsymbol{p}-\frac{\boldsymbol{k}}{2}\right)\end{pmatrix}\begin{pmatrix}0 & \Delta'(-k)\\ \Delta'^+(k) & 0\end{pmatrix}\right]$$
$$-\frac{1}{g}\sum_k \Delta'^+(k)\,\Delta'(k)$$
$$= \frac{1}{2}\sum_k\sum_{\omega,\boldsymbol{p}}^{T}\left[\left(\left(\omega+\frac{\nu}{2}\right)^2 + E^2\left(\boldsymbol{p}+\frac{\boldsymbol{k}}{2}\right)\right)\left(\left(\omega-\frac{\nu}{2}\right)^2 + E^2\left(\boldsymbol{p}-\frac{\boldsymbol{k}}{2}\right)\right)\right]^{-1}$$
$$\times\left\{\left(\omega^2-\frac{\nu^2}{4}+\xi\left(\boldsymbol{p}+\frac{\boldsymbol{k}}{2}\right)\xi\left(\boldsymbol{p}-\frac{\boldsymbol{k}}{2}\right)\right)\left(\Delta'^+(k)\,\Delta'(k)+\Delta'(-k)\,\Delta'^+(-k)\right)\right.$$
$$\left. - |\Delta_0|^2\left(\Delta'^+(k)\,\Delta'^+(-k) + \Delta'(k)\,\Delta'(-k)\right)\right\} - \frac{1}{g}\sum_k \Delta'^+(k)\,\Delta'(k). \qquad (4.111)$$

For temperatures close to zero, the $\sum_\omega$ may be performed as $\int d\omega/2\pi$ with the result*)

$$L_{11}(k) = L_{22}(k) = \sum_{\omega,\boldsymbol{p}} l_{11}(p \mid k) - \frac{1}{g} = \sum_{\boldsymbol{p}} \frac{EE'+\xi\xi'}{2EE'}\,\frac{E+E'}{(E+E')^2+\nu^2} - \frac{1}{g}$$
$$(4.112)$$
$$L_{12}(k) = L_{21}(k) = \sum_{\omega,\boldsymbol{p}} l_{12}(p \mid k) = -|\Delta_0|^2 \sum_{\boldsymbol{p}} \frac{1}{2EE'}\,\frac{E+E'}{(E+E')^2+\nu^2}$$

where we have set

$$E \equiv E\left(p - \frac{k}{2}\right); \quad E' \equiv E\left(p + \frac{k}{2}\right)$$
$$\xi = \xi\left(p - \frac{k}{2}\right); \quad \xi' \equiv \xi\left(p + \frac{k}{2}\right) \tag{4.113}$$

for brevity. Notice that due to the gap equation (4.83) at $T = 0$:

$$\frac{1}{g} = \sum_{p} \frac{1}{2E(p)}. \tag{4.114}$$

The energies of fundamental excitations are obtained by diagonalizing the action $\mathscr{A}_2[\Delta'^+, \Delta']$ and searching for zero eigenvalues of the matrix $L(k)$ via

$$L_{11}(k)\, L_{22}(k) - L_{12}(k)^2 = 0. \tag{4.115}$$

Since $L_{11}(k) = L_{22}(k)$ this amounts to the two equations

$$L_{11}(k) = \pm L_{12}(k). \tag{4.116}$$

These equations can be solved for small k. Expanding to forth order in ν and k one obtains (using (4.114)) [*13*]

$$L_{11}(k) = -\frac{m^2 v_F}{4\pi^2}\left(1 + \frac{\nu^2}{3\Delta_0^2} + \frac{v_F^2 \boldsymbol{k}^2}{9\Delta_0^2} - \frac{v_F^2\nu^2\boldsymbol{k}^2}{30\Delta_0^4} - \frac{\nu^4}{20\Delta_0^4} - \frac{v_F^4\boldsymbol{k}^4}{100\Delta_0^4}\right) + \cdots$$
$$L_{12}(k) = -\frac{m^2 v_F}{4\pi^2}\left(1 - \frac{\nu^2}{6\Delta_0^2} - \frac{v_F^2 \boldsymbol{k}^2}{18\Delta_0^2} + \frac{v_F^2\nu^2\boldsymbol{k}^2}{45\Delta_0^4} + \frac{\nu^4}{30\Delta_0^4} + \frac{v_F^4\boldsymbol{k}^4}{150\Delta_0^4}\right) + \cdots \tag{4.117}$$

such that the first of equ. (4.116) has the small k_0, $\boldsymbol{k}$ solution ($k_0 = -i\nu$)

$$k_0 = \pm c|\boldsymbol{k}|\,(1 - \gamma \boldsymbol{k}^2); \qquad c \equiv \frac{v_F}{\sqrt{3}}, \quad \gamma = \frac{v_F^2}{45\Delta_0^2}. \tag{4.118}$$

The other eqn. (4.116) can be solved for small $\boldsymbol{k}$ and all $i\nu$ directly. Using (4.112) and (4.114) one can write $-L_{11}(k) - L_{12}(k) = 0$ as*)

$$\sum_{p} \left[\frac{1}{2E} + \frac{(\Delta_0^2 - EE' - \xi\xi')\,(E + E')}{2EE'\big((E + E')^2 + \nu^2\big)}\right] = 0. \tag{4.119}$$

For small $\boldsymbol{k}$ this leads to the energies [*13*]

$$k_0^{(n)} = 2\Delta_0 + \Delta_0 \left(\frac{v_F \boldsymbol{k}}{2\Delta_0}\right)^2 z_n \tag{4.120}$$

with z_n being the solutions of the integral equation

$$\int_{-1}^{1} dx \int_{-\infty}^{\infty} dy\, \frac{x^2 - z}{x^2 + y^2 - z} = 0. \tag{4.121}$$

---

*) For $T \neq 0$ each result appears with a factor $\frac{1}{2}\left(\text{th}\,\frac{E}{2T} + \text{th}\,\frac{E'}{2T}\right)$ to which one has to add ouce more the whole expression with E' replaced by $-E'$.

Setting $e^t = (\sqrt{1-z}+1)/(\sqrt{1-z}-1)$ this is equivalent to

$$\operatorname{sh} t + t = 0 \tag{4.122}$$

which has infinitely many solutions t_n starting with

$$t_1 = 2.251 + i\,4.212 \tag{4.123}$$

and tending asymptotically to

$$t_n \approx \ln\left[\pi(4n-1)\right] + i\left(2\pi n - \frac{\pi}{2}\right). \tag{4.124}$$

The excitation energies are

$$k_0{}^{(n)} = 2\Delta_0 - \frac{v_F{}^2}{4\Delta_0}\boldsymbol{k}^2 \frac{1}{\operatorname{sh}^2 t_n/2}. \tag{4.125}$$

Of these only the first one*) lies on the second sheet and may have observable consequences while the others are hiding under lower and lower sheets of the two-particle branch cut from $2\Delta_0$ to ∞ (which is logarithmic due to the dimensionality of the surface of the Fermi sea at $T = 0$).

V. Plasmons, Pairs, and the ^{3}He System

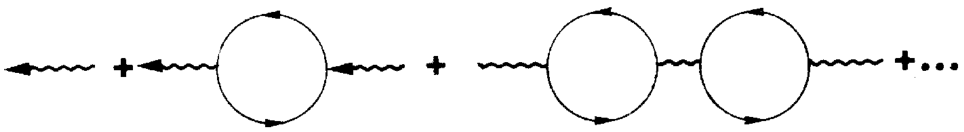
V.1. General Considerations

Under many circumstances, the two-body potential $V(x, x')$ will consist of several pieces favouring different collective excitations

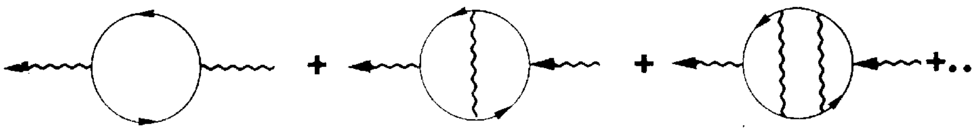
$$V(x, x') = \sum_i V_i(x, x'). \tag{5.1}$$

Thus V may have a long-range part supporting plasma oscillations and, in addition, a strong short-range contribution giving rise to tightly bound pairs. It is obvious that in such situations it is convenient to eliminate each potential V_i separately by the introduction of different collective fields. Only then has a perturbation expansion a chance of showing fast convergence.

Also, for one fundamental potential there may be different domains in T, μ, V with different collective phenomena being dominant. Thus a system of electrons will, at lower density, not be governed by plasmons due to ring graphs



but corrections of the type



will become increasingly important. The path integral formalism has no formal problem in incorporating such effects. One simply performs, in the grand-canonical action, an artificial splitting

$$V(x, x') = V_1(x, x') + V_2(x, x') \tag{5.2}$$

*) at $k_0{}^{(1)} \approx 2\Delta_0 + (.24 - .30i)\, v_F{}^2/4\Delta^2\, \boldsymbol{k}^2$

with an arbitrary $V_1(x, x')$ which may depend on μ, T, V and defines

$$V_2(x, x') \equiv V(x, x') - V_1(x, x').$$

Then V_1 may be turned into plasmons, V_2 into pairs. The full final answer should not depend on the parameters characterizing the splitting (5.2). But at every given order in the collective perturbation theory there will be an optimal set of these parameters minimizing the free energy.

Certainly, physical intuition and experience has to guide the selection of V_1 and general rules have yet to be worked out.

V.2. The ^{3}He System

As an illustration of how to treat the situation (5.1) we would like to discuss the Fermi liquid ^{3}He. This system may be described by a model action

$$\mathscr{A}[\psi^+, \psi] \equiv \mathscr{A}_0[\psi^+, \psi] + \mathscr{A}_{\text{int}}[\psi^+, \psi] = \int dx \psi^+(x) \left(i\partial_t - \xi(-i\nabla)\right) \psi(x) + \mathscr{A}_{\text{int}}[\psi^+, \psi] \quad (5.3)$$

with $\mathscr{A}_{\text{int}}$ being the sum of several terms [13—18]: There is first a Zeeman coupling of the magnetic moments $\gamma \approx 2.04 \times 10^4$ (gauss s)$^{-1}$ of the ^{3}He atoms to an external field

$$\mathscr{A}_z[\psi^+, \psi] = \gamma \int dx \psi^+(x) \frac{\sigma}{2} \psi(x) \boldsymbol{H}^{\text{ext}}. \quad (5.4)$$

Among each other, the magnetic moments have a dipole interaction[15])

$$\mathscr{A}_d[\psi^+, \psi] = -\frac{\gamma^2}{2} \int dx\, dx' \frac{\left(\delta_{\mu\nu} - 3 \frac{(x - x')_\mu}{|\boldsymbol{x} - \boldsymbol{x}'|} \frac{(x - x')_\nu}{|\boldsymbol{x} - \boldsymbol{x}'|_\nu}\right)}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \psi^+(x) \frac{\sigma_\mu}{2} \psi(x) \psi^+(x') \frac{\sigma_\nu}{2} \psi(x')$$
$$\times \delta(t - t') \quad (5.5)$$

where Greek labels $\mu, \nu = 1, 2, 3$ have been used to denote spin-one indices (not to be confused with the four-vector labels of Ch. II).

The main interaction among the atoms is due to van-der-Waals forces [*14, 15*]

$$\mathscr{A}_i[\psi^+, \psi] = -\frac{1}{2} \int dx\, dx' \psi^{+\alpha}(x) \psi^{+\beta}(x') g(x - x') \psi_\beta(x') \psi_\alpha(x) \quad (5.6)$$

giving rise to the most important phenomenon in ^{3}He: the formation of $S = 1$ P-wave pairs. A complete understanding of this phenomenon requires the knowledge of the interatomic potential (see Fig. VIII). This has a sharp rise at about 2,5 Å[16]) thus preventing the atoms from getting any closer than this distance. At about 3 Å there is a minimum of energy -10 °K. From then on the potential goes to zero from below with an R^{-6} power law. A treatment of the full microscopic interaction has not yet been attempted. The problem is quite difficult since the average distance of ≈ 4 Å between the atoms is only a little beyond the point of deepest potential. Thus one is really confronted with a strong-coupling problem. The only feature giving a chance to make the problem tractable, theoretically, is the powerful screening effect in the Fermi liquid. The atoms being very mobile form shielding clouds around any source screening away large potentials. In particular, the atoms themselves are dressed with such a cloud

[15]) Repeated indices are understood to be summed.

[16]) This is about twice the Fermi wave length.

thereby forming quasi-particles. These in turn have a much smaller residual interactions than the original atoms. In fact, in the normal state, the ^{3}He liquid behaves in many respects like a weakly interacting Fermi system ($c_v \sim T$, $\chi \sim$ const. $\varkappa \sim T^{-2}$ as T becomes small (but remains above the super-liquid transition). If this picture is to be consistent, the effective mass of the quasi-particles should include the screening cloud.

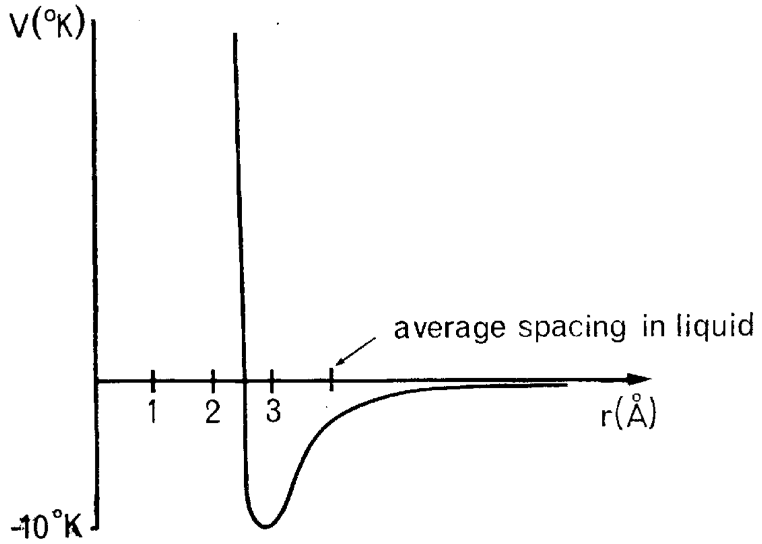


Fig. VIII. The potential between two ^{3}He atoms. For $r \lesssim 2.5$ Å the potential has essentially a hard core

This is indeed so. If the specific heat is calculated for free ^{3}He particles, the mass spectrum on the Fermi sea must be taken as

$$E(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m_{\text{eff}}}$$

with

$$m_{\text{eff}} \equiv c_v \frac{3}{p_F T} \frac{\hbar^3}{k_B{}^2} \approx 3 - 6\, m_{^3\text{He}}\, (p \approx 0 - 34 \text{ bar}).$$

Thus one is encouraged to investigate this system by considering the Fermi fields ψ in the action (5.3) directly as quasi-particles. The screened interaction is simplified and may be weak enough to permit a perturbative calculation. The simplest approximation sharing the important feature of the interatomic potential of giving an attraction in the $S = 1$ P-wave is

$$g(\boldsymbol{x} - \boldsymbol{x}') = -\frac{3g}{4p_F{}^2} \nabla^2 \delta^3(\boldsymbol{x} - \boldsymbol{x}'). \tag{5.7}$$

The factor $3/4p_F{}^2$ has been chosen to make the resulting pair theory most similar to that of Ch. IV. This potential will turn out to explain many of the super-liquid properties of ^{3}He.

There are, however, some features which cannot be accounted for by this potential. In particular, the strong paramagnetic susceptibility of the liquid in its normal state remains unexplained. The reason is that the screened potential still shows considerable deviations from (5.7): There are many molecular field effects which, when working with the potential (5.7), must be accounted for separately. It has become customary to introduce the strongest correction of this type in the form of a paramagnon interaction among the quasi-particles:

$$\mathscr{A}_I[\psi^+\psi] = I \int dx \psi^+(x) \frac{\sigma_\mu}{2} \psi(x)\, \psi^+(x) \frac{\sigma_\mu}{2} \psi(x). \tag{5.8}$$

It is generally believed that the four pieces of interaction $\mathscr{A}_z$, $\mathscr{A}_d$, $\mathscr{A}_i$, $\mathscr{A}_I$ contain the most important microscopic forces in ³He [29] (using m_{eff}).
Before we begin it is useful to perform a few manipulations on the pairing interaction. After some partial integrations, $\mathscr{A}_i$ can be rewritten as

$$\mathscr{A}_i \equiv \mathscr{A}_{i_1} + \mathscr{A}_{i_2} = \frac{1}{2}\frac{3g}{4p_F^2}\int dx[\psi^{+\alpha} i\overset{\leftrightarrow}{\nabla}_i \psi^{+\beta}\psi_\beta i\overset{\leftrightarrow}{\nabla}_i\psi_\alpha + \partial_i(\psi^{+\alpha}\psi_\beta)\,\partial_i(\psi^{+\beta}\psi_\alpha)]. \quad (5.9)$$

The coupling in the first term takes obviously place in a P-wave, due to the gradients in the relative coordinate. Statistics force the spins to be in a triplet state. This is seen by using the Fierz identities

$$\delta_{\alpha\delta}\delta^{\beta\gamma} = \frac{1}{2}\,C^+_{\alpha\delta}C^{\beta\gamma} + \frac{1}{2}\,(\sigma^\mu C^+)_{\alpha\delta}\,(C\sigma^\mu)^{\beta\gamma} \quad (5.10)$$

together with the anticommutativity of the fields ψ among each other to rewrite

$$\psi_\beta i\overset{\leftrightarrow}{\nabla}_i\psi_\alpha = (\sigma^\mu C^+)_{\beta\alpha}\psi i\overset{\leftrightarrow}{\nabla}_i C\frac{\sigma^\mu}{2}\,\psi \quad (5.11)$$

such that the coupling is now manifestly P wave triplet:

$$\psi^{+\alpha} i\overset{\leftrightarrow}{\nabla}_i\psi^{+\beta}\psi_\beta i\overset{\leftrightarrow}{\nabla}_i\psi_\alpha = 2\left(\psi^+ i\overset{\leftrightarrow}{\nabla}_i\,\frac{\sigma^\mu}{2}\,C^+\psi^+\right)\left(\psi i\overset{\leftrightarrow}{\nabla}_i C\,\frac{\sigma^\mu}{2}\,\psi\right). \quad (5.12)$$

Another Fierz transformation

$$\delta_\alpha{}^\gamma\delta^\beta{}_\delta = \frac{1}{2}\,\delta_\alpha{}^\beta\delta_\delta{}^\gamma + \frac{1}{2}\,(\sigma^\mu)_\alpha{}^\beta(\sigma^\mu)_\delta{}^\gamma \quad (5.13)$$

brings the second term in (5.9) to the form

$$\partial_i(\psi^{+\alpha}\psi_\beta)\,\partial_i(\psi^{+\beta}\psi_\alpha) = \frac{1}{2}\,\partial_i(\psi^+\psi)\,\partial_i(\psi^+\psi) + 2\partial_i\left(\psi^+\frac{\sigma^\mu}{2}\,\psi\right)\partial_i\left(\psi^+\frac{\sigma_\mu}{2}\,\psi\right). \quad (5.14)$$

Thus $\mathscr{A}_{i_{1,2}}$ become

$$\mathscr{A}_{i_1} = \frac{3g}{4p_F^2}\int dx\left[\psi^+ i\overset{\leftrightarrow}{\nabla}_i\,\frac{\sigma^\mu}{2}\,C^+\psi^+\psi i\overset{\leftrightarrow}{\nabla}_i C\,\frac{\sigma^\mu}{2}\,\psi\right] \quad (5.15)$$

$$\mathscr{A}_{i_2} = \frac{3g}{4p_F^2}\int dx\left[\frac{1}{4}\,\partial_i(\psi^+\psi)\,\partial_i(\psi^+\psi) + \frac{1}{2}\,\partial_i\left(\psi^+\frac{\sigma^\mu}{2}\,\psi\right)\partial_i\left(\psi^+\frac{\sigma^\mu}{2}\,\psi\right)\right]. \quad (5.16)$$

The first part, $\mathscr{A}_{i_1}$, has exactly the quartic form which can conveniently be used to introduce a pair field. The second part, $\mathscr{A}_{i_2}$, on the other hand, offers itself to a plasmon type of treatment. This term vanishes in the long-wavelength limit and will be neglected in these notes.

V.3. The Pair Field

We are now ready to introduce the most important collective variable by eliminating the first part of the interactions $\mathscr{A}_{i_1}$, according to formula (4.1) (now in the local case

$V(x, x') \propto \delta(x - x'))$

$$\exp\left\{\frac{i}{2}\frac{3g}{4p_F^2}\int dx[\psi^{+\beta} i\overleftrightarrow{\nabla}_i \psi^{+\alpha}\psi_\alpha i\overleftrightarrow{\nabla}_i \psi_\beta]\right\}$$

$$= \text{const.}\int \mathscr{D}A^+\mathscr{D}A \exp\left\{-\frac{i}{2}\int dx\left[A^{+\alpha\beta}(x)\, A_{\beta\alpha}(x)\frac{1}{3g} - A^{+\alpha\beta}(x)\,\psi_\alpha i\hat{\overleftrightarrow{\nabla}}\psi_\beta\right.\right.$$
$$\left.\left. - \psi^{+\alpha} i\hat{\overleftrightarrow{\nabla}}\psi^{+\beta}A_{\alpha\beta}(x)\right]\right\} \tag{5.17}$$

where we have introduced the dimensionless derivative $\hat{\overleftrightarrow{\nabla}} \equiv 1/2p_F\, \overleftrightarrow{\nabla}$, for convenience. The pair field, therefore, corresponds from the equations of constraint analogous to (4.4), to the following composite fields

$$\begin{aligned} A_{\beta\alpha}(x) &= 3g\psi_\alpha(x)\, i\hat{\overleftrightarrow{\nabla}}\psi_\beta(x), \\ A^+(x)^{\alpha\beta} &= 3g\psi^{+\alpha}(x)\, i\hat{\overleftrightarrow{\nabla}}\psi^{+\beta}(x). \end{aligned} \tag{5.18}$$

The fields are symmetric in the spin indices due to Fermi statistics. Because of (5.11) one may prefer to use the alternative form:

$$\begin{aligned} A_\mu(x) &= \frac{1}{2}\,\text{tr}(C\sigma_\mu A) = \frac{3g}{2p_F}\psi(x)\, i\overleftrightarrow{\nabla} C\frac{\sigma_\mu}{2}\,\psi(x) \\ A_\mu^+(x) &= \frac{1}{2}\,\text{tr}(A^+\sigma_\mu C^+) = \frac{3g}{2p_F}\,\psi^+(x)\, i\overleftrightarrow{\nabla}\,\frac{\sigma_\mu}{2}\, C^+\psi^+(x). \end{aligned} \tag{5.19}$$

Let us restrict our attention for a moment only to an action $\mathscr{A}_0 + \mathscr{A}_{i_1}$. Inserting (5.17) in the generating functional one obtains expressions of the form (4.2), (4.3), but in a local version rather than bilocal. Introducing again the fermion fields f and their sources j as in (4.5) the f dependent part of the action reads

$$\int dx\, dx' \left\{\frac{1}{2}\, f_\alpha^+(x)\begin{pmatrix}\big(i\partial_t - \xi(-iV)\big)\,\delta(x-x')\,\delta_{\alpha\beta} & A_{\alpha\beta}(x)\, i\hat{\overleftrightarrow{\nabla}} \\ A_{\alpha\beta}(x)\, i\hat{\overleftrightarrow{\nabla}} & \big(i\partial_t + \xi(iV)\big)\,\delta(x-x')\,\delta_{\alpha\beta}\end{pmatrix} f_\beta(x)\right.$$
$$\left. + \big(f^+(x)\, j(x) + \text{h.c.}\big)\,\delta(x - x')\right\} \tag{5.20}$$

and integrating out Df renders, for $j = 0$, the collective action

$$\mathscr{A}[A^+A] = -\frac{i}{2}\,\text{tr}\log(iG_A^{-1}) - \frac{1}{2}\int dx\, \text{tr}_{\text{spin}}\big(A^+(x)\, A(x)\big)\frac{1}{3g} \tag{5.21}$$

with the propagator

$$G_A(x, x') = \begin{pmatrix}\big(i\partial_t - \xi(-iV)\big)\,\delta\delta_{\alpha\beta} & A_{\alpha\beta}(x)\, i\hat{\overleftrightarrow{\nabla}} \\ A_{\alpha\beta}^+(x)\, i\hat{\overleftrightarrow{\nabla}} & \big(i\partial_t + \xi(iV)\big)\,\delta\delta_{\alpha\beta}\end{pmatrix}^{-1} \tag{5.22}$$

where the derivatives $\hat{\overleftrightarrow{\nabla}}$ ignore the fields $A_{\alpha\beta}(x)$. The expanded action (5.21) takes the form analogous to (4.14) and (4.32)

$$\mathscr{A}[A^+A] = i\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\,\text{tr}\,\text{tr}_{\text{spin}}[G_0 i\hat{\overleftrightarrow{\nabla}} A\tilde{G}_0 i\hat{\overleftrightarrow{\nabla}} A]^n - \frac{1}{2}\int dx\,\text{tr}_{\text{spin}}\big(A^+(x)\, A(x)\big)\frac{1}{3g} \tag{5.23}$$

where the derivatives act only on the Green's functions before and after and not on the $A(x)$ fields.

The calculation of the lowest powers in $\boldsymbol{A}$ is hardly different from that of the superconductor in Section IV-2. Consider first the quadratic term $\mathscr{A}_2$. In analogy with (4.32) one has

$$\mathscr{A}_2[\boldsymbol{A}^+\boldsymbol{A}] = -i \int dx\,dx' \left\{[G_0(x, x')\, i\overleftrightarrow{\nabla}_i \tilde{G}_0(x', x)\, i\overleftrightarrow{\nabla}_j] - \frac{1}{3g}\,\delta(x - x')\,\delta_{ij}\right\} \times \frac{1}{2}\,\mathrm{tr}_{\text{spin}}\left(A_i^+(x)\,A_j(x')\right) \tag{5.24}$$

where the right derivative of $\overleftrightarrow{\nabla}_j$ is meant to act on the x variable of the left $G_0(x, x')$ due to the descendence of this expression from the functional trace (5.23).
Expanding again

$$\boldsymbol{A}(x)' = \boldsymbol{A}(x) + (t' - t)\,\dot{\boldsymbol{A}}(x) + (x' - x)^i\,\partial_i \boldsymbol{A}(x) + \cdots \tag{5.25}$$

one can rewrite this action in momentum space as

$$\mathscr{A}_2[\boldsymbol{A}^+, \boldsymbol{A}] = \sum_k \mathrm{tr}_{\text{spin}}\,\{A_i^+(k)\,L^{ij}(k)\,A_j(k)\} \tag{5.26}$$

with an expression analogous to (4.37)

$$\begin{aligned} L^{ij}(\nu, \boldsymbol{k}) &= -\sum_{\omega,\boldsymbol{p}}^{T} \frac{1}{i(\omega + \nu) - \xi(\boldsymbol{p} + \boldsymbol{k})}\,\frac{1}{i\omega + \xi(\boldsymbol{p})}\,\frac{(2p + k)^i\,(2p + k)^j}{4p_F^2} - \frac{1}{3g}\,\delta^{ij} \\ &= \frac{1}{2}\sum_{\boldsymbol{p}} \frac{(2p + k)^i\,(2p + k)^j/4p_F^2}{\xi(\boldsymbol{p} + \boldsymbol{k}) + \xi(\boldsymbol{p}) - i\nu}\left(\mathrm{th}\,\xi(\boldsymbol{p} + \boldsymbol{k})/2T + \mathrm{th}\,\xi(\boldsymbol{p})/2T\right) - \frac{1}{3g}\,\delta^{ij}. \end{aligned} \tag{5.27}$$

Expanding in powers of ν, k one obtains the lowest order result

$$\begin{aligned} L^{ij}(0, \boldsymbol{0}) &= \frac{1}{3}\,\delta^{ij}\,\frac{1}{2}\sum_{\boldsymbol{p}} \frac{\boldsymbol{p}^2/p_F^2}{\xi(\boldsymbol{p})}\,\mathrm{th}\,\xi(\boldsymbol{p})/2T - \frac{1}{3g}\,\delta^{ij} \\ &\approx \frac{1}{3}\,\delta^{ij}\quad L(0) \approx \frac{1}{3}\,\delta^{ij}\quad N(0)\left(1 - \frac{T}{T_c}\right) \equiv \mu_A(T) \end{aligned} \tag{5.28}$$

with $L(0)$ from the superconductor formula (4.38). The chemical potential μ_A of the A field as given by (5.28) and (4.38) vanishes at $T = T_c$ of (4.39).[17]
The full ν, $\boldsymbol{k}$ dependence is calculated in complete analogy with the superconductor expression (4.37). There is only the minor modification of the factor $(p + k/2)^i\,(p + k_F/2)^j/p_F^2$ inside the sum. But since the $\boldsymbol{p}$ integral is sensitive only to $p \approx p_F$ and since $p^i \ll p_F$, this factor is simply $\hat{p}^i\hat{p}^j$ with the small error T/T_F. Thus the final expression (4.47) takes the form now

$$\begin{aligned} L^{ij}(\nu, \boldsymbol{k}) &\approx L^{ij}(0, \boldsymbol{0}) + 2N(0)\mathrm{Re}\sum_{n=0}^{\infty} \frac{1}{2^n\pi^n T^n}\,(1 - 2^{-(n+1)})\,\zeta(n + 1) \\ &\quad\times \int \frac{d\hat{p}}{4\pi}\left[|\nu| - i\left(v_F\hat{\boldsymbol{p}}\boldsymbol{k} + \frac{\boldsymbol{k}^2}{2m}\right)\right]^n \hat{p}^i\hat{p}^j \\ &\approx L^{ij}(0, \boldsymbol{0}) + N(0)\,\mathrm{Re}\int \frac{d\hat{p}}{4\pi}\left\{\psi\left(\frac{1}{2}\right)\right. \\ &\quad\left. - \psi\left(\frac{1}{2} + \left[|\nu| - i\left(v_F\hat{\boldsymbol{p}}\boldsymbol{k} + \frac{\boldsymbol{k}^2}{2m}\right)\right]\Big/4\pi T\right)\right\}\hat{p}^i\hat{p}^j. \end{aligned} \tag{5.29}$$

[17]) In ^{3}He, the critical temperature is of the order of 2.5 m °K.

As a consequence, the low-frequency and long-wavelength expansion, analogous to (4.51), emerges as before with the following slight changes: The terms linear in ν receive a factor

$$\int \frac{d\hat{p}}{4\pi}\, \hat{p}^i \hat{p}^j = \frac{1}{3}\, \delta^{ij}. \tag{5.30}$$

The terms proportional to $k_k k_l$, on the other hand, are accompanied by a tensor

$$\int \frac{d\hat{p}}{4\pi}\, \hat{p}^i \hat{p}^j \hat{p}^k \hat{p}^l = \frac{1}{15}\, (\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{5.31}$$

rather than (4.50). This amounts to replacing in the $\boldsymbol{k}^2$ terms (4.51) according to

$$\frac{1}{3}\, \boldsymbol{k}^2\delta^{ij} \to \frac{1}{3} \times \frac{1}{5}\, (\boldsymbol{k}^2\delta^{ij} + 2k^i k^j). \tag{5.32}$$

Thus the free, quadratic part, of the action of ^{3}He reads:

$$\begin{aligned} \mathscr{A}_2[A^+, A] \approx \frac{1}{6} \sum_k \mathrm{tr}_{\mathrm{spin}}\, A_i^+(k) \Big\{ N(0) \Big[\Big(1 - \frac{T}{T_c}\Big) \\ - \frac{\pi}{8T}\, |\nu| - \frac{1}{2}\, \frac{1}{5}\, \frac{7}{8\pi^2 T^2}\, \zeta(3)\, v_F^2 (\boldsymbol{k}^2\delta^{ij} + 2k^i k^j) \Big] \Big\} A_j(k) \end{aligned} \tag{5.33}$$

where $1/2\, \mathrm{tr}_{\mathrm{spin}}\, A_i^+(k)\, A_j(k)$ may also be written in terms of the fields (5.19) as $A^+_{\mu i}(\boldsymbol{k})\, A_{\mu j}(\boldsymbol{k})$. The quartic term finally becomes in the same approximation as (4.57)

$$\begin{aligned} \mathscr{A}_4[\boldsymbol{A}^+, \boldsymbol{A}] \approx -\frac{i}{2} \int dx\, \mathrm{tr}_{\mathrm{spin}}\, \big(A^{+i}(x)\, A^j(x)\, A^{+k}(x)\, A^l(x)\big)\, T \sum_{\omega_n} \\ \times \int d^3x_2\, d^3x_3\, d^3x_4 [G_0(\boldsymbol{x} - \boldsymbol{x}_2, \omega_n)\, i\hat{\overleftrightarrow{\nabla}}_i \tilde{G}_0(\boldsymbol{x}_2 - \boldsymbol{x}_3, \omega_n)\, i\hat{\overleftrightarrow{\nabla}}_j \\ \times G_0(\boldsymbol{x}_3 - \boldsymbol{x}_4, \omega_n)\, i\hat{\overleftrightarrow{\nabla}}_k \tilde{G}_0(\boldsymbol{x}_4 - \boldsymbol{x}, \omega_n)\, i\hat{\overleftrightarrow{\nabla}}_l]. \end{aligned} \tag{5.34}$$

With $\boldsymbol{p}$ only close to the Fermi momentum, and due to the rotational invariance of the integral, one obtains, using (5.31), the result

$$\mathscr{A}_4 \approx -\frac{\beta}{2}\, \frac{1}{15} \int dx\, \mathrm{tr}_{\mathrm{spin}}\, (2A_i^+ A_i A_j^+ A_j + A_i^+ A_j A_i^+ A_j) \tag{5.35}$$

with the same β as in (4.56). In terms of the field variables (5.19) this becomes

$$\begin{aligned} \mathscr{A}_4 \approx +\frac{\beta}{2}\, \frac{1}{15} \int dx [A^+_{\mu i} A_{\nu j} A^+_{\mu i} A_{\nu j} - 2(A^+_{\mu i} A_{\mu i})^2 - 2A^+_{\mu i} A_{\mu j} A^+_{\nu i} A_{\nu j} - 2A^+_{\mu i} A_{\nu i} A^+_{\nu j} A_{\mu j} \\ + 2A^+_{\mu i} A_{\nu i} A^+_{\mu j} A_{\nu j}]. \end{aligned} \tag{5.36}$$

One can establish contact with the general analysis of Refs. 17—19 in which the classical phenomenological couplings are defined via the free energy expansion:

$$\begin{aligned} F = \int d^3x \Big[-\mu_A\, A^+_{\mu i} A_{\mu i} + \frac{1}{2}\, K_L\, \partial_i A^+_{\mu i} \partial_j A_{\mu j} + \frac{1}{2}\, K_T\, |\varepsilon_{ijk}\, \partial_j A_{\mu k}|^2 \\ + \beta_1 A^+_{\mu i} A_{\nu j} A^+_{\mu i} A_{\nu j} + \beta_2 (A^+_{\mu i} A_{\mu i})^2 + \beta_3 A^+_{\mu i} A_{\mu j} A^+_{\nu i} A_{\nu j} + \beta_4 A^+_{\mu i} A_{\nu i} A^+_{\nu j} A_{\mu j} \\ + \beta_5 A^+_{\mu i} A_{\nu i} A^+_{\mu j} A_{\nu j} \Big] + \cdots \end{aligned} \tag{5.37}$$

The results (5.33) and (5.36) imply the well-known relations

$$\mu_A = \frac{1}{3} N(0) \log \frac{T}{T_c} \approx \frac{1}{3} N(0) \left(1 - \frac{T}{T_c}\right) \tag{5.38}$$

$$-2\beta_1 = \beta_2 = \beta_3 = \beta_4 = -\beta_5 = \beta/15 = \frac{2}{5} N(0) \frac{\xi_0}{v_F^2} = \frac{1}{60\pi^4} \frac{7\zeta(3)}{8} \frac{p_F^3}{T_F T_c^2} \tag{5.39}$$

and (compare the definition of the coherence length ξ_0 in (4.54)):

$$K_T = 2 \frac{N(0)}{5} \xi_0^2 = \frac{1}{15\pi^4} \frac{7\zeta(3)}{8} \frac{T_F}{T_c^2} p_F \approx 6 \times 10^{-4} \frac{T_F}{T_c^2} p_F \tag{5.40}$$

$$K_L = 3K_T = 6 \frac{N(0)}{5} \xi_0^2. \tag{5.41}$$

The derivative terms can also be written in the form:

$$F_{\text{kin}} = \int d^3x \left\{\frac{1}{2} K_1 \partial_i A_{\mu j}^+ \partial_i A_{\mu j} + \frac{1}{2} K_2 \partial_i A_{\mu j}^+ \partial_j A_{\mu i} + \frac{1}{2} K_3 \partial_i A_{\mu i}^+ \partial_j A_{\mu j}\right\} \tag{5.42}$$

such that, up to surface contributions

$$\begin{gathered} K_1 = K_T \\ K_2 + K_3 = K_L - K_T = 4 \frac{N(0)}{5} \zeta_0^2. \end{gathered} \tag{5.43}$$

Notice that for fixed size of the order parameter $A_{\mu i}$ (i.e. $\sum_{\mu i} A_{\mu i}^+ A_{\mu i} = \text{const}$), F_{kin} describes the change in free energy due to spatial distortions of the field lines. Therefore it is often referred to as "bending energy".
The phenomenological parameters K_i, β_i are directly measurable. Particularly interesting, from the theoretical point of view, is the determination of satellite frequencies in nuclear magnetic resonance experiments. It appears that they correspond to spin waves trapped in domain walls (solitons). The ratio $\varkappa = 2K_1/(K_2 + K_3)$ found from the analysis of such data agrees quite well with the prediction of (5.42). The coefficients β_i will be discussed in Sect. V.5. and V.6.

V.4. The Magnetic Interactions

The dipole interaction $\mathcal{A}_d$ and the Zeeman term $\mathcal{A}_z$ can both be treated with the same formulas. For this we note that an interaction

$$\mathcal{A} = \int dx \left(\gamma\psi^+ \frac{\sigma^i}{2} \psi \varepsilon^{ijk} \nabla_i A_k + \frac{1}{8\pi} A_i \nabla^2 A_i\right)$$

with a fluctuating quasistatic magnetic field produces exactly the dipole interaction $\mathcal{A}_d$ of (5.5) by integrating this field out in the path integral:

$$\begin{aligned} &\int \mathcal{D}\boldsymbol{A} \exp\left\{i \int dx \left(\gamma\psi^+ \frac{\sigma^i}{2} \psi \varepsilon^{ijk} \nabla_i A_k + \frac{1}{8\pi} A_i \nabla^2 A_i\right)\right\} \\ &= \text{const.} \exp\left\{-i \frac{\gamma^2}{2} \int dx\, dx' \psi^+(x) \frac{\sigma^i}{2} \psi(x)\, \psi^+(x') \frac{\sigma^j}{2} \psi(x')\, D_{ij}(\boldsymbol{x} - \boldsymbol{x}')\, \delta(t - t')\right\}. \end{aligned} \tag{5.44}$$

From the functional formula

$$\mathrm{D}_{ij}(x-x') \equiv -i\overline{H_i(x)\, H_j}(x') = -\varepsilon_{ilm}\varepsilon_{jl'm}\,\partial_l\partial_{l'}\frac{4\pi}{\nabla^2}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$$
$$= (\nabla^2\delta_{ij} - \nabla_i\nabla_j)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$= \left(\delta_{ij} - 3\frac{(x-x')_i\,(x-x')_j}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|^3} - \frac{8\pi}{3}\delta_{ij}\delta^3(\boldsymbol{x}) \qquad (5.45)$$

such that $\mathscr{A}_d$ is indeed reproduced.

In addition, there is a contact interaction which, however, cannot become active due to the short range repulsion in the potential between the He atoms. (See the discussion after equ. (5.61).)

But then $\mathscr{A}_d + \mathscr{A}_z$ can be written as

$$\mathscr{A}_{d+z} = \int dx\left\{\gamma\psi^+\frac{\sigma^i}{2}\,\psi(H_i^{\text{ext}} + H_i^f) - \frac{1}{8\pi}\,H_i^{f2}\right\} \qquad (5.46)$$

where $H_i^f = \varepsilon_{ijk}\,\partial_j A_k$ denotes the fluctuating H field. Setting $H \equiv H^{\text{ext}} + H^f$, the action modifies the matrix $A(x, x')$ of (4.5) by adding $\gamma\,\sigma_i/2\,H_i$ in the diagonal entries such that the propagator (5.22) becomes[18])

$$G_{\boldsymbol{A},\boldsymbol{H}}(x, x') = \begin{pmatrix} \left(i\partial_t - \xi(-i\nabla) + \gamma\frac{\boldsymbol{\sigma}}{2}\boldsymbol{H}\right)\delta & A_{\mu i}(x)\,\sigma^\mu i\overset{\leftrightarrow}{\nabla}_i \\ A^+_{\mu i}(x)\,\sigma^\mu i\overset{\leftrightarrow}{\nabla}_i & \left(i\partial_t + \xi(i\nabla) + \gamma\frac{\boldsymbol{\sigma}}{2}\boldsymbol{H}\right)\delta \end{pmatrix}^{-1}. \qquad (5.47)$$

The expansion part of (5.23) now reads

$$\mathscr{A}[\boldsymbol{A}_\mu^+, \boldsymbol{A}_\mu] = \frac{i}{2}\sum_{n=0}^{\infty}\frac{(-i)^n}{n}\,\mathrm{tr}\,\mathrm{tr}_{\text{spin}}\left[\boldsymbol{G}_0\begin{pmatrix}\gamma\frac{\boldsymbol{\sigma}}{2}\boldsymbol{H} & A_{\mu i}\sigma^\mu i\overset{\leftrightarrow}{\nabla}_i \\ A^+_{\mu i}\sigma^\mu i\overset{\leftrightarrow}{\nabla}_i & \gamma\frac{\boldsymbol{\sigma}}{2}\boldsymbol{H}\end{pmatrix}\right]^n. \qquad (5.48)$$

The purely magnetic terms of the form H^n give rise to plasma type of corrections to the propagation of the A field. These corrections are of order α and can be neglected with respect to the much stronger paramagnon effects described by the interaction (5.8) which will be discussed in Sect. V.7. Since it is quite obvious how the calculation would proceed, in the light of chapter III, we shall not write down the results explicitly.

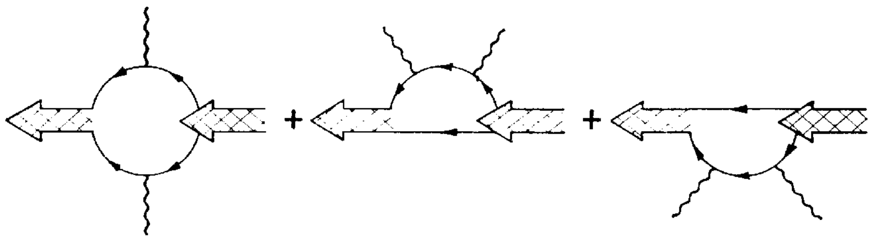


Fig. IX. The lowest order coupling of the pair field to the magnetic field $H^{\text{ext}} + H^f$ arises from these diagrams

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[18]) Here we have slightly changed the 2×2 matrix space of (5.22) by using the Fermi fields $f = \begin{pmatrix}\psi(x)\\ C^+\psi^+(x)\end{pmatrix}$, $f^+ = (\psi^+(x)\;\psi(x)C)$ in (4.5) rather than $f = \begin{pmatrix}\psi(x)\\ \psi^+(x)\end{pmatrix}$.

The mixed terms describe the coupling of H to the pair field A. The lowest, second, order contribution in H as well as A has the form

$$\mathscr{A}[A^+, A, H]|_{A^+AH^2\text{part}} = \frac{i}{8} \operatorname{tr} \operatorname{tr}_{\text{spin}} \left[\mathbf{G}_0 \begin{pmatrix} \gamma \frac{\sigma}{2} H & \sigma^\mu i\overset{\leftrightarrow}{\nabla}_i A_{\mu i} \\ \sigma^\mu i\overset{\leftrightarrow}{\nabla}_i A_{\mu i} & \gamma \frac{\sigma}{2} H \end{pmatrix} \right]^4 \Bigg|_{\Delta^+\Delta H^2\text{part}}$$

$$= i \frac{\gamma^2}{8} \operatorname{tr} \operatorname{tr}_{\text{spin}} \{\tilde{G}_0 \sigma^\mu i\overset{\leftrightarrow}{\nabla}_i A^+_{\mu i} G_0 \sigma^\lambda H_\lambda G_0 \sigma^\nu i\overset{\leftrightarrow}{\nabla}_j A_{\nu j} \tilde{G}_0 \sigma^\varkappa H_\varkappa$$
$$+ \tilde{G}_0 \sigma^\mu i\overset{\leftrightarrow}{\nabla}_i A^+_{\mu i} G_0 \sigma^\lambda H_\lambda G_0 \sigma^\varkappa H_\varkappa G_0 \sigma^\nu i\overset{\leftrightarrow}{\nabla}_j A_{\nu j}$$
$$+ \tilde{G}_0 \sigma^\mu i\overset{\leftrightarrow}{\nabla}_i A^+_{\mu i} G_0 \sigma^\nu i\overset{\leftrightarrow}{\nabla}_j A_{\nu j} \tilde{G}_0 \sigma^\lambda H_\lambda \tilde{G}_0 \sigma^\varkappa H_\varkappa\} \tag{5.49}$$

corresponding to the diagrams pictured in Fig. IX.

Again $i\overset{\leftrightarrow}{\nabla}_i \cdot i\overset{\leftrightarrow}{\nabla}_j$ can be set $1/3\, \delta_{ij}$ with little error. For very small space time variations of A, H, the fields can all be pulled out of the traces leaving each functional trace in the form

$$T \sum_{\omega_n p} \frac{1}{i\omega_n + \xi(\boldsymbol{p})} \frac{1}{i\omega_n - \xi(\boldsymbol{p})} \frac{1}{i\omega_n \mp \xi(\boldsymbol{p})} \frac{1}{i\omega_n \pm \xi(\boldsymbol{p})}. \tag{5.50}$$

The integrals over $\int_{-\infty}^{\infty} d\xi$ can be performed leaving via residues

$$N(0)\, 2T \sum_{\omega_n > 0} \frac{\pi}{2\omega_n^3} \begin{Bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix} = N(0) \frac{1}{\pi^2 T^2} \frac{7\zeta(3)}{8} \begin{Bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix}. \tag{5.51}$$

Collecting the tensors arising from the spin traces gives a factor

$$4\, |A_{\mu i} H_\mu|^2 \tag{5.52}$$

together with the upper result. Thus the lowest order magnetic interaction becomes

$$\mathscr{A}_{A^+AH^2} = -\frac{1}{3} N(0) \frac{1}{2\pi^2 T^2} \frac{7\zeta(3)}{8} \gamma^2 \int |A_{\mu i} H_\mu|^2\, d^4x. \tag{5.53}$$

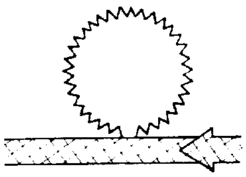
If Fermi liquid corrections to the Landau molecular fields are taken into account (see Sect. V.7) each H field is multiplied by an enhancement factor $(1 + Z_0/4)^{-2}$. The corrected interaction can be written as [*14*—*19*]

$$\mathscr{A}_{A^+AH^2} = -g_z \int |A_{\mu i} H_\mu|^2\, d^4x \tag{5.54}$$

with

$$g_z = N(0) \frac{1}{\pi^2 T^2} \frac{7\zeta(3)}{6 \cdot 8} \gamma^2 \left(1 + \frac{Z_0}{4}\right)^{-2} = N(0) \frac{\xi_0^2}{v_F^2} \gamma^2 \left(1 + \frac{Z_0}{4}\right)^{-2}. \tag{5.55}$$

For magnetic fields fluctuating strongly over small distances, the approximation after (5.49) is no longer valid. This is the case if the effect of the dipole coupling upon the pair fields is to be calculated. As we see from (5.53), a single photon loop



(5.56)

would give a linearly divergent contribution

$$\mathscr{A} = ig_z \int dx A^{+}_{\mu i}(x)\, A_{i\nu}(x)\, D_{\mu\nu}(0)\, \delta(0) \tag{5.57}$$

with the photon propagator at $x = 0$:

$$D_{\mu\nu}(0) = -4\pi \int \frac{d^3k}{(2\pi)} \frac{\boldsymbol{k}^2\delta_{\mu\nu} - k_\mu k_\nu}{\boldsymbol{k}^2} = -\frac{8\pi}{3}\, \delta_{\mu\nu} N(0) \int d\xi. \tag{5.58}$$

This strong short-distance divergence is smoothed out to only the standard logarithmic divergence of the gap equation which characterizes the spatial extension of the pair wave function and renders a natural ultraviolet cutoff to the above diagram. To see this, more care has to be taken in the calculation of (5.49) by leaving the fluctuating fields H inside the trace. If the contractions of the magnetic fields are included immediately, the following diagrams are to be calculated:

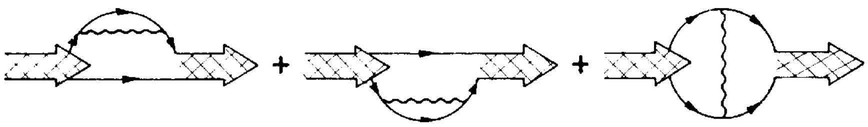


Fig. X. By closing the fluctuating part of the magnetic lines in Fig. IX to a loop, one obtains the change of free energy due to the magnetic dipole interaction among the ^{3}He atoms in a Cooper pair

The first two of these amount to electromagnetic corrections to the self-energy of the He atoms. They do not have to be calculated since their effect can be accounted for by using physical input values for atomic mass and chemical potential everywhere. The third graph represents the analytic expression:

$$\mathscr{A}_d \approx i\frac{\gamma^2}{8} \sum_{\substack{\boldsymbol{p}\boldsymbol{p}' \\ \omega\omega'}}^{T} \left\{\frac{1}{i\omega' - \xi(\boldsymbol{p}')} \frac{1}{i\omega - \xi(\boldsymbol{p})} \frac{1}{i\omega + \xi(\boldsymbol{p})} \frac{1}{i\omega' + \xi(\boldsymbol{p}')} \frac{p_l' p_m}{p_F^2}\, iD_{ij}(\boldsymbol{p} - \boldsymbol{p}')\right\}$$
$$\times\, \mathrm{tr}(\sigma^\mu\sigma^i\sigma^\nu\sigma^j) \int d^4x A^{+}_{\mu l}(x)\, A_{\nu m}(x).$$

Performing the frequency sum gives

$$\mathscr{A}_d \approx -\frac{\gamma^2}{8} \sum_{\substack{\boldsymbol{p}\boldsymbol{p}' \\ \omega\omega'}}^{T} \left\{\frac{1}{2\xi(\boldsymbol{p}')}\, \mathrm{th}\, \frac{\xi(\boldsymbol{p}')}{2T}\, \frac{1}{2\xi(\boldsymbol{p})}\, \mathrm{th}\, \frac{\xi(\boldsymbol{p})}{2T}\, \frac{p_l' p_m}{p_F^2}\, D_{ij}(\boldsymbol{p} - \boldsymbol{p}')\right\}$$
$$\times\, 2(\delta^{\mu i}\delta^{\nu j} + \delta^{\mu j}\delta^{\nu i} - \delta^{\mu\nu}\delta^{ij}) \int d^4x A^{+}_{\mu l}(x)\, A_{\nu m}(x).$$

Since the momentum integrals are to be cut off at $\xi = \omega_D \ll \mu$ the momenta remain all in the neighbourhood of the Fermi surface. To this approximation the propagator $D_{ij}(\boldsymbol{p} - \boldsymbol{p}')$ depends only on the directions of p, p'. Similarly $p_l' p_m / p_F^2 \approx \hat{p}_l' \hat{p}_m$. Then one can perform the integrals over the magnitudes of p, p' at fixed directions using the formula

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\xi(\boldsymbol{p})} \operatorname{th} \frac{\xi(\boldsymbol{p})}{2T} \approx \frac{N(0)}{4\pi} \log\left(\frac{\omega_D}{T_c} \frac{2e^\gamma}{\pi}\right) \int d^2\hat{p}$$

with the ultraviolet cutoff frequency ω_D being the same as in the calculation of $L(0)$ in (5.28). This brings $\mathscr{A}_d$ to the form

$$\mathscr{A}_d \approx -\frac{\gamma^2}{8}\left[\frac{N(0)}{4\pi} \log\left(\frac{\omega_D}{T_c} \frac{2e^\gamma}{\pi}\right)\right]^2 (\delta^{\mu i}\delta^{\nu j} + \delta^{\mu j}\delta^{\nu i} - \delta^{\mu\nu}\delta^{ij})$$
$$\times 8\left[\int d\hat{p}' d\hat{p}\hat{p}_l'\hat{p}_m D_{ij}(\hat{p}' - \hat{p})\right] \int dx A^+_{\mu l}(x)\, A_{\nu m}(x). \tag{5.59}$$

In the final integrals over the directions some care is in order. According to equ. (5.45), the propagator

$$D_{ij}(\boldsymbol{k}) = -4\pi(\delta_{ij} - \hat{k}_i\hat{k}_j)$$

contains two contributions: one long-range dipole force and a δ-function force. In momentum space this splitting amounts to

$$D_{ij}(\boldsymbol{k}) = -4\pi\left(\frac{1}{3}\,\delta_{ij} - \hat{k}_i\hat{k}_j\right) - \frac{8\pi}{3}\,\delta_{ij} \tag{5.60}$$

respectively. Now we remember that in the approximation (5.15) for the pairing force, the short-range repulsion of the interatomic potential, which sets in at about 0.25 nm, has been completely neglected. This "hard core" prevents the atoms from getting close enough to "feel" the δ function part of the dipole force. As a consequence, only the first traceless part of $D_{ij}(\boldsymbol{k})$ has to be included in (5.59). Now the directional integration in the brackets of (5.59) can be done in a straight forward manner. From symmetry arguments it is obvious that it must be a combination of Kronecker symbols

$$a(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) + b\delta_{ij}\delta_{lm}. \tag{5.61}$$

The tracelessness in the indices i, j ensures $b = -2/3a$. The constant a is fixed by setting $l = i$, $m = j$ and summing over both.
Then the integral becomes

$$\int d\hat{p}'\, d\hat{p}\hat{p}_l'\, \hat{p}_m D_{lm}(\widehat{p' - p}) = -4\pi \int d\hat{p}'\, d\hat{p}\left(\frac{1}{3}\,\hat{p}'\hat{p} - \hat{p}'(\widehat{\hat{p}' - \hat{p}})\, \hat{p}(\widehat{\hat{p}' - \hat{p}})\right)$$
$$= -4\pi\,\frac{(4\pi)^2}{2}\int dz\left(\frac{1}{3}\,z - \frac{1}{2}\,(1 - z)\right) = \frac{(4\pi)^3}{2}$$

while (5.61) gives $10\,a$. Hence $a = \pi/5\,(4\pi)^2$. Inserting this with (5.61) into (5.59) one finds the collective form of the dipole interaction

$$\mathscr{A}_d \approx -\gamma^2\left[N(0)\,\log\left(\frac{\omega_D}{T_c}\frac{2e^\gamma}{\pi}\right)\right]^2 \frac{\pi}{10}\left(\delta_{\mu i}\delta_{\nu j} + \delta_{\mu j}\delta_{\nu i} - \frac{2}{3}\,\delta_{\mu\nu}\delta_{ij}\right) \int dx A^+_{\mu i}(x)\, A_{\nu j}(x). \tag{5.62}$$

Since the photon exchanged in the diagram may travel, on the average, a few atomic distances, the numeric factor in front is expected to receive an enhancement due to the same molecular field effects occuring in the magnetic energy (5.54)[19]. It may

[19]) See Sect. V.7 for a general discussion.

be shown that the corrected dipole coupling is [14]

$$g_d \approx \gamma^2 \left[N(0) \log\left(\frac{\omega_D}{T_c} \frac{2e^\gamma}{\pi}\right)\right]^2 \frac{\pi}{10} \frac{1}{1+\frac{Z_0}{2}}. \tag{5.63}$$

V.5. Discussion of the Collective Action

The different terms in the collective action of ^3He may be ensembled together in the form:

$$\mathscr{A} = -\frac{\pi N(0)}{24T_c} \int dx A^{+}_{\mu i} \partial_t A_{\mu i} - \int dt F + \mathscr{A}_z + \mathscr{A}_d \tag{5.64}$$

where F is the free energy expression of (5.37) and $\mathscr{A}_z$, $\mathscr{A}_d$ are the magnetic interactions (5.54) and (5.62). If we neglect these for a moment the collective action can be seen to be invariant under the group $U(1) \times SU(2) \times O(3)$ of separate rotations in phase, spin, and orbital space, just as the original action (5.3). (For the nontrivial groups one just has to observe the separate invariant contractions of Greek and Latin labels in equ. (5.37)). Now consider small oscillations in the $A_{\mu i}$ field. Just as in the case of a superconductor, these are stable for $T > T_c$ with T_c of (4.39) due to the negative chemical potential

$$\mu_A = \frac{1}{3} N(0)\left(1 - \frac{T}{T_c}\right) < 0; \quad T > T_c.$$

For $T < T_c$ the sign of μ_A changes, making the point $A_{\mu i} = 0$ a maximum. Fluctuations will drive the field to a new minimum at non-zero value. Physically, this corresponds to the formation of a Bose condensate in the ground state. This goes on until stabilization is achieved, usually via the quartic terms in F. A complete discussion of the minimal surface of the complicated energy (5.37) is not yet available [20]. Among the known types of stable minima, two are apparently observed in ^3He at no magnetic field:

$$A^0_{\mu i} = \sqrt{\frac{3}{2}} \Delta_A d_\mu (\Phi_i^{(1)} + i\Phi_i^{(2)}); \quad \Delta_A \equiv \sqrt{\frac{\mu_A}{6\beta_{245}}} \tag{5.65}$$

$$A^0_{\mu i} = \Delta_B R_{\mu i}(\hat{n}, \theta)\, e^{i\varphi}; \quad \Delta_B \equiv \sqrt{\frac{\mu_A}{6\beta_{12} + 2\beta_{345}}}. \tag{5.66}$$

Here, β_{ik}, β_{ijk} are abbreviations for $\beta_i + \beta_k$, $\beta_i + \beta_j + \beta_k$, respectively, $R_{\mu i}(\hat{n}, \theta)$ is an arbitrary 3×3 rotation matrix around an axis $\hat{n}$ by an angle θ and d_μ, $\Phi_i^{(1)}$, $\Phi_i^{(2)}$ are unit vectors with $\Phi_i^{(1)}$ and $\Phi_i^{(2)}$ being orthogonal to each other. The first minimum has been identified with the A phase, the second with the B phase of ^3He (see Fig. XI for the phase diagram). The two unit vectors can also be replaced by one: $\boldsymbol{l} = \boldsymbol{\Phi}^{(1)} \times \boldsymbol{\Phi}^{(2)}$ and a phase φ indicating the azimutal angle of $\boldsymbol{\Phi}^{(1)}$ in the plane orthogonal to $\boldsymbol{l}$.

Once the vacuum expectation of the field $A_{\mu i}$ has settled at the new values (5.65) and (5.66) of minimal energy, the original $U(1) \times SU(2) \times O(3)$ symmetry of the action is obviously broken. Certain infinitesimal group transformations do not leave any more the system invariant but transform it into another one of the same energy with a different orientation of the ground state. It is a well-known consequence of this situation that the system has necessarily[20]) low-energy excitations associated with such degree of freedom of the original symmetry (Nambu-Goldstone bosons). For, if one performs a symmetry transformation on the values (5.65) and (5.66) with a very smooth dependence on space, say in form of a wave-like modulation, the system as a whole is transformed into itself but it now has a larger energy. From the symmetry of the action,

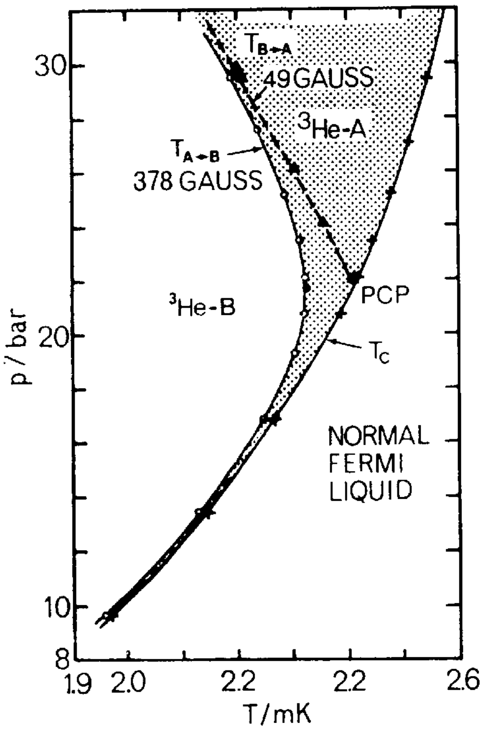


Fig. XI. The phase diagram of ^{3}He plotted for different temperatures and pressures

this energy must go to zero as the wave-length of modulations becomes longer and longer. The above symmetry group has $1 + 3 + 3$ parameters. Hence, there are at most seven Goldstone bosons. In the A phase, there are five of them, consisting of two transverse vibrations of $\boldsymbol{d}$ and $\boldsymbol{l}$ vectors (spin and orbital waves) and one torsional oscillation of $\boldsymbol{\Phi}^{(1)}$ and $\boldsymbol{\Phi}^{(2)}$ around $\boldsymbol{l}$ (longitudinal zero-sound). In the B phase, there are four modes, two transverse vibrations of the $\hat{n}$ vector, an oscillation in the rotation angle, and a zero-sound mode in the angle φ. The subgroup of transformations which do *not* change the vacuum expectation values (5.65), (5.66) remain as a residual symmetry of the spontaneously broken theory.

Let us now turn on the dipole coupling $\mathscr{A}_d$ of (5.62). Since it contracts spin and spatial indices, the groups $SU(2) \times O(3)$ are no more independent symmetries. Only the joint total angular momentum is conserved. As a consequence, the Goldstone bosons associated with the group transformations orthogonal to these, i.e. those in which spin and orbit are rotated in opposite direction, are no more massless. Explicitly, the symmetry breaking dipole actions reads in the two different phases

$$\mathscr{A}_d = 3g_d\Delta_A^2 \left[(\boldsymbol{dl})^2 - \frac{1}{3}\right] \tag{5.67}$$

$$\mathscr{A}_d = -8g_d\Delta_B^2 \left[\left(\cos\theta + \frac{1}{4}\right)^2 - \frac{1}{2}\right]. \tag{5.68}$$

[20]) If no zero-mass gauge particles are around which hybridize with this state as in the case of the superconductor, where the photon eats up the would-be Goldstone boson becoming one massive particle with an additional longitudinal degree of polarization.

The effect of the dipole energy is to bend $\boldsymbol{d}$ parallel or antiparallel to $\boldsymbol{l}$ in the A phase and to force the angle θ to arc cos $(-1/4) \approx 104^0$ in the B phase. The interesting feature about ^{3}He is that the dipole energy is very small. Inserting the microscopic values (5.39) into (5.65) or (5.66) one can estimate the energy density to be $\approx 10^{-3}(1 - T/T_c) \times$ erg/cm^3 near T_c. For this reason, oscillations in $\boldsymbol{d} \cdot \boldsymbol{l}$ and θ have quite small energies and can easily be excited. Also, due to this low cost in energy, nonlinear effects can be observed in which $\boldsymbol{d}$, $\boldsymbol{l}$ are parallel in one and anti-parallel in another domain. The domain walls act like pseudoparticles ("solitons") and can form traps for other collective excitations of the system. There is a rich set of phenomena which can be calculated theoretically and tested experimentally [*21*]. A more detailed discussion of such phenomena will be presented in Sect. V.8. Here we just mention that the nontrivial topology of the symmetry group $U(1) \times SU(2) \times O(3)$ gives rise to many different singularities among the solutions of minimal energy [*22*]. The mathematical possibilities are similar as in the pseudoparticle solutions studied recently in field theory [*23*]. Contrary to the field theoretic situation, however, the ^{3}He System offers the physical advantage of being able to prepare definite singularity structures in the laboratory. The principal means of enforcing a specific configuration in the pair field $A_{\mu i}$ are the following:

1) Magnetic fields. They couple to the system via the action (5.54). In the A phase, the energy density can be written as

$$f^A = 3g_z\Delta_A^2(\boldsymbol{d} \cdot \boldsymbol{H})^2 \tag{5.69}$$

such that the d vector points in the plane orthogonal to the magnetic field. The external magnetic field is quite efficient. The field strength at which this orientation effect exceeds the dipole energy is only about 35 Oe. In the B phase, the magnetic energy is constant

$$f^B = g_z\Delta_B^2\boldsymbol{H}^2 \tag{5.70}$$

and has no directional dependence to lowest order perturbation theory. To higher order, however, such an effect does appear since the magnetic field causes a distortion of the isotropic gap [*24*]. Its energy is negative:

$$f_B = -g_z'\Delta_B^2(\hat{n}\boldsymbol{H})^2 \tag{5.71}$$

and pulls the $\hat{n}$ vector in the direction of the magnetic field.

2) Walls. On the boundaries of a container the super-liquid certainly becomes normal such that $A_{\mu i}$ has to vanish. If the walls are sufficiently flat, the derivative terms (5.42) of the free energy will ensure that the direction of the field $A_{\mu i}$ does not change when approaching the boundary and only its magnitude does. Writing close to such a boundary

$$A_{\mu i}(x) = A^0_{\mu i}\lambda(x) \tag{5.72}$$

the derivative energy is

$$f_{\text{kin}} = \frac{1}{2} K_1|A^0_{\mu i}|^2 (\nabla\lambda)^2 + \frac{1}{2} K_{23}A^{0+}_{\mu i}A^0_{\mu j}\nabla_i\lambda\nabla_j\lambda. \tag{5.73}$$

Inserting the form of the A phase this becomes

$$f_{\text{kin}} = \frac{3}{4} \Delta_A^2[(2K_1 + K_{23}) (\nabla\lambda)^2 - K_{23}(\boldsymbol{l} \cdot \nabla\lambda)^2]. \tag{5.74}$$

As a result of the microscopic calculation (see (5.43)) the constant K_{23} is

$$K_{23} \equiv 2K_T = 4 \frac{N(0)}{5} \xi_0^2 > 0. \tag{5.75}$$

Hence the derivative energies are minimized if $\boldsymbol{l}$ points in the direction parallel or antiparallel to the gradient, i.e. orthogonal to the walls. In the B phase, the derivative terms give

$$f_{\text{kin}} = \frac{1}{2} \Delta_B^2 (3K_1 + K_{23}) (\nabla\lambda)^2 \tag{5.76}$$

and there is again no directional effect. As in the case of the magnetic energy, however, higher order effects *do* cause orientation energies [*25*]. Since these are quite small (a factor 10^2 to 10^3 times smaller than the small dipole energy) the walls give rise to quite a complex set of phenomena in the B phase.

3) Currents. If the system is passed by a superfluid current, the vector in the A phase points parallel or antiparallel to the flow. This can simply be seen by putting a plane wave

$$A_{\mu i} = \sqrt{\frac{3}{2}} \Delta_A d_\mu (\Phi_i^{(1)} + i\Phi_i^{(2)})\, e^{i\boldsymbol{qx}} \tag{5.77}$$

into the system corresponding to a super-flow of velocity $\boldsymbol{v} = \hbar\boldsymbol{q}/2m_{\text{eff}}$. The derivative energy terms become

$$f_{\text{kin}} = \frac{3}{4} \Delta_A^2 \boldsymbol{q}^2 [(2K_1 + K_{23}) - K_{23}(\boldsymbol{l}\cdot\hat{\boldsymbol{q}})^2]. \tag{5.78}$$

The current at which this energy becomes comparable to the dipole energy (5.67) is about 0.25 cm/s.*)

In the B phase this effect is again absent to lowest order since F_{kin} is isotropic:

$$f_{\text{kin}} = \frac{1}{2} \Delta_B^2 \boldsymbol{q}^2 (3K_1 + K_{23}). \tag{5.79}$$

There is, however, a higher order effect that *does* turn $\hat{n}$ parallel to the flow [*14*].

4) Electric fields. There are theoretical calculations [*26*] that an electric field causes an orientation energy

$$f^A = +\frac{4\alpha^2\boldsymbol{E}^2}{\gamma^2} g_d \left[\frac{1}{3} A_{\mu i}^+ A_{\mu i} - A_{\mu i}^+ A_{\mu j} \hat{E}_i \hat{E}_j\right] \tag{5.80}$$

and hence forces $\boldsymbol{d}$ orthogonal to $\boldsymbol{E}$ (g_d is the constant (5.63) appearing in the magnetic energy). At fields of the order of 10^5 V/cm the factor in parenthesis is of order unity such that this energy becomes as strong as the magnetic dipolar energy. Experimentally, this effect has not been found at the predicted magnitude [*27*].

The full study of the phenomena in ^{3}He is still in the beginning, both theoretically and experimentally. On the experimental side, there is the challenging problem of investigating a macroscopic quantum system of many degrees of freedom in the collective field $A_{\mu i}$. On the theoretical side, the system offers the exciting possibility of testing many ideas about classical solutions of non-linear field equations and their quantum corrections. There is the rich set of phenomena associated with different singular structures. Domain walls represent two-dimensional surfaces which have been studied in field theory as "solitons" (see Sect. V.8. for a discussion). Pinching the field lines of the $\boldsymbol{d}$ vector in a strong magnetic field gives control over vortex lines or linear singularities. These objects are similar to those considered in string theories of elementary particles.

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*) For a discussion of super-flow in ^{3}He-A see H. Kleinert, Y. R. Lin-Liu, and K. Maki, Paper presendet at the 1978 Conference on Low Temperature Physics in Grenoble, J. Phys. (Paris) **6**, C6—59 (1978), and USC preprint, March 1978. Also: H. Kleinert, *Collective Field Theory of Superliquid ^{3}He* and *The Two Supercurrents in ^{3}He A*, Berlin preprints, Sept. 1978.

Embedding the system, finally, in a sphere creates a point singularity somewhere inside or on the boundary, since the l-vectors are all orthogonal to the wall. This structure is related to the monopoles and instantons in Yang-Mills theories (see Sect. V.9).
The interwining of theoretical ideas in this field of research and current studies of non-abelian field theories may provide interesting new insights for either side.

V.6. Far below T_c

If the temperature drops far below T_c, the vacuum expectation values around which $A_{\mu i}$ oscillates will become too large to permit an expansion (5.23) of the action in $A_{\mu i}$. Instead, one has to allow for a non-zero value $A^0_{\mu i}$ directly in the free propagator:

$$\mathbf{G}_{A^0}(x-x')_{\alpha\beta} = \begin{pmatrix} \big(i\partial_t - \xi(-i\nabla)\big)\,\delta\delta_{\alpha\beta} & A^0_{\alpha\beta} i\overset{\leftrightarrow}{\nabla} \\ A^{0+}_{\alpha\beta} i\overset{\leftrightarrow}{\nabla} & \big(i\partial_t + \xi(i\nabla)\big)\,\delta\delta_{\alpha\beta} \end{pmatrix}^{-1}. \tag{5.81}$$

The off-diagonal terms cause a linear potential in $\mathscr{A}[A^+, A]$ which is removed as usual by satisfying the gap equation:

$$A^{0i}_{\alpha\beta} = 3g\,\mathrm{tr}\left(i\overset{\leftrightarrow}{\nabla}{}^i \mathbf{G}_{A^0}(x, y)_{\alpha\beta}\,\frac{\tau^-}{2}\right)\bigg|_{x=y}. \tag{5.82}$$

The solutions of this equation are straight-forward if one restricts oneself to unitary gaps

$$\hat{p}A^{0+}_{\alpha\beta}\,\hat{p}A^0_{\beta\gamma} = \frac{1}{2}\,\mathrm{tr}(\hat{p}A^{0+}\,\hat{p}A^0)\,\delta_{\alpha\gamma} = \hat{p}A^{0+}_{\mu i}\,\hat{p}A^0_{\mu i}\delta_{\alpha\gamma}.$$

This condition is fulfilled for A fields of the form (5.65) and (5.66) for $T \approx T_c$ corresponding to the two important phases A and B. One may, therefore, expect the gaps to satisfy this condition also for $T \ll T_c$. Then the propagator (5.81) can be written in momentum space as

$$\mathbf{G}_{A^0}(x)_{\alpha\beta} = -\sum_{\omega,\boldsymbol{p}}^{T} e^{-ipx}\,\frac{1}{\omega^2 + \xi^2(\boldsymbol{p}) + \left|A^0_{\mu i}\,\dfrac{p^i}{p_F}\right|^2}\begin{pmatrix} \big(i\omega + \xi(\boldsymbol{p})\big)\,\delta_{\alpha\beta} & -A^0_{\alpha\beta}\boldsymbol{p}/p_F \\ -A^{0+}_{\alpha\beta}\boldsymbol{p}/p_F & \big(i\omega - \xi(\boldsymbol{p})\big)\,\delta_{\alpha\beta} \end{pmatrix} \tag{5.83}$$

and the gap equation reads

$$\begin{aligned} A^0_{\mu i} &= 3g\sum_j \sum_{\omega,\boldsymbol{p}}^{T} \frac{p^i p^j}{p_F^2}\,\frac{1}{\omega^2 + \xi^2(\boldsymbol{p}) + \left|A^0_{\mu l}\,\dfrac{p^l}{p_F}\right|^2}\,A^0_{\mu j} \\ &= 3g\sum_j \sum_{\boldsymbol{p}} \frac{p^i p^j}{p_F^2}\,\mathrm{th}\,\frac{E(\boldsymbol{p})}{2T}\,\frac{A^0_{\mu j}}{2E(\boldsymbol{p})}; \quad E(\boldsymbol{p}) \equiv \sqrt{\xi^2(\boldsymbol{p}) + \left|A_{\mu l}\,\frac{p^l}{p_F}\right|}. \end{aligned} \tag{5.84}$$

Summing over directions and magnitudes of p separately gives

$$A^0_{\mu i} \approx 3g\,\frac{N(0)}{4\pi}\sum_j \int d^2\hat{p}\,\hat{p}^i\hat{p}^j \int_0^{\omega_D} d\xi\,\frac{1}{E(\boldsymbol{p})}\,\mathrm{th}\,\frac{E(\boldsymbol{p})}{2T}\,A^0_{\mu j}. \tag{5.85}$$

According to the general discussion in Sect. IV.1., equ. (5.84) can be interpreted as the bound-state equation for two quasiparticles of energy

$$E(\boldsymbol{p}) = \sqrt{\xi^2(\boldsymbol{p}) + \left|A^0_{\mu i}\,\frac{p^i}{p_F}\right|^2}. \tag{5.86}$$

Thus, due to the derivative terms in the potential (5.15) the energy gap in the quasi-particle spectrum becomes dependent on the momentum. In the final version of (5.85) the gap is relevant only at small $\xi < \omega_D$ i.e. for $\boldsymbol{p}$ close to the Fermi surface and therefore notices only the direction on the Fermi sphere.
In analogy with the minima (5.65), (5.66) there are now two solutions to the gap equation which are valid here for $T \ll T_c$. Inserting in the B phase the unitary ansatz:

$$A^0_{\mu i} = \Delta_B e^{i\varphi} R_{\mu i}(\hat{n}, \theta) \tag{5.87}$$

renders an isotropic gap $|A^0_{\mu i}\hat{p}_i|^2 = \Delta_B{}^2$ with Δ_B satisfying the same equation as the superconductor, i.e. close to T_c

$$\Delta_B(T) \approx \sqrt{\frac{8}{7\zeta(3)}}\, \pi T_c \left(1 - \frac{T}{T_c}\right)^{1/2} \tag{5.88}$$

and close to $T = 0$

$$\Delta_B(T) \approx \Delta_B(0) - \sqrt{2\pi T \Delta_B(0)}\; e^{-\Delta_B(0)/T}. \tag{5.89}$$

Notice that the value close to T_c can be written in terms of the constants μ_{pair}, β as $\sqrt{\mu_{\text{pair}}/\beta}$ which is the same as (5.66) since due to (5.38), (5.39) one has

$$\Delta_B = \sqrt{\frac{\mu_{\text{pair}}}{\beta}} = \sqrt{\frac{\frac{1}{3}\,\mu_{\text{pair}}}{5\beta/15}} = \sqrt{\frac{\mu_A}{5\beta_2}} = \sqrt{\frac{\mu_A}{6\beta_2 + 2\beta_{345}}}. \tag{5.90}$$

For the A phase the ansatz

$$A^0_{\mu i} = \sqrt{\frac{3}{2}}\, \Delta_A d_\mu (\Phi_i^{(1)} + i\Phi_i^{(2)})$$

leads to

$$\begin{aligned} E(\boldsymbol{p}) &= \sqrt{\xi^2(\boldsymbol{p}) + \frac{3}{2}\,\Delta_A{}^2\, |(\boldsymbol{\Phi}^{(1)} + i\boldsymbol{\Phi}^{(2)})\, \hat{\boldsymbol{p}}|^2} = \sqrt{\xi^2(\boldsymbol{p}) + \frac{3}{2}\Delta_A{}^2(\boldsymbol{l}\times \hat{\boldsymbol{p}})^2} \\ &= \sqrt{\xi^2(\boldsymbol{p}) + \frac{3}{2}\,\Delta_A{}^2 \sin^2 \Theta} \end{aligned} \tag{5.91}$$

where Θ is the direction between $\boldsymbol{l}$ and $\hat{\boldsymbol{p}}$. Inserting this into (5.85) and orienting $\Phi^{(1)}$ in x and $\Phi^{(2)}$ in y-direction gives the gap equation

$$\begin{aligned} 1 &= 3g\,\frac{N(0)}{4\pi} \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\Theta \int_0^{\omega_D} d\xi\, \frac{\sin^2\Theta \cos\varphi e^{i\varphi}}{\sqrt{\xi^2 + \frac{3}{2}\,\Delta_A{}^2 \sin^2\Theta}}\, \text{th}\, \frac{\sqrt{\xi^2 + \frac{3}{2}\,\Delta_A{}^2 \sin^2\Theta}}{2T} \\ &= \frac{3g}{4}\, N(0) \int_{-1}^{1} dz \int_0^{\omega_D} d\xi\, \frac{1 - z^2}{\sqrt{\xi^2 + \frac{3}{2}\Delta_A{}^2(1 - z^2)}}\, \text{th}\, \frac{\sqrt{\xi^2 + \frac{3}{2}\,\Delta_A{}^2(1 - z^2)}}{2T}. \end{aligned} \tag{5.92}$$

The behaviour of Δ_A as a function of temperature is a little different from the isotropic situation: For $T \approx T_c$ there is again a square root type of singularity obtained by expanding the first expression (5.84) in powers of $|A^0_{\mu l} p^l/p_F|^2 = 3/2\Delta_A^2 \sin^2 \Theta$

$$(\Phi_i^{(1)} + i\Phi_i^{(2)}) \approx 3g \sum_{\omega,\boldsymbol{p}}^{T} \hat{p}^i \hat{p}^j (\Phi_j^{(1)} + i\Phi_j^{(2)}) \times \left(\frac{1}{\omega^2 + \xi^2(\boldsymbol{p})} - \frac{3}{2} \Delta_A^2 \sin^2 \Theta \frac{1}{(\omega^2 + \xi^2(\boldsymbol{p}))^2} + \cdots \right). \tag{5.93}$$

Performing frequency sums and integrals over ξ at fixed direction gives

$$\Phi_i^{(1)} + i\Phi_i^{(2)} \approx 3g \frac{N(0)}{4\pi} \sum_{\hat{p}} \hat{p}^i \hat{p}^j (\Phi_j^{(1)} + i\Phi_j^{(2)}) \times \left\{ \log \left(\frac{\omega_D}{T_c} 2 \frac{e^\gamma}{\pi} \right) + \left(1 - \frac{T}{T_c} \right) - \frac{7}{8} \zeta(3) \frac{1}{\pi^2 T_c^2} \frac{3}{2} \Delta_A^2 \sin^2 \Theta \right\}. \tag{5.94}$$

The first term is isotropic and is calculated via

$$\sum_{\hat{p}} \hat{p}^i \hat{p}^j = \frac{4\pi}{3} \delta_{ij}. \tag{5.95}$$

For the evaluation of the second term put again $\Phi^{(1)}$ in x- and $\Phi^{(2)}$ in y-direction. Then the x-component of (5.94) yields

$$1 \approx gN(0) \left\{ \log \left(\frac{\omega_D}{T_c} 2 \frac{e^\gamma}{\pi} \right) + \left(1 - \frac{T}{T_c} \right) \right\} - \frac{7}{8} \zeta(3) \frac{1}{\pi^2 T_c^2} \frac{3}{2} \Delta_A^2 \, 3g \frac{N(0)}{4\pi} \int_0^{2\pi} d\varphi \int_{-1}^{1} d \cos \Theta \sin^2 \Theta \cos \varphi e^{i\varphi} \sin^2 \Theta. \tag{5.96}$$

Performing the integral and taking advantage of the $T = T_c$ gap equation $\Delta_A(T_c) = 0$ gives

$$\Delta_A^2(T) \approx \frac{2}{3} \frac{5}{4} \frac{8}{7\zeta(3)} \pi^2 T_c^2 \left(1 - \frac{T}{T_c} \right). \tag{5.97}$$

Thus Δ_A can be written as

$$\Delta_A(T) = \sqrt{\frac{5}{6}} \sqrt{\frac{\mu_{\text{pair}}}{\beta}} = \sqrt{\frac{5}{6}} \Delta_B(T); \quad T \approx T_c \tag{5.98}$$

which coincides with (5.66) since

$$\Delta_A = \sqrt{\frac{5}{6} \frac{\mu_{\text{pair}}}{\beta}} = \sqrt{\frac{1}{3} \frac{\mu_{\text{pair}}}{6\beta/15}} = \sqrt{\frac{\mu_A}{6\beta_2}} = \sqrt{\frac{\mu_A}{6\beta_{245}}}. \tag{5.99}$$

For $T \to 0$ the gap tends to a constant $\Delta_A(0)$ as determined from (5.92) by expanding

$$\text{th} \frac{E(\boldsymbol{p})}{2T} \approx 1 - 2e^{-E(\boldsymbol{p})/T}.$$

At $T=0$ one has

$$1 = 3g\frac{N(0)}{4}\int\limits_{-1}^{1} dz \int\limits_{0}^{\omega_D} d\xi \frac{1-z^2}{\sqrt{\xi^2+\frac{3}{2}\Delta_A^2(1-z^2)}}$$

$$= \frac{3g}{4}N(0)\left\{\frac{4}{3}\log\frac{2\omega_D}{\sqrt{\frac{3}{2}}\Delta_A} - \frac{1}{2}\int\limits_{-1}^{1} dz(1-z^2)\log(1-z^2)\right\}. \tag{5.100}$$

Using the gap equation at $T=T_c$ this yields

$$\Delta_A(0) = \sqrt{\frac{2}{3}}\,T_c\pi e^{-\gamma}e^{c} \tag{5.101}$$

with

$$c \equiv -\frac{3}{8}\int\limits_{-1}^{1} dz(1-z^2)\log(1-z^2) = \frac{5}{6} - \log 2\,. \tag{5.102}$$

If T is non-zero but small this gap is approached as follows:

$$-1+\frac{\Delta_A(T)}{\Delta_A(0)} \approx -2\frac{3}{4}\int\limits_{0}^{\omega_D} d\xi \int\limits_{-1}^{1} d\cos\Theta\sin^2\Theta \frac{1}{\sqrt{\xi^2+\frac{3}{2}\Delta_A^2\sin^2\Theta}}$$

$$\times\exp\left\{-\sqrt{\xi^2+\frac{3}{2}\Delta_A^2\sin^2\Theta}\Big/2T\right\}$$

$$\approx -\sqrt[4]{\frac{27}{8}}\frac{\sqrt{2\pi\Delta_A(0)\,T_c}}{\Delta_A(0)}\int\limits_{0}^{1} d\cos\Theta\sin^2\Theta\exp\left\{-\sin\Theta\sqrt{\frac{3}{2}}\Delta_A(0)/T\right\}$$

$$\approx -\sqrt[4]{\frac{27}{8}}\,\frac{8}{3}\,\frac{\sqrt{2\pi\Delta_A(0)\,T_c}}{\Delta_A(0)}\left(\frac{T}{\Delta_A(0)}\right)^4. \tag{5.103}$$

Thus, contrary to the exponential behaviour in the isotropic case there is now a power. The reason is the absence of the gap in the direction of $\boldsymbol{l}$.

It is useful to picture the physical consequence of the gap anisotropy in the A phase. The electrons on the Fermi sphere prefer the regions of smallest gap and will, at fixed temperature T, populate mostly $\Theta \approx 0$. Thus the vector $\boldsymbol{l}$ has the physical interpretation of pointing in the direction of largest electron population. If $\boldsymbol{l}$ is bent stiftly away from its direction the electrons will follow only after some finite time delay. Thus the energy will instantaneously increase. This gives rise to a restoring force and consequently to oscillations. The collisions in the electron gas trying to reinstall the new thermal distribution will damp such oscillations ("Cross-Anderson viscosity"). This effect has been seen experimentally [*28*].

Finally, let us give a physical interpretation of the direction of the gap parameter

$$\hat{d}_\mu \equiv |\widehat{A^0_{\mu i}p^i}|\,. \tag{5.104}$$

It can easily be realized that $\hat{d}_\mu$ determines the direction of the quantization axis in which the spin triplet of the Cooper pairs has a zero spin projection $S_3 = 0$. This follows directly by writing this vector in spinor form

$$\hat{d}_{\alpha\beta} = \hat{d}_\mu (C\sigma_\mu)_{\alpha\beta}. \tag{5.105}$$

If, for example, $\hat{d}_\mu$ points in z direction, the matrix $d_{\alpha\beta}$ becomes

$$\hat{d}_{\alpha\beta} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta}. \tag{5.106}$$

Remembering now that $A_{\alpha\beta}$ is proportional to the Cooper pair wave function at $q = 0$

$$\langle \psi_\alpha \psi_\beta \mid B(q = 0) \rangle$$

one sees the wave function to have the spin form $|\uparrow\downarrow + \downarrow\uparrow\rangle$, which is $S = 1$, $S_3 = 0$. It is possibly easier to visualize the orthogonal direction: If $\hat{d}$ points in the x, y plane with an angle φ against the x axis, the matrix $\hat{d}$ has the form

$$d_{\alpha\beta} = \begin{pmatrix} d_1 + id_2 & 0 \\ 0 & -d_1 + id_2 \end{pmatrix} = \begin{pmatrix} e^{i\varphi} & \\ & -e^{-i\varphi} \end{pmatrix}$$

corresponding to a bound state wave function

$$e^{i\varphi}(|\uparrow\uparrow\rangle - e^{-2i\varphi} |\downarrow\downarrow\rangle)$$

i.e. there is a coherent superposition of spin-up and spin-down pairs in the ground state (equal-spin pairing (ESP)).
In the B phase, the spin wave function depends on the direction of the Fermi momentum, the vector $\hat{d}$ being equal to the vector $\hat{p}$ after a rotation by the matrix $R(\hat{n}, \theta)$. Thus $\hat{d}$ gives the direction of zero spin projection for every $\hat{p}$. In the A phase, the direction $\hat{d}_\mu$ coincides with d_μ such that d_μ denotes directly the quantization axis with no spin projection.

V.7. Possible Stabilization of A Phase by Paramagnon Effects

At $T < T_c$ there are two solutions to the gap equation corresponding to the two phases A and B of He^3. The free-energy of these phases is obtained by inserting the field values (5.65), (5.66) into (5.37). In this way one finds the (minimal) energy densities

$$f^A_{\min} = -\frac{{\mu_A}^2}{4\beta_{245}} \tag{5.107}$$

$$f^B_{\min} = -\frac{3}{2}\frac{{\mu_A}^2}{6\beta_{12} + 2\beta_{345}}. \tag{5.108}$$

Using the coefficients β of equ. (5.39) derived from the pure pairing force these energies become

$$-\frac{1}{4}\frac{{\mu_A}^2}{\beta/15} \quad \text{and} \quad -\frac{3}{10}\frac{{\mu_A}^2}{\beta/15}$$

respectively. Thus with the forces included until now, the B phase is slightly more stable than the A phase. Experimentally, this is not true: If ^3He is cooled below T_c

between melting pressure and 20 bar, the transition occurs to the A phase. It is present belief [14, 29] that this stabilization is caused by the strong spin-fluctuations known to exist in ^{3}He. They are generated by magnetic effects due to the strong exchange forces and their microscopic parametrization is conventionally given in the form of the "paramagnon" interaction (5.7) [29].
A full discussion of the "paramagnon" philosophy and its experimental manifestations would go beyond these lecture notes.
Here we limit ourselves to a derivation of the corresponding corrections to the paramagnetic susceptibility, in order to estimate the size of the coupling constant I, and to the subsequent modification of the magnetic interaction $\mathscr{A}_z$ of (5.54).
The quartic paramagnon interaction is eliminated from the total action by adding the term

$$-\frac{I}{4}\left(P_\mu(x)-\psi^+(x)\,\sigma_\mu\psi(x)\right)^2=-\frac{I}{4}\,P_\mu{}^2+IP_\mu\psi^+\frac{\sigma_\mu}{2}\,\psi-\frac{I}{4}\,(\psi^+\sigma_\mu\psi)^2 \tag{5.109}$$

and integrating the generating functional over the auxiliary paramagnon field $\boldsymbol{P}$. Thus $I\boldsymbol{P}$ couples in the same way as $\gamma(\boldsymbol{H}^{\text{ext}}+\boldsymbol{H}^{\text{f}})$ and can simply be added to these fields in the matrix $\mathbf{G}^{-1}$ of (5.47). As a consequence, the collective action (5.48) is the same as before except with $\gamma\boldsymbol{H}$ replaced by $\gamma\boldsymbol{H}+I\boldsymbol{P}$ everywhere and $-I/4\,\boldsymbol{P}^2$ added. The most important effect of these modifications appears in the paramagnetic susceptibility. Consider first the lowest order susceptibility as obtained from the $\boldsymbol{H}^2$ term in (5.48). This reads

$$\mathscr{A}[A^+A\boldsymbol{H}]|_{H^2\text{term}}=-\frac{i}{2}\,\gamma^2\,\text{tr}\,\text{tr}_{\text{spin}}\left(G_0\,\frac{\sigma\boldsymbol{H}}{2}\,G_0\,\frac{\sigma\boldsymbol{H}}{2}\right)=-\frac{i}{4}\,\gamma^2\,\text{tr}\,(G_0\boldsymbol{H}G_0\boldsymbol{H}) \tag{5.110}$$

with the definition of the susceptibility

$$\mathscr{A}|_{H^2\text{term}}\equiv\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\chi(k)\,|\boldsymbol{H}(k)|^2 \tag{5.111}$$

one has the lowest order result

$$\chi^0(k^0,\boldsymbol{k})=-\frac{i}{2}\,\gamma^2\int d^4x e^{ikx}G_0(x)\,G_0(-x)\,. \tag{5.112}$$

This expression was given in equ. (3.17) for small k^0 and fixed $k^0/|\boldsymbol{k}|$ as

$$\begin{aligned}\chi^0(k^0,\boldsymbol{k})&=-\frac{\gamma^2}{4}\,\pi(-ik^0,\boldsymbol{k})\\&=\frac{\gamma^2}{2}\,N(0)\left(1-\frac{\tilde{\varrho}}{2}\log\left|\frac{\tilde{\varrho}+1}{\tilde{\varrho}-1}\right|-i\,\frac{\pi}{2}\,|\tilde{\varrho}\Theta|\,(1-|\tilde{\varrho}|)\right)\\&\equiv\frac{\gamma^2}{2}\,N(0)\,f(k^0,\boldsymbol{k})\,.\end{aligned} \tag{5.113}$$

For k^0 much smaller than $p_F|\boldsymbol{k}|/m$ and $|\boldsymbol{k}|\ll p_F$ the result of the integration (3.17) is instead:

$$f(k^0,\boldsymbol{k})\approx1-\frac{\boldsymbol{k}^2}{12p_F{}^2}+i\,\frac{\pi}{2}\,\frac{mk^0}{p_F|\boldsymbol{k}|}\,. \tag{5.114}$$

The substitution $\gamma \boldsymbol{H} \to \gamma \boldsymbol{H} + I\boldsymbol{P}$ and the addition of $-I/4\,\boldsymbol{P}^2$ in (5.111) renders the action

$$\mathscr{A} = \frac{1}{4} N(0) \int \frac{d^4k}{(2\pi)^4} f(k)\, |\gamma \boldsymbol{H}(k) + I\boldsymbol{P}(k)|^2 - \frac{I}{4} \int \frac{d^4k}{(2\pi)^4} |\boldsymbol{P}(k)|^2. \qquad (5.115)$$

The term quadratic in $\boldsymbol{P}$ determines the paramagnon propagator as

$$\overbrace{P_\mu P_\nu} = \delta_{\mu\nu} \frac{i}{-\dfrac{I}{2} + \dfrac{I^2}{\gamma^2} \chi^0(k^0, \boldsymbol{k})} \approx -\frac{2}{I} \frac{i\delta_{\mu\nu}}{1 - IN(0)\left(1 - \dfrac{\boldsymbol{k}^2}{12p_F^2} + i\dfrac{\pi}{2}\dfrac{mk^0}{p_F|\boldsymbol{k}|}\right)}. \qquad (5.116)$$

This does not correspond to a stable quasi-particle due to the nonvanishing imaginary part. There is, however, a definite peak in the imaginary part at the energy

$$k^0 = \frac{2}{\pi}(1 - \bar{I}) \frac{p_F}{m} |\boldsymbol{k}|; \quad \bar{I} \equiv IN(0) \qquad (5.117)$$

with a width of the same order of magnitude such that the quasiparticle concept does have an approximate meaning. If $\bar{I} \approx 1$, which will turn out to be the case experimentally, the velocity of this excitation approaches zero. The paramagnon couples directly to the magnetic field, via

$$\mathscr{A} = \frac{I}{\gamma} \int \frac{dk}{(2\pi)^4} \chi^0(k^0, \boldsymbol{k})\, H^+(k^0, \boldsymbol{k})\, P(k^0, \boldsymbol{k}). \qquad (5.118)$$

As a consequence, all magnetic properties receive strong paramagnon corrections. Consider first the susceptibility itself. The quadratic action (5.115) in the generating functional can be integrated in DP_μ. The result is found, as usual, by completing the square

$$\mathscr{A}_2 = \int \frac{d^4k}{(2\pi)^4} \left\{ -\frac{I}{4} \left| P_\mu(k) - \frac{N(0) f(k)}{1 - \bar{I} f(k)} \gamma H_\mu(k) \right|^2 (1 - \bar{I} f(k)) \right.$$
$$\left. + \frac{\bar{I} N(0) f^2(k)}{1 - \bar{I} f} \frac{\gamma^2}{4} |\boldsymbol{H}(k)|^2 + \frac{\gamma^2}{4} N(o) f(k) |\boldsymbol{H}(k)|^2 \right\}. \qquad (5.119)$$

The complete square in parenthesis leads to an irrelevant constant in the integral of $Z[\eta^+, \eta]$. The remaining H^2 term can be written as

$$\mathscr{A}_2|_{H^2} = \frac{\gamma^2}{4} N(0) \int \frac{dk}{(2\pi)^4} f(k) \left(1 + \frac{\bar{I} f(k)}{1 - \bar{I} f(k)}\right) |\boldsymbol{H}(k)|^2. \qquad (5.120)$$

Thus the susceptibility of the normal liquid corrected by paramaquon effects becomes

$$\chi(k^0, \boldsymbol{k}) = \frac{\gamma^2}{2} N(0) \frac{f(k^0, \boldsymbol{k})}{1 - \bar{I} f(k^0, \boldsymbol{k})}. \qquad (5.121)$$

For static homogeneous fields: $f = 1$ and $\bar{I}$ can be determined to be roughly 0.9 in ^{3}He. The parameter $\bar{I}$ is related to Landau's phenomenological Fermi liquid parameter Z_0 as $\bar{I} = -Z_0/4$. Remember that in Landau's theory this parameter determines the molecular field strength associated with the induced spin polarization density $\boldsymbol{S}$:

$$\boldsymbol{H}_{\text{mol}} = -\frac{1}{2\gamma N(0)} Z_0 \boldsymbol{S}. \qquad (5.122)$$

Consider now the paramagnon correction to the magnetic energy in the presence of a collective field $A_{\mu i}$. Again, $\gamma \boldsymbol{H}$ has to be replaced by $\gamma \boldsymbol{H} + I\boldsymbol{P}$. If one restrict to slowly varying fields all terms quadratic in the paramagnon field may be collected as follows:

$$\mathscr{A} = \int \frac{d^4k}{(2\pi)^4} \left\{ \left(\frac{N(0)}{4} f(k)\, \delta_{\mu\nu} - \frac{g_z}{\gamma^2} A^+_{\mu i}(0)\, A_{\nu i}(0) \right) \big(\gamma H_\mu(k) + IP_\mu(k)\big) \big(\gamma H_\nu(k) + IP_\nu(k)\big)^+ - \frac{I}{4} |\boldsymbol{P}(k)|^2 \right\}. \tag{5.123}$$

If $A^+_{\mu i}(0)\, A_{\nu i}(0)$ would be proportional to $\delta_{\mu\nu}$, say $|A|^2\, \delta_{\mu\nu}$, then completing again the square would lead to exactly the same expression as (5.20) except with $N(0)$ replaced by $N(0)\,(1 - a)$, (also in $\bar{I} = IN(0)$) where

$$a = \frac{g_z}{\gamma^2} |A|^2 \frac{4}{fN(0)}$$

Thus the corrected action quadratic in H an A would read

$$\mathscr{A}_2|_{A^2, H^2} = \frac{\gamma^2}{4} N(0) \int \frac{dk}{(2\pi)^4} \frac{f(k)\,(1-a)}{1 - \bar{I} f(k)\,(1-a)} |\boldsymbol{H}(k)|^2 .$$

Expanding this to lowest order in $|A|^2$ (i.e. a) one would find

$$\mathscr{A}_2|_{H^2} = \frac{\gamma^2}{4} N(0) \int \frac{dk}{(2\pi)^4} \left\{ \frac{f(k)}{1 - \bar{I} f(k)} |\boldsymbol{H}(k)|^2 - a \frac{f(k)}{\big(1 - \bar{I} f(k)\big)^2} |\boldsymbol{H}(k)|^2 \right\}. \tag{5.124}$$

The first term is again the corrected susceptibility calculated before. The second term gives the magnetic energy $\mathscr{A}_z$ with paramagnon corrections

$$\mathscr{A}_z|_{\text{Corr}} = -g_z \int \frac{dk}{(2\pi)^4} \frac{1}{\big(1 - \bar{I} f(k)\big)^2} |A(0)\, \boldsymbol{H}(k)|^2 . \tag{5.125}$$

In reality, however, $A^+_{\mu i} A_{\nu i}$ is a matrix, but it can easily be seen that all steps can again be done as before. Thus the paramagnon correction consists mereley in multiplying the original result (5.54) by $\big(1 - \bar{I} f(k)\big)^{-2}$ with $f = 1$ for a static homogeneous magnetic field. The calculation of the paramagnon correction to the quartic terms of the free energy is quite involved. One obviously has to calculate the term of order $H^2 A^2$ in the action, replace $\gamma \boldsymbol{H}$ by $\gamma \boldsymbol{H} + I\boldsymbol{P}$ and proceed as before. The result can be found only numerically [*29*]. The corrections to the coefficients β_i of (5.39) are:

$$\begin{aligned} \beta_1 &= -(1 + 0.1\delta)\, \beta/30; & \beta_2 &= (2 + 0.2\delta)\, \beta/30 \\ \beta_3 &= (2 - 0.05\delta)\, \beta/30; & \beta_4 &= (2 - 0.55\delta)\, \beta/30 \\ \beta_5 &= -(2 + 0.7\delta)\, \beta/30 \end{aligned} \tag{5.126}$$

where the parameter δ is related to an exchange of two paramagnons between two collective particles,

$$\delta = \frac{150\pi^2}{7\zeta(3)} \frac{T_c}{\mu} \int_0^{2p_F} \frac{dq}{2p_F} \left(\frac{\bar{I}}{1 - \bar{I} + \bar{I} \dfrac{g^2}{12 p_F{}^2}} \right)^2 . \tag{5.127}$$

The δ corrections change the minimal free energies to

$$F^A_{\min} = -\frac{1}{4}\frac{\mu_A{}^2}{\beta/15}\frac{2}{2-1.05\delta}$$
$$F^B_{\min} = -\frac{1}{4}\frac{\mu_A{}^2}{\beta/15}\frac{6}{5-1.00\delta}. \tag{5.128}$$

One now sees that the energy of the A phase is lower than that of the B phase as soon as δ exceeds 0.47. In reality, the value of δ calculated from the above formula is merely of the correct order of magnitude if $\bar{I}$ is chosen to agree with the susceptibility data.
The paramagnon explanation is not completely satisfactory in that a corresponding correction to the quadratic terms in the free energy are quite large[21]) as compared with what comes from the initial pairing force. It is gratifying to note, however, that the excellent agreement of the relation $K_1 = (K_2 + K_3)/2$ with experiment which is natural consequence of the pairing force remains valid approximatly since the corrections to $K_{1,2,3}$ are of the order of a few percent only.
Besides, there are inconsistencies. If the free energy (5.37) is used to calculate the specific heats one finds the jumps at T_c of*

$$\frac{\Delta c^A}{c_{\text{normal}}} \approx 1.19\,\frac{2}{\bar{\beta}_{245}}$$
$$\frac{\Delta c^B}{c_{\text{normal}}} \approx 1.43\,\frac{5}{3\bar{\beta}_{12}+\bar{\beta}_{345}} \tag{5.129}$$

in A and B phase where $\bar{\beta}_i \equiv \beta_i \cdot 30/\beta$. The experimental numbers for the left hand side are $2 \pm 0,08$ and $1.9 \pm 0,08$, respectively [*14*] such that

$$\bar{\beta}_{245} \approx 1,2 \pm 0,05; \qquad 3\bar{\beta}_{12} + \bar{\beta}_{345} \approx 3,75 \pm 0,15. \tag{5.130}$$

The parametrization (5.126) gives

$$\bar{\beta}_{245} = (2 - 1,05\delta); \qquad 3\bar{\beta}_{12} + \bar{\beta}_{345} = (5 - 1,00\delta) \tag{5.131}$$

and therefore $\delta \approx 0.76 \pm 0.05$ and 1.25 ± 0.15, respectively. The second value is above 0.47 where the B phase becomes unstable.
For a general approach to strong-coupling corrections the reader is referred to Ref. [*30*].

V.8. Spin Dynamics

After incorporation of the paramagnon corrections, the action takes a sufficiently convenient form to study the behaviour of the spin density operators $\psi^+(x)\,\sigma_\mu/2\,\psi(x)$ in the liquid. Also here, the path integral method permits a quite straightforward derivation of the equations of motion which are known as Leggett's equations [*14*]. For this purpose, remember that the information on all Green's functions involving spin densities $\psi^+(x)\,\sigma_\mu/2\,\psi(x)$ is contained in the dependence of the generating functional Z on the external magnetic field H^{ext}: Due to the coupling (5.46) in the original Lagrangian, arbitrarily many factors $\psi^+(x)\,\sigma_\mu/2\,\psi(x)$ can be generated by functional differentiation with respect to $\gamma\delta H^{\text{ext}}$, for example:

$$\langle 0|\, T\left(\psi^+(x)\frac{\sigma_\mu}{2}\psi(x)\,\psi^+(x')\frac{\sigma_\nu}{2}\psi(x')\right)|0\rangle = -\frac{1}{\gamma^2}\frac{\delta^2 Z}{\delta H_\mu{}^{\text{ex}}(x)\,\delta H_\nu{}^{\text{ext}}(x')}. \tag{5.132}$$

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21) The autor is grateful to Prof. K. Maki for several clarifying discussions of this point.
*) The numeric factors are $10/7\zeta(3) \approx 1.19$ and $12/7\zeta(3) \approx 1.43$.

This certainly remains true after having gone from fundamental fields $\psi(x)$ to collective fields $A_{\mu i}(x)$. In order to extract the physical consequences of this dependence on H^{ext} in the *collective* action $\mathscr{A}[A_{\mu i}, \boldsymbol{H}^{\text{ext}}]$ it is useful to convert it into an explicit dependence on the spin density operators which will, from now on, be abbreviated by

$$s_\mu(x) \equiv \psi^+(x) \frac{\sigma_\mu}{2} \psi(x). \tag{5.133}$$

Such a conversion is formally achieved by determining an action $\mathscr{A}[A_{\mu i}, s_\nu]$ such that the functional integral

$$Z[\boldsymbol{H}^{\text{ext}}] = \int \mathrm{D}A_{\mu i}^+ \mathrm{D}A_{\mu i} \mathrm{D}s_\mu \exp\left\{i\mathscr{A}[A_{\mu i}, s_\nu] + i \int dx \gamma H_\mu{}^{\text{ext}} s_\mu\right\} \tag{5.134}$$

reproduces the original form

$$Z[\boldsymbol{H}^{\text{ext}}] = \int \mathrm{D}A_{\mu i}^+ \mathrm{D}A_{\mu i} \exp\{i\mathscr{A}[A_{\mu i}, \boldsymbol{H}^{\text{ext}}]\} \tag{5.135}$$

after integrating out $\mathrm{D}s_\mu$. That this procedure renders a correct description of the spin dynamics is seen as follows: Due to the functional integral, s_μ is indeed a fluctuating quantum field. Moreover, since all Green's functions of s_μ may be generated by functional differentiation with respect to $H_\mu{}^{\text{ext}}$, the correlation functions of s_μ must coincide with those of the composite fields $\psi^+(x)\, \sigma_\mu/2\, \psi(x)$. As a consequence, the *quantum* fields $\hat{s}_\mu$ are identical with the spin density operators and commute at equal times, according to

$$[\hat{s}_\mu(\boldsymbol{x}, t), \hat{s}_\nu(\boldsymbol{x}', t)] = i\varepsilon_{\mu\nu\lambda}\hat{s}_\lambda(\boldsymbol{x}, t)\, \delta^3(x - x') \tag{5.136}$$

thereby forming the spin rotation group. Since the operators $\psi^+(x)\, \sigma_\mu/2\, \psi(x)$ also generate, by commutation, rotations on the spin indices of all *other* composite fields, it is straightforward to derive the commutation rules with the collective quantum fields $A_{\mu i}$:

$$\left[\int d^3x \hat{s}_\mu(\boldsymbol{x}, t), A_{\nu i}(\boldsymbol{x}', t)\right] = i\varepsilon_{\mu\nu\lambda} A_{\lambda i}(\boldsymbol{x}', t). \tag{5.137}$$

In order to prove this statement formally, one has to introduce an external field $J_{\mu i}$, generating Green's functions of $3g/2p_F\, \psi^+ i\overleftrightarrow{\nabla}_i\, \sigma_\mu/2\, \psi$ via an additional action:

$$\Delta\mathscr{A} = \int dx \left[\psi^+(x)\, i\overleftrightarrow{\nabla}_i \frac{\sigma_\mu}{2} \psi^+(x)\, J_{\mu i}(x) + \text{h.c.}\right]. \tag{5.138}$$

After having performed the transition to collective coordinates this current enters in the tr log term via the effective replacement (compare (5.17), (5.22))

$$A_{\mu i} \to A_{\mu i} + J_{\mu i}. \tag{5.139}$$

Moreover, by a corresponding shift in the functional integral, this current can be moved completely into the contact term such that it amounts, in the collective action, to adding

$$\Delta\mathscr{A} = -\frac{1}{3g} \int dx\, (J_{\mu i}^+ J_{\mu i} - J_{\mu i}^+ A_{\mu i} - J_{\mu i} A_{\mu i}^+). \tag{5.140}$$

By functional differentiation with respect to $\gamma\delta H^{\text{ext}}$ and $1/3g\, \delta J_{\mu i}$ it is obvious that the original commutator

$$\left[\int d^3x \psi^+(\boldsymbol{x}, t) \frac{\sigma_\mu}{2} \psi(\boldsymbol{x}, t),\, \psi^+(\boldsymbol{x}', t)\, i\overleftrightarrow{\nabla}_i \sigma_\nu \psi^+(\boldsymbol{x}', t)\right] = i\varepsilon_{\mu\nu\lambda} \psi^+(x', t)\, \overleftrightarrow{\nabla} i_i \sigma_\lambda \psi^+(x, t) \tag{5.141}$$

goes exactly over into (5.137).

Consider now the collective-field action up to second order in $\boldsymbol{H}^{\text{ext}}$ after the inclusion of the paramagnon effects:

$$\mathscr{A}[A_{\mu i}, \boldsymbol{H}^{\text{ext}}] = \mathscr{A}_0[A_{\mu i}] + \int \frac{dk}{(2\pi)^4} \chi(k) \frac{1}{2} |\boldsymbol{H}^{\text{ext}}(k)|^2$$
$$- g_z \int \frac{dk}{(2\pi)^4} \frac{1}{\left(1 - \bar{I}f(k)\right)^2} |A_{\mu i} H_\mu{}^{\text{ext}}|^2. \qquad (5.142)$$

Here $\mathscr{A}_0$ denotes the remaining collective action at $H^{\text{ext}} = 0$. To this approximation, the spin dependent action (5.134) is readily determined to be

$$\mathscr{A}[A_{\mu i}, s_\nu] = \mathscr{A}_0[A_{\mu i}] - \frac{1}{2} \gamma^2 \int \frac{dk}{(2\pi)^4} \chi^{-1}(k)$$
$$\times \left(|s_\nu(k)|^2 + \frac{4}{f(k)\, N(0)} \frac{g_z}{\gamma^2} \frac{1}{1 - \bar{I}f(k)} A_{\mu i}^+ A_{\nu i} s_\mu(k)\, s_\nu^+(k)\right). \quad (5.143)$$

This action allows a direct study of the spin motion. In order to simplify the discussion, let us take the temperature very close to T_c. Then the last term can be neglected since it is of the order $(\Delta/T)^2 \sim (1 - T/T_c)$ as compared to the one before. If one simplifies further by considering only isotropic spin motion in the sample, then only the total spin

$$S_\mu(x) \equiv \int d^3x s_\mu(\boldsymbol{x}, t) \qquad (5.144)$$

occurs in the action which commutes as usual angular momentum. In a unit volume, $\mathscr{A}$ can be written as

$$\mathscr{A}[A_{\mu i}, s_\nu] \approx \mathscr{A}_0[A_{\mu i}] - \frac{1}{2} \gamma^2 \int dt\, \chi^{-1} S_\nu{}^2(t). \qquad (5.145)$$

Together with the external current piece in (5.134) this amounts to a spin dependent energy

$$\hat{H}_{\text{spin}} \approx \frac{1}{2} \gamma^2 \chi^{-1} S_\nu{}^2 - \gamma H_\nu{}^{\text{ext}} S_\nu. \qquad (5.146)$$

Notice that classically (i.e. in the temperature average) variation with respect to δS gives

$$\boldsymbol{S} = \gamma^{-1} \chi \boldsymbol{H} \qquad (5.147)$$

such that the magnetization $\boldsymbol{M} = \chi \boldsymbol{H}$ in related to the spin field by $\boldsymbol{M} = \gamma \boldsymbol{S}$.
The Hamiltonian may be quantized in accordance with (5.136) which, for the total spin, amounts to the usual commutation rules

$$[S_\mu, S_\nu] = i\varepsilon_{\mu\nu\lambda} S_\lambda. \qquad (5.148)$$

Remembering now that the first term of the action (5.145) has the form (5.64), with $\mathscr{A}_z = 0$, the full Hamiltonian reads:

$$\hat{H} \approx \hat{H}_{\text{spin}} + F + F_d + \cdots \qquad (5.149)$$

where F is the Ginzburg-Landau free energy (5.37) and F_d is the dipole energy corresponding to (5.62):

$$F_d = g_d \int d^3x \left(A^+_{\mu\mu} A_{\nu\nu} + A^+_{\mu\nu} A_{\nu\mu} - \frac{2}{3} A^+_{\mu\nu} A_{\mu\nu} \right). \tag{5.150}$$

The dots indicate higher time and spatial derivative terms. For T close to T_c, the time derivative terms are of order $(\Delta/T)^2 \sim 1 - T/T_c$ as compared with the time derivative terms implied by (5.146) and may be neglected. Now the equations of motion for S_i can be calculated by straightforward commutation:

$$\dot{\boldsymbol{S}} = i[\hat{H}\boldsymbol{S}] = \gamma \boldsymbol{S} \times \boldsymbol{H}^{\text{ext}} + i[F_d \boldsymbol{S}]. \tag{5.151}$$

The last terms contains only the dipole energy since this is the only term not being invariant under separate spin and orbital rotations. Using the commutator (5.137), this term can be written as

$$i[F_d S_i] = g_d \varepsilon_{ikl}\, 2\, \mathrm{Re}\, (A^+_{kl} A_{jj} + A^+_{kj} A_{jl}). \tag{5.152}$$

In a vanishing magnetic field, the second time derivative of S_i can be computed by one more commutation

$$\ddot{S}_i = -\big[[S_i F_d]\, S_j\big] \gamma^2 \chi^{-1} S_i \equiv -\Omega^2_{ij} S_j \tag{5.153}$$

where the frequency tensor Ω^2_{ij} is defined as

$$\begin{aligned} \Omega^2_{ij} \equiv g_d \gamma^2 \chi^{-1}\, 2\, \mathrm{Re}\, [A^+_{ij} A_{kk} &+ A^+_{ik} A_{kj} \\ &- \delta_{ij} (A^+_{kk} A_{ll} + A^+_{kl} A_{lk}) \\ &+ \varepsilon_{ikl} \varepsilon_{jmn} (A^+_{kl} A_{mn} + A^+_{kn} A_{ml})]. \end{aligned} \tag{5.154}$$

For small vibrations around an equilibrium position, this tensor may be evaluated at the equilibrium values of S_i. In the presence of an external magnetic field H^{ext}, the equations of motion (5.153) have to be supplemented with the standard precession term $\omega_L \times \dot{\boldsymbol{S}}$ on the left hand side ($\omega_L \equiv \gamma \boldsymbol{H}^{\text{ext}}$).
If $\boldsymbol{H}^{\text{ext}}$ points in z direction and Ω^2_{ij} is diagonal in the x, y, z coordinate frame, this leads to a longitudinal precession frequency around the z axis

$$\omega_l = \Omega^2_{zz} \tag{5.155}$$

while there are two different transverse oscillations with frequencies

$$\omega_{t1,2} = \frac{1}{2} \{(\omega_L{}^2 + \Omega^2_{xx} + \Omega^2_{yy}) \pm [(\omega_L{}^2 + \Omega^2_{xx} + \Omega^2_{yy})^2 - 4\Omega^2_{xx}\Omega^2_{yy}]^{1/2}\}^2. \tag{5.156}$$

If one of the diagonal values vanishes, say $\Omega_{yy} = 0$, i.e. if there is no harmonic driving force for small rotations around this direction, one of these frequencies vanishes, say $\omega_{t2} = 0$, and the other becomes simply:

$$\omega_t = \gamma^2 \boldsymbol{H}^{\text{ext}^2} + \Omega^2_{xx}. \tag{5.157}$$

The matrix elements Ω^2_{ij} can easily be given for the A and B phases by considering directly the dipole energies in the forms (5.67) and (5.68). In the A phase, if $\boldsymbol{H}^{\text{ext}}$ points in z direction and $\boldsymbol{l}$ is pinned, say, in y direction by external walls, the dipole energy is

$$f_d = -3 g_d \Delta_A{}^2 \cos^2 \varphi \tag{5.158}$$

where φ is the angle between $\boldsymbol{d}$ and $\boldsymbol{l}$. Thus, one has $\Omega^2_{yy} = 0$ and for rotations around x and z

$$\Omega^2_{xx} = \Omega^2_{zz} \equiv \Omega_A{}^2 = 6\chi\gamma^{-2}\,\Delta_A{}^2 g_d. \tag{5.159}$$

As a result, the longitudinal and transverse spin resonances in the presence of an external $\boldsymbol{H}^{\text{ext}}$ field oscillate with frequencies Ω_A and $\sqrt{\gamma^2 H^{\text{ext}^2} + \Omega_A{}^2}$, respectively (experimentally, Ω_A is of the order of 50 kHz).
In the B phase, on the other hand, the external field pulls the $\hat{n}$-vector of (5.66) in the z direction (see equ. (5.71)). The dipole energy

$$f_d = 8g_d\,\Delta_B{}^2 \left[\left(\cos\theta + \frac{1}{4}\right)^2 - \frac{1}{4}\right] \tag{5.160}$$

gives rise to a non-vanishing frequency if θ vibrates around arc cos (— 1/4). For small deviations

$$\delta^2 f_d = 8g_d\,\Delta_B{}^2 \frac{15}{16}\,(\delta\theta)^2. \tag{5.161}$$

Since θ is the angle of rotation around $\hat{n} = \hat{z}$ there is only one non-vanishing resonance for longitudinal excitation:

$$\omega_l{}^2 \equiv \Omega_B{}^2 = \Omega^2_{zz} = 15\chi\gamma^{-2}\,\Delta_B{}^2 g_d \tag{5.162}$$

for the spin precessing around the z axis. Notice: $\Omega_B{}^2/\Omega_A{}^2 = \frac{5}{2}\,\chi_B\Delta_B{}^2/\chi_A\Delta_A{}^2$. Experimentally, this ratio is close to 5/2. Since $\chi_B \approx \chi_A$ this implies $\Delta_B \approx \Delta_A$.
For transverse excitations, on the other hand, the normal resonance frequency $\omega_t = \gamma H^{\text{ext}}$ is unchanged as the liquid makes its transition from the normal to the B phase.

V.9. Solitons and Satellites

For completeness, we now present a brief discussion of singular field configurations and of possible ways of detecting them experimentally. With the order parameter being a complex 3×3 matrix $A_{\mu i}$ there exist many such configurations which are stable and

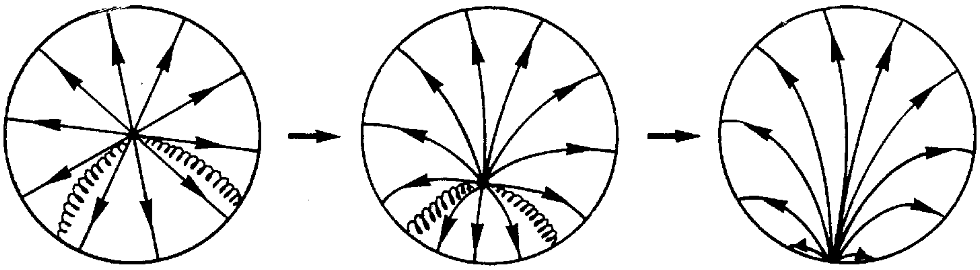

Fig. XII. The singular line a cylinder (disgyration) tries to escape into the dist... ted configuration on the right (Mermin-Ho vortex) in order to avoid spending condensation energy

topologically quite distinct. In principle, the sample may contain planar, linear, or point-like singularities. For small enough temperatures $T \ll T_c$, the liquid tends to avoid true singularities in its interior. The reason is that at a point at which the direction of the 18 component vector $A_{\mu i}$ is undefined it must vanish altogether and this implies that there the liquid must be normal. If T is far enough below T_c, the normal liquid has a high energy density. As a consequence, it tries to escape into energetically less costly configurations in which the size of the order parameter, $A^+_{\mu i}A_{\mu i} = 3\Delta^2$ stays constant throughout the volume (namely exactly at the values given by (5.65), (5.66) at which the free energy F is minimal). This is achieved by bending the field lines until the singularities have all moved to the boundaries of the container. There they can be acco-

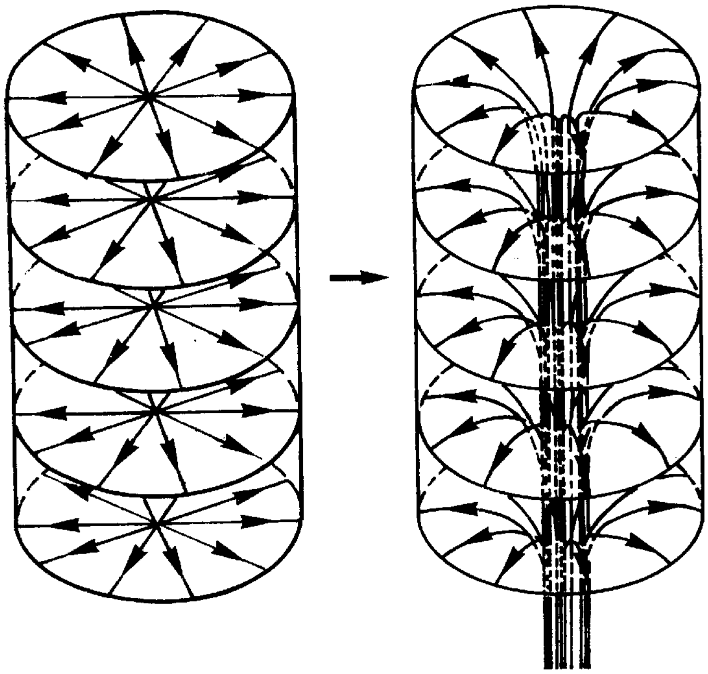

Fig. XIII. The point singularity in a radial l texture is drawn to the wall by two singular lines whose energy is proportional to their length. The lines are singularities in what is defined as superliquid velocity $v_{si} = \Phi^{(1)} \nabla_i \Phi^{(2)}$. Its curl on the surface equa s the curvature such that the integral over a sphere gives two flux units ($\int d\boldsymbol{s}\, \boldsymbol{v}_s = -\int (\nabla \times \boldsymbol{v}_s)\, d\boldsymbol{a} = -\int da\, \varkappa \to -2 \times 2\pi$)

modated with the least additional cost of energy. From equ. (5.43) we know that also bending of the field lines raises the energy density but this amount is much smaller than what would be required for bringing the liquid from superliquid to the normal phase (for $T \ll T_c$).

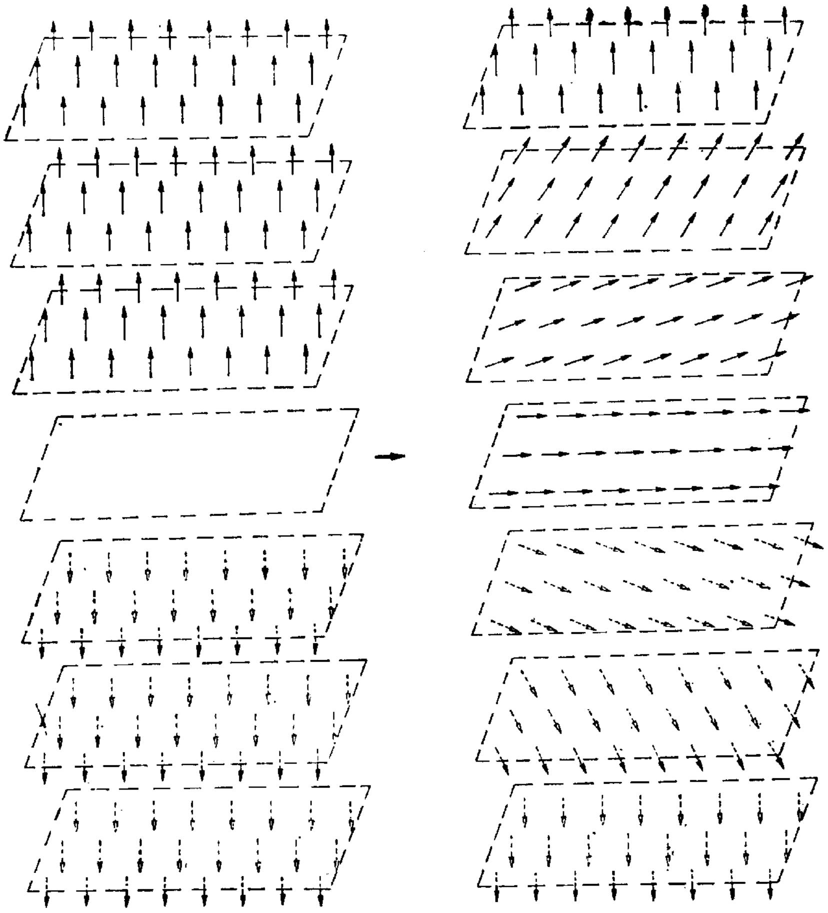

Fig. XIV a

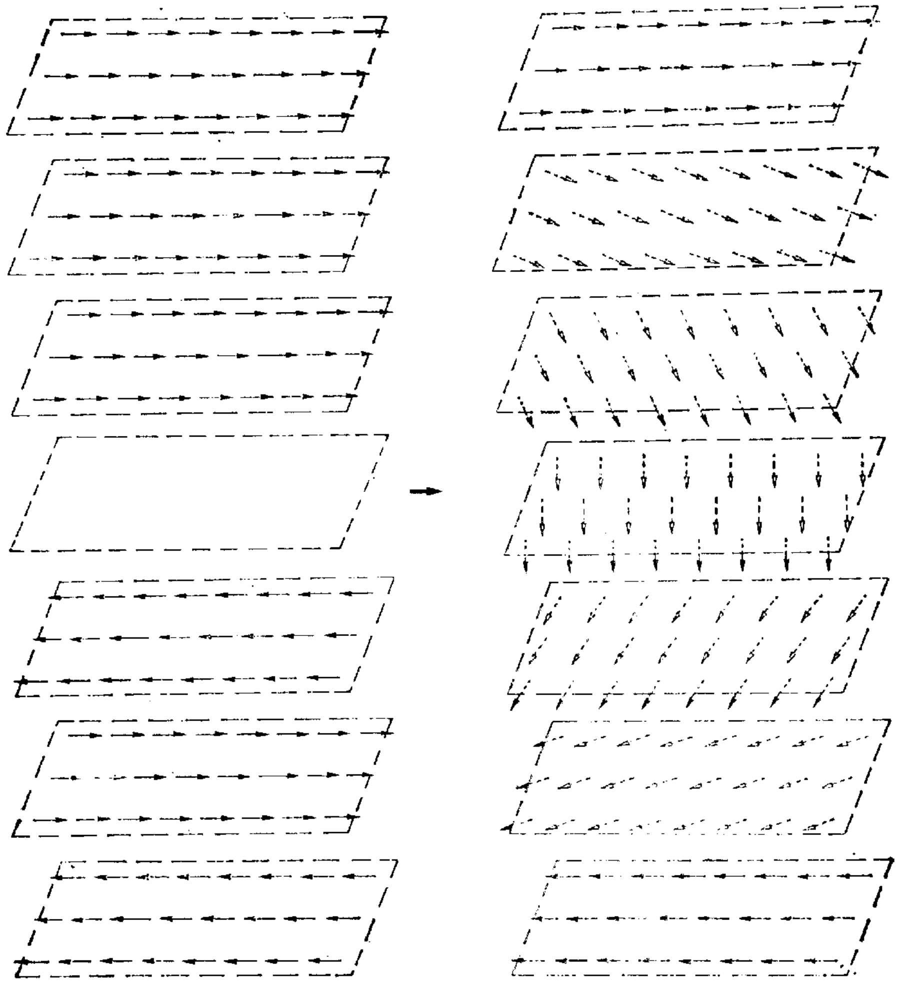

Fig. XIV b

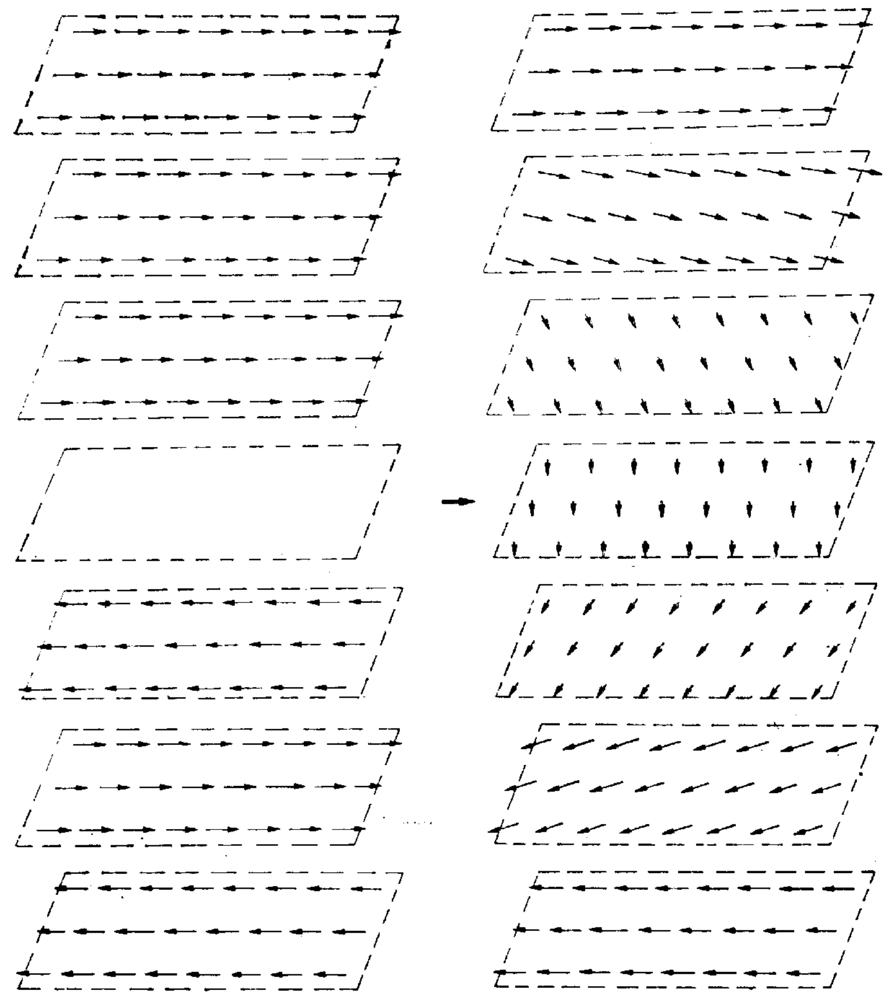

Fig. XIV c

Fig. XIV. Planar singularities are smoothed out to splay (a), bend (b), or twist-like domain walls (c)

Examples for smoothed out singular field configurations are given in Figs. XII, XIII, XIV which display $\boldsymbol{l}$ field lines of ^{3}He in the A phase. Since the $\boldsymbol{l}$ vectors are always orthogonal to the walls, there must be a singularity in the spherical vessel which is avoided by the field lines distorting until the singularity reaches the boundary [22] (see Fig. XII). In a cylindrical vessel, the singular line along the z-axis is regularized by the field lines turning parallel to the axis when approaching the center [34] (see Fig. XIII). The distortion of planar $\boldsymbol{l}$ or $\boldsymbol{d}$ field configurations may, in principle, proceed in three ways (see Figs. XIVa—c). In the situation a) the field lines turn from forward to backward in one of the planes parallel to the x-axis ("splay"); in b) or c) the distortion may proceed in the $x - z$ plane ("bend") or in the $x - y$ plane ("twist"), respectively [35].
Apart from these simple situations there may exist topologically more involved field configurations. In a torus [36], for example, the vectors $\boldsymbol{\Phi}^{(1)}$, $\boldsymbol{\Phi}^{(2)}$ may turn once around the $\boldsymbol{l} = \boldsymbol{\Phi}^{(1)} \times \boldsymbol{\Phi}^{(2)}$ direction when going around the circumference [36] at the surface the $\boldsymbol{l}$ vector always pointing outward. A parallel $\boldsymbol{l}$ field may contain a cylindrial region with $\boldsymbol{l}$ field lines running in the opposite direction, the wall forming a bend or twist like domain [36, 37]. There may be vortex ring-like objects which move through the liquid with finite energy and momentum [38]. The possibilities are certainly many and the liquid will, in general, be crowded with all of them at the same time unless experimentalists learn how to prepare samples with only one specific type of would-be-singularities [39, 40]. The many ways of enforcing desired configurations were already discussed in Sect. V.5. (walls, external fields, currents, etc.). With more experience, these should be sufficient to bring the liquid under control.
Apart from would-be-singularities enforced by external conditions, there are others whose properties are governed completely by intrinsic parameters of the liquid. As a matter of fact, they can be used to measure experimentally certain phenomenological coefficients in the Ginzburg-Landau expansion. The prime examples consist in planar domain walls arising from the presence of the dipole energy (5.67) or (5.68). Their size is controlled by the ratio $\sqrt{K_{23}/8g_d} \approx 10^{-3}$ cm (see (5.63) and (5.43)). In addition, they form traps for spin waves and thereby give rise to satellite frequencies accompanying nuclear magnetic resonances. The measured frequency ratio determines directly the ratio among the coefficients in the bending energy (5.42)

$$\varkappa \equiv 2K_1/(K_2 + K_3).$$

As an illustration, let us study a structure of this type for ^{3}He in the A-phase. The situation becomes most simple by turning on a magnetic field $\boldsymbol{H}$ in z-direction. This assures the $\boldsymbol{d}$ vectors to stay in the $x - y$ plane (see equ. (5.69)). There they may be parametrized by

$$\boldsymbol{d} = \sin\psi \cdot \hat{\boldsymbol{x}} + \cos\psi \cdot \hat{\boldsymbol{y}}. \tag{5.163}$$

The dipole force (5.67) tends to align the $\boldsymbol{l}$ vectors parallel or anti-parallel to the fields. Thus also the $\boldsymbol{l}$ vectors want to stay in the x — y-plane and may be parameterized similarly via

$$\boldsymbol{l} = \sin\chi\hat{\boldsymbol{x}} + \cos\chi\hat{\boldsymbol{y}}.$$

With this $\boldsymbol{l}$, the complex vector $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{(1)} + i\boldsymbol{\Phi}^{(2)}$ can have the general form

$$\boldsymbol{\Phi} = e^{i\varphi}(-\cos\chi\hat{\boldsymbol{x}} + \sin\chi\hat{\boldsymbol{y}} + i\hat{\boldsymbol{z}}). \tag{5.164}$$

We expect that χ will be equal to ψ or $\psi + \pi$ for most portions of the liquid. The change from one configuration to the other should occur only inside narrow domain walls. Notice, that this parametrization of $\boldsymbol{d}$ and $\boldsymbol{l}$ limits the domain configurations to the pure "twist" type (Fig. XIVc).

With the order parameter (5.65), the free bending energy density becomes

$$f \cong \frac{K_{23}}{2}\left\{\partial_i A^+_{\mu i}\,\partial_j A_{\mu j} + \frac{\varkappa}{2}\,\partial_i A^+_{\mu j}\,\partial_{i\mu} A_{\mu j}\right\}$$

$$\cong \frac{3}{4}\,K_{23}\,\Delta_A{}^2\left\{|\boldsymbol{\Phi}\,\partial\,d_\mu|^2 + |\,\partial\,\boldsymbol{\Phi}|^2 + \varkappa\left(\left|\frac{1}{2}\,\partial_i \Phi_j\right|^2 + |\partial_i d_\mu|^2\right)\right\}. \tag{5.165}$$

Here, we have combined K_2 and K_3 terms since they differ only by a pure divergence which can be neglected in a bulk sample. The ratio $\varkappa \equiv 2K_1/K_{23}$ equals one for the weak coupling BCS values (5.43). We shall keep this parameter in the calculation since the phenomena to be presented now are sensitive to it and may be used for its experimental measurement.

Inserting the above planar parametrizations for $\boldsymbol{l}$ and $\boldsymbol{d}$ into the bending energy, one has

$$f \cong \frac{3}{4}\,K_{23}\,\Delta_A{}^2\left\{(\varkappa + 1)\,(\partial\psi)^2 - (\partial_l\psi)^2 + (\varkappa + 1)\,(\partial\varphi)^2 - (\partial_l\varphi)^2\right.$$

$$\left. + \frac{\varkappa}{2}\,(\partial\chi)^2 + (\partial_l\chi)^2 - 2\partial_z\varphi\,\partial_l\chi\right\} \tag{5.166}$$

where ∂_l denotes the derivative in the direction of $\boldsymbol{l}$:

$$\partial_l = \boldsymbol{l}\boldsymbol{\partial} = \sin\chi\,\partial_x + \cos\chi\partial_y. \tag{5.167}$$

Of the remaining part of the free energy, only the dipole expression (5.67) contributes which amounts to adding

$$f_d = \frac{1}{2\xi^2}\sin^2(\chi - \psi) \tag{5.168}$$

inside the curly brackets with the coherence length parameter being defined as

$$\xi^2 \equiv K_{23}/8g_d \tag{5.169}$$

which is of the order of 10^{-3} cm. It is this dipole energy term which gives rise to domain walls. Their thickness is governed by the length parameter ξ.

In order to proceed, let us now limit the consideration to plane waves where all fields depend only on $s = \boldsymbol{k}\cdot\boldsymbol{x}$, where $\boldsymbol{k}$ is some unit vector. Then the free energy is proportional to[22])

$$f \propto \left\{(\varkappa + 1 - a^2)\,\psi_s{}^2 + (\varkappa + 1 - a^2)\,\varphi_s{}^2 + \left(\frac{\varkappa}{2} + a^2\right)\chi_s{}^2\right.$$

$$\left. - 2k_z a\varphi_s\chi_s + \frac{1}{2\xi^2}\sin^2(\chi - \psi)\right\} \tag{5.170}$$

where we have set $a \equiv \boldsymbol{l}\boldsymbol{k}$. A quadratic completion brings this to the form

$$f \propto \left\{(\varkappa + 1 - a^2)\,\psi_s{}^2 + (\varkappa + 1 - a^2)\left(\varphi_s - \frac{k_z a}{\varkappa + 1 - a^2}\,\chi_s\right)^2\right.$$

$$\left. + \frac{1}{2}\left(\varkappa + 2a^2 - 2\,\frac{k_z{}^2a^2}{\varkappa + 1 - a^2}\right)\chi_s{}^2 + \frac{1}{2\xi^2}\sin^2(\chi - \psi)\right\}. \tag{5.171}$$

Minimization of this requires the dependence

$$\varphi_s = \frac{k_z a}{\varkappa + 1 - a^2}\,\chi_s. \tag{5.172}$$

In order to minimize the remainder of the energy density one may introduce $v \equiv \chi - \psi$ and $u = \chi + 2(\varkappa + 1 - a^2)^2/[(\varkappa + 1 - a^2)(\varkappa + 2a^2) - 2k_z^2a^2]\,\psi$ and finds the diagonal form

$$f = f_u + f_v \tag{5.173}$$

with[22])

$$f_u = \frac{1}{2}\,\frac{(\varkappa + 1 - a^2)(\varkappa + 2a^2) - 2k_z^2a^2}{(\varkappa + 1 - a^2)(3\varkappa + 2) - 2k_z^2a^2}\,u_s^2, \tag{5.174a}$$

$$f_v = (\varkappa + 1 - a^2)\,\frac{(\varkappa + 1 - a^2)(\varkappa + 2a^2) - 2k_z^2a^2}{(\varkappa + 1 - a^2)(3\varkappa + 2) - 2k_z^2a^2}\,v_s^2 + \frac{1}{2\xi^2}\,\mathrm{sh}^2\,v. \tag{5.174b}$$

Variations in u give $u = \text{const}$.
A non-trivial minium of the second part cannot be given for general $\boldsymbol{k}$. The expression simplifies, however, if $\boldsymbol{k}$ runs in z direction (i.e. if the domain walls is orthogonal to the H field). This might be enforced experimentally by choosing a narrow cylindrical vessel with the axis along the H field. Since a domain wall tends to minimize its surface energy, it is expected to run across the cylinder. Notice, however, that the diameter has to be large enough as to avoid distorting effects from the boundary conditions at the cylinder wall. With $\boldsymbol{k}$ in z direction, the second part of the energy becomes

$$f_v = \frac{\varkappa(\varkappa + 1)}{3\varkappa + 2}\,v_z^2 + \frac{1}{2\xi^2}\sin^2 v \tag{5.175}$$

and can be minimized by the well-known classical soliton solution

$$\tan v_{\mathrm{sol}}/2 = e^{\pm z/\xi_{\mathrm{sol}}}; \quad \sin v_{\mathrm{sol}} = \mathrm{ch}^{-1}\,z/\xi_{\mathrm{sol}}$$
$$\cos v_{\mathrm{sol}} = \mp\mathrm{th}\,z/\xi_{\mathrm{sol}} \tag{5.176}$$

where $\xi_{\mathrm{sol}} \equiv \sqrt{2\varkappa(\varkappa + 1)/(3\varkappa + 2)}\,\xi$. The energy per unit surface of the domain wall is[22])

$$F/\sigma = \int dz f = \frac{1}{\xi^2}\int dz\,\mathrm{ch}^{-1}\,z/\xi_{\mathrm{sol}} = 2\xi_{\mathrm{sol}}\xi^{-2} = 2\sqrt{\frac{2\varkappa(\varkappa + 1)}{3\varkappa + 2}}\,\xi^{-1}. \tag{5.177}$$

Due to the joint distortion of $\boldsymbol{d}$ and $\boldsymbol{l}$ vectors this soliton may be called composite [21].
For general direction $\boldsymbol{k}$, a similar solution is expected but it can no longer be calculated exactly. One may, however, convince oneself that a small $k_\perp$ would raise the energy. Thus, if solitons are produced in arbitrary directions they are expected to align towards the z direction under emission of spin-wave radiation.
The physical situation described by the solution (5.176) is quite simple. For $z \to -\infty$ the vectors $\boldsymbol{l}$ and $\boldsymbol{d}$ start out in the same direction, say y. As z runs through the origin the relative angle changes from zero to π. Since $u \equiv \chi + 2(\varkappa + 1)/\varkappa\,\psi$ remains constant, χ and ψ turn opposite to each other, the first $2(\varkappa + 1)/\varkappa$ times as fast as the second, until they are pointing antiparallel to one another. In the final position, at $z \to \infty$, $\boldsymbol{l}$ points at an angle $2(\varkappa + 1)/(3\varkappa + 2)\,\pi$, while $\boldsymbol{d}$ has moved into the opposite direction $-\varkappa/(3\varkappa + 2)\,\pi$. Since the theoretical weak coupling value is $\varkappa = 1$, the motion of $\boldsymbol{l}$ is expected to be roughly four times as far to the right as the motion of $\boldsymbol{d}$ to the left (see Fig. XV).
How can one detect the presence of such a domain wall? In order to answer this question, one observes that for small vibrations a soliton configuration represents a potential

[22]) Up to the proportionality factor $3/4K_{23}\Delta_A^2 = n/8m(1 - T/T_c)$ from (5.166).

well. Consider the free energy density

$$f = \alpha_z^2 + \Omega \sin^2 \alpha \tag{5.178}$$

with a soliton

$$\tan \alpha_{\text{sol}}/2 = e^{z\Omega}.$$

Inserting small oscillations around this solution

$$\alpha = \alpha_{\text{sol}} + \delta \tag{5.179}$$

one finds

$$\delta^2 f = 2\Omega^2 \frac{1}{\text{ch}^2 z\Omega} + \delta_z^2 + \Omega^2 \left(1 - 2\frac{1}{\text{ch}^2 z\Omega}\right) \delta^2. \tag{5.180}$$

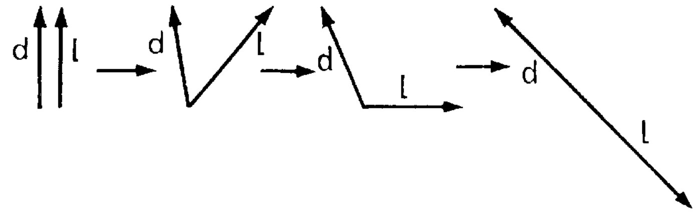

Fig. XV. In the composite soliton (5.176) the l vector twists about four times as far as the d vector. The soliton is energetically quite close to the pure l type

The minimization of this amounts to a standard Schrödinger problem [*41*]:

$$\left[-\partial_x^2 + \left(\Omega^2 - \frac{V_0}{\text{ch}^2 za}\right)\right] \delta(z) = \omega^2 \delta(z). \tag{5.181}$$

The spectrum of this equation consists of a continuum with asymptotic momenta k and eigenvalues

$$\omega_k^2 = \Omega^2 + k^2 \tag{5.182}$$

and of bound states for $n = 0, 1, \ldots, s$

$$\delta_n(z) \propto (\text{ch } za)^{n-s} F_{2,1}\left(-n, 2s - n; s - n + 1; \frac{1}{2}(1 - \text{th } za)\right) \tag{5.183}$$

with eigenvalues

$$\omega_n^2 = \Omega^2 - a^2(s - n)^2. \tag{5.184}$$

The parameter s is

$$s = \frac{1}{2}\left[-1 + \sqrt{1 + 4V_0/a^2}\right]. \tag{5.185}$$

Comparing the bound-state values with that of the $k = 0$ continuum state, one finds the ratio

$$R_n = 1 - \frac{a^2}{\Omega^2}(s - n)^2. \tag{5.186}$$

It is a pleasant property of ^{3}He that such a ratio can apparently be detected experimentally. For this, one adds a small oscillating magnetic field $\boldsymbol{H}_l$ longitudinal to the external field $\boldsymbol{H}$. This is capable of exciting oscillations of the $\boldsymbol{d}$ vector in the $x - y$ plane. If the small changes of the angle ψ are denoted by f, say $\psi = \psi_{\text{sol}} + f$, the angles

u and v vibrate around the soliton configuration according to

$$u = u_{sol} + 2\frac{\varkappa + 1}{\varkappa} f; \quad v = v_{sol} - f. \tag{5.187}$$

Inserting this into (5.174) and (5.175), one finds the change in free energy

$$\delta^2 f \propto \frac{1}{2}\left\{2(\varkappa + 1) f_z^2 + \xi^{-2}\left(1 - \frac{2}{\mathrm{ch}^2 z/\xi_{sol}}\right) f^2\right\}. \tag{5.188}$$

As a consequence, the parameter s is found to be

$$s = \frac{1}{2}\left[-1 + \sqrt{1 + 4\xi_{sol}^2/(\varkappa + 1)\,\xi^2}\right]$$
$$= \frac{1}{2}\left[-1 + \sqrt{1 + \frac{8\varkappa}{3\varkappa + 2}}\right]. \tag{5.189}$$

For $\varkappa \approx 1$, this is about 0.3 and we find exactly one bound state with a wave function

$$f_0 \propto \frac{1}{(\mathrm{ch}\, z/\xi_{sol})^s}. \tag{5.190}$$

Its eigenvalue corresponds to a ratio

$$R_0 = 1 - 2(\varkappa + 1)\,\xi^2\xi_{sol}^{-2}s^2 = 1 - 2(\varkappa + 1)\,\frac{3\varkappa + 2}{2\varkappa(\varkappa + 1)}\,s^2$$
$$= \frac{1}{2\varkappa}\left[\sqrt{(11\varkappa + 2)(3\varkappa + 2)} - (5\varkappa + 2)\right]. \tag{5.191}$$

It is gratifying that this ratio is independent of the knowledge of the size of the time derivative term in the action. All that has to be known about this term is that it is dominated by the lowest possible form $\dot{f}^2$ (see Sect. V.8.). Thus the above value of R determines the ratio of the squares of the resonance frequencies observed in longitudinal magnetic excitations. For $\varkappa = 1 + \varepsilon \approx 1$ the ratio comes out to be

$$R_0 \approx \frac{1}{2}\left(\sqrt{65} - 7\right)\left(1 - \frac{5}{6}\varepsilon\right)^2 \approx (0.728)^2\left(1 - \frac{5}{6}\varepsilon\right)^2. \tag{5.192}$$

There exists an experiment which detects a satellite frequency shifted by a factor 0.74 below the normal nuclear magnetic resonance line [27]. Thus, there is a good chance that this line is due to spin waves trapped in domain walls of the type discussed here. Taking this identification serious, the coefficients in the bending energy are found to have a ratio $\varkappa = 2K_1/K_{23}$ only 2% smaller than the weak coupling prediction (5.43).
An alternative experiment may be performed by applying a transverse oscillating magnetic field $\boldsymbol{H}_t$ (i.e. orthogonal to the external field $\boldsymbol{H}$) say in x direction. Then the $\boldsymbol{d}$ vector will start tipping toward the z direction. If one parametrizes such a tipping $\boldsymbol{d}$ vector field by

$$\boldsymbol{d} = \sin\psi \cos g\,\hat{\boldsymbol{x}} + \cos\psi \cos g\,\hat{\boldsymbol{y}} + \sin g\,\hat{\boldsymbol{z}} \tag{5.193}$$

and inserts this into the original free energy (5.165) one finds the quadratic deviation from the extremal soliton configuration:

$$\delta^2 f = \frac{1}{2}\left\{2[(\varkappa + 1)(\partial g)^2 - (\partial_l g)^2] + \left[\xi^{-2}\left(1 - \sin^2(\chi - \psi)\right) - 2(\varkappa + 1)(\partial\psi)^2\right] g^2. \tag{5.194}$$

For the composite soliton (5.176) this amounts to

$$\delta^2 f = \frac{1}{2}\left\{2(\varkappa + 1)\, g_z^2 + \xi^{-2}\left(1 - \frac{4\varkappa + 2}{3\varkappa + 2}\,\frac{1}{\operatorname{ch}^2 z/\xi_{\text{sol}}}\right) g^2\right\}. \tag{5.195}$$

This energy density has an s value

$$s = \frac{\varkappa}{3\varkappa + 2}$$

and therefore exactly one bound state (for $\varkappa > 0$) with a ratio

$$R_0 = 2\,\frac{\varkappa + 1}{3\varkappa + 2} \approx (0.896)^2 \quad \text{for } \varkappa \approx 1. \tag{5.196}$$

One may easily check that the $\boldsymbol{g}$ vibrations do not interfere with the previously calculated f modes by using

$$\boldsymbol{d} = \sin(\psi + f)\cos g\,\hat{\boldsymbol{x}} + \cos(\psi + f)\cos g\,\hat{\boldsymbol{y}} + \sin g\,\hat{\boldsymbol{z}} \tag{5.197}$$

and observing that $\delta^2 f$ is really the sum of the expressions (5.188) and (5.194) with no mixed fg term.

There is an experiment [*27*] in which a satellite is observed for a transverse nuclear magnetic resonance. Its frequency, however, is by a factor 0.835 lower than the main line. While the discrepancy between 0.896 and 0.835 is quite small, it is significant as far as the value of $\varkappa$ is concerned. As we see from (5.196), a value of $\varkappa \approx 6$ would be necessary to bring the ratio to the observed value in contrast to the longitudinal value $\varkappa \approx 1$ which agrees so well with the weak coupling prediction. The resolution of this difficulty may consist in the following argument [*39*]: While the longitudinal experiment was performed in a vertical cylinder (i.e. parallel to the z-axis) such that the formation of a twist line domain wall is most probable, the transverse experiment had the axis turned horizontally with the vibrating field $\boldsymbol{H}_t$ along the axis. Since the domain wall is expected to span across the cylinder, the k vector should be in the xy plane with a splay or bend like distortion of the field lines. The corresponding minimum of (5.174) can only be found numerically. The same thing holds for the small oscillations trapped in the domain wall. The result for $\varkappa \approx 1$ is $R_t \approx (0.82)^2$ and compares much more favourably with the experimental number $(0.835)^2$.

A short discussion is in order concerning the systematic creation and dynamics of solitons in the liquid. Consider a sample between two plates parallel to the $x - z$ plane such that $\boldsymbol{l}$ and $\boldsymbol{d}$ vectors run uniformly from wall to wall in y-direction. This corresponds to the $\psi = 0$, $\chi = 0$, $\varphi =$ const. solution of (5.170). Suppose now, an external magnetic field in z direction with a shape $H_0(z) \equiv \omega_0(z)/\gamma$ is suddenly turned off. Then the $\boldsymbol{d}$ vector will start rotating around the z axis with the velocity

$$\dot{\psi}(z, 0) \equiv \omega_0(z). \tag{5.198}$$

This can be seen by commuting the Hamiltonian (5.146) with (5.170): Since a spin rotation around the z-axis changes ψ, one has

$$[S_3, \psi] = -i \tag{5.199}$$

and therefore

$$\dot{\psi}(z, 0) = i[\hat{H}_{\text{spin}}\psi] = \gamma^2\chi^{-1}S_3 - \gamma H_3 = -\gamma(H_3 - \chi^{-1}M_3) \tag{5.200}$$

which says that the speed of ψ is determined by the instantaneous discrepancy between the magnetic field and the corresponding magnetization. Turning off part of H_3 by going to $H_3 - H_0(z)$ will cause exactly the discrepancy $\omega_0(z)/\gamma$ setting ψ into motion. In order to study the time evolvement of $\psi(z, t)$ one has to add a kinetic term to (5.170). From the discussion of (5.158) it is obvious that this term must have the form

$$f^d_{\text{kin}} \propto \frac{1}{2\xi^2 \Omega_A{}^2} \dot{\psi}^{-2} \tag{5.201}$$

allowing for vibrations of frequency Ω_A around the equilibrium position. A corresponding term must be added for the $\boldsymbol{l}$ field. The discussion simplifies due to the fact that the time scale of vibrations in $\boldsymbol{l}$ is usually about 10^2 times slower than in $\boldsymbol{d}$. Therefore, one may assume $\boldsymbol{l}$ to remain in its equilibrium position for the whole process of generation and the solitons will be of the pure $\boldsymbol{d}$ type. The free energy is therefore (adding (5.170) and (5.201))

$$f^d \propto \frac{1}{2\xi^2 \Omega_A{}^2} [\dot{\psi}^2 + c^2 \psi_z{}^2 + \Omega_A{}^2 \sin^2 \psi] \tag{5.202}$$

with $c^2 \equiv 2\xi^2 \Omega_A{}^2(\varkappa + 1)$ and with the initial conditions

$$\psi(z, 0) \equiv 0, \qquad \dot{\psi}(z, 0) = \omega_0(z). \tag{5.203}$$

The solution of this problem can be given exactly by means of the inverse scattering method [*42*]. A first estimate is possible for $\omega_0 \gg \Omega_A$. Then the last term can be neglected and ψ has the general form:

$$\psi(z, t) = f(z + ct) - f(z - ct) \tag{5.204}$$

where $f(z) = \omega_0(z)/2c$ such that

$$\psi(z, t) = \frac{1}{2c} \int_{z-ct}^{z+ct} \omega_0(z') \, dz'. \tag{5.205}$$

Suppose $H_0(z)$ has the box form $H_0 \theta(a^2 - z^2)$, i.e. $\omega_0(z') = \omega_0 \theta(a^2 - z'^2)$. Then $\psi(z, t)$ at fixed t has the shape of a trapezoid whose sides rise linearly from the corners at $z = \mp(a + ct)$ to $z = \mp(a - ct)$ with a height value $\omega_0 a/c$. As t exceeds a/c, the height does not grow any more and remains at its maximal value $\omega_0 a/c$. The right and left sides keep spreading apart with velocity c. If one divides the resulting trapezoids by lines $\psi = n\pi$; $n = 1, 2, 3, \ldots$ then every intersection gives the approximate position of a soliton on the right-hand side and of an antisoliton on the left-hand side, both sets moving apart from each other (see Fig. XVI). Thus, one has the approximate number of soliton-antisoliton pairs:

$$N_{\text{pairs}} \approx \omega_0 a/c = \frac{\omega_0}{\Omega_A} \frac{a}{\xi}. \tag{5.206}$$

With $\xi \approx 10^{-3}$ cm one can create many hundred solitons in a vessel of a few centimeter diameter (say $a \approx 1$ cm).

The exact solution is somewhat different: A fraction of the soliton-antisoliton pairs does not separate but forms bound states (breather modes), which annihilate after a short time ($\approx 10^{-4}$ s) due to spin diffusion. The emerging free pairs differ from the estimate (5.206) by a factor [*39*]:

$$\frac{1}{2} \sqrt{1 - \left(\frac{\Omega_A}{\omega_0}\right)^2} + \frac{1}{\pi N_{\text{pairs}}} \operatorname{arc\,cos} \frac{\Omega_A}{\omega_0}.$$

The cloud of pure $\boldsymbol{d}$ solitons created in this fashion spreads with velocity $c = 2\xi\Omega_A \approx 100$ cm/s. The cloud will be slowed down by spin diffusion [*43*] and come to a halt within about 10^{-4} s, i.e. after 10^{-2} cm. During this whole process the $\boldsymbol{l}$ vectors with their characteristic time of 10^{-2} s do not have time to move. Now that the $\boldsymbol{d}$ solitons have stopped the coupling with the $\boldsymbol{l}$ vectors can become active (see the orbital viscosity discussion in Ref. [*28*]). From the derivative terms in (5.140) one sees that distorting the $\boldsymbol{l}$ vectors is $(\varkappa/2 + a^2)/(\varkappa + 1 - a^2)$ times easier[23]) than distorting the $\boldsymbol{d}$ vectors. For a wave in

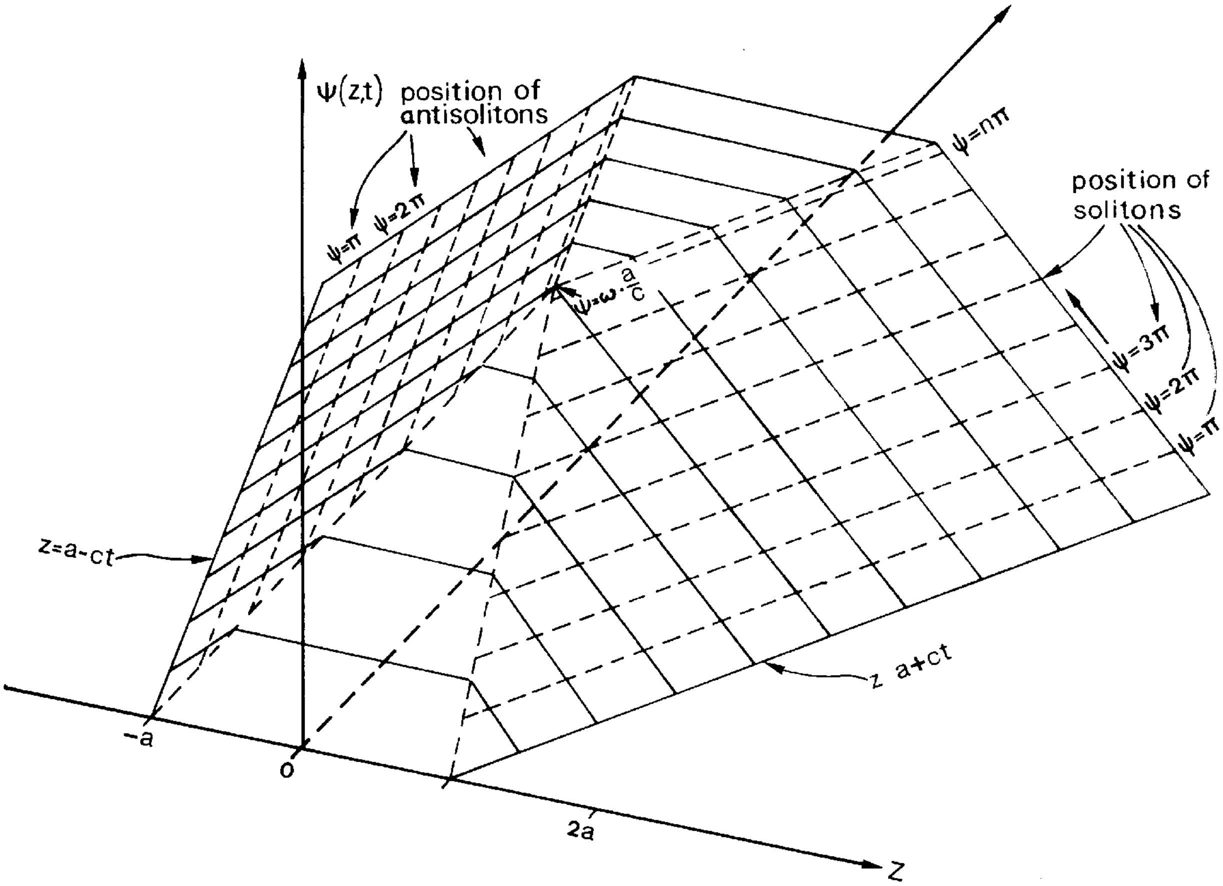

Fig. XVI. The figure illustrates the generation of soliton-antisoliton pairs. As time proceeds, the trapezoidal profile of $\psi(zt)$ becomes wider linearly and higher until the top hits $a\omega_0$. After this only the right and left sides keep spreading with velocity c. The approximate position of each pair is obtained by the intersections with the vertical lines at $\psi = n\pi$, $n = 1, 2, 3, \ldots$

z direction this becomes $\varkappa/2(\varkappa + 1)$. Moreover, from (5.175) one concludes that the joint bending of $\boldsymbol{d}$ and $\boldsymbol{l}$ vectors lies even lower, namely by a factor $\varkappa/(3\varkappa + 2)$ ($\approx 1/5$). As a consequence, the pure $\boldsymbol{d}$ solitons will relax into the composite soliton. The time in which this happens may be estimated to be $\approx 10^{-2}(1 - T/T_c)^{1/2}$ s[24]).

Finally let us take a look at the possibility of the liquid containing pure $\boldsymbol{l}$ solitons. They may arise by conversion from $\boldsymbol{d}$ solitons if the direction of $\boldsymbol{l}$ is driven parallel to, say, the y-axis by an additional external force, for example a current along the y-axis with a velocity of the order cm/s (see (5.78)) or by an electric field (if that coupling is found to be present, see (5.80)). Then the energy will come mainly from

$$f^l \propto \left(\frac{\varkappa}{2} + a^2\right) \chi_s{}^2 + \frac{1}{2\xi_{\text{tot}}^2} \sin^2 \chi \tag{5.207}$$

[23]) Due to the $\sin^2$ term in f increasing the size of the domain wall at the same rate, this factor enters the final energy density of a soliton only with a square root (see equ. (5.177)).

[24]) Notice that even if d solitons would be stable, they would not give rise to any interesting observable satellite frequency: Going through the chain of arguments after (5.179) one finds for the free energy (5.188): $R = 0$: This zero-frequency made corresponds to an infinitesimal translation of the soliton.

with $1/2\xi_{tot}^2$ including the external directional force (i.e. $= 1/2\xi^2 + q^2$ where q is the flow velocity). If $\boldsymbol{k}$ is in z direction, this gives a pure twist soliton

$$\tan \frac{\chi_{sol}}{2} = e^{\pm z/\sqrt{\varkappa}\,\xi_{tot}} \tag{5.208}$$

of energy density $F/\sigma \propto 2\sqrt{\varkappa}\,\xi_{tot}^{-1}$. If $\boldsymbol{k}$ has an arbitrary direction, then $a^2 = (k_1 \sin \chi + k_2 \cos \chi)^2$ and F/σ can be minimized approximately by an ansatz

$$\sin \chi = \mathrm{ch}^{-1}\,\eta s, \qquad \cos \chi = \mathrm{th}\,\eta s \tag{5.209}$$

such that $a^2 = (k_1{}^2 + 2k_1k_2 \,\mathrm{sh}\,\eta s + k_2{}^2 \,\mathrm{sh}^2\,\eta s)/\mathrm{ch}^2\,\eta s$.
Using the formula:

$$\int ds\, \mathrm{sh}^{2\mu}\,\eta s\, \mathrm{ch}^{-2\nu}\,\eta s = \frac{1}{\eta}\,\frac{\Gamma\left(\frac{1}{2}+\mu\right)\Gamma(\nu-\mu)}{\Gamma\left(\nu+\frac{1}{2}\right)} \tag{5.210}$$

one finds the energy per unit surface:

$$F/\sigma \propto \int ds \left\{\left[\frac{\varkappa}{2} + (k_1{}^2 + 2k_1, k_2 \,\mathrm{sh}\,\eta s + k_2{}^2\,\mathrm{sh}^2\,\eta s)\,\mathrm{ch}^{-2}\,\eta s\right]\eta^2\frac{1}{\mathrm{ch}^2\,\eta s} + \frac{1}{2\xi^2}\left(\frac{1}{\mathrm{ch}^2\,\eta s}\right)\right\}$$

$$\propto \eta\left[\varkappa + \frac{4}{3}\,k_1{}^2 + \frac{2}{3}\,k_2{}^2\right] + \frac{1}{\eta\xi_{tot}^2} \equiv \eta X[\boldsymbol{k}] + \frac{1}{\eta\xi_{tot}^2} \tag{5.211}$$

which is minimal at

$$\eta = \xi_{tot}^{-1}[X(\boldsymbol{k})]^{-1/2} \tag{5.212}$$

with a value $F/\sigma \propto 2\xi_{tot}^{-1}[\chi(\boldsymbol{k})]^{1/2}$. This value is for pure bend $(\boldsymbol{k} = (1, 0, 0))$

$$F/\sigma \propto 2\xi_{tot}^{-1}\sqrt{\varkappa + \frac{4}{3}} \tag{5.213}$$

and for pure splay $(\boldsymbol{k} = (0, 1, 0))$

$$F/\sigma \propto 2\xi_{tot}^{-1}\sqrt{\varkappa + \frac{2}{3}}\,. \tag{5.214}$$

Thus the splay energy is lower than the bend energy, both being larger than the twist value. The quality of the approximation (5.209) with (5.213) and (5.214) is quite remarkable. The exact kink solutions differ only by a few per cent [*44*].
Since $\boldsymbol{l}$ solitons can be stabilized, one may investigate what satellite frequencies they give rise to. In a longitudinal magnetic field one finds from (5.170) for small vibrations $\psi = \delta$:

$$\delta^2 f \propto \left\{[\varkappa + 1 - (k_1{}^2 + 2k_1, k_2\,\mathrm{sh}\,\eta s + k_2{}^2\,\mathrm{sh}^2\,\eta s)\,\mathrm{ch}^{-2}\,\eta s]\,\delta_s{}^2 + \frac{1}{2\xi^2}\left(1 - \frac{2}{\mathrm{ch}^2\,\eta s}\right)\delta^2.\right. \tag{5.215}$$

This may be minimized approximately with a normalized wave function

$$\delta = (\mathrm{ch}\,\eta s)^{-\nu}\left[\frac{1}{\eta}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)}{\Gamma\left(\nu+\frac{1}{2}\right)}\right]^{-1/2}. \tag{5.216}$$

Inserting this into (5.215) and integrating according to (5.210) gives

$$\delta^2 F/\sigma = \frac{1}{2\nu + 1}\left\{\eta^2\nu^2\left[\varkappa + 1 - (2k_1^2\nu + 3k_2^2)\frac{1}{2\nu + 3}\right] + \frac{1}{2\xi^2}(1 - 2\nu)\right\}. \quad (5.217)$$

With the value η taken from (5.212) this amounts to a satellite frequency $\omega_l^2 = R_l \Omega_A^2$ with

$$R_l = \frac{1}{2\nu + 1}\left\{\frac{\xi^2}{\xi_{\text{tot}}^2}\nu^2 X^{-1}(\boldsymbol{k})\left[\varkappa + 1 - (2k_1^2\nu + 3k_2^2)\frac{1}{2\nu + 3}\right] + (1 - 2\nu)\right\}. \quad (5.218)$$

Of this expression, the minimum has to be taken as a function of ν. This can be done only numerically. If one neglects the current, i.e. $\xi_{\text{tot}} \approx \xi$, one finds for a pure bend soliton $(\boldsymbol{k} = (1, 0, 0))$ the minimum at $\nu \approx 0.855$ of $R_l \approx 0.117$ and for the pure splay case $(\boldsymbol{k} = (0, 1, 0))$: $\nu \approx 0.675$, $R_l \approx 0.156$. For the pure twist soliton $(\boldsymbol{k} = (0, 0, 1))$, the minimum lies at $\nu = (\sqrt{3} - 1)/2$ with $R_l = 2(\sqrt{3} - 1) \approx 0.464$ which can also be verified via an exact solution.

For transverse excitation one has to take the kinetic term from (5.194) (with $\psi = 0$):

$$\delta^2 f = (\varkappa + 1 - a^2)\, g_s^2 + \frac{1}{2\xi^2}(1 - \sin^2\chi)\, g^2. \quad (5.219)$$

Now the R_t ratio is

$$R_t = \frac{1}{2\nu + 1}\left\{\frac{\xi^2}{\xi_{\text{tot}}^2}\nu^2 X^{-1}(\boldsymbol{k})\left[\varkappa + 1 - (2k_1^2\nu + 3k_2^2)\frac{1}{2\nu + 3}\right] + 1 - \nu\right\}. \quad (5.220)$$

Neglecting the external current, i.e. $\xi_{\text{tot}} \approx \xi$, this yields satellite frequencies at minimal values

$R_t \approx 0.687$ (for $\nu \approx 0.48$)

$R_t \approx 0.684$ (for $\nu \approx 0.44$)

$R_t = 2(\sqrt{2} - 1) \approx 0.828$ (for $\nu = (\sqrt{2} - 1)/2 \approx 0.21$)

caught in pure bend, splay, and twist solitons, respectively, the latter case being again exactly soluble.

Notice that these values should not be very different from those of composite solitons under the same geometric circumstance. In the composite soliton the $\boldsymbol{l}$ vector is doing 4/5 of the bending and is, therefore, not really far from an almost pure $\boldsymbol{l}$ soliton. Neglecting the difference, for simplicity, one is tempted to compare the previously discussed satellite frequency observed in transverse excitations with the bend or splay results of a pure $\boldsymbol{l}$ soliton which is in both cases

$$R \approx (0.83)^2. \quad (5.221)$$

As was said before, the horizontal position of the cylinder would indeed favour either one of the domain walls such that the agreement with experiment may be significant.

The identification of the observed satellite resonances with the calculated trapping frequencies in a soliton-like domain wall is certainly not unproblematic. The liquid is expected to be crowded with "would-be" singularities of many sorts, all of which act as potential wells for stray spin waves. Much more work will be necessary in order to find safe ways of preparing specific kinds of singularities in the laboratory. Only then can one attach significance to the comparison of theory with experiment.

VI. Exactly Soluble Models

In order to gain more insight into the mechanism which allows the original fundamental theory to become replaced by the collective field theory, without any approximation, it is useful to study soluble models.

VI.1. The Pet Model

Consider the extremely simple case of a fundamental theory

$$H = (a^+a)^2/2 \tag{6.1}$$

where a^+ denotes the creation operator of either a boson or a fermion. In the first case the spectrum is

$$E_n = \frac{n^2}{2} \quad \text{for} \quad |n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n\,|0\rangle \tag{6.2}$$

in the second

$$\begin{aligned} E_0 &= 0 \quad \text{for} \quad |0\rangle \\ E_1 &= \frac{1}{2} \quad \text{for} \quad |1\rangle = a^+|0\rangle . \end{aligned} \tag{6.3}$$

The Lagrangian corresponding to H is

$$\mathscr{L}(t) = a^+(t) i\, \partial_t a(t) - \big(a^+(t)\, a(t)\big)^2/2 \tag{6.4}$$

and the generating functional of all Green's functions becomes

$$Z[\eta^+, \eta] = \langle 0|\, T \exp\left[i \int dt(\eta^+ a + a^+\eta)\right] |0\rangle = N \int \mathrm{D}a^+\mathrm{D}a \exp\left[i \int dt(\mathscr{L} + \eta^+ a + a^+\eta)\right]. \tag{6.5}$$

A collective field may be introduced via the formula

$$\exp\left[-i \int dt\big(a^+a(t)\big)^2/2\right] = \int \mathrm{D}\varrho(t) \exp\left[\frac{i}{2} \int dt\big(\varrho^2(t) - 2\varrho(t)\, a^+a(t)\big)\right] \tag{6.6}$$

or by adding to (6.5) in the exponent $i/2 \int dt\big(\varrho(t) - a^+a(t)\big)^2$ and integrating functionally over the ϱ field.
Hence, the generating functional Z can be rewritten as

$$Z[\eta^+, \eta] = N \int \mathrm{D}a^+\mathrm{D}a\mathrm{D}\varrho \exp\left[i \int dt \left\{a^+ i\, \partial_t a(t) - \varrho(t)\, a^+a(t) + \frac{\varrho^2}{2} + \eta^+ a + a^+\eta\right\}\right]. \tag{6.7}$$

The collective field describes the particle density: Functional derivation of the action in (6.7) displays the dependence

$$\varrho(t) = a^+(t)\, a(t). \tag{6.8}$$

Integrating out the a^+, a fields gives

$$\begin{aligned} Z[\eta^+, \eta] &= N \int \mathrm{D}\varrho \exp\left\{i\mathscr{A}[\varrho] - \int dt\, dt'\eta^+(t)\, G_\varrho(t, t')\, \eta(t')\right\} \\ \mathscr{A}[\varrho] &= \pm i \operatorname{tr} \log\,(iG_\varrho^{-1}) + \frac{\varrho^2}{2}, \end{aligned} \tag{6.9}$$

where G_ϱ denotes the propagator of the fundamental particles in a classical $\varrho(t)$ field

$$\big(i\,\partial_t - \varrho(t)\big)\,G_\varrho(t, t') = i\delta(t - t'). \tag{6.10}$$

The Green's function can be solved by introducing an auxiliary field

$$\varphi(t) = \int^{t} \varrho(t')\,dt'.$$

Then

$$G_\varrho(t, t') = e^{-i\varphi(t)} e^{i\varphi(t')}\,G_0(t - t') \tag{6.11}$$

with G_0 being the free-field propagator of the fundamental particles. At this point one has to specify the boundary condition on G_0. For this let us recall that the generating functional describes the amplitude for vacuum to vacuum transitions in the presence of the source fields η^+, η. The propagation of the free particles must take place in the same vacuum. If a_0^+, a_0 describes a free particle it follows that

$$G_0(t - t') = \langle 0|\,T\big(a_0(t)\,a_0^+(t')\big)\,|0\rangle = \Theta(t - t') \tag{6.12}$$

and therefore, due to (6.11):

$$G_\varrho(t, t') = e^{-i\varphi(t)} e^{i\varphi(t')}\,\Theta(t - t'). \tag{6.13}$$

Knowing this we can readily calculate the tr log term in (6.9). The functional derivative is certainly

$$\frac{\delta}{\delta\varrho(t)}\,\{\pm i\,\mathrm{tr}\log\,(iG_\varrho^{-1})\} = \mp G_\varrho(t, t')|_{t'=t+\varepsilon} = 0, \tag{6.14}$$

where the $t' \to t$ limit is specified such that the field $\varrho(t)$ couples, in (6.7), to

$$a^+(t)\,a(t) = \pm T\big(a(t)\,a^+(t')\big)\big|_{t'=t+\varepsilon} \triangleq \pm G_\varrho(t, t')|_{t'=t+\varepsilon}.$$

Hence, the Θ function in (6.13) makes the functional derivative vanish and the tr log becomes an irrelevant constant. The generating functional is then simply

$$Z[\eta^+, \eta] = N \int \mathrm{D}\varphi(t)\,\exp\left[\frac{i}{2}\int dt\dot\varphi(t)^2 - \int dt\,dt'\eta^+(t)\,\eta(t')\,e^{-i\varphi(t)}e^{i\varphi(t')}\Theta(t - t')\right] \tag{6.15}$$

where

$$\mathrm{D}\varrho = \mathrm{D}\varphi\,\det\big(\dot\delta(t - t')\big) = \mathrm{const}\;\mathrm{D}\varphi.$$

has been used.

Observe that it is $\varphi(t)$ which becomes a convenient dynamical plasmon variable, not $\varrho(t)$ itself.

The original theory has been transformed into a new one involving plasmons of zero mass. At this point we take advantage of equivalence between functional and quantized operator formulation by considering the plasmon action in the exponent of (6.15) directly as a quantum field theory. The first term may be associated with a Lagrangian

$$\mathscr{L}_0(t) = \frac{1}{2}\,\dot\varphi(t)^2 \tag{6.16}$$

describing free plasmons.

The Hilbert space of the corresponding Hamiltonian $H = p^2/2$ consists of plane waves which are eigenstates of the functional momentum operator $p = -i\,\partial/\partial\varphi$:

$$\{\varphi \mid p\} = \frac{1}{\sqrt{2\pi}}\, e^{ip\varphi} \tag{6.17}$$

normalized according to

$$\int_{-\infty}^{\infty} d\varphi \{p \mid \varphi\}\,\{\varphi \mid p'\} = \delta(p - p'). \tag{6.18}$$

In the operator version (2.6) then, the generating functional reads

$$Z[\eta^+, \eta] = \frac{1}{\{0 \mid 0\}}\,\{0|\, T \exp\left[-\int dt\, dt' \eta^+(t)\, \eta(t')\, e^{-i\varphi(t)} e^{i\varphi(t')} \Theta(t - t')\right] |0\} \tag{6.19}$$

where $\varphi(t)$ are free field operators.
Notice that it is the zero-functional momentum state between which Z is taken. Due to the norm (6.18) there is an infinite normalization factor which has formally been taken out.
We can now trace the generation of all Green's functions of fundamental particles by forming functional derivatives with respect to η^+, η. First

$$\langle 0|\, Ta(t)\, a^+(t')\, |0\rangle = -\left.\frac{\delta^2 Z}{\delta\eta^+(t)\,\delta\eta(t')}\right|_{\eta^+,\eta=0} = \frac{1}{\{0 \mid 0\}}\,\{0|\, e^{-i\varphi(t)} e^{i\varphi(t')}\, |0\}\; \Theta(t - t'). \tag{6.20}$$

Inserting the time translation operator

$$e^{iHt} = e^{i\frac{p^2}{2}t} \tag{6.21}$$

the matrix element (6.20) becomes

$$\frac{1}{\{0 \mid 0\}}\,\{0|\, e^{-i\frac{p^2}{2}} e^{-i\varphi(0)}\, e^{-i\frac{p^2}{2}(t-t')}\, e^{i\varphi(0)}\, e^{-i\frac{p^2}{2}t'}\, |0\} = \frac{1}{\{0 \mid 0\}}\,\{0|\, e^{-i\varphi(0)} e^{-i\frac{p^2}{2}(t-t')} e^{i\varphi(0)}\, |0\}. \tag{6.22}$$

But the state $e^{i\varphi(0)}|0\}$ is an eigenstate of p with momentum $p = 1$ such that (6.22) equals

$$\frac{1}{\{0 \mid 0\}}\,\{1 \mid 1\}\, e^{-i(t-t')/2} = e^{-i(t-t')/2} \tag{6.23}$$

and the Green's function (6.20) becomes

$$\langle 0|T\, a(t)\, a^+(t')\, |0\rangle = e^{-i(t-t')/2}\, \Theta(t - t'). \tag{6.24}$$

This coincides exactly with the result of a calculation within the fundamental fields $a^+(t)$, $a(t)$:

$$\begin{aligned}\langle 0|\, Ta(t)\, a^+(t')\, |0\rangle &= \Theta(t - t')\, \langle 0|\, e^{i(a^+a)^2 t/2} a(0)\, e^{-\frac{i}{2}(a^+a)^2(t-t')}\, a^+(0)\, e^{-i(a^+a)^2 t'/2} |0\rangle \\ &= \Theta(t - t')\, e^{-i(t-t')/2}. \end{aligned} \tag{6.25}$$

Observe that nowhere in the calculation has Fermi or Bose statistics been used. This becomes relevant for higher Green's functions. Expanding the exponent (6.19) to n'th order gives

$$Z^{[n]}[\eta^+, \eta] = \frac{1}{\{0 \mid 0\}} \frac{(-)^n}{n!} \int dt_1\, dt_1' \dots dt_n\, dt_n' \eta^+(t_1)\, \eta(t_1') \dots \eta^+(t_n)\, \eta(t_n')$$
$$\times \{0|\, T e^{-i\varphi(t_1)} e^{i\varphi(t_1')} \dots e^{-i\varphi(t_n)} e^{i\varphi(t_n')} |0\}\, \Theta(t_1 - t_1') \dots \Theta(t_n - t_n'). \tag{6.26}$$

The Green's function

$$\langle 0|\, T a(t_1) \cdot \ldots \cdot a(t_n)\, a^+(t_n') \cdot \ldots \cdot a^+(t_1')\, |0\rangle \tag{6.27}$$

is obtained by forming the derivative $(-i)^{2n}\, \delta^{2n}\mathscr{Z}[\eta^+\eta]/\delta\eta^+(t_1) \cdot \ldots \cdot \delta\eta^+(t_n)\, \delta\eta(t_n') \cdot \ldots \cdot \delta\eta(t)$. There are $(n!)^2$ contributions due to the product rule of differentiation, $n!$ of them being identical thereby cancelling the factor $1/n!$ in (6.26). The others correspond, from the point of view of combinatorics, to all Wick contractions of (6.27), each contraction being associated with a factor $e^{-i\varphi(t)} e^{i\varphi(t')}\, \Theta(t - t')$. In addition, the Grassmann nature of source fields η causes a minus sign to appear if the contractions deviating by an odd permutation from the natural order $11', 22', 33', \dots$ For example

$$\langle 0|\, T a(t_1)\, a(t_2)\, a^+(t_2')\, a^+(t_1')\, |0\rangle$$
$$= \langle 0|\, T a(t_1)\, a(t_2)\, a^+(t_2')\, a^+(t_1')\, |0\rangle \pm \langle 0|\, T a(t_1)\, a(t_2)\, a^+(t_2')\, a^+(t_1')\, |0\rangle$$
$$= \frac{1}{\{0 \mid 0\}} \{0|\, T e^{-i\varphi(t_1)} e^{-i\varphi(t_2)} e^{i\varphi(t_2')} e^{i\varphi(t_1')}\, |0\} \left(\Theta(t_1 - t_1')\, \Theta(t_2 - t_2') \pm \Theta(t_1 - t_2')\, \Theta(t_2 - t_1')\right) \tag{6.28}$$

where the upper sign holds for bosons, the lower for fermions. The lower sign enforces the Pauli exclusion principle: If $t_1 > t_2 > t_2' > t_1'$ the two contributions cancel reflecting the fact that no two fermions $a^+(t_2')\, a^+(t_1')$ can be created successively on the particle vacuum. For bosons one may insert again the time translation operator (6.21) and complete sets of states $\int dp |p\}\{p| = 1$ with the result:

$$\frac{1}{\{0 \mid 0\}} \int dp\, dp' \{0|\, e^{-i\varphi(0)} e^{-i\frac{p^2}{2}(t_1 - t_2)} e^{-i\varphi(0)} e^{-i\frac{p^2}{2}(t_2 - t_2')} e^{i\varphi(0)} e^{-i\frac{p^2}{2}(t_2' - t_1')} e^{i\varphi(0)}\, |0\}$$
$$= e^{-i(t_1 - t_2)/2} e^{-i2(t_2 - t_2')} e^{-i(t_2' - t_1')/2}. \tag{6.29}$$

where $\{0|\, e^{-i\varphi(0)} |p\} = \delta(1 - p)$, $\{p|\, e^{-i\varphi(0)} |p'\} = \delta(p + 1 - p')$ has been used. This again agrees with an operator calculation like (6.25).

We now understand how the collective quantum field theory works in this model. Its Hilbert space is very large consisting of states of *all* functional momenta $|p\rangle$. When it comes to calculating the Green's functions of the fundamental fields, however, only a small portion of this Hilbert space is used. A fermion can make plasmon transitions back and forth between ground state $|0\}$ and the momentum one state $|1\}$ only, due to the anticommutativity of the fermion source fields η^+, η. Bosons, on the other hand, can connect all states of integer momentum $|n\}$. No other states are ever reached. The collective basis is overcomplete as far as the description of the underlying system is concerned. Strong selection rules, $p \to p \pm 1$, together with the source statistics make sure that only a small subspace becomes involved in the dynamics of the fundamental system. That such a projection is compatible with unitarity is ensured by the cnnseriation law $a^+a = \text{const}$. In higher dimensions, there have to be infinitely many conservation laws (one for every space point).

Actually, in the boson case, the overcompleteness can be removed by defining the collective Lagrangian in (6.15) on a cyclic variable, i.e. one takes (6.16) on $\varphi \in [0, 2\pi)$ and

extends it periodically. The path integral (6.15) is then integrated accordingly. In this case the Hilbert space would be grated containing only integer momenta $p = 0, \pm 1, \pm 2, \ldots$ coinciding with the multi-boson states.
The following observations may be helpful in understanding the structure of the collective theory: It may sometimes be convenient to build all Green's functions not on the vacuum state $|0\rangle$ but on some other reference state $|R\rangle$ for which we may choose the excited state $|n\rangle$. In the operator language this amounts to a generating functional

$$ {}^{n}Z[\eta^+, \eta] = \langle n| \, T \exp\left[i \int dt(\eta^+ a + a^+\eta)\right] |n\rangle. \tag{6.30}$$

This would reflect itself in the boundary condition of G_0 for bosons

$$\begin{aligned} {}^{n}G_0(t - t') &= \langle n| \, T\big(a_0(t)\, a_0^+(t')\big) \, |n\rangle \\ &= (n + 1)\, \Theta(t - t') + n\Theta(t' - t). \end{aligned} \tag{6.31}$$

For fermions, only $n = 1$ would be an alternative with

$$ {}^{1}G_0(t - t') = \langle 1| \, T\big(a_0(t)\, a_0^+(t')\big) \, |1\rangle = -\Theta(t' - t). \tag{6.32}$$

As a consequence of (6.31) or (6.32), formula (6.14) would become

$$\frac{\delta}{\delta\varrho(t)} \{\pm i \operatorname{tr} \log (iG_\varrho^{-1})\} = -\begin{Bmatrix} n \\ 1 \end{Bmatrix}. \tag{6.33}$$

Integrating this functionally gives

$$\pm i \operatorname{tr} \log (iG_\varrho^{-1}) = -\begin{Bmatrix} n \\ 1 \end{Bmatrix} \int_{-\infty}^{\infty} \varrho(t)\, dt \tag{6.34}$$

such that the functional form of (6.30) reads, according to (6.9):

$$\begin{aligned} {}^{\left\{ n \atop 1 \right\}}Z[\eta^+, \eta] = &\int \mathrm{D}\varphi \exp\left[i \int dt \left(\frac{\dot\varphi^2}{2} - \begin{Bmatrix} n \\ 1 \end{Bmatrix} \dot\varphi\right) dt\right] \\ &\times \exp\left[-\int dt\, dt' \eta^+(t)\, \eta(t')\, e^{-i\varphi(t)} e^{i\varphi(t')} \left[\begin{Bmatrix} n+1 \\ 0 \end{Bmatrix} \Theta(t - t') \right.\right. \\ &\left.\left. + \begin{Bmatrix} n \\ -1 \end{Bmatrix} \Theta(t' - t)\right]\right]. \end{aligned} \tag{6.35}$$

Now the collective Lagrangian is

$$\begin{aligned} \mathscr{L}(t) &= \frac{\dot\varphi^2}{2} - \begin{Bmatrix} n \\ 1 \end{Bmatrix} \dot\varphi \\ &= \frac{1}{2}\left(\dot\varphi - \begin{Bmatrix} n \\ 1 \end{Bmatrix}\right)^2 - \frac{1}{2}\begin{Bmatrix} n^2 \\ 1 \end{Bmatrix} \end{aligned} \tag{6.36}$$

with the functional canonical momentum

$$p = \dot\varphi - \begin{Bmatrix} n \\ 1 \end{Bmatrix}$$

the Hamiltonian takes the form

$$H = \left(\dot{\varphi} - \begin{Bmatrix} n \\ 1 \end{Bmatrix}\right) \dot{\varphi} - \mathscr{L}$$
$$= \frac{\dot{\varphi}^2}{2} = \frac{\left(p + \begin{Bmatrix} n \\ 1 \end{Bmatrix}\right)^2}{2}. \tag{6.37}$$

Thus the spectrum is the same as before but the momenta are shifted by n (or 1) units accounting for the fundamental particles contained in the reference state $|R\rangle$ of (6.30). In the collective quantum field theory, this reference state corresponds now to functional momentum zero:

$$\begin{Bmatrix} n \\ 1 \end{Bmatrix} Z[\eta^+, \eta] = \frac{1}{\{0 \mid 0\}} \{0| \, T \exp\left[-\int dt\, dt' \eta^+(t)\, \eta(t')\, e^{-i\varphi(t)} e^{i\varphi(t')}\right.$$
$$\left. \times \left[\begin{Bmatrix} n+1 \\ 0 \end{Bmatrix} \Theta(t-t') + \begin{Bmatrix} n \\ -1 \end{Bmatrix} \Theta(t'-t)\right]\right] |0\}. \tag{6.38}$$

In fact, the one-particle Green's function becomes

$$\begin{Bmatrix} n \\ 1 \end{Bmatrix} G(t, t') = -\frac{\delta^2}{\delta\eta^+(t)\, \delta\eta(t')} \begin{Bmatrix} n \\ 1 \end{Bmatrix} Z[\eta^+, \eta]$$
$$= \frac{1}{\{0 \mid 0\}} \{0| \, T e^{-i\varphi(t)} e^{i\varphi(t')} |0\} \left[\begin{Bmatrix} n+1 \\ 0 \end{Bmatrix} \Theta(t-t') + \begin{Bmatrix} n \\ -1 \end{Bmatrix} \Theta(t'-t)\right]. \tag{6.39}$$

Inserting the time translation operator corresponding to (6.37) this yields for $t > t'$

$$\begin{Bmatrix} n \\ 1 \end{Bmatrix} G(t, t') = \exp\left[-i \begin{Bmatrix} n + 1/2 \\ 3/2 \end{Bmatrix} (t-t')\right] \begin{Bmatrix} n+1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} (n+1) \exp[-i(n+1/2)(t-t')] \\ 0 \end{Bmatrix} \tag{6.40}$$

and for $t < t'$

$$\begin{Bmatrix} n \\ 1 \end{Bmatrix} G(t, t') = \exp\left[-i \begin{Bmatrix} n - 1/2 \\ 1/2 \end{Bmatrix} (t-t')\right] \begin{Bmatrix} n \\ -1 \end{Bmatrix} = \begin{Bmatrix} n \exp[-i(n-1/2)(t-t')] \\ -e^{-i(t-t')/2} \end{Bmatrix} \tag{6.41}$$

in agreement with a direct operator calculation.

The appearance of the additional derivative term $\dot{\varphi}$ in the Lagrangian (6.36) can be understood in an alternative fashion. The reference state $|n\rangle$ of ${}^nZ$ in (6.30) can be generated in the original generating functional by applying successively derivatives $-\delta^2/\delta\eta^+(t)\, \delta\eta(t')$, letting $t' \to -\infty$, $t \to \infty$ and absorbing an infinite phase $\exp[-i\Delta E \times (2\infty)]$ into the normalization constant where ΔE is the energy difference between $|n\rangle$ and $|0\rangle$:

$${}^nZ[\eta^+, \eta]|_{\eta^+=\eta=0} \propto \frac{\delta^n}{\left(\delta\eta^+(+\infty)\right)^n} \frac{\delta^n}{\left(\delta\eta(-\infty)\right)^n} {}^0Z[\eta^+, \eta]|_{\eta^+=\eta=0}.$$

Each such pair of derivatives brings down a Green's function

$$e^{-i\varphi(t)} e^{i\varphi(t')}\, \Theta(t-t') = \exp\left[-i \int_{t'}^{t} \dot{\varphi}(t'')\, dt''\right] \Theta(t-t').$$

As $t' \to -\infty$, $t \to \infty$ this becomes for n factors

$$\exp\left[-in \int_{-\infty}^{\infty} \dot{\varphi}(t)\, dt\right]$$

in agreement with the derivative term in (6.35).
While the functional Schrödinger picture is useful in understanding what happens in the Hilbert space of the collective field theory, it is quite awkward to apply to more than one dimension, in particular to the relativistic situation where the time does not play a special role. A more direct and easily generalizable method for the evaluation of fermion propagators in the collective theory consists in the following procedure: One brings the products of exponentials in (6.26) to normal order by using Wick's, contraction formula in the functional form (2.31). Let the "charges" of the incoming and outgoing fermions be $q_i = +1$ and $q_i = -1$, respectively.
Then the matrix element to be calculated in (6.26) are

$$\{0|\, T \exp\left[i \sum_i q_i \varphi(t_i)\right] |0\} = \{0|\, T \exp\left[i \int dt \varphi(t) \sum_i q_i \delta(t - t_i)\right] |0\}$$

where we have numbered the times as $t_1, t_2, t_3, t_4, \ldots$ rather than $t_1, t_1', t_2, t_2', \ldots$ etc.
Now from (2.31) one has

$$\begin{aligned} \{0|\, T e^{i\sum q_i \varphi(t_i)} |0\} &= \exp\left[-\frac{1}{2} \int dt\, dt' \sum_i q_i \delta(t - t_i)\, \overline{\varphi(t)\, \varphi(t')} \sum_j q_j \delta(t - t_j)\right] \\ &\quad \times \{0|\, T : \exp\left[i \int dt \varphi(t) \sum_i q_i \delta(t' - t_i)\right] : |0\} \\ &= \exp\left[-\frac{1}{2} \sum_{ij} q_i q_j \overline{\varphi(t_i)\, \varphi(t_j)}\right]. \end{aligned} \qquad (6.42)$$

The propagator of φ is well defined only after introducing a small regulator mass $\varkappa$:

$$\begin{aligned} \overline{\varphi(t)\, \varphi(t')} &= \int \frac{dE}{2\pi} \frac{i}{E^2 - \varkappa^2 + i\varepsilon} e^{-iE(t-t')} \\ &= \frac{1}{2\varkappa} e^{-i\varkappa|t-t'|} = \frac{1}{2\varkappa} - \frac{i}{2} |t - t'| + O(\varkappa). \end{aligned}$$

Then the right-hand side of (6.42) becomes for small $\varkappa$

$$\exp\left[-\left(\sum_i q_i\right)^2 \frac{1}{4\varkappa}\right] \exp\left[\frac{i}{4} \sum_{ij} q_i q_j |t_i - t_j|\right].$$

As $\varkappa \to 0$ this expression vanishes unless the sum of all charges is zero: $\sum_i q_i = 0$.
Thus one finds the general result for (6.26):

$$\{0|\, T \exp\left[i \sum_{q_i} q_i \varphi(t_i)\right] |0\} = \delta_{\sum_i q_i, 0} \exp\left[\frac{i}{2} \sum_{i>j} q_i q_j\, |t_i - t_j|\right]. \qquad (6.43)$$

In particular, the two point function (6.20) agrees with the Schrödinger calculation (6.24).

VI.2. The Generalized BCS Model in a Degenerate Shell

A less trivial but completely transparent example is provided by the BCS degenerate-shell model used in nuclear physics to describe the energy levels of some nuclei in which pairing forces are dominant (for example Sn and Pb isotopes [31]). For the understanding of the structure of the collective theory it will be useful to consider at first both bosons and fermions as well as a more general interaction and impose the restriction to fermions and to the particular BCS pairing force at a later stage. This more general Hamiltonian reads

$$H = H_0 + H_{\text{int}} = \varepsilon \sum_{i=1}^{\Omega} (a_i^+ a_i + b_i^+ b_i) - \frac{V}{2} \left\{ \sum_{i,j} a_i^+ b_i^+ b_j a_j \right\}$$
$$\pm \frac{V}{4} g \left[\sum_i (a_i^+ a_i + b_i^+ b_i) \pm \Omega \right]^2. \tag{6.44}$$

where $g = 0$ reduces to the actual BCS model in the case of fermions. The model can be completely solved by introducing quasi-spin operators

$$L^+ = \sum_{i=1}^{\Omega} a_i^+ b_i^+ \quad L^- = \sum_{i=1}^{\Omega} b_i a_i = (L^+)^+$$

$$L_3 = \frac{1}{2} \left\{ \sum_i (a_i^+ a_i + b_i^+ b_i) \pm \Omega \right\} = \frac{1}{2} \sum_i a_i^+ a_i \pm b_i b_i^+ = \frac{1}{2} \{N \pm \Omega\} \tag{6.45}$$

where N counts the total number of particles. These operators generate the group $SU(1, 1)$ or $SU(2)$ for bosons or fermions, respectively:

$$[L_3, L^\pm] = \pm L^\pm$$
$$[L^+, L^-] = \mp 2 L_3 \tag{6.46}$$

using

$$L^+ L^- = \boldsymbol{L}^2 \mp L_3 \pm L_3^2$$

we can write

$$H = 2\varepsilon L_3 \mp \varepsilon\Omega - V(\boldsymbol{L}^2 \pm L_3^2 \mp g L_3^2)$$
$$= 2\varepsilon L_3 - V\left(\boldsymbol{L}^2 \pm (1 - g) L_3^2\right) \mp \varepsilon\Omega. \tag{6.47}$$

Notice that the interaction term is $SU(1, 1)$ or $SU(2)$ symmetric for $g = 1$. The irreducible representations of the algebra (6.45) consist of states

$$|n[\Omega, \nu]\rangle = N_n (L^+)^n \, |0[\Omega, \nu]\rangle \tag{6.48}$$

where the seniority label ν denotes the presence of ν unpaired particles a_i^+ or b_j^+, i.e. those which are orthogonal to the configurations $(L^+)^n \, |0\rangle$. For $\nu = 0$ the spectrum of L_3 in an irreducible representation is

$$\pm\frac{\Omega}{2}, \quad \pm\frac{\Omega}{2} + 1, \quad \pm\frac{\Omega}{2}, + 2, \ldots \tag{6.49}$$

This continues ad infinitum for bosons due to the non-compact topology of $SU(1, 1)$ while it terminates for fermions at $\Omega/2$ corresponding to a finite spin $\Omega/2$. The invariant Casimir operator

$$\boldsymbol{L}^2 \equiv L_1^2 + L_2^2 \mp L_3^2 \tag{6.50}$$

characterizing the representation has the eigenvalue $\Omega/2(1 \mp \Omega/2)$ showing in the fermion case again the quasi-spin $\Omega/2$. If ν unpaired particles are added to the vacuum, the eigenvalues start at $\pm(\Omega + \nu)/2$. Thus the quasi-spin is reduced to $(\Omega - \nu)/2$. If $\nu = \Omega$

unpaired fermions are present, the state is quasi-spin symmetric, for example:

$$|0[\Omega, \Omega]\rangle = b_1^+ b_2^+ \cdot \ldots \cdot b_\Omega^+ |0\rangle . \qquad (6.51)$$

Due to the many choices of unpaired particles with the same total number the levels show considerable degeneracies and one actually needs another label for their distinction. This has been dropped for brevity.

On the states $|n[\Omega\nu]\rangle$ the energies are from (6.47) and using $N = 2n + \nu$:

$$E = \varepsilon(N \pm \Omega) - V\left[\frac{\Omega \pm \nu}{2}\left(1 \mp \frac{\Omega \pm \nu}{2}\right) \pm \frac{(1-g)}{4}(N \pm \Omega)^2\right] \mp \varepsilon\Omega . \qquad (6.52)$$

A typical level scheme for fermions of $\Omega = 8$ with $\varepsilon = 0$ is displayed on Fig. XVII. If the single particle energy ε is non-vanishing, the scheme is distorted via a linear dependece on L_3 lifting the right- and depressing the left-hand side.

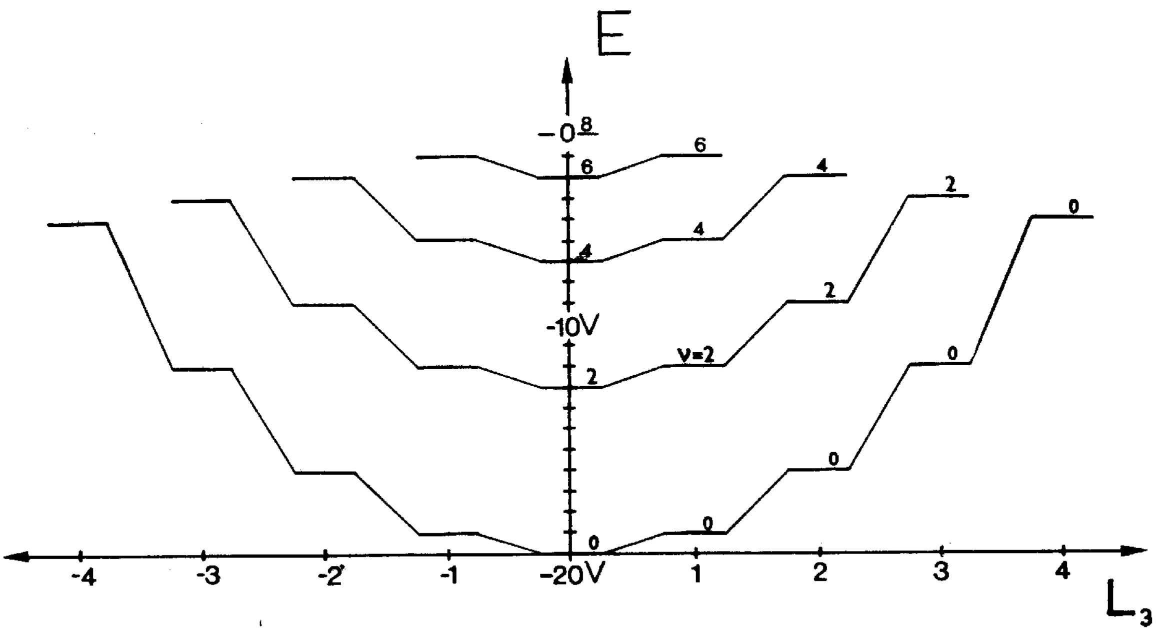

Fig. XVII. The figure shows the level scheme of the BCS model in a single degenerate shell of multiplicity $\Omega = 8$. The abscissa denotes the third component of quasi-spin. The index ν at each level stands for the number of unpaired particle ("seniority")

For an attractive potential and given total particle number N, the state with $\nu = 0$ is the ground state with the higher seniorities having higher energies:

$$E_{N\Omega\nu} - E_{N\Omega 0} = V\left(\Omega \mp 1 \pm \frac{\nu}{2}\right)\nu . \qquad (6.53)$$

The Lagrangian of the model is from (6.44)

$$\mathscr{L}(t) = \sum_i \left(a_i^+(t)(i\partial_t - \varepsilon)\, a_i(t) + b_i^+(t)\,(i\partial_t - \varepsilon)\, b_i(t)\right)$$
$$+ \frac{V}{2}\left\{\sum_{i,j} a_i^+ b_i^+ b_j a_j\right\} \mp \frac{V}{4}\, g \left\{\sum_i \left(a_i^+ a_i \pm b_i b_i^+\right)\right\}^3 \qquad (6.54)$$

and the generating functional:

$$Z[\eta^+, \eta, \lambda^+, \lambda] = \int \prod_i \mathrm{D}a_i^+\mathrm{D}a_i\mathrm{D}b_i^+\mathrm{D}b_i \times \exp\left[i\int dt\left\{\mathscr{L} + \sum_i \eta_i^+a_i + a_i^+\eta_i + \lambda_i^+b_i + b_i^+\lambda_i\right\}\right]. \tag{6.55}$$

The quartic terms in the exponential can be removed by introducing a complex field $S = S_1 + iS_2$ and a real field S_3', adding

$$-V\left\{\left|S(t) - \sum_i a_i^+b_i^+\right|^2 \mp g\left[S_3'(t) - \frac{1}{2}\sum_i (a_i^+a_i \pm b_ib_i^+)\right]^2\right\} \tag{6.56}$$

and integrating Z functionally over $\mathbf{D}S = \mathrm{D}S_1\mathrm{D}S_2\mathrm{D}S_3$.
The addition of (6.56) changes $\mathscr{L}$ to:

$$\mathscr{L}(t) = \sum_i \{a_i^+(i\partial_t - \varepsilon \mp gVS_3')\,a_i \mp b_i(i\partial_t + \varepsilon \pm gVS_3')\,b_i^+\} + VS^+\sum_i a_i^+b_i^+ + \sum_i b_ia_iVS - V(|S|^2 \mp gS_3'^2) \pm \varepsilon\Omega. \tag{6.57}$$

By using the more convenient two-spinor notation for fundamental fields and sources

$$f_i \equiv \begin{pmatrix} a_i \\ b_i^+ \end{pmatrix}; \quad f_i^+ \equiv (a_i^+, b_i)$$
$$j_i \equiv \begin{pmatrix} \eta_i \\ \lambda_i^+ \end{pmatrix}; \quad j_i^+ \equiv (\eta_i^+, \lambda_i) \tag{6.58}$$

the generating functional can be rewritten as

$$Z[j^+j] = \int \prod_i \mathrm{D}f_i^+\mathrm{D}f_i\mathbf{D}S\ \exp\left[i\int dt\left\{\mathscr{L} + \sum_i (j_i^+f_i + f_i^+j_i)\right\}\right] \tag{6.59}$$

with

$$\mathscr{L} = \sum_{i=1}^{\Omega} f_i^+(t)\begin{pmatrix} i\partial_t - \varepsilon \mp gVS_3' & VS^+ \\ VS & \mp(i\partial_t + \varepsilon \pm gVS_3') \end{pmatrix} f_i(t) - V(|S|^2 \mp gS_3'^2) \pm \varepsilon\Omega. \tag{6.60}$$

Now the fundamental fields f_i^+, f_i can be integrated out yielding the collective action [32]

$$\mathscr{A}[\mathbf{S}] = \pm i\,\mathrm{tr}\log(iG_{\mathbf{S}}^{-1}) - V(S_1^2 + S_2^2 \mp gS_3'^2) \pm \varepsilon\Omega \tag{6.61}$$

where $G_{\mathbf{S}}$ is the matrix collecting the Green's functions of the particles in the external field $\mathbf{S}$

$$G_{\mathbf{S}}(t, t')_{ij} = \begin{pmatrix} \overline{a_i(t)\,a_j^+(t_i')} & \overline{a_i(t)\,b_j(t_i')} \\ \overline{b_i^+(t)\,a_j^+(t')} & \overline{b_i^+(t)\,b_j(t')} \end{pmatrix}. \tag{6.62}$$

Its equation of motion, multiplied by $\begin{Bmatrix} \sigma^3 \\ 1 \end{Bmatrix}$

$$\begin{pmatrix} i\partial_t - \varepsilon \mp gVS_3' & VS^+ \\ \mp VS & i\partial_t + \varepsilon \pm gVS_3' \end{pmatrix} G_{\mathbf{S}}(t, t') = i\begin{Bmatrix} \sigma^3 \\ 1 \end{Bmatrix}\delta(t - t') \tag{6.63}$$

may be solved by an Ansatz

$$G_S(t, t') = U^+(t)\, G_0(t, t')\, U(t') \tag{6.64}$$

where G_0 is a solution of (6.63) for $S = 0$, $S_3' = 0$, $\varepsilon = 0$. Before we proceed it is useful to absorb ε and g into S_3' by defining the more symmetric variable

$$\mp S_3 = \mp g S_3' - \frac{\varepsilon}{V}. \tag{6.65}$$

Then equ. (6.63) reads

$$\left(i\partial_t + V \begin{Bmatrix} -iS_2 \\ S_1 \end{Bmatrix} \sigma^1 + V \begin{Bmatrix} iS_1 \\ -S_2 \end{Bmatrix} \sigma^2 \mp V S_3 \sigma^3\right) U^+(t)\, G_0 U(t') = i \begin{Bmatrix} \sigma^3 \\ 1 \end{Bmatrix} \delta(t - t'). \tag{6.66}$$

It is solved if U satisfies $U^+ \begin{Bmatrix} \sigma^3 \\ 1 \end{Bmatrix} U = \begin{Bmatrix} \sigma^3 \\ 1 \end{Bmatrix}$ and the differential equation

$$i\dot{U}^+(t)\, U^+(t)^{-1} = -V \left(\begin{Bmatrix} -iS_2 \\ S_1 \end{Bmatrix} \sigma^1 + \begin{Bmatrix} iS_1 \\ S_2 \end{Bmatrix} \sigma^2 \mp V S_3 \sigma^3 \right). \tag{6.67}$$

The condition $U^+ \begin{Bmatrix} \sigma^3 \\ 1 \end{Bmatrix} U = \begin{Bmatrix} \sigma^3 \\ 1 \end{Bmatrix}$ can be met by parametrizing U in terms of Euler angles

$$U(t) = e^{i\alpha \frac{\sigma_3}{2}}\, e^{\left\{ \begin{smallmatrix} -\tilde{\beta} \\ i\beta \end{smallmatrix} \right\} \frac{\sigma_2}{2}}\, e^{i\gamma \frac{\sigma_3}{2}}. \tag{6.68}$$

As should be expected from the above discussion of the operators L_i, the matrices U form a subgroup of the Lorentz group $SL(2, C)$. In the Bose case this subgroup is $SU(1, 1)$ in the Fermi case $SU(2)$. The equ. (6.66) implies the differential equations for the Euler angles

$$\begin{aligned} \tilde{\omega}_1 &\equiv \dot{\tilde{\beta}} \sin\gamma + \dot{\alpha}\, \mathrm{sh}\, \tilde{\beta} \cos\gamma = 2VS_1 \\ \tilde{\omega}_2 &\equiv \dot{\tilde{\beta}} \cos\gamma - \dot{\alpha}\, \mathrm{sh}\, \tilde{\beta} \sin\gamma = 2VS_2 \\ \tilde{\omega}_3 &\equiv \dot{\alpha}\, \mathrm{ch}\, \tilde{\beta} + \dot{\gamma} = 2VS_3 \end{aligned} \tag{6.69}$$

and

$$\begin{aligned} \omega_1 &\equiv -\dot{\beta} \sin\gamma + \dot{\alpha} \sin\beta \cos\gamma = -2VS_1 \\ \omega_2 &\equiv \dot{\beta} \cos\gamma + \dot{\alpha} \sin\beta \sin\gamma = -2VS_2 \\ \omega_3 &\equiv \dot{\alpha} \cos\beta + \dot{\gamma} = -2VS_3. \end{aligned} \tag{6.70}$$

The left-hand sides of (6.70) are recognized as the standard Euler equations for the angular velocities ω_i in a body-fixed reference frame.
The upper equations follow from the lower by replacing $\beta \to -i\tilde{\beta}$, $S_1 \to -iS_2$, $S_2 \to iS_1$, $S_3 \to -S_3$. Since this transition can be done at any later stage it is convenient to avoid the clumsy distinction of different cases and focus attention to the Fermi case only.
In the Fermi case, the matrix $U(t)$ is unitary and coincides with the well-known representation matrices $\mathrm{D}^{1/2}_{m'm}(\alpha\beta\gamma)$ of the rotation group.[25] The equs. (6.70) now correspond to the kinematic problem of finding the positions of a rigid body given the angular velocities $\omega_i = -2VS_i$.

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[25]) For the conventions see: A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press.

They can be solved formally as

$$U(t) = T \exp\left[-i \int_{-\infty}^{t} 2V\mathbf{S}\sigma \, dt'\right]. \tag{6.71}$$

Given this $U(t)$ we can now proceed to evaluate the tr log term in (6.61). By differentiation with respect to S we find:

$$\frac{\delta}{\delta S_k(t)} [-i \operatorname{tr} \log (iG_S^{-1})] = V \sum_i \operatorname{tr} \left(\sigma^k G_S{}^{ii}(t, t')\right)\big|_{t'=t+\varepsilon}. \tag{6.72}$$

The right-hand side can be calculated in terms of Euler angles by inserting (6.68). In addition one has to choose the reference state for $Z[\eta^+, \eta]$ by specifying the boundary condition on G_0. Since G_0 represents the same matrix of Green's functions as (6.62), except with free oscillators $a_0{}^+$, $b_0{}^+$ of zero energy, this is easily done. Let us choose as reference state $|R\rangle$ one of the quasi-spin symmetric states of seniority $\nu = \Omega$, say (6.51). Then G_0 has to have the form

$$G_0{}^{ij}(t, t') = \begin{pmatrix} \Theta(t-t') & 0 \\ 0 & \Theta(t-t') \end{pmatrix} \delta^{ij}. \tag{6.73}$$

As a consequence $G_0{}^{ij}(t, t')|_{t'=t+\varepsilon} = 0$ such that also (6.72) vanishes and $-i \operatorname{tr} \log (iG_S{}^{-1})$ becomes an irrelevant constant.
Hence the generating functional in the quasi-spin symmetric reference state (6.51) is

$${}^R Z[j^+ j] = \int \mathrm{D}\mathbf{S} \exp\left[i \int dt V \mathbf{S}(t)^2 - \int dt dt' \Theta(t-t') \sum_i j_i{}^+(t) \, U^+(t) \, U(t') \, j_i(t')\right]. \tag{6.74}$$

As in the case of the trivial model it is now convenient to change variables and integrate directly over the Euler angles $\alpha\beta\gamma$ rather than $S_1 S_2 S_3$. Using the derivatives

$$\begin{aligned} -\frac{1}{2V} \frac{\delta S_i(t)}{\delta q_j(t')} &\equiv A(t)_{ij} \delta(t-t') + B(t)_{ij} \dot{\delta}(t-t') \\ &= \begin{pmatrix} 0 & \dot\alpha \cos\beta \cos\gamma & -\dot\beta \cos\gamma - \dot\alpha \sin\beta \sin\gamma \\ 0 & \dot\alpha \cos\beta \sin\gamma & -\dot\beta \sin\gamma + \dot\alpha \sin\beta \cos\gamma \\ 0 & -\dot\alpha \sin\beta & 0 \end{pmatrix}_{ij} \\ &\quad \times \delta(t-t') + \begin{pmatrix} \sin\beta\cos\gamma & -\sin\gamma & 0 \\ \sin\beta \sin\gamma & \cos\gamma & 0 \\ \cos\beta & 0 & 1 \end{pmatrix}_{ij} \dot{\delta}(t-t') \end{aligned} \tag{6.75}$$

one calculates the functional determinant as the determinant of the second matrix B. This can be seen most easily by multiplication with the constant (functional) matrix $\int dt' \Theta(t' - t'')$ which diagonalizes the $\dot{\delta}(t-t')$ and brings the δ term completely to the right of the (functional) diagonal: $\delta\Theta = \Theta$. The determinant of such a matrix equals the determinant of the diagonal part only. Thus, up to an irrelevant factor, one has

$$\mathrm{D}\mathbf{S} = \text{const } \mathrm{D}\alpha \mathrm{D}\beta \mathrm{D}\gamma \sin\beta \tag{6.76}$$

corresponding to the standard measure of the rotation group. Inserting now (6.70) into (6.74) we find

$$Z[j^+, j] = \int D\alpha D \cos\beta D\gamma \exp\left[i\int dt\left\{-\frac{1}{4V}\left(\omega_1^2 + \omega_2^2 + \frac{1}{g}(\omega_3 - 2\varepsilon)^2\right) - \varepsilon\Omega\right]\right.$$
$$\times \exp\left[i\int dt\,dt'\Theta(t-t')\sum_i j_i^+(t)\,U^+(t)\,U(t')\,j_i(t')\right]. \tag{6.77}$$

The collective Lagrangian becomes:

$$\mathscr{L} = -\frac{1}{4V}\left(\omega_1^2 + \omega_2^2 + \frac{1}{g}(\omega_3 - 2\varepsilon)^2\right) - \varepsilon\Omega$$
$$= -\frac{1}{4V}\left\{(\dot\beta^2 + \dot\alpha^2\sin^2\beta) + \frac{1}{g}(\dot\gamma + \dot\alpha\cos\beta)^2\right\} + \frac{\varepsilon}{Vg}(\dot\gamma + \dot\alpha\cos\beta) - \frac{\varepsilon^2}{Vg} - \varepsilon\Omega. \tag{6.78}$$

This has the standard form

$$\mathscr{L} = \frac{1}{2}\dot q^i g_{ij}(q)\,\dot q^j + a_i(q)\,\dot q^i - v(q) \tag{6.79}$$

with the metric

$$g_{ij}(q) = -\frac{1}{2V}\begin{pmatrix} \sin^2\beta + \frac{1}{g}\cos^2\beta & 0 & \frac{1}{g}\cos\beta \\ 0 & 1 & 0 \\ \frac{1}{g}\cos\beta & 0 & \frac{1}{g} \end{pmatrix} \tag{6.80}$$

$$g^{ij}(q) \equiv \left(g^{-1}(q)\right)^{ij} = -2V\frac{g}{\sin^2\beta}\begin{pmatrix} \frac{1}{g} & 0 & -\frac{1}{g}\cos\beta \\ 0 & 1 & 0 \\ -\frac{1}{g}\cos\beta & 0 & \sin^2\beta + \frac{1}{g}\cos^2\beta \end{pmatrix} \tag{6.81}$$

of determinant

$$g \equiv \det(g_{ij}) = -\frac{1}{8V^3}\frac{1}{g}\sin^2\beta$$

in the space labelled again by $q^i \equiv (\alpha, \beta, \gamma)$.
The Hamiltonian in such a curved space is given by [33]

$$H = H_1 + H_2 + H_3 + v(q) + \frac{1}{2}a^i a_i(q) \tag{6.82}$$

with

$$H_1 = -\frac{1}{2}g^{-1/2}\frac{\partial}{\partial q^i}\left(g^{1/2}g^{ij}\frac{\partial}{\partial q^j}\right) \tag{6.83}$$

$$H_2 = \frac{i}{2} g^{-1/2} \left[\frac{\partial}{\partial q^i} g^{1/2} g^{ij} a_j(q) \right] \tag{6.83}$$

$$H_3 = i a_i(q)\, g^{ij} \frac{\partial}{\partial q^j}. \tag{6.84}$$

Here we find H_1 as the standard asymmetric-top Hamiltonian,

$$H_1 = V\left(\frac{\partial^2}{\partial \beta^2} + \cot\beta \frac{\partial}{\partial \beta} + (g + \cot\beta)\frac{\partial^2}{\partial \gamma^2} + \frac{1}{\sin^2\beta}\frac{\partial^2}{\partial \alpha^2} - 2\frac{\cos\beta}{\sin^2\beta}\frac{\partial^2}{\partial\alpha\,\partial\gamma}\right). \tag{6.85}$$

Since

$$a_i = \frac{\varepsilon}{Vg} (\cos\beta, 0, 1) \tag{6.86}$$

the second part, H_2, vanishes and the third part becomes

$$H_3 = -2\varepsilon i \partial_\gamma. \tag{6.87}$$

The resulting Hamiltonian is exactly the Schrödinger version of the quasi-spin form (6.47) with

$$L^\pm = e^{\pm i\gamma}\left[\pm \partial_\beta + \cot\beta\, i\, \partial_\alpha - i \frac{1}{\sin\beta} \partial_\gamma\right]$$
$$L_3 = -i\partial_\gamma. \tag{6.88}$$

The eigenfunctions of H coincide with the rotation matrices

$$D^j_{m'm}(\alpha, \beta, \gamma) = e^{i(\alpha m' + \gamma m)} d^j_{m'm}(\beta). \tag{6.89}$$

The energy eigenvalues of H_1 are well-known

$$E^1_{jm} = -V[j(j+1) - m^2(1-g)] \tag{6.90}$$

such that the full energies are

$$E_{jm} = 2\varepsilon m - V[j(j+1) - (1-g)\, m^2] + \varepsilon\Omega. \tag{6.91}$$

This coincides with the fermion part of the spectrum (6.52) if m, j are set equal to

$$m = (N - \Omega)/2, \quad j = \frac{\Omega - \nu}{2} \tag{6.92}$$

as is necessary due to (6.45), (6.50).

For $g = 1$, $\varepsilon = 0$ the spectrum is degenerate as the Lagrangian (6.78) is rotationally invariant. It may be worth mentioning that in this case the Lagrangian can also be written as a standard σ-model in the time dimension. In order to see this use $i\dot{U}^+U = -iU^+\dot{U} = \omega_i\sigma_i/2$ to bring (6.71) to the form

$$\mathscr{L} = -\frac{1}{4V}(\omega_1^2 + \omega_2^2 + \omega_3^2) = -\frac{1}{2V} \operatorname{tr}(\dot{U}^+U\; U^+\dot{U}).$$

If one now defines σ and π fields as

$$U = \sigma + i\boldsymbol{\pi}\cdot\boldsymbol{\sigma},$$

where $\sigma^2 + \pi^2 = 1$ due to unitarity of U, the Lagrangian takes the familiar expression:

$$\mathscr{L} = -\frac{1}{V}(\dot{\sigma}^2 + \pi^2). \tag{6.93}$$

It is instructive to exhibit the original quasi-spin operators and their algebra within the collective Lagrangian. For this we add a coupling to external currents

$$\Delta H = -2V \int L_i(t)\, l_i(t)\, dt,$$

to the Hamiltonian (6.44) where L_i are the operators (6.45). In the Lagrangian (6.57) this amounts to

$$\Delta\mathscr{L}(t) = 2VL_i(t)\, l_i(t), \tag{6.94}$$

which modifies (6.60) by adding the matrix

$$Vf^+(t)\begin{pmatrix} l_3 & l^+ \\ l & l_3 \end{pmatrix} f(t). \tag{6.95}$$

This has the effect of replacing

$$S_i \to \tilde{S}_i \equiv S_i + l_i$$

in the tr log term of (6.61).
Performing a shift in the integration $\mathrm{D}\boldsymbol{S} \to \mathrm{D}(\boldsymbol{S} + \boldsymbol{l})$ we can also write

$$\mathscr{A}[\boldsymbol{S}, \boldsymbol{l}] = +i \operatorname{tr}\log(iG_S{}^{-1}) - V\left((S_1 - l_1)^2 + (S_2 - l_2)^2 + \frac{1}{g}\left(S_3 + \frac{\varepsilon}{V} - l_3\right)^2\right). \tag{6.96}$$

The Green's functions involving angular momentum operators can now be generated by differentiating

$$Z[0, 0, l_i] = \int \mathrm{D}\boldsymbol{S} \exp\{i\mathscr{A}[\boldsymbol{S}, \boldsymbol{l}]\}$$

with respect to δl_i:

$$L_i \triangleq -\frac{i}{2V}\frac{\delta}{\delta l_i}. \tag{6.97}$$

In the reference state $|R\rangle$ where the tr log term vanishes, $-i/2V\, \delta/\delta l_{\frac{1}{2}}$, $-i/2V\, \delta/\delta l_3$ generate from (6.94) the fields $S_{\frac{1}{2}} - l_{\frac{1}{2}}$, $(S_3 + \varepsilon/V - l_3)/g$
in the functional integral.
In the fermion case, this implies for $\boldsymbol{l} = 0$, using equs. (6.68)

$$\begin{aligned} L^{\pm} &= -\frac{1}{2V}(\omega_1 \pm i\omega_2) = -\frac{1}{2V}(\pm i\dot{\beta} + \dot{\alpha}\sin\beta)\, e^{\pm i\gamma} \\ L_3 &= -\frac{1}{2Vg}(\omega_3 - 2\varepsilon) = -\frac{1}{2Vg}(\dot{\alpha}\cos\beta + \dot{\gamma} - 2\varepsilon) \end{aligned} \tag{6.98}$$

which are exactly the angular momenta of the Lagrangian (6.78) with moments of inertia

$$I_{\frac{1}{2}} = -\frac{1}{2V}, \quad I_3 = -\frac{1}{2Vg}. \tag{6.99}$$

Inserting the canonical momenta of (6.78)

$$\begin{aligned}
P_\alpha &= -\frac{1}{2V}\left(\dot\alpha \sin^2\beta + \frac{1}{g}(\dot\gamma + \dot\alpha\cos\beta - 2\varepsilon)\cos\beta\right) \\
&= -\frac{1}{2V}\dot\alpha\sin^2\beta + \cos\beta p_\gamma = -i\partial_\alpha \\
P_\beta &= -\frac{1}{2V}\dot\beta = -i\sin^{-1/2}\beta\,\partial_\beta\sin^{1/2}\beta = -i\partial_\beta - \frac{i}{2}\cot\beta \\
P_\gamma &= -\frac{1}{2Vg}(\dot\gamma + \dot\alpha\cos\beta - 2\varepsilon) = -i\partial_\gamma
\end{aligned} \tag{6.100}$$

we recover the differential operators (6.88).
The quasi-spinalgebra can now be verified by applying the derivatives:

$$-\frac{1}{4V^2}\left(\frac{\delta}{\delta l_j(t+\varepsilon)}\frac{\delta}{\delta l_i(t)} - \frac{\delta}{\delta l_i(t+\varepsilon)}\frac{\delta}{\delta l_j(t)}\right)Z\bigg|_{l=0} = \frac{1}{2V}\varepsilon_{ijk}\frac{\delta}{\delta l_k}Z\bigg|_{l=0}. \tag{6.101}$$

What would have happened in this model if we had not chosen the symmetric reference state $|R\rangle$ to specify the boundary condition on G_0? Consider for example the vacuum state $|0\rangle$. Then the Green's function becomes for $\boldsymbol{S} = 0$:

$$G_0^{ij}(t,t') = \begin{pmatrix}\Theta(t-t') & 0 \\ 0 & -\Theta(t'-t)\end{pmatrix}\delta^{ij} \tag{6.102}$$

rather than (6.73). In this case *there is* a contribution of $-i\,\mathrm{tr}\log(iG_S^{-1})$ since from (6.72) and (6.64):

$$\frac{\delta}{\delta S_i}[-i\,\mathrm{tr}\log(iG_S^{-1})] = -V\Omega\,\mathrm{tr}\left(\sigma^i U^+(t)\frac{-1+\sigma^3}{2}U(t')\right)\bigg|_{t'=t}. \tag{6.103}$$

Now (6.68) implies

$$U^+(t)\,\sigma^3 U(t) = \cos\beta\sigma_3 + \sin\beta(\cos\gamma\sigma_1 + \sin\gamma\sigma_2)$$

yielding for the right hand side of (6.103) the expressions

$$-V\Omega\begin{Bmatrix}n_1\\n_2\\n_3\end{Bmatrix} \equiv -V\Omega\begin{Bmatrix}\sin\beta\cos\gamma\\ \sin\beta\sin\gamma\\ \cos\beta\end{Bmatrix}. \tag{6.104}$$

Observe that due to the differential equations (6.70) the unit vector n_i can be found to satisfy the equation of motion

$$\dot{\boldsymbol{n}} = 2V\boldsymbol{n}\times\boldsymbol{S}. \tag{6.105}$$

We can now proceed and find $-i\,\mathrm{tr}\log iG_S^{-1}$ by functionally integrating (6.103). We shall do so in terms of the Euler variables $\alpha\beta\gamma$. Using (6.103), (6.104), (6.75), and the chain rule of differentiation

$$\begin{aligned}
\frac{\delta}{\delta q_j(t')}[-i\,\mathrm{tr}\log iG_S^{-1}] &= \sum_i\int dt\,\frac{\delta S_i(t)}{\delta q_j(t')}\frac{\delta}{\delta S_i(t)}[-i\,\mathrm{tr}\log iG_S^{-1}] \\
&= -V\Omega\sum_i\int dt n_i(t)\frac{\delta S_i(t)}{\delta q_j(t')}
\end{aligned} \tag{6.106}$$

we find

$$\frac{\delta}{\delta q_j(t)}[-i \operatorname{tr} \log iG_S^{-1}] = \frac{\Omega}{2}\sum_i \int dt\big(n_i(t)\, A_{ij}(t)\, \delta(t-t') + n_i(t)\, B_{ij}(t)\, \dot\delta(t-t')\big)$$

$$= \frac{\Omega}{2}\left[(0, 0, -\dot\beta \sin\beta(t'))_j + \int dt\big(1, 0, \cos\beta(t)\big)_j\, \dot\delta(t-t')\right]. \tag{6.107}$$

Partial integration renders for the second part in brackets

$$\big(1, 0, \cos\beta(t)\big)\,\delta(t-t')\Big|_{t=-\infty}^{t=\infty} + \big(0, 0, \dot\beta \sin\beta(t')\big). \tag{6.108}$$

With the boundary condition $\cos\beta(\pm\infty) = 1$ one has therefore

$$\frac{\delta}{\delta(\alpha,\beta,\gamma)(t)}[-i \operatorname{tr} \log iG_S^{-1}] = \frac{\Omega}{2}(1, 0, 1)\,[\delta(\infty - t) - \delta(-\infty - t)]. \tag{6.109}$$

This pure boundary contribution can immediately be functionally integrated with the result:

$$-i \operatorname{tr} \log iG_S^{-1} = \frac{\Omega}{2}\int_{-\infty}^{\infty}\big(\dot\alpha(t) + \dot\gamma(t)\big)\, dt. \tag{6.110}$$

Hence the exponent of the generating functional $Z[j^+j]$ on the reference state $|0\rangle$ becomes

$$i\int dt\left\{-\frac{1}{4V}\left(\omega_1^2 + \omega_2^2 + \frac{1}{g}(\omega_3 - 2\varepsilon)^2\right) + \frac{\Omega}{2}(\dot\alpha + \dot\gamma) - \varepsilon\Omega\right\}$$

$$-\int dt\, dt' \sum_i j_i^+(t)\left\{U^+(t)\frac{1+\sigma^3}{2}U(t')\,\Theta(t-t') - U^+(t)\frac{1-\sigma^3}{2}U(t')\,\Theta(t'-t)\right\} j_i(t') \tag{6.111}$$

ather than (6.77). As in the case of the Pet model in the last section, the Hamiltonian s changed quite trivially. The canonical momenta P_α, P_γ become

$$P_\alpha = -\frac{1}{2V}\left[\dot\alpha \sin^2\beta + \frac{\cos\beta}{g}(\dot\gamma + \dot\alpha\cos\beta - 2\varepsilon)\right] + \frac{\Omega}{2}$$

$$= -\frac{1}{2V}\dot\alpha\sin^2\beta + \cos\beta p_\gamma - \frac{\Omega}{2}(\cos\beta - 1) = -i\partial_\alpha \tag{6.112}$$

$$P_\gamma = -\frac{1}{2Vg}(\dot\gamma + \dot\alpha\cos\beta - 2\varepsilon) + \frac{\Omega}{2} = -i\partial_\gamma.$$

The additional term can be removed by multiplying all eigenfunctions belonging to (6.112) by a phase $\exp[-i\Omega/2(\alpha + \gamma)]$ thereby reducing them to the previous case. In the present context it is really superfluous to discuss such trivial surface terms. We are doing this only because these terms do become important at that moment at which the transition to the true BCS model is made by letting $g \to 0$.
This will be discussed in the next section.

VI.3. The Hilbert Space of Generalized BCS Model

Let us now study in which fashion the Hilbert of all rotational wave functions imbeds the fermion theory. For this consider the generation of Green's functions by functional derivation of ${}^{R}Z[j^+, j]$, with the reference state $|R\rangle$ being the quasi-spin symmetric one (6.49), for simplicity.
The resulting one-particle Green's function will have to coincide with

$$G^{ij}_{mm'}(t, t') = \langle 0| \, b_\Omega \cdot \ldots \cdot b_1 \begin{pmatrix} Ta_i(t)\, a_j^+(t') & Ta_i(t)\, b_j(t') \\ Tb_i^+(t)\, a_j^+(t') & Tb_i^+(t)\, b_j(t') \end{pmatrix}_{mm'} b_1^+ \cdot \ldots \cdot b_\Omega^+ \,|0\rangle. \tag{6.113}$$

If we differentiate (6.77) accordingly, we find

$$G^{ij}_{mm'}(t, t') = \int \mathrm{D}\alpha \mathrm{D} \cos \beta \mathrm{D}\gamma \delta^{ij} \big(U^+(t)\, U(t')\big)_{mm'}\, \Theta(t - t') \exp\left[i \int dt \mathscr{L}(t)\right]. \tag{6.114}$$

This can be calculated most easily by going to the Schrödinger picture

$$G^{ij}_{mm'}(t, t') = \sum_k \{R|\, D^{1/2}_{km}\big(\alpha\beta\gamma(t)\big)\, D^{1/2}_{km'}\big(\alpha\beta\gamma(t')\big)\, |R\}\, \delta^{ij} \Theta(t - t'). \tag{6.115}$$

Since the reference state is symmetric, it must be associated with the wave function $\{\alpha\beta\gamma(t) \mid R\} = \mathrm{D}^0_{00}\big(\alpha\beta\gamma(t)\big) \equiv 1/\sqrt{8\pi^2}$

$$E_R \equiv E_{0;0} = \varepsilon \Omega. \tag{6.116}$$

Inserting the time translation operator[26])

$$\mathrm{D}\big(\alpha\beta\gamma(t)\big) = e^{iHt} \mathrm{D}\big(\alpha\beta\gamma(0)\big)\, e^{-iHt} \tag{6.117}$$

with H in the differential form (6.82) one finds a phase

$$e^{i\Delta E(t-t')}, \tag{6.118}$$

where ΔE is the energy difference between the state $|jm\rangle = |1/2\; 1/2\rangle$ and the reference state $|R\rangle = |0, 0\rangle$

$$\Delta E = E_{1/2\,1/2} - E_{0,0} = \varepsilon - V\left(\frac{1}{2} + \frac{g}{4}\right) \tag{6.119}$$

and an integral

$$\sum_k \int d\alpha\, d \cos \beta d\gamma \{R \,|\, \alpha\beta\gamma\}\, D^{1/2*}_{km}(\alpha\beta\gamma)\, D^{1/2}_{km'}(\alpha\beta\gamma)\, \{\alpha\beta\gamma \mid R\} = \delta_{mm'}. \tag{6.120}$$

This coincides exactly with the result one would obtain from (6.113) by using the original operator (6.44) and observing the energy spectrum (6.52).
Notice that the orthogonality relation together with the Grassmann algebra ensure the validity of the anticommutation rules among the operators. For higher Green's functions the functional derivatives amount again to the contractions as in (6.28), except that now the contractions are associated with

$$\begin{aligned} \overline{f_{mi}(t)\, f_{m'j}}(t') &= D^{1/2}_{mm'}\big(U^+(t)\, U(t')\big)\, \Theta(t - t')\, \delta^{ij} \\ &= \sum_k D^{1/2*}_{km}\big(U(t)\big)\, D^{1/2}_{km'}\big(U(t)\big)\, \Theta(t - t')\, \delta^{ij} \end{aligned} \tag{6.121}$$

where $(f_{1/2\,i}, f_{-1/2\,i})$ stands for (a_i, b_i^+).

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[26]) The Schrödinger angles $\alpha\beta\gamma$ coincide with the time dependent angles $\alpha(t)$, $\beta(t)$, $\gamma(t)$ at $t = 0$.

We can now proceed and construct the full Hilbert space by piling up operators a_i^+ or b_j on the reference state $|R\rangle = b_1^+ \cdots b_\Omega^+|0\rangle$. First we shall go to true vacuum state of a^+, b^+: $|0\rangle$, i.e. we shall calculate ${}^0Z[j^+, j]$ in this state. For this we obviously have to bring down successively $b_1^+(\infty) \cdot \ldots \cdot b_\Omega^+(\infty)\, b_\Omega(-\infty) \cdot \ldots \cdot b_1(-\infty)$ by forming the functional derivatives:

$$Z^0[0, 0] \propto \frac{\delta^{2\Omega}}{\delta j_{-1/2,1}(\infty) \cdot \ldots \cdot \delta j^+_{-1/2,1}(-\infty)}\, {}^RZ[j^+, j]\Big|_{j=0} \tag{6.122}$$

in the functional (6.77). Of the resulting $n!$ contractions, only one combination survives, since all indices i, j are different and the Kronecker δ^{ij} permits only one set of contractions. The result is

$${}^0Z[0, 0] = N \int \mathrm{D}\alpha \mathrm{D} \cos\beta \mathrm{D}\gamma \exp\left[i \int dt \mathscr{L}(t)\right] \left[D^{1/2}_{-1/2\,-1/2}\big(U^+(\infty)\, U(-\infty)\big)\right]^\Omega . \tag{6.123}$$

But from the coupling rules of angular momenta and the group property one has:

$$\begin{aligned}\left[D^{1/2}_{-1/2\,-1/2}\big(U^+(\infty)\, U(-\infty)\big)\right]^\Omega &= D^{\Omega/2}_{-\Omega/2\,-\Omega/2}\big(U^+(\infty)\, U(-\infty)\big)\\ &= \sum_k D^{\Omega/2*}_{k\,-\Omega/2}\big(U(\infty)\big)\, \mathrm{D}_{k-\Omega/2}\big(U(-\infty)\big).\end{aligned} \tag{6.124}$$

Going to the Schrödinger picture and inserting the time translation operator (6.117) one finds an infinite phase $\exp\,[i(E_R - E_0)\, 2\infty]$ which can be absorbed in the normalization factor N. Here $E_0 = E_{\Omega/2,-\Omega/2}$ is the energy of the ground state $|0\rangle$ which has $|jm\rangle = |\Omega/2 - \Omega/2\rangle$. The eigenfunction $D(\alpha, \beta, \gamma)$ now appear both at $t = 0$ and the functional (6.123) becomes in the Schrödinger picture

$${}^0Z[0, 0] = \sum_{k=-\Omega/2}^{\Omega/2} \int d\alpha\, d\beta\, d\gamma \sin\gamma \{0k \mid \alpha\beta\gamma\}\, \{\alpha\beta\gamma \mid 0k\} \tag{6.125}$$

with the vacuum wave functions

$$\{\alpha\beta\gamma \mid 0, k\} = D^{\Omega/2}_{k,-\Omega/2}(\alpha\beta\gamma) = e^{i(k\alpha - \Omega\gamma/2)}\, d^{\Omega/2}_{k,-\Omega/2}(\beta) . \tag{6.126}$$

It is easy to verify, how an additional unpaired particle a^+, added to the vacuum, decreases $\Omega/2 \to (\Omega - 1)/2$ and raises the third component of quasi-spin by 1/2 unit. Differentiating (6.75) by $-\delta^2/\delta j_{1/2\,1}(\infty)\, \delta j^+_{1/2\,1}(-\infty)$ in addition to (6.122) one finds a different set of contractions. Picturing them within the original fermion language, there are

$$\begin{aligned}&\langle R|\, T\big(b_1^+(+\infty) \cdot \ldots \cdot b_\Omega^+(+\infty)\, a_1(+\infty)\, a_1^+(-\infty)\, b_\Omega(-\infty) \cdot \ldots \cdot b_1(-\infty)\big)\, |R\rangle\\ &= \langle R|\, T\big(b_1^+(\infty) \cdot \ldots \cdot b_\Omega^+(\infty)\, a_1(\infty)\, a_1^+(-\infty)\, b_\Omega(-\infty) \cdot \ldots \cdot b_1(-\infty)\big)\, |R\rangle\\ &\quad + \langle R|\, T\big(b_1^+(\infty) \cdot \ldots \cdot b_\Omega^+(\infty)\, a_1(\infty)\, a_1^+(-\infty)\, b_\Omega(-\infty) \cdot \ldots \cdot b_1(-\infty)\big)\, |R\rangle .\end{aligned} \tag{6.127}$$

These render under the functional integral (6.123)

$$\begin{aligned}&\left[D^{1/2}_{-1/2\,-1/2}\big(U^+(\infty)\, U(-\infty)\big)\right]^\Omega D^{1/2}_{1/2\,1/2}\big(U^+(\infty)\, U(-\infty)\big)\\ &\quad - \left[D^{1/2}_{-1/2\,-1/2}\big(U^+(\infty)\, U(-\infty)\big)\right]^{\Omega-1} D^{1/2}_{-1/2\,1/2}\big(U^+(\infty)\, U(-\infty)\big)\, D^{1/2}_{1/2\,-1/2}\big(U^+(\infty)\, U(-\infty)\big)\\ &= D^{\Omega/2}_{-\Omega/2\,-\Omega/2}\big(U^+(\infty)\, U(-\infty)\big)\, D^{1/2}_{1/2\,1/2}\big(U^+(\infty)\, U(-\infty)\big)\\ &\quad - D^{(\Omega-1)/2}_{-(\Omega-1)/2\,-(\Omega-1)/2}\big(U^+(\infty)\, U(-\infty)\big)\, D^{1/2}_{-1/2\,1/2} D^{1/2}_{1/2\,-1/2}\end{aligned} \tag{6.128}$$

Employing the explicit formulas

$$\begin{aligned} D^{\Omega/2}_{-\Omega/2\,-\Omega/2}(\alpha\beta\gamma) &= e^{-\Omega(\alpha+\gamma)/2}\left(\cos\frac{\beta}{2}\right)^{\Omega} \\ D^{1/2}_{1/2\,1/2}(\alpha\beta\gamma) &= e^{(\alpha+\gamma)/2}\cos\frac{\beta}{2} \\ D^{1/2}_{-1/2\,1/2}(\alpha\beta\gamma)\,D^{1/2}_{1/2\,-1/2}(\alpha\beta\gamma) &= -\sin^2\frac{\beta}{2} \end{aligned} \tag{6.129}$$

the r.h.s. of (6.128) becomes

$$\begin{aligned} &= e^{-\Omega(\alpha+\gamma)/2}\left(\cos\frac{\beta}{2}\right)^{\Omega} e^{(\alpha+\gamma)/2}\cos\frac{\beta}{2} \\ &\quad + e^{-(\Omega-1)(\alpha+\gamma)/2}\left(\cos\frac{\beta}{2}\right)^{\Omega-1}\sin^2\frac{\beta}{2} \\ &= e^{-(\Omega-1)(\alpha+\gamma)/2}\left(\cos\frac{\beta}{2}\right)^{\Omega-1} = D^{(\Omega-1)/2}_{-(\Omega-1)/2,-(\Omega-1)/2}(\alpha\beta\gamma) \end{aligned} \tag{6.130}$$

and therefore, in analogy to (6.123), (6.125)

$$\begin{aligned} {}^{a_1^+|0\rangle}Z[j^+,j]\,|_{j=0} &= N\int \mathrm{D}\alpha\mathrm{D}\cos\beta\mathrm{D}\gamma D^{(\Omega-1)/2}_{-(\Omega-1)/2,-(\Omega-1)/2}(\alpha\beta\gamma)\exp\left(i\int \mathscr{L}dt\right) \\ &= \sum_{k=-(\Omega-1)/2}^{(\Omega-1)/2}\int dt\alpha\, d\cos\beta\, d\gamma\{a_1\,k\mid\alpha\beta\gamma\}\,\{\alpha\beta\gamma\mid a_1{}^+k\} \end{aligned} \tag{6.131}$$

with the Schrödinger wave functions

$$\{\alpha\beta\gamma\mid a_1{}^+k\}\equiv D^{(\Omega-1)/2}_{k,-(\Omega-1)/2}(\alpha\beta\gamma). \tag{6.132}$$

In a similar fashion we may work our way through the whole Hilbert space!

VI.4. The BCS Model

Consider now the Hamiltonian (6.44) with only the pairing force, i.e. $g = 0$. In the generating functional on the symmetric state (6.74), this forces a δ-functional to appear in the integral

$$\begin{aligned} {}^{R}Z[j^+,j] &= \int \mathrm{D}\alpha\mathrm{D}\cos\beta\mathrm{D}\gamma\;\delta(\omega_3-2\varepsilon)\exp\left[i\int dt\left\{-\frac{1}{4V}(\omega_1{}^2+\omega_2{}^2)-\varepsilon\Omega\right\}\right] \\ &\quad\times\exp\left[-\int dt\,dt'\Theta(t-t')\sum_i j_i{}^+(t)\,U^+(t)\,U(t')\,j_i(t')\right]. \end{aligned} \tag{6.133}$$

The δ functional ensures the differential equation

$$\dot\gamma+\cos\beta\dot\alpha = 2\varepsilon \tag{6.134}$$

according to which α becomes a function dependent on γ and β. Performing the functional integral $\mathrm{D}\alpha$ gives

$$\begin{aligned} {}^{R}Z[j^+,j] &= \int \mathrm{D}\alpha\mathrm{D}\beta\tan\beta\exp\left[i\int dt\left\{-\frac{1}{4V}\left(\dot\beta^2+(\dot\gamma-2\varepsilon)^2\tan^2\beta\right)-\varepsilon\Omega\right\}\right. \\ &\quad\left.-\int dt\,dt'\sum_i j_i{}^+(t)\,G_S{}^{ii}(t,t')\,j_i(t)\right]. \end{aligned} \tag{6.135}$$

Introducing the variable of (6.104):

$$\begin{Bmatrix} n \\ n^+ \end{Bmatrix} = n_1 \pm i n_2 = \sin \beta e^{\pm i\gamma} \qquad (6.136)$$

this can also be written as

$$^RZ[j^+, j] = \int \frac{DnDn^+}{1-|n|^2} \exp\left[i \int dt \left\{-\frac{1}{4V}\frac{|\dot n - 2i\varepsilon n|^2}{1-|n|^2} - \varepsilon\Omega\right\} - \int dt dt' \sum_i j_i^+(t)\, G_S^{ii}(t,t')\, j_i(t)\right]. \qquad (6.137)$$

In the limit $g \to 0$, the calculation of the angular momentum operators (6.98) becomes quite different: Adding the external sources l_i, the limit $g \to 0$ produces in the action (6.96) a δ-functional

$$\delta(\omega_3 - 2\varepsilon + 2Vl_3) \qquad (6.138)$$

with a Lagrangian

$$\mathscr{L} = -\frac{1}{4V}|\omega + 2Vl|^2 \qquad (6.139)$$

where

$$\begin{Bmatrix} \omega \\ \omega^* \end{Bmatrix} = \omega^\pm = (\pm i\dot\beta + \dot\alpha \sin\beta)\, e^{i\gamma}. \qquad (6.140)$$

Respecting the δ-functional (6.138) amounts to eliminating the variable α via

$$\dot\gamma + \cos\beta\dot\alpha = -2Vl_3 + 2\varepsilon. \qquad (6.141)$$

Hence:

$$\omega^\pm = \left(\pm i\dot\beta - (\dot\gamma - 2\varepsilon + 2Vl_3)\tan\beta\right) e^{i\gamma}. \qquad (6.142)$$

Differentiation of (6.96) with respect to $\delta/\delta l_i$ according to (6.97) renders the angular momenta

$$\begin{aligned} L^\pm &= -\frac{1}{2V}\,\omega^\pm\Big|_{l=0} = -\frac{1}{2V}\left(\pm i\dot\beta - (\dot\gamma - 2\varepsilon)\tan\beta\right) e^{i\gamma} \\ L_3 &= -\frac{1}{2V}(\dot\gamma - 2\varepsilon)\tan^2\beta. \end{aligned} \qquad (6.143)$$

From the Lagrangian in (6.137) we read off the canonical momenta

$$\begin{aligned} P_\beta &= -\frac{1}{2V}\dot\beta = -i\partial_\beta - \frac{i}{2}\cot\beta \\ P_\gamma &= -\frac{1}{2V}(\dot\gamma - 2\varepsilon)\tan^2\beta = -i\partial_\gamma \end{aligned} \qquad (6.144)$$

such that

$$\begin{aligned} L^\pm &= e^{\pm i\gamma}(\pm\partial_\beta + \cot\beta\, i\,\partial_\gamma) \\ L_3 &= -i\,\partial_\gamma \end{aligned} \qquad (6.145)$$

which are the standard differential operators on spherical harmonies $Y_{lm}(\beta, \gamma)$. Observe now that in the BCS model the generating functional in the vacuum state $|0\rangle$ receives an essential modification. The reason is that due to (6.134) the surface term (6.110) becomes a dynamical object

$$-i \operatorname{tr} \log iG_S^{-1} = \frac{\Omega}{2} \int_{-\infty}^{\infty} (\dot{\alpha} + \dot{\gamma})\, dt = \frac{\Omega}{2} \int_{-\infty}^{\infty} \left[\dot{\gamma}\left(1 - \frac{1}{\cos\beta}\right) + \frac{2\varepsilon}{\cos\beta}\right] dt. \qquad (6.146)$$

Expressing γ in terms of the field (6.136) gives

$$-i \operatorname{tr} \log iG_S^{-1} = i\frac{\Omega}{4} \int_{-\infty}^{\infty} [(\dot{n} - 2i\varepsilon n)\, n^+ - \text{h.c.}]\, \frac{\sqrt{1 - |n|^{2^{-1}}} - 1}{|n|^2} + \varepsilon\Omega. \qquad (6.147)$$

Adding this to the exponent in (6.137) renders the generating functional in the vacuum [32]

$$ {}^0Z[j^+j] = \int \frac{Dn^+Dn}{1 - |n|^2} \exp\left[i \int dt \left\{-\frac{1}{4}\, \frac{|\dot{n} - 2i\varepsilon n|^2}{1 - |n|^2} + i\,\frac{\Omega}{4}\, [(\dot{n} - 2i\varepsilon n)\, n^+ - \text{h.c.}] \right.\right.$$
$$\left.\left. \times \frac{\sqrt{1 - n^{2^{-1}}} - 1}{|n|^2}\right\}\right] \exp\left[-\int dt\, dt' \sum_i j_i^+(t)\, G_n{}^{ii}(t, t')\, j_i(t_i')\right]. \qquad (6.148)$$

The Green's function G_n coincides with that of (6.111) except that the relation (6.134) has to be used to express α in terms of γ everywhere, i.e. $U(t) = D(\alpha\beta\gamma)$ becomes

$$U_{m'm}(t) = D^{1/2}_{m'm}\big(\alpha(\gamma), \beta, \gamma\big) = \exp\left[-im' \int_0^t \frac{\dot{\gamma} - 2\varepsilon}{\cos\beta}\, dt' + m\gamma(t)\right] d^{1/2}_{m'm}\big(\beta(t)\big). \qquad (6.149)$$

Appendix A: The Propagator of the Bilocal Pair Field

Consider the Bethe-Salpeter equation (4.17) with a potential λV instead of V

$$\Gamma = -i\lambda V G_0 G_0 \Gamma. \qquad (A.1)$$

Take this as an eigenvalue problem in λ at fixed energy-momentum $q = (q^0, \boldsymbol{q}) \equiv (E, \boldsymbol{q})$ of the bound states. Let $\Gamma_n(P \mid q)$ be all solutions, with eigenvalues $\lambda_n(q)$. Then the convenient normalization of Γ_n is:

$$-i \int \frac{d^4P}{(2\pi)^4}\, \Gamma_n^+(P \mid q)\, G_0\left(\frac{q}{2} + P\right) G_0\left(\frac{q}{2} - P\right) \Gamma_{n'}(P \mid q) = \delta_{nn'}. \qquad (A.2)$$

If all solutions are known, there is a corresponding completeness relation (the sum may comprise an integral over a continuous part of the spectrum)

$$-i \sum_n G_0\left(\frac{q}{2} + P\right) G_0\left(\frac{q}{2} - P\right) \Gamma_n(P \mid q)\, \Gamma_n^+(P' \mid q) = (2\pi)^4\, \delta^4(P - P'). \qquad (A.3)$$

This completeness relation makes the object given in (4.26) the correct propagator of Δ. In order to see this write the free Δ action $\mathscr{A}_2[\Delta^+\Delta]$ as

$$\mathscr{A}_2 = \frac{1}{2}\, \Delta^+ \left(\frac{1}{\lambda V} + iG_0 \times G_0\right) \Delta \qquad (A.4)$$

where we have used λV instead of V.

The propagator of Δ would have to satisfy

$$\left(\frac{1}{\lambda V} + iG_0 \times G_0\right) \overline{\Delta\Delta^+} = i. \tag{A.5}$$

Performing this calculation on (4.22) one has, indeed, by virtue of (A.1) for Γ_n, λ_n:

$$\begin{aligned}
&\left(\frac{1}{\lambda V} + iG_0 \times G_0\right) \times \left\{-i\lambda \sum_n \frac{\Gamma_n \Gamma_n^+}{\lambda - \lambda_n(q)}\right\} \\
&= -i\lambda \sum_n \frac{\frac{1}{\lambda V} \Gamma_n \Gamma_n^+ + iG_0 \times G_0 \Gamma_n \Gamma_n^+}{\lambda - \lambda_n(q)} = i\lambda \sum_n \frac{\left(-\frac{\lambda_n(q)}{\lambda} + 1\right)}{\lambda - \lambda_n(q)} (-iG_0 \times G_0 \Gamma_n \Gamma_n^+) \\
&= i \left(-i \sum_i G_0 \times G_0 \Gamma_n \Gamma_n^+\right) = i.
\end{aligned} \tag{A.6}$$

Notice that the expansion of the propagator in powers of λ

$$\overline{\Delta\Delta^+} = i \sum_k \left(\sum_n \left(\frac{\lambda}{\lambda_n(q)}\right)^k \Gamma_n \Gamma_n^+\right) \tag{A.7}$$

corresponds to the graphical sum over one, two, three, etc. exchanges of the potential λV. For $n = 1$ this is immediately obvious due to (A.1):

$$i \sum_n \frac{\lambda}{\lambda_n(q)} \Gamma_n \Gamma_n^+ = \sum \frac{\lambda}{\lambda_n(q)} \lambda_n(q) V G_0 \times G_0 \Gamma_n \Gamma_n^+ = i\lambda V. \tag{A.8}$$

For $n = 2$ one can rewrite, using the orthogonality relation,

$$i \sum_n \left(\frac{\lambda}{\lambda_n(q)}\right)^2 \Gamma_n \Gamma_n^+ = \sum_{n,n'} \frac{\lambda}{\lambda_n(q)} \Gamma_n \Gamma_n^+ G_0 \times G_0 \Gamma_{n'} \Gamma_{n'}^+ \frac{\lambda}{\lambda_{n'}(q)} = \lambda V G_0 \times G_0 \lambda V \tag{A.9}$$

which displays the exchange of two λV terms with two particles propagating in between. The same procedure applies at any order in λ. Thus the propagator has the expansion

$$\overline{\Delta\Delta^+} = i\lambda V - i\lambda V G_0 \times G_0 i\lambda V + \cdots. \tag{A.10}$$

If the potential is instantaneous, the intermediate $\int dP_0/2\pi$ can be performed replacing

$$G_0 \times G_0 \to i \frac{1}{E - E_0(\boldsymbol{P} \mid q)} \tag{A.11}$$

where

$$E_0(\boldsymbol{P} \mid q) = \xi\left(\frac{\boldsymbol{q}}{2} + \boldsymbol{P}\right) + \xi\left(\frac{\boldsymbol{q}}{2} - \boldsymbol{P}\right)$$

is the free particle energy which may be considered as the eigenvalue of an operator H_0. In this case the expansion (A.10) reads

$$\overline{\Delta\Delta^+} = i\left(\lambda V + \lambda V \frac{1}{E - H_0} \lambda V + \cdots\right) = i\lambda V \frac{E - H_0}{E - H_0 - \lambda V}. \tag{A.12}$$

We see it related to the resolvent of the complete Hamiltonian as

$$\Box\!\!\!\Delta\Delta^{+} = i\lambda V(R\lambda V + 1) \tag{A.13}$$

where

$$R \equiv \frac{1}{E - H_0 - \lambda V} = \sum_n \frac{\psi_n \psi_n^{+}}{E - E_n} \tag{A.14}$$

with ψ_n being the Schrödinger amplitudes in standard normalization. We can now easily determine the normalization factor N in the connection between Γ_n and the Schrödinger amplitude ψ_n. Equ. (A.2) gives in the instantaneous case

$$\int \frac{d^3P}{(2\pi)^3} \Gamma_n^{+}(\boldsymbol{P} \mid q) \frac{1}{E - H_0} \Gamma_{n'}(\boldsymbol{P} \mid q) = \delta_{nn'}. \tag{A.15}$$

Inserting ψ from (4.22) renders

$$\frac{1}{N^2} \int \frac{d^3P}{(2\pi)^3} \psi_n^{+}(\boldsymbol{P} \mid q) (E - H_0) \psi_{n'}(\boldsymbol{P} \mid q) = \delta_{nn'}. \tag{A.16}$$

But since

$$(E - H_0) \psi = \lambda V \psi \tag{A.17}$$

this is also

$$\frac{1}{N^2} \int \frac{d^3P}{(2\pi)^3} \psi_n^{+}(\boldsymbol{P} \mid q) \lambda V \psi_{n'}(\boldsymbol{P} \mid q) = \delta_{nn'}. \tag{A.18}$$

For ψ_n wave functions in standard normalization the integral expresses the differential

$$\lambda \frac{dE}{d\lambda}$$

as can be seen by using perturbation theory on the infinitesimal potential $d\lambda V$. Hence

$$N_n^2 = \lambda \frac{dE_n}{d\lambda}. \tag{A.19}$$

For a typical calculation of a resolvent, the reader is referred to Schwinger's treatment [*45*] of the Coulomb problem. His result may directly be used for a propagator of electron hole pairs bound to excitons.

Appendix B: Fluctuations around the Composite Field

Here we show that the quantum mechanical fluctuations around the *classical* equations of motion (3.4)

$$\varphi(x) = \int dy V(x, y) \psi^{+}(y) \psi(y) \tag{B.1}$$

or (4.4)

$$\Delta(y, x) = V(x - y) \psi(x) \psi(y) \tag{B.2}$$

are quite simple to calculate. For this let us compare the Green's functions of $\varphi(x)$ or $\Delta(x, y)$ with those of the composite operators on the right-hand side of equs. (B.1) or (B.2). The Green's functions or φ or Δ are generated by adding external currents

$\int dx\varphi(x)\, I(x)$ or $1/2 \int dx\, dy(\Delta(y, x)\, I^+(x, y) + \text{h.c.})$ to the final actions (3.8) or (4.7), respectively, and by forming functional derivatives $\delta/\delta I$. The Green's functions of the composite operators, on the other hand, are obtained by adding

$$\int dx\Big(\int dy V(x, y)\, \psi^+(y)\, \psi(y)\Big)\, K(x)$$

or

$$\frac{1}{2}\, dx\, dy V(x - y)\, \psi(x)\, \psi(y)\, K^+(x, y) + \text{h.c.}$$

to the original actions (3.3) or (4.3), respectively, and by forming functional derivatives $\delta/\delta K$. It is obvious that the sources K can be included in the final actions (3.8) and (4.7) by simply replacing

$$\varphi(x) \to \varphi'(x) = \varphi(x) - \int dx' K(x')\, V(x', x)$$

or

$$\Delta(x, y) \to \Delta'(x, y) = \Delta(x, y) - K(x, y)\,.$$

If one now shifts the functional integrations to these new translated variables and drops the irrelevant superscript "prime", the actions can be rewritten as

$$\mathscr{A}[\varphi] = \pm i \operatorname{tr} \log (iG_\varphi^{-1}) + \frac{1}{2} \int dx\, dx' \varphi(x)\, V^{-1}(x, x')\, \varphi(x') + i \int dx\, dx' \eta^+(x)\, G_\varphi(x, x')\, \eta(x)$$
$$+ \int dx\varphi(x)\, \big(I(x) + K(x)\big) + \frac{1}{2} \int dx\, dx' K(x)\, V(x, x')\, K(x') \qquad \text{(B.3)}$$

or

$$\mathscr{A}[\Delta] = \pm \frac{i}{2} \operatorname{tr} \log (i\boldsymbol{G}_\Delta^{-1}) + \frac{1}{2} \int dx\, dx' |\Delta(x, x')|^2\, \frac{1}{V(x, x')}$$
$$+ \frac{i}{2} \int dx\, dx' j^+(x)\, \boldsymbol{G}_\Delta(x, x')\, j(x')$$
$$+ \frac{1}{2} \int dx\, dx' \big[\Delta(y, x)\, \big(I^+(x, y) + K^+(x, y)\big) + \text{h.c.}\big]$$
$$+ \frac{1}{2} \int dx\, dx' |K(x, x')|^2\, V(x, x')\,. \qquad \text{(B.4)}$$

In this form the actions display clearly the fact that derivatives with respect to the sources K or I coincide exactly, except for all possible insertions of the direct interaction V. For example, the propagators of the plasmon field $\varphi(x)$ and of the composite operator $\int dy V(x, y)\, \psi^+(y)\, \psi(y)$ are related by

$$\overbrace{\varphi(x)\, \varphi(x')} = - \frac{\delta^2 Z}{\delta I(x)\, \delta I(x')} = V^{-1}(x, x') - \frac{\delta^2 Z}{\delta K(x)\, \delta K(x')}$$
$$= V^{-1}(x, x') + \Big(\int dy V(x, y)\, \overbrace{\psi^+(y)\, \psi(\varphi)\Big)\Big(\int dy' V(x', y')}\, \psi^+(y')\, \psi(y')\Big) \qquad \text{(B.5)}$$

in agreement with (3.5). Similarly, one finds for the pair fields:

$$\overbrace{\Delta(x, x')\, \big(\Delta}(y, y')\big)^+ = \delta(x - y)\, \delta(x' - y')\, iV(x - x')$$
$$+ \big(V(x', x)\, \overbrace{\psi(x')\, \psi(x)\big)\big(V(y', y)\, \psi^+}(y)\, \psi^+(y')\big)\,. \qquad \text{(B.6)}$$

Notice that the latter relation is manifestly desplayed in the representation (A.10) of the propagator Δ. Since

$$\overbrace{\Delta\Delta^{+}} = iV + VG^{(4)}V$$

one has from (B.6)

$$(V\overbrace{\psi\psi)\,(\psi^{+}}\psi^{+}V) = V \cdot G^{(4)}V \tag{B.7}$$

which is correct, remembering that $G^{(4)}$ is the full four-point Green's function. In the equal-time situation at instantaneous potential, $G^{(4)}$ is replaced by the resolvent R.

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