STABILITY OF HELICAL TEXTURES IN ^{3}He-A IN THE PRESENCE OF SUPERFLOW

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We analyze the stability of a variety of uniform textures in superfluid ^{3}He-A in the presence of superflow p. We find below T_{ins} a class of stable helical $\hat{l}$ textures, where $\hat{l}$ is no longer aligned with p but winds around it with a constant pitch.

In a recent letter Bhattacharyya et al. [1] analyzed the stability of uniform texture in the presence of uniform superflow and showed that the uniform texture with the gap anisotropy axis $\hat{l}$ parallel to the superflow, though stable near $T = T_c$, becomes unstable in the dipole-locked limit at low temperatures, where

$$\kappa\,[\equiv K_b\rho_0(\tfrac{1}{2}\rho_s'' + c_0)^{-2}] < 1\,, \tag{1}$$

and K_b, ρ_0, ρ_s and c_0 are coefficients in the texture free energy (see eq. (2)). The purpose of this letter is to determine the stable (uniform) texture in the region $\kappa < 1$.

We will limit ourselves in the following to only the z-dependent textures, where z is the direction of the superflow velocity. The free energy of ^{3}He-A in the dipole-locked limit is given [2,3] as

$$\begin{aligned} f = \tfrac{1}{2}\int \mathrm{d}z\,\{&(\rho_s - \rho_0\cos^2\beta)(\alpha_z + \cos\beta\,\gamma_z)^2 \\ &- 2c_0(\alpha_z + \cos\beta\,\gamma_z)\cos\beta\sin^2\beta\,\gamma_z \\ &+ (K_b\cos^2\beta + K_s\sin^2\beta)\,\beta_z^2 \\ &+ (K_b\cos^2\beta + K_t\sin^2\beta)\sin^2\beta\,\gamma_z^2\}\,, \end{aligned} \tag{2}$$

where the coefficients ρ_s, ρ_0, etc., are introduced by Mermin–Ho [2]. Here $\hat{l}$ and $\hat{\mathbf{\Delta}}$ are parameterized as

$$\hat{l} = (\sin\beta\cos\gamma,\ \sin\beta\sin\gamma,\ \cos\beta)\,,$$

$$\begin{aligned} \hat{\mathbf{\Delta}} = \mathrm{e}^{-\mathrm{i}\alpha}(&-\sin\gamma - \mathrm{i}\cos\beta\cos\gamma\,, \\ &\cos\gamma - \mathrm{i}\cos\beta\sin\gamma,\ \mathrm{i}\sin\beta)\,, \end{aligned} \tag{3}$$

and α, β, γ are the eulerian angles, which describe the spatial orientation of $\hat{\mathbf{\Delta}}$.

Since α is a cyclic coordinate, the z component of superflow p is completely uniform and can be used to eliminate α_z from f, by

$$\begin{aligned} \partial f/\partial\alpha_z \equiv p = &(\rho_s - \rho_0\cos^2\beta)(\alpha_z + \cos\beta\,\gamma_z) \\ &- c_0\sin^2\beta\cos\beta\,\gamma_z\,. \end{aligned} \tag{4}$$

Then we have

$$\begin{aligned} g = f - \int p\alpha_z\,\mathrm{d}z = \tfrac{1}{2}\int \mathrm{d}z\,\{&B(s)\,\beta_z^2 \\ &+ G(s)\,\gamma_z^2 - A(s)^{-1}p^2 + 2pH(s)\,\gamma_z\}\,, \end{aligned} \tag{5}$$

where

$$A(s) = \rho_s'' + \rho_0 s\,, \qquad B(s) = K_b(1-s) + K_s s\,,$$

$$G(s) = \{K_b(1-s) + K_t s - c_0^2 A^{-1} s(1-s)\}s\,,$$

$$H(s) = (1 - c_0 A^{-1} s)(1-s)^{1/2}\,, \tag{6}$$

and $s = \sin^2\beta$.

The dynamics of $\hat{l}$ is determined by the Cross–Anderson equation [4]

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$$-\mu \sin^2\beta\,\gamma_t = \delta g/\delta\gamma\,, \quad -\mu\beta_t = \delta g/\delta\beta\,, \qquad (7a,b)$$

where μ is the orbital viscosity.

First let us find stationary solutions, which satisfy $\gamma_t = \beta_t = 0$ (i.e., the static $\hat{l}$ texture). $\delta g/\delta\gamma = 0$ is automatically satisfied for any uniform texture with constant β and γ_z, while $\delta g/\delta\beta = 0$ allows in addition to the trivial solution with $\beta = 0$ (we call this texture I), other solutions for nonvanishing β. For these solutions γ_z is the function of β as given by

$$(\gamma_z)_\pm/p = (G')^{-1}(-H' \pm \sqrt{\Delta})\,, \qquad (8)$$

$$\Delta = (H')^2 + G'(A^{-1})'\,,$$

where the prime means the derivative in s. For $\Delta > 0$, we have stationary solutions with real $(\gamma_z)_\pm$. In particular Δ can be calculated for $s = 0$ and $s = 1$ as

$$\Delta(0) = (\tfrac{1}{2} + c_0/\rho_s'')^2(1-\kappa)\,,$$

$$\lim_{s\to 1}(1-s)\,\Delta(s) = \tfrac{1}{4}(1 - c_0\rho_s''^{-1})^2\,. \qquad (9)$$

From these we can conclude that for $\kappa > 1$, there is only one stationary region III near $s = 1$ ($\beta_2 < \beta < \pi/2$), while for $\kappa < 1$, there appears another stationary region II near $s = 0$ ($0 < \beta < \beta_1$). These stationary regions are shown schematically in fig. 1, where $\kappa = 1$ corresponds to the point $\epsilon = 0.1$. Two threshold values of β (β_1 and β_2) are determined by

$$\Delta = 0\,. \qquad (10)$$

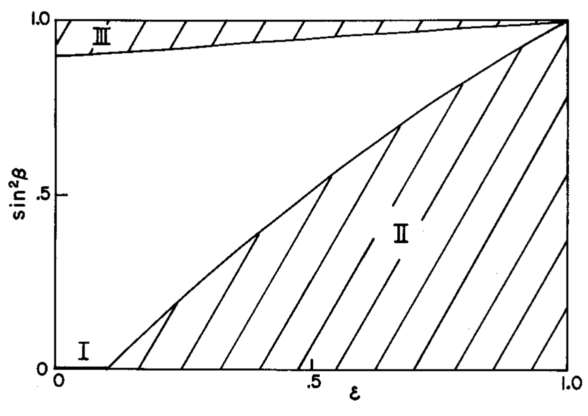

Fig. 1. The three regions (I, II, and III) where stationary solutions ($\delta g/\delta\gamma = 0$, $\delta g/\delta\beta = 0$) exist are shown. ϵ = 0, 0.1, and 1 corresponds to $T = T_c$, T_{ins}, and 0, respectively.

In particular in the vicinity of $\kappa = 1$, β_1 is given by

$$\sin^2\beta_1 = (1-\kappa)/(2U) + O(1-\kappa)^2\,, \qquad (11)$$

where

$$U = \rho_0/\rho_s'' + K_t/K_b - c_0^2/(\rho_s'' K_b)$$
$$- (\rho_0 + \tfrac{3}{4}\rho_s'')\cdot(\tfrac{1}{2}\rho_s'' + c_0)^{-1}\,. \qquad (12)$$

Secondly, we have tested the stability of these solutions against small perturbations; assuming that $\gamma_z = c + \delta\gamma_z$ and $\beta = \beta_0 + \delta\beta$, eqs. (7a) and (7b) are solved for $\delta\gamma_z, \delta\beta \propto e^{-\lambda t + ikz}$, where c is $(\gamma_z)_\pm$ given in eq. (8) as functions of β_0. $\lambda > 0$ implies the stability of the solution. λ is easily found to be

$$\lambda_\pm = \mu^{-1}\{(B + Gs^{-1})k^2 + L$$
$$\pm ([(B + Gs^{-1})k^2 + L]^2 - 4BGs^{-1}k^4 - 4Dk^2)^{1/2}\}\,, \qquad (13)$$

where

$$L = 2s(1-s)[G''(s)c^2 + 2pH''c - (A^{-1})''p^2]\,,$$

$$D = GL - 4s(1-s)p^2(H'^2 + G'(A^{-1})')\,. \qquad (14)$$

Since we have $B, G, A > 0$, the stability criteria are

$$L > 0\,, \quad D > 0\,. \qquad (15)$$

The second condition can be rewritten as

$$K = \left(\frac{\partial^2 g}{\partial\beta^2}\right)\left(\frac{\partial^2 g}{\partial\gamma_z^2}\right) - \left(\frac{\partial^2 g}{\partial\beta\partial\gamma_z}\right)^2 > 0\,, \qquad (16)$$

where K is the gaussian curvature of g in the β–γ_z space. In the light of the stability criteria, we find

(a) The region I ($\beta = 0$) is stable only for $\kappa > 1$ and becomes unstable for $\kappa < 1$.

(b) There are two distinct stable branches in the region II, II$_+$ ($\beta_+ < \beta < \beta_1$) and II$_-$ ($\beta_- < \beta < \beta_1$) with $\gamma_z = (\gamma_z(\beta))_+$ and $(\gamma_z(\beta))_-$, respectively, for $\kappa < 1$.

(c) The region III is never stable.

The result (a) confirms earlier analysis [1,5] ‡1. On the other hand for $\kappa < 1$ (which will be realized at low temperatures [5]), there appear new stable textures, where the $\hat{l}$ vector winds around the z-axis in a form of helix.

We have studied these helical solutions in greater

‡1 See footnote on next page.

detail within a simplified model where only ρ_0 deviates significantly from the G–L values

$$(1-\epsilon)^{-1}\rho_0 = \rho_s'' = c_0 = \tfrac{2}{5} K_{t,b,s}\,, \qquad (17)$$

where $\epsilon = 0$ corresponds to the G–L limit. Within this model β_1 and β_2 are given exactly by

$$\sin^2\beta_{1,2}(\epsilon)$$

$$= \{3 + 6\epsilon \mp (1-\epsilon)\sqrt{17-10\epsilon}\}(8+\epsilon^2)^{-1}\,, \qquad (18)$$

which is shown in fig. 1. In fig. 1, the shaded areas are three regions of stationary solutions. ϵ may be considered as a parameter scaled with the temperature; $\epsilon = 0$ at $T = T_c$, $\epsilon = 0.1$ at $T = T_{ins}$ and $\epsilon = 1$ at $T = 0$, where T_{ins} is the instability temperature ($\kappa = 1$), where the parallel $\hat{l}$ texture to the superflow becomes unstable. Making use of the stability criteria (15), the stable regions II_+ and II_- are determined numerically within the same model and shown in fig. 2. The corresponding $c = (\gamma_z^0)_\pm$ are also shown in the same figure. We

[‡1] In refs. [1] and [5] the stability is analyzed around a constant α_z. In this case general texture fluctuation is decribed in terms of $\delta\alpha_z$, $\delta\beta$, and $\delta\gamma_z$. However, it is shown that $\delta f = \frac{1}{2}A(s)(\delta p)^2 + \delta g$, where δf, δp, and δg are the fluctuation in f, p and g, respectively. Furthermore, δp is a linear combination of $\delta\alpha_z$, $\delta\beta$, and $\delta\gamma_z$ (see eq. (4)). Therefore the stability region is independent of whether p is fixed or not fixed. In the case of the parallel alignment the situation with fixed p is equivalent to that with fixed v_s, the superfluid velocity. However, in a more general context these two conditions are not necessarily equivalent.

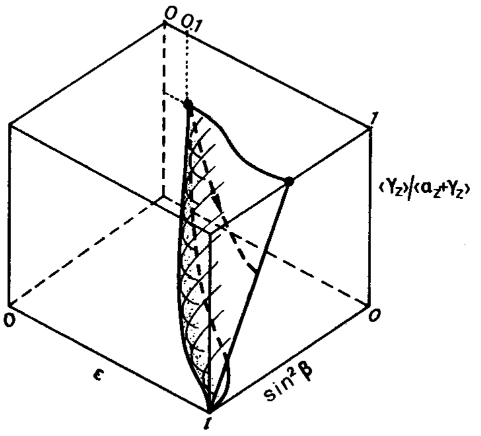

Fig. 2. The stable region of the helical textures is shown. The helical texture appears for all temperature below T_{ins} ($\epsilon = 0.1$). The broken curve with an arrow indicates the helical texture with $\langle\gamma_z\rangle/\langle\alpha_z+\gamma_z\rangle = 3/5$.

note that two regions II_+ and II_- form a continuous sheet in this presentation. In the limit κ approaches 1, the stability regions shrink linearly, both $\beta_+(\epsilon)$ and $\beta_-(\epsilon)$ converging towards $\beta_1(\epsilon)$. In this limit the unique value of $\beta_1(\epsilon)$ has been given already in eq. (11), without our simplifying assumption (18). For the model (17),

$$U = \tfrac{2}{5} + O(\kappa - 1) \quad \text{implying} \quad \sin^2\beta = \tfrac{5}{4}(1-\kappa)\,. \qquad (19)$$

When $T < T_{ins}$, as shown in fig. 2, we have a class of helical textures with a constant tilt angle β and the corresponding pitch $\gamma_z(\beta)$, which are locally stable. If liquid ^{3}He-A in a long container is cooled *slowly* through $T = T_{ins}$, the resulting helical $\hat{l}$ texture should be the one corresponding with the pitch γ_z given at the point $\kappa = 1$; $\gamma_z = (\gamma_z)_1 \equiv \frac{1}{2}p(\rho_s'' + 2c_0)/(K_b\rho_s'')$. This is because just above T_{ins}, the thermal fluctuation gives rise to distribution of γ_z,

$$P(\gamma_z) \propto \exp\{-(K_b/2KT)\langle s\rangle V[\gamma_z - (\gamma_z)_1]^2\}\,,$$

where T, V, and $\langle s\rangle$ are the temperature, the volume of the container and the thermal average of s, respectively. This implies that γ_z fluctuates around $(\gamma_z)_1$ above T_{ins}. In other words, the slow cooling proceeds along the horizontal line drawn in fig. 2, with increasing β as the temperature decreases.

In the case of superflow in a torus, the constancy of γ_z follows from the following argument. The uniqueness of the condensate order parameter in ^{3}He-A requires

$$\sin\beta\, e^{-i\alpha} = \sin\beta'\, e^{-i\alpha'}\,,$$

$$(1+\cos\beta)\, e^{-i(\alpha+\gamma)} = (1+\cos\beta')\, e^{-i(\alpha'+\gamma')}\,,$$

$$(1-\cos\beta)\, e^{-i(\alpha-\gamma)} = (1-\cos\beta')\, e^{-i(\alpha'-\gamma')}\,, \qquad (20)$$

where primed α, β and γ are those obtained after circling around the torus along a closed path in the torus. When β never passes $\beta = \pi$ (or $\beta = 0$) in the torus, eq. (20) reduces to the conditions

$$\beta = \beta'\,, \quad \alpha - \alpha' = 2\pi n\,, \qquad \gamma - \gamma' = 2\pi m\,, \qquad (21)$$

where n and m are integers. In the case of helical solutions in a torus, these integers are the topological conserved quantities, implying constant γ_z when $\kappa < 1$; when the system is cooled smoothly, γ_z and α_z are constant. On the other hand, in the case of a long cylinder with open ends, γ_z ($\equiv c$) may relax to the value with the minimum g, when P is fixed. The local stability of the helical solutions appear to contradict the conjectured instability of the superflow by Bhattacharyya et al. [1]. Analyzing their dipole-unlocked case, within the present framework, we have discovered peculiar features, which are quite different from those in the dipole-locked case below T_{ins}. First of all, $\Delta(s) > 0$ for all β in the dipole-unlocked cases, implying the existence of a stationary solution with arbitrary β. Furthermore, we find $D < 0$ for all β; none of these solutions are stable. Therefore, we believe that their dipole-unlocked case does show intrinsic instability of any texture with uniform superflow, unlike the dipole-locked case below T_{ins}.

As already pointed out by Cross and Liu [5], the correct analysis of the orbital dynamics does not show the instability of Hall and Hook [6] nor the existence of the orbitary solitary wave.

After completing this work we have received a preprint [7] from A.L. Fetter, who has shown the stability of the helical texture below T_{ins}, although his analysis is limited to the vicinity $\kappa = 1$.

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