COLLECTIVE EXCITATIONS OF $^3$He B IN THE PRESENCE OF SUPERFLOW

Hagen KLEINERT
Department of Physics, University of California, Berkeley, CA 94720, USA
and Institute of Theoretical Physics, Department of Physics, Stanford University,
Stanford, CA 94305, USA

Received 3 April 1980

The energy levels of collective excitations of $^3$He B are given in the presence of superflow. There is level splitting due to the distortion of the energy gap which should be observable experimentally.

The recently observed BCS-like behavior of the critical currents of superfluid $^3$He B at zero pressure [1] has led to a renewed theoretical interest in the flow properties of this phase. A complete Ginzburg–Landau type of study close to $T_c$ [2] and a BCS calculation for all $T$ neglecting gap distortion have been available [3] for some time. The results have now been improved by calculating all BCS properties including the effects of gap distortion for all temperatures [4]. It turns out that the maximal distortion of transverse to longitudinal gap $\gamma^2 \equiv 1 - \gamma^2 \equiv 1 - \Delta_T^2/\Delta_F^2$, which is $4/9$ for $T \leq T_c$ at the critical velocity $v_c/v_0 = \frac{1}{3}(5/3)^{\frac{1}{2}}(1 - T/T_c)^{\frac{1}{2}}$ ($v_0 = 1/2m^*\xi_0 \approx 6.3$ cm/s at zero pressure, $m^* =$ effective mass, $\xi_0 =$ coherence length), gradually decreases to zero for $T \to 0$.

It is the purpose of this note to point out how the distortion of the gap manifests itself in the energy spectrum of collective excitations. We find that the well-known $J = 2$ levels $\omega^2 = \frac{\xi}{3} \Delta_F^2$ and $\omega^2 = \frac{12}{5} \Delta_F^2$ split up into three branches each, the $\pm J_\delta$ levels remaining degenerate. The results should be observable in sound experiments [5], in particular at zero pressure where the line widths are small.

Consider the collective action [6,7] in the weak coupling limit

$$\mathcal{A} = -\frac{1}{2} \text{Tr} \log \left( \begin{array}{c} i \partial_t - \xi + p \cdot v \\
\frac{\sigma_A \Lambda_{\alpha} \Lambda_{\beta} \Lambda_{\gamma}}{2 \rho_F} \end{array} \right) \frac{\sigma_A \Lambda_{\alpha} \Lambda_{\beta} \Lambda_{\gamma}}{2 \rho_F} \frac{1}{i \partial_t + \xi + p \cdot v} - \frac{1}{2} \int d^3 x \int d^3 p \left| A_{\alpha \beta \gamma} \right|^2,$$

(1)

where $\xi$ stands for $-\vec{v}^2/2m - \mu$ and $\sigma_A \Lambda_{\alpha} \Lambda_{\beta} \Lambda_{\gamma}$ are the Pauli matrices. The trace runs over $4 \times 4$ matrix indices as well as space–time variables. A constant flow velocity is enforced on the average by means of an external source coupling to the particle current which appears in the diagonal terms as $p \cdot v$ [8]. Apart from the full quantum mechanics, the partition function obtained by summing over all fluctuating field configurations $Z = \Sigma \{ A_{\alpha \beta \gamma} \} \exp \{ i \mathcal{A} \{ A_{\alpha \beta \gamma} \} \}$ describes the thermodynamics at a fixed temperature $T$ by Wick rotating all energy integrations and grating them into Matsubara frequencies $\omega_n = (2n + 1)\pi T$. For $T \leq T_c$, there is equilibrium at a non-zero constant value ($T^2 = 1$) $A_{\alpha \beta \gamma} = \Delta_0 \Lambda_{\alpha} \Lambda_{\beta} \Lambda_{\gamma} + (c - 1)\Lambda^2 \Lambda_3$ which gives rise to an energy gap in the quasiparticle spectrum: $E = (\xi^2 + \left| A_{\alpha \beta \gamma} \right|^2)^{1/2} \equiv \{ \xi^2 + \Delta_0^2(1 - r^2) \}^{1/2}$. The two gap values $\Delta_1$, $\Delta_2 \equiv \Delta_0 (1 - r^2)^{1/2}$ are found from the simultaneous solution of a longitudinal and a transversal gap equation [4].

1 On sabbatical leave from Institut für Theoretische Physik, Freie Universität, Berlin, 1 Berlin 33; supported in part by Deutsche Forschungsgemeinschaft.
\[
\log \frac{T}{T_c} = \int_{-1}^{1} \frac{dz}{2} \left\{ \frac{3z^2}{2} (1 - z^2) \right\} \gamma, \tag{2}
\]

where \( \gamma \) denotes the function

\[
\gamma = \int_{-\infty}^{\infty} \frac{d\xi}{4E} \left\{ \text{th}[E + (p_{F} v_{F})/2T] + (z \to -z) \right\} - [\Delta_{\parallel} = 0]. \tag{3}
\]

For \( T \leq T_c \) one finds \([4]\) \( r^2 = 2(p_{F} v_{F})/2T = 3v^2 / 2T \). For \( T = 0 \), \( r \) stays at zero up to \( p_{F} v_{F} / \Delta_{\perp} = 1 \), i.e. practically up to the critical current \([4]\).

Consider now small time dependent fluctuations in the complex order parameter \( A_{\alpha i} = A_{\alpha i}^0 + \Delta_{\alpha i} \delta A_{\alpha i} (r) = A_{\alpha i}^0 + \Delta_{\alpha i} (r) \). Expanding the action \( \mathcal{A} \) up to quadratic order we find with \( d_i(r) = \sum_{\nu_B} \exp(-i\nu_B r) \delta d_i(r) \):

\[
i \mathcal{A} = -\frac{F_{\text{free}}}{T} - \frac{F_{\text{cond}}}{4\pi^2} \rho \frac{V}{T} \sum_{\nu_B} (\lambda_{\nu B}^1 \nu_B) \left\{ d_i(r)^2 - [\lambda_{\nu B}^1 \nu_B] |d_i(r)|^2 \right\} \]

\[
+ \Re \{ \delta A_{\alpha i} (r) \delta A_{\beta j} (r) \} - \left( e^2 \omega_{1} + \omega_{2} \right) \delta A_{\alpha i} (r) \delta A_{\beta j} (r) \}
\]

\[
+ \left( \omega_{1} + \omega_{2} - e^2 \omega_{3} \right) \delta A_{\alpha i} (r) \delta A_{\beta j} (r) + 2 \omega_{1} \omega_{2} \delta A_{\alpha i} (r) \delta A_{\beta j} (r) + \left( \omega_{1} + \omega_{2} + e^2 \omega_{3} \right) \delta A_{\alpha i} (r) \delta A_{\beta j} (r) \} \]. \tag{4}

Here \( \omega_{1}, \omega_{2}, \omega_{3} \) are angular projections of the dynamical generalization of Yoshida's function \( \phi \):

\[
\omega_{1,2,3} = \frac{1}{2} \int_{-1}^{1} \frac{dz}{2} \left\{ 3z^2 (1 - z^2), \frac{3}{2} (1 - z^2), 3z^4 \right\} \phi, \tag{5}
\]

with

\[
\phi = \frac{\Delta_{\perp}^2}{2} \int_{-\infty}^{\infty} \frac{d\xi}{E} \frac{1}{E^2 - (\omega + i\xi)^2/4} \left\{ \text{th}[E + (p_{F} v_{F})/2T] + (z \to -z) \right\}. \tag{6}
\]

The functions \( \lambda_{\parallel}^1, \lambda_{\parallel}^2 \) are the integrals

\[
\lambda_{\parallel}^1 = \int_{-1}^{1} \frac{dz}{2} \left\{ 3z^2 \left( \frac{3}{2} (1 - z^2) \right) \right\} (1 - r^2 z^2 - \omega^2/4\Delta_{\perp}^2) \phi, \tag{7}
\]

and can be expressed in terms of \( \omega_{1,2,3} \) as

\[
\omega_{1} = 2 \omega_{1} + \omega_{2} \omega_{3} - 2(\omega^2/4\Delta_{\perp}^2)(2\omega_{1} + \omega_{3}), \quad \omega_{1} = e^2 \omega_{1} + 2\omega_{2} - 2(\omega^2/4\Delta_{\perp}^2)(\omega_{1} + \omega_{2}).
\]

In writing eq. (6) we have continued analytically to physical frequencies \( \omega \) by merely replacing the Matsubara frequencies \( \omega_{n} \) by \(- (\omega + i\xi)^2/4\). Notice that all these functions depend on \( \omega \) only via \( w^2 = \omega^2/4\Delta_{\perp}^2 \) which will henceforth be used as natural variable.

The expression (4) can be diagonalized in the spaces \( r_{1,1}, r_{2,2}, r_{3,3}, (r_{1,2}, r_{2,3}), (r_{1,3}, r_{3,1}), (r_{2,3}, r_{3,2}) \) and the corresponding imaginary parts. In these spaces, the sum in eq. (4) is composed of the following matrices:

\[
R = \begin{pmatrix}
3\sigma_{2} - 2w^2(\omega_{1} + \omega_{2}) & \omega_{2} & 2\omega_{1} \\
\omega_{2} & 3\sigma_{2} - 2w^2(\omega_{1} + \omega_{2}) & 2\omega_{1} \\
2\omega_{1} & 2\omega_{1} & 2\sigma_{2} - 2w^2(\omega_{1} + \omega_{3})
\end{pmatrix},
\]

156
\[ R^{12} = \begin{pmatrix} a_2 - 2w^2(a_1 + 2a_2) & 0 \\ a_2 & a_2 - 2w^2(a_1 + 2a_2) \end{pmatrix}, \quad R^{13} = \begin{pmatrix} 2\sigma_1 - 2w^2(2\sigma_1 + \sigma_3) & 2\sigma_1 \\ 2\sigma_1 & 2w^2(2\sigma_1 + \sigma_3) \end{pmatrix}, \quad R^{23} = \begin{pmatrix} 2\sigma_1 - 2w^2(2\sigma_1 + \sigma_3) & 2\sigma_1 \\ 2\sigma_1 & 2w^2(2\sigma_1 + \sigma_3) \end{pmatrix}, \]

\[ I = \begin{pmatrix} 2\sigma_2 + \sigma_2 - 2w^2(a_1 + 2a_2) & -\sigma_2 \\ -\sigma_2 & 2\sigma_2 + \sigma_2 - 2w^2(a_1 + 2a_2) \end{pmatrix}, \quad I^{12} = \begin{pmatrix} 2\sigma_2 + \sigma_2 - 2w^2(a_1 + 2a_2) & -\sigma_2 \\ -\sigma_2 & 2\sigma_2 + \sigma_2 - 2w^2(a_1 + 2a_2) \end{pmatrix}, \]

\[ I^{13} = \begin{pmatrix} (2\sigma_1 + c^2\sigma_3) - 2w^2(2\sigma_1 + \sigma_3) & -2\sigma_1 \\ -2\sigma_1 & 4\sigma_2 - 2w^2(2\sigma_1 + \sigma_3) \end{pmatrix}, \quad I^{23} = \begin{pmatrix} (2\sigma_1 + c^2\sigma_3) - 2w^2(2\sigma_1 + \sigma_3) & -2\sigma_1 \\ -2\sigma_1 & 4\sigma_2 - 2w^2(2\sigma_1 + \sigma_3) \end{pmatrix}. \]

The collective frequencies are determined by the values \( w \) at which the determinants vanish. They are found as

\[ w^2_1 = \sigma_2/(\sigma_1 + 2\sigma_2), \]
\[ w^2_3 = \frac{\sigma_2}{\sigma_1 + 2\sigma_2} + \frac{1}{2} \frac{c^2\sigma_3 + \sigma_1}{2\sigma_1 + \sigma_3} = \frac{\alpha}{2(\sigma_1 + 2\sigma_2)(\sigma_1 + \sigma_3)}, \]
\[ \alpha \equiv [(\sigma_1 + 2\sigma_2)^2 - (\sigma_1 + \sigma_3)2\sigma_2] + \frac{8c^2\sigma_1^2(\sigma_1 + 2\sigma_2)(\sigma_1 + \sigma_3)}{1}. \]

For \( R \):

\[ R^{12}: \quad w^2_{12} = \begin{pmatrix} 0 \\ \sigma_2/(\sigma_1 + 2\sigma_2) \end{pmatrix}, \quad (1, -1, 0) \equiv \{22\} + \{2 -2\}, \]
\[ (1, 1) \equiv \{22\} - \{2 -2\}, \]
\[ (1, -1/c) \equiv \{11\} \pm \{1 -1\}, \]
\[ R^{13}: \quad w^2_{13} = \begin{pmatrix} 0 \\ \sigma_1/(\sigma_1 + 2\sigma_2)(\sigma_1 + \sigma_3) \end{pmatrix} \]
\[ [(2\sigma_2 + 1)\sigma_1 + 2\sigma_2 + c^2\sigma_3], \quad (1, -1/c) \equiv \{21\} \pm \{2 -1\} + \ldots, \]
\[ (1, 1, c) \equiv \{00\} + \ldots, \]
\[ R^{23}: \quad w^2_{23} = \begin{pmatrix} 0 \\ 2\sigma_2/(\sigma_1 + 2\sigma_2) \end{pmatrix}, \quad (1, -1, 0) \equiv \{22\} + \{2 -2\}, \]
\[ (1, 1) \equiv \{20\}, \quad \text{Eq. (8)}. \]

For \( I \):

\[ I^{12}: \quad w^2_{12} = \begin{pmatrix} c^2\sigma_1 + \sigma_2 \\ \sigma_1 + 2\sigma_2 \end{pmatrix}, \quad (1, -1) \equiv \{10\}, \]
\[ (1, 1) \equiv \{22\} - \{2 -2\}, \]
\[ I^{13}: \quad w^2_{13} = \begin{pmatrix} c^2\sigma_1 + \sigma_2 \\ \sigma_1 + 2\sigma_2 \end{pmatrix}, \quad (1, -1/c) \equiv \{11\} \pm \{1 -1\} + \ldots, \]
\[ \{11\} \pm \{1 -1\} + \ldots, \]
\[ I^{23}: \quad w^2_{23} = \begin{pmatrix} \sigma_2 \\ \sigma_1 + 2\sigma_2 \end{pmatrix}, \quad (1, 1) \equiv \{22\} - \{2 -2\}, \]
\[ \alpha \equiv [(\sigma_1 + \sigma_3)2\sigma_2 - (\sigma_1 + c^2\sigma_3)(\sigma_1 + 2\sigma_2)]^2 + 4c^2\sigma_1^2(\sigma_1 + \sigma_3)(\sigma_1 + 2\sigma_2) \]}^{1/2}. \]
Behind each eigenvalue we have written down the eigenvector if it is simple as well as the $|J3\rangle$ content. The symbol $\ldots$ indicates that for $c \neq 1$ there is mixing among states with $\pm J3$. For small current or for all $\nu \Delta_{v}/\Delta_{k} \ll 1$ at $T = 0$, the distortion parameter $r$ is zero, $F$ is independent of $x$, and we see from eq. (10) that $\sigma_{1} : \sigma_{2} : \sigma_{3} = 1 : 2 : 3$. Consequently, we recover the well-known frequencies

$$\omega/\Delta_{k} = (\sqrt{8/5}, \sqrt{8/5}, 1)^{R}, \quad (0, \sqrt{8/5})^{R}_{12}, \quad (0, \sqrt{8/5})^{R}_{13}, \quad (0, \sqrt{12/5}, \sqrt{12/5})^{T}, \quad (2, \sqrt{12/5})^{R}_{12}, \quad (2, \sqrt{12/5})^{R}_{13},$$

(9)

now valid also in the presence of currents at $T = 0$.

Since gap distortion is strongest close to $T_{c}$ let us now consider this regime. Here $\phi$ has the simple form $\phi = (\pi \Delta_{k}/4T)(1 - r^{2})^{1/2}$, such that $\sigma_{1,2,3}$ can be calculated, apart from the factor $\pi \Delta_{k}/4T$ as

$$\sigma_{1} = (3/4r^{2})\{(-\frac{3}{4}(1 - w^{2}) + r^{2}(1 - w^{2})/l + (\frac{3}{4}(1 - w^{2}) - \frac{7}{4}r^{2})r(1 - w^{2} - r^{2})/l\},$$

$$\sigma_{2} = \frac{2}{3}(3/4r^{2})\{(1 - w^{2})^{2} - \frac{7}{8}r^{2}(1 - w^{2} - r^{2})/l + (-1 - w^{2} + 2r^{2})r(1 - w^{2} - r^{2})/l\},$$

$$\sigma_{3} = \frac{2}{3}(3/4r^{2})\{[(1 - w^{2})^{2} - \frac{7}{8}r^{2}(1 - w^{2} - r^{2})/l - (1 - w^{2}) + \frac{7}{8}r^{2})r(1 - w^{2} - r^{2})/l\}.$$  

(10)

Here, $l$ denotes the function $rf_{-1}(dz/2)(1 - r^{2})^{1/2} = \arcsin [r(1 - w^{2})^{1/2}]$. By inserting the values (9) into the right-hand side of eqs. (10) and iterating eqs. (8) a few times, the values of $\omega$ converge rapidly against the correct eigenvalue at any $r^{2} \approx r_{c}^{2} = 5/9$. The results are displayed in fig. 1, together with the $T = 0$ lines.

It is quite simple to include Fermi liquid corrections [10]: in all equations one has to use, instead of $\nu$, the local velocity $\nu_{s}$, which is $\nu$ modified by the molecular field $\frac{1}{2}F_{1}^{2}\rho_{n}^{3}$, i.e., $[1 + \frac{1}{2}F_{1}^{2}\rho_{n}^{3}(\nu^{*})]\nu^{*} = \nu$, where $\rho_{n}^{3}$ is the density of the normal component and $F_{1}^{2}$ is the standard Landau parameter of current-current coupling [10].

It is hoped that the splitting between the $\sqrt{12/5}$ $\Delta_{k}$ modes will soon be detected in sound experiments. A similar discussion of the collective modes in the A phase is not yet possible since no field configuration is known, in the

---

Fig. 1. The collective frequencies are shown, at infinite wavelength, as a function of $(\omega^{2}/\nu_{s}^{2})(1 - T/T_{c})^{-1}$. The dashed lines are the limits $T = 0$. For $T \lesssim T_{c}$, there is appreciable splitting between levels of different $J3$. The angular momenta are displayed on the right-hand side of each curve. For $T = T_{c}$, the gap distortion $r^{2} = 1 - \Delta_{k}^{2}/\Delta_{k}^{2}$ is related to the superfluid velocity by $r^{2} = 3 \times (\nu^{2}/\nu_{0}^{2})(1 - T/T_{c})^{-1}$ ($\nu_{0} = 1/2m^{3/2}t_{0} \approx 6.5$ cm/s at zero pressure). The curves are drawn up to the critical velocity.
presence of a current, which is stable under infinitesimal static fluctuations, except for very small velocities limited by the dipole interaction ($v \leq 0.1 \text{ mm/s}$) [11,12]. Only in this very restricted region are there two types of stable helical textures around which dynamic fluctuations may be studied. More details will be presented elsewhere [13].

**Note added.** After completion of this work a paper appeared by Tewordt and Schopohl [14], which discusses the line splitting due to a strong magnetic field.

**References**