NEW SYMMETRIES AND CONSTANTS OF THE MOTION FROM DYNAMICAL GROUPS ☆

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It is shown that every dynamical group (sometimes also called spectrum generating group) gives rise to a proper Noether symmetry group of the action. For each generator there is a constant of the motion. Those which do not commute with the hamiltonian but connect states of different energy contain an explicit time dependence when expressed as a function of the Heisenberg variables $p(t)$, $q(t)$ which ensures their conservation. If the hamiltonian is in the Lie algebra, this time dependence is given by a simple "rotation" matrix in the adjoint representation. The statements are illustrated by exhibiting the conserved symmetry operators for the bound state problem with electric and magnetic charges.

With resurging interest in the quantum mechanics of particles in monopole fields [1], group theory will be helpful in solving the dynamical problem [2]. After having exhausted the groups commuting with the hamiltonian the question arises whether there are higher symmetries which connect levels of different energies but still leave the action invariant ‡1. Their generators would be constants of the motion with an explicit time dependence when expressed as functions of the canonical variables $p(t)$, $q(t)$. Most desirable would be a symmetry group which contains the whole spectrum of physical states in one of its irreducible representations with the hamiltonian and $q(t)$ being among the generators. In this case the trajectory would be given by a time dependent group "rotation" matrix in the adjoint representation.

It has been known for a long time that many quantum mechanical systems do possess a group with the property of generating the spectrum. Since part of the generators do not commute with the hamiltonian they have been called non-invariance, dynamical, or spectrum generating groups [3,4] ‡2. What seems to have gone unnoticed is that they are associated with a group of proper symmetry transformations in the Noether sense [6]. It is really this property which, if the hamiltonian is one of the generators, leads to a full solution of the dynamical problem in terms of a "rotation" matrix.

In order to see this we observe that for N degrees of freedom there are $2N$ constants of the motion which are simply the Schrödinger operators p_S, q_S:

$$p_S \equiv p(0) = e^{-iHt}\, p(t)\, e^{iHt} ,$$

$$q_S \equiv q(0) = e^{-iHt}\, q(t)\, e^{iHt} . \tag{1}$$

These equations are non-trivial since the right-hand sides must be calculated explicitly by commuting $H(p(t), q(t))$ with $p(t)$, $q(t)$, i.e.

$$p_S = P(t, p(t), q(t)) ,$$

$$q_S = Q(t, p(t), q(t)) . \tag{2}$$

Of course, with p_S, q_S also any function

$$F(p_S, q_S) = F(p(0), q(0)) = e^{-iHt} F(p(t), q(t))\, e^{iHt}$$
$$= F(P(t, p(t), q(t)), Q(t, p(t), q(t)))$$
$$\equiv J(t, p(t), q(t)) \tag{3}$$

is a constant of the motion. By Noether's theorem [6], the infinitesimal transformations generated by J via

$$\delta q(t) \equiv i[J(t, p(t), q(t)), q(t)] ,$$

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‡1 Throughout this work we will call an action invariant if it has this property upto surface terms.

‡2 The dynamical group O(4,2) for the hydrogen atom was developed in ref. [4]. This work was generalized to the motion in Coulomb *and* monopole field (plus centrifugal barrier) by Barut and Bornzin [5].

$$\delta p(t) \equiv \mathrm{i}[J(t, p(t), q(t)), p(t)] \,, \tag{4}$$

must leave the action invariant. Conversely, given a transformation (4) in the form

$$\delta q(t) = f(t, q(t), \dot{q}(t)) \,, \tag{5}$$

i.e. with $p(t)$ expressed in terms of $\dot{q}(t)$ via $p = \partial L/\partial \dot{q}$, the lagrangian changes only by a total time derivative

$$\delta L = (\mathrm{d}/\mathrm{d}t)\, \Delta(q(t), \dot{q}(t)) \tag{6}$$

But due to the equations of motion, δL can always be evaluated as

$$\delta L(q(t), \dot{q}(t)) = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) = \frac{\mathrm{d}}{\mathrm{d}t}(p\,\delta q) \,, \tag{7}$$

such that comparison with (6) proves

$$J(t) = p(t)\,\delta q(t) - \Delta(q(t), \dot{q}(t)) \tag{8}$$

to be a conserved quantity which, if $\dot{q}(t)$ is expressed in terms of $p(t)$, $q(t)$, becomes precisely the operator $J(t, p(t), q(t))$ generating $\delta q(t)$ via (4).

In general, the evaluation of the equations of motion (2) is a difficult problem such that realizing p_S, q_S to be conserved quantities is of no use. It may, however, happen that there is a finite number of functions $F_a(p(t), q(t))$ which under commutation with H are transformed linearly among each other:

$$[H, F_a(p(t), q(t))] = \mathrm{i}F_b(p(t), q(t))\, h_{ba} \,. \tag{9}$$

In this case we can calculate the explicit time dependence of the constants of the motion J_a of (3) as

$$J_a \equiv J_a(t, p(t), q(t)) = \mathrm{e}^{-\mathrm{i}Ht} F_a(p(t), q(t))\, \mathrm{e}^{\mathrm{i}Ht}$$

$$= F_b(p(t), q(t))\, R_{ba}(t) \,, \tag{10}$$

where the transformation matrix is simply the exponential

$$R(t) = \mathrm{e}^{ht} \,. \tag{11}$$

Since the J_a are constants of the motion we may invert (10) as

$$F_a(p(t), q(t)) = R(-t)_{ba}\, J_b \tag{12}$$

and obtain the full time dependence of F_a from $R(t)$.

If $q(t)$ can be extracted from $F_a(p(t), q(t))$ by purely algebraic manipulations, the dynamical problem is solved.

The solubility of the oscillator may be viewed as a consequence of this situation: Here $H = \frac{1}{2}(p^2 + q^2)$ and the quantities p_S, q_S themselves can be used as the functions F_1, F_2 which are transformed into each other via (9):

$$[H, p(t)] = \mathrm{i}q(t), \quad [H, q(t)] = -\mathrm{i}p(t) \,. \tag{13}$$

The corresponding conserved quantities are found from (10) as

$$J_1 \equiv p(t)\cos t + q(t)\sin t \,,$$

$$J_2 \equiv -p(t)\sin t + q(t)\cos t \,, \tag{14}$$

which may be inverted to express the motion of $p(t)$, $q(t)$ in terms of the constants J_1, J_2. Conversely, commuting J_1, J_2 with q via (4) we find the symmetry transformations

$$\delta_1 q(t) = \cos t \,, \qquad \delta_2 q(t) = -\sin t \,,$$

$$\delta_1 p(t) = -\sin t \,, \qquad \delta_2 p(t) = -\cos t \,, \tag{15}$$

which leave the action invariant:

$$\delta_{\substack{1\\2}} L = -\frac{\mathrm{d}}{\mathrm{d}t} q(t) \begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\Delta \,. \tag{16}$$

The corresponding Noether currents (8) are again J_1, J_2 of (14).

Of course, the oscillator and its zero frequency limit, the free particle, are the only cases where $p(t)$, $q(t)$ themselves form a closed set under the commutation rules (9). There are, however, several physical systems, where some combinations of p_S, q_S are known which, together with the hamiltonian, form a Lie algebra of a group [3–5]. In fact, the oscillator itself is an example of this; the functions $F_1 = \frac{1}{4}(p^2 - q^2)$, $F_2 = \frac{1}{4}(pq + qp)$, satisfy, together with $F_3 = \frac{1}{2}H$, the algebra of O(2,1): ($[F_1, F_2] = -\mathrm{i}F_3$, $[F_2, F_3] = \mathrm{i}F_1$, $[F_3, F_1] = \mathrm{i}F_2$). Thus eqs. (9) read $[H, F_1] = 2\mathrm{i}F_2$, $[H, F_2] = -2\mathrm{i}F_1$, and the constants of the motion (10) become $J_1 = \cos 2t\, F_1 + \sin 2t\, F_2$, $J_2 = -\sin 2t\, F_1 + \cos 2t\, F_2$, $J_3 = \frac{1}{2}H$ and form a *symmetry* group O(2,1).

Let us now consider a non-trivial problem: The motion of an electrically and magnetically charged particle (e_1, g_1) around a fixed center with (e_2, g_2). The Coulomb problem is a special case of this [4,5]. The hamiltonian reads

$$H=\pi^2/2m-\alpha/r\,, \tag{17}$$

where $\alpha=e_1e_2+g_1g_2, \mu\equiv e_1g_2-e_2g_1, \boldsymbol{\pi}=p-\mu A$. The problem has a simple dynamical group if there is an additional centrifugal barrier $\mu^2/2mr^2$. It is generated by the following combinations of $\boldsymbol{\pi}(t), x(t)$:

$$F_{ij}=\epsilon_{ijk}(x\times\boldsymbol{\pi}-\mu x)_k=\epsilon_{ijk}F_k\,,$$
$$F_{i5}-F_{i4}=x_i\,,$$
$$F_{i5}+F_{i4}=x\boldsymbol{\pi}^2-2\boldsymbol{\pi}(x\cdot\boldsymbol{\pi})+(2\mu/r)F_k+(\mu^2/r^2)x\,,$$
$$F_{i6}=r\pi_i\,,$$
$$F_{56}-F_{46}=r\,,$$
$$F_{56}+F_{46}=r\boldsymbol{\pi}^2+\mu^2/r\,,$$
$$F_{56}=x\cdot\boldsymbol{\pi}-\mathrm{i}\,, \tag{18}$$

which commute according to the rules of O(4,2) [4,5]:

$$[F_{ab},F_{ac}]=\mathrm{i}g_{aa}F_{bc}\,,\quad g=\mathrm{diag.}(1,1,1,1,-1,-1)\,. \tag{19}$$

The hamiltonian with centrifugal term may be expressed as

$$2mH=(F_{56}+F_{46}-2m\alpha)/(F_{56}-F_{46})\,. \tag{20}$$

Due to the awkward quotient form this does not yet allow for a simple calculation of constants of the motion via (9). There is, however, an auxiliary dynamical problem for which this can be done.

Consider the path integral representation of the probability amplitude of the system

$$K(x_bt_b,x_at_a)=\int\frac{\mathcal{D}^3p\,\mathcal{D}^3x}{(2\pi)^3}\times\exp\left[\mathrm{i}\int_{t_a}^{t_b}\left(p\dot{q}-\frac{\boldsymbol{\pi}^2}{2m}+\frac{\alpha}{r}-\frac{\mu^2}{2mr^2}\right)\mathrm{d}t\right]. \tag{21}$$

We can make p_0, t additional dynamical variables by parametrizing the paths in terms of a new pseudo-time s which is related to t via

$$t-t_a=\int_0^s \mathrm{d}s'\,r(s')\,, \tag{22}$$

The the amplitude can be written as [7]

$$K(x_bt_b,x_at_a)=r_b\int_0^\infty \mathrm{d}s\int_{x_a}^{x_b}\frac{\mathcal{D}^3p\,\mathcal{D}^3x}{(2\pi)^3}\int_{t_a}^{t_b}\frac{\mathcal{D}p_0\,\mathcal{D}t}{2\pi}\times\exp\left[\mathrm{i}\int_0^s[px'-p_0t'-r(H-p_0)]\,\mathrm{d}s\right]. \tag{23}$$

In this way the original problem reduces to the zero pseudo-energy problem with respect to the auxiliary hamiltonian $\mathcal{H}=r(H-p_0)$, which drives the motion of all variables $p(s), q(s), p_0(s), t(s)$ along the s axis. But inserting (18) we see that $\mathcal{H}$ is indeed contained in the Lie algebra:

$$\mathcal{H}=(2m)^{-1}(F_{56}+F_{46})-(F_{56}-F_{46})p_0-\alpha\,. \tag{24}$$

Using (19) we can now easily determine h of (9), and find 15 *conserved symmetry* operators J_{ab} of (10) via the "rotation" matrix $R_{p_0}(s)$. Notice that the variables $p_0(s)$, $t(s)$ are not part of the Lie algebra but we see from (24) that also they follow condition (9):

$$\mathrm{i}[\mathcal{H},p_0(s)]=0\,,$$
$$\mathrm{i}[\mathcal{H},t(s)]=(F_{56}-F_{46})(s)=r(s)\,, \tag{25}$$

such that we immediately find $p_0(s)$ to be a constant of the motion (as it should, $p_0\equiv E$) and $t(s)$ may be evaluated from (22) after having solved for $r(s)=F_{46}-F_{56}$ from (12). Since $x(s)$ itself is contained in the Lie algebra, its motion is given by a pure group "rotation" matrix in the adjoint representation.

It should be noted that the exponentiation problem simplifies greatly [4] by observing that, with $\theta\equiv\frac{1}{2}\log(2m|p_0|)$,

$$\mathrm{e}^{-\mathrm{i}\theta F_{45}}\mathcal{H}\,\mathrm{e}^{\mathrm{i}\theta F_{45}}=\sqrt{2|p_0|/m}\,F_{56}\,,\quad p_0>0\,,$$
$$\mathrm{e}^{-\mathrm{i}\theta F_{45}}\mathcal{H}\,\mathrm{e}^{\mathrm{i}\theta F_{45}}=\sqrt{2|p_0|/m}\,F_{46}\,,\quad p_0<0\,. \tag{26}$$

This reveals the bound and continuous states as the discrete and continuous eigenvectors of F_{56}, F_{46}, respectively, within the same irreducible representation (18). The transformation is non-unitary due to the dependence of the "angle" θ on the energy. This is linked to the fact that only *together* the two parts of the spectrum form a complete set of states.

If the additional potential $\mu^2/2mr^2$ is omitted, the hamiltonian becomes

$$\mathcal{H}=(2m)^{-1}(F_{56}+F_{46})-(F_{56}-F_{46})p_0-\alpha-\mu^2/2m(F_{56}-F_{46})\,, \tag{27}$$

and the full O(4,2) equations of motion (10) are again very hard to solve. There is, however, another group O(2,1) generated by

$$\bar{F}_{56} - \bar{F}_{46} = r ,$$
$$\bar{F}_{56} + \bar{F}_{46} = r\boldsymbol{\pi}^2 ,$$
$$\bar{F}_{45} = \boldsymbol{x} \cdot \boldsymbol{\pi} - \mathrm{i} , \tag{28}$$

for which the auxiliary hamiltonian reads

$$\mathcal{H} = (2m)^{-1}(\bar{F}_{56} + \bar{F}_{46}) - (\bar{F}_{56} - \bar{F}_{46})\, p_0 - \alpha , \tag{29}$$

such that at least the motion of the radial coordinate $r(s)$ can be found from a simple 3 X 3 "rotation" matrix ‡3.

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‡3 This solution is related to but different from ref. [2]. There, the square $r^2(t)$ is obtained by a group transformation in the adjoint representation (which is linear in t). Here, $r(s)$ is found directly but $s(t)$ involves one further integration, eq. (22).

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