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# Surface Energy and Textural Boundary Conditions Between A and B Phases of $^3$He

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*The behavior of the order parameter* $A_{ai}(x)$ *is investigated for a planar interface between* $^3$*He-A and* $^3$*He-B. Boundary conditions are derived and the surface energy is calculated in excellent agreement with experiment. Our energy is 30% lower than a previous estimate which assumed the presence of a planar phase in the transition region.*

## 1. INTRODUCTION

A necessary step toward the explanation of the nucleation problem between the A and B phases of superliquid $^3$He is the understanding of the energetics at the AB interface. Experimentally, the surface tension of the interface is[1,2]

$$\sigma_{AB} \simeq 0.7 \xi f_c^B \tag{1}$$

where $\xi = \xi_0(1 - T/T_c)^{1/2}$ is the temperature-dependent coherence length† and $f_c^B$ is the condensation energy density of the B phase. Neglecting the curvature of the interface, Cross[1] found the theoretical result (see Fig. 1)

$$\sigma_{AB}^{\text{Cross}} \simeq 1.1 \xi f_c^B \tag{2}$$

His analysis assumes the presence of a planar phase at the middle of the interface. Probably due to the good agreement between (1) and (2), the planar phase has played an import role in subsequent discussions of the AB interface.[3]

It is the purpose of this study to point out that there exists a direct path between the A and B phases in the functional space of the order parameter

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†$\xi_0 = [7\zeta(3)/48\pi^2]^{1/2} \hbar v_F^* / k_B T_c$, where $v_F^*$ is the Fermi velocity of the quasiparticles and $T_c$ is the critical temperature. At $\rho = 0, 21, 34.36$ bar, $\xi_0 = 559, 149, 127$ Å (using $m^*/m = 3.01, 5.17, 5.5$).

which has considerably lower surface energy,

$$\sigma_{\mathrm{AB}} \simeq 0.77 \xi f_c^{\mathrm{B}} \tag{3}$$

in excellent agreement with the experimental results (see Fig. 1).

The minimization of the energy stored in the interface leads to boundary conditions for the textures on both sides. These are found to be the same as in Cross' analysis in spite of his path not being minimal.

## 2. LINEAR INTERPOLATION BETWEEN A AND B PHASES

Consider the simplest possible interpolation between the two phases

$$\begin{gathered} \Delta_{ai}(x) = \lambda(x) A_{ai} + \kappa(x) B_{ai} \\ A_{ai} \equiv \Delta_{\mathrm{A}} d_a \Phi_i, \qquad B_{ai} \equiv \Delta_{\mathrm{B}} R_{ai}(\hat{n}, \theta)\, e^{i\varphi}, \\ 0 \leq \lambda,\ \kappa \leq 1 \end{gathered} \tag{4}$$

where $\Phi_i = \phi_i^{(1)} + i\phi_i^{(2)}$, the $d_a$ are the orbital and spin directional parameters of $^3$He-A, and $R_{ai}(\hat{n}, \theta)$ is the rotation matrix characterizing the texture in the B phase. Let the interface lie in the $yz$ plane. The problem consists in finding a path $\lambda(x)$, $\kappa(x)$ which connects phase A at $(\lambda, \kappa) = (1, 0)$ to phase B at $(\lambda, \kappa) = (0, 1)$ while minimizing the free energy per unit surface

$$F = \int dx (f_b + f_p) \tag{5}$$

where $f_b$ is the bending energy and $f_p$ the nonderivative potential energy. For simplicity, we shall study only the neighborhood of the polycritical point (PCP) such that the Ginzburg–Landau approximation can be assumed for the bending energy $f_b$

$$f_b = \tfrac{1}{2} K (\partial_x \Delta_{ai}^* \partial_x \Delta_{ai} + 2 \partial_x \Delta_{ax}^* \partial_x \Delta_{ax}) \tag{6}$$

with

$$K = \tfrac{2}{5} N(0) \xi_0^2 = \frac{3}{10} \frac{\rho}{m^* T_{\mathrm{F}}^*} \xi_0^2 = \frac{3}{5} \frac{\rho}{(m^*)^2} \frac{1}{(v_{\mathrm{F}}^*)^2} \xi_0^2 \tag{7}$$

The potential energy $f_p$ is given by

$$f_p = \mu I_0 + \beta_0 \sum_{n=1}^{5} \beta_n I_n \tag{8}$$

with the invariants

$$\begin{aligned} I_0 &= \mathrm{tr}\,(\Delta^{\dagger}\Delta), & I_1 &= |\mathrm{tr}\,(\tilde{\Delta}\Delta)|^2 \\ I_2 &= [\mathrm{tr}\,(\Delta^{\dagger}\Delta)]^2, & I_3 &= \mathrm{tr}\,[(\Delta^{\dagger}\Delta)(\Delta^{\dagger}\Delta)^*] \\ I_4 &= \mathrm{tr}\,(\Delta^{\dagger}\Delta)^2, & I_5 &= \mathrm{tr}\,[(\Delta\Delta^{\dagger})(\Delta\Delta^{\dagger})^*] \end{aligned} \tag{9}$$

The weak coupling BCS values of $\mu$ and $\beta_n$ are

$$\begin{aligned} \mu^{\mathrm{BCS}} &= \frac{1}{2}\,\frac{\rho}{(m^*)^2}\,\frac{1}{(v_{\mathrm{F}}^*)^2}\left(1-\frac{T}{T_c}\right) = \frac{5}{6\xi^2}K \\ -2\beta_1^{\mathrm{BCS}} &= \beta_2^{\mathrm{BCS}} = \beta_3^{\mathrm{BCS}} = \beta_4^{\mathrm{BCS}} = -\beta_5^{\mathrm{BCS}} = 1 \\ \beta_0 &= \frac{3}{5}\,\frac{\rho}{(m^*)^2}\,\frac{\xi_0^2}{(v_{\mathrm{F}}^*)^4} \end{aligned} \tag{10}$$

where $m^*$ is the effective mass of $^3$He and $\rho/m^*$ is the particle density.

It is well known that the very existence of the A phase requires strong coupling corrections even close to $T_c$. The dominant effects are due to paramagnon exchanges (with coupling strength $\bar{I}$),[4] leading to a corrected parametrization as

$$\begin{aligned} \beta_1 &= \beta_1^{\mathrm{BCS}}(1+\tfrac{1}{10}\delta), & \beta_2 &= \beta_2^{\mathrm{BCS}}(1+\tfrac{1}{10}\delta) \\ \beta_3 &= \beta_3^{\mathrm{BCS}}(1-\tfrac{1}{40}\delta), & \beta_4 &= \beta_4^{\mathrm{BCS}}(1-\tfrac{55}{200}\delta), \\ \beta_5 &= \beta_5^{\mathrm{BCS}}(1+\tfrac{7}{20}\delta) \end{aligned} \tag{11}$$

with

$$\delta = \frac{150\pi^2}{7\zeta(3)}\,\frac{T_c}{T_{\mathrm{F}}}\int_0^{q_c}\frac{dq}{2p_{\mathrm{F}}}\left(\frac{\bar{I}}{1-\bar{I}+\bar{I}q^2/12p_{\mathrm{F}}^2}\right)^2$$

This parametrization is roughly consistent with experiments on the jumps of specific heats (see Ref. 5 for a review) if $\mu^{\mathrm{BCS}}$ receives a small correction which at melting pressure is

$$\mu \simeq 1.084\mu^{\mathrm{BCS}} \tag{12}$$

The correction can probably be neglected at zero pressure and also, it is hoped, at the polycritical point.

The condensation energies of the phases A and B are given by*

$$\begin{aligned} f_c^{\mathrm{A}} &= \mu\Delta_{\mathrm{A}}^2 = f_c/\beta_{245}, \\ f_c^{\mathrm{B}} &= \tfrac{3}{2}\mu\Delta_{\mathrm{B}}^2 = f_c/(\beta_{12}+\tfrac{1}{3}\beta_{345}) \end{aligned} \tag{13}$$

with the Ginzburg–Landau condensation energy

$$f_c = \tfrac{5}{24}\rho\left(1-\frac{T}{T_c}\right)^2 \frac{1}{2\xi_0^2(m^*)^2} \tag{14}$$

At the AB phase transition point the energies $f_c^{\mathrm{A}}$ and $f_c^{\mathrm{B}}$ are equal, which determines the parameter $\delta$ of Eq. (11) as

$$\delta = \delta_{\mathrm{AB}} \equiv 20/43 \tag{15}$$

such that the corrected $\beta$ values become

$$\beta_1 = -0.52, \qquad \beta_2 = 1.05, \qquad \beta_3 = 0.99, \qquad \beta_4 = 0.87, \qquad \beta_5 = -1.16 \tag{16}$$

Let us now study the energy along our path between the A and B phases.

With the interpolation (4) depending only on $x$, the bending energy can be written as

$$\begin{aligned} f_b = \tfrac{1}{2}K[&2\Delta_{\mathrm{A}}^2(\boldsymbol{\partial}\lambda)^2 + 5\Delta_{\mathrm{B}}^2(\boldsymbol{\partial}\kappa)^2 \\ &+ \Delta_{\mathrm{A}}\Delta_{\mathrm{B}}(\mathbf{d}^R\cdot\tilde{\boldsymbol{\Phi}} + \text{h.c.})\,\boldsymbol{\partial}\lambda\cdot\boldsymbol{\partial}\kappa + 2\Delta_{\mathrm{A}}^2|\boldsymbol{\partial}\lambda\cdot\boldsymbol{\Phi}|^2 \\ &+ 2\Delta_{\mathrm{A}}\Delta_{\mathrm{B}}(\mathbf{d}^R\cdot\boldsymbol{\partial}\lambda)(\boldsymbol{\partial}\kappa\cdot\tilde{\boldsymbol{\Phi}}^* + \text{h.c.})] \end{aligned} \tag{17}$$

where we have introduced for convenience the modified directional parameters of the A phase

$$d_i^R = d_a R_{ai}(\hat{n}, \theta), \qquad \tilde{\Phi}_i = \Phi_i\, e^{-i\varphi} \tag{18}$$

which contain the information on the orientation of the A texture relative to the B texture. For the potential energy we find

$$\begin{aligned} f_p = \mu\{&\Delta_{\mathrm{A}}^2\lambda^2(\lambda^2-2) + \tfrac{3}{2}\Delta_{\mathrm{B}}^2\kappa^2(\kappa^2-2) \\ &+ \Delta_{\mathrm{A}}\Delta_{\mathrm{B}}\lambda\kappa(\lambda^2+\kappa^2-1)(\mathbf{d}^R\cdot\tilde{\boldsymbol{\Phi}} + \text{h.c.}) \\ &+ \Delta_{\mathrm{A}}^2\Delta_{\mathrm{B}}^2\lambda^2\kappa^2[2|\mathbf{d}^R\cdot\boldsymbol{\Phi}|^2(2\beta_1+\beta_{235}) \\ &+ ((\mathbf{d}^R\cdot\tilde{\boldsymbol{\Phi}})^2 + \text{h.c.})\beta_{24} + 8\beta_{24} + 4\beta_{235}]\} \end{aligned} \tag{19}$$

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*Notice that

$$\beta_0\Delta_{\mathrm{A}}^4 = (\mu/4\beta_{245})\Delta_{\mathrm{A}}^2 = f_c/4\beta_{245}^2 = f_c^{\mathrm{A}}/4\beta_{245}$$

$$\beta_0\Delta_{\mathrm{B}}^4 = [\mu/6(\beta_{12}+\tfrac{1}{3}\beta_{345})]\Delta_{\mathrm{B}}^2 = f_c/9(\beta_{12}+\tfrac{1}{3}\beta_{345})^2 = f_c^{\mathrm{B}}/9(\beta_{12}+\tfrac{1}{3}\beta_{345})$$

$$\tfrac{1}{2}K\Delta_{\mathrm{A}}^2 = \tfrac{3}{5}\xi^2 f_c^{\mathrm{A}}, \qquad \tfrac{1}{2}K\Delta_{\mathrm{B}}^2 = \tfrac{2}{5}\xi^2 f_c^{\mathrm{B}}$$

where we have made use of the gap sizes

$$\Delta_{\mathrm{A}}^2=\mu/4\beta_{245}\beta_0, \qquad \Delta_{\mathrm{B}}^2=\mu/6(\beta_{12}+\tfrac{1}{3}\beta_{345})\beta_0 \tag{20}$$

to simplify the expression. Close to the transition temperature $T_{\mathrm{AB}}$, the energies $f_c^{\mathrm{A}}$ and $f_c^{\mathrm{B}}$ differ only by a very small amount and we may write

$$\Delta_{\mathrm{B}}^2/\Delta_{\mathrm{A}}^2=\tfrac{2}{3}(1+\varepsilon) \tag{21}$$

where $\varepsilon$ is given by

$$1+\varepsilon=\frac{\beta_{245}}{\beta_{12}+\frac{1}{3}\beta_{345}}=\frac{6}{5}\,\frac{1-(21/40)\delta}{1-\frac{1}{5}\delta} \tag{22}$$

Putting (21) into the potential energy (19), we get, up to first order in $\varepsilon$

$$\begin{aligned} v\equiv f_p/f_c^{\mathrm{B}}+1=1&+\lambda^2(\lambda^2-2)(1-\varepsilon)+\kappa^2(\kappa^2-2)\\ &+2\sqrt{\tfrac{2}{3}}c_1(1-\tfrac{1}{2}\varepsilon)\lambda\kappa(\lambda^2+\kappa^2-1),\\ &+\lambda^2\kappa^2(1-\varepsilon)\rho(c_1,c_2) \end{aligned} \tag{23}$$

where we have normalized the potential energy by dividing it by the condensation energy $f_c^{\mathrm{B}}$ of phase B and added 1 such that the energy starts counting at $-f_c^{\mathrm{B}}$. Thus for the A phase $v_{\mathrm{A}}=\varepsilon$ and for the B phase $v_{\mathrm{B}}=0$. The function $\rho(c_1, c_2)$ depends on the directional cosines of the textures

$$c_1=\mathbf{d}^R\cdot\tilde{\mathbf{\Phi}}^{(1)}, \qquad c_2=\mathbf{d}^R\cdot\tilde{\mathbf{\Phi}}^{(2)}, \qquad c_3=\mathbf{d}^R\cdot\mathbf{l} \tag{24}$$

and it is given explicitly by

$$\begin{aligned} \rho(c_1,c_2)=(1/6\beta_{245})[&2(c_1^2+c_2^2)(2\beta_1+\beta_{235})\\ &+2(c_1^2-c_2^2)\beta_{24}+8\beta_{24}+4\beta_{235}] \end{aligned} \tag{25}$$

The effect of $\varepsilon$ is to cause an asymmetry between the potential minima. There is also an accompanying slight distortion in the shape of the potential which may be neglected for present considerations[1]. Therefore we may write the potential energy approximately as sum of a symmetric part

$$\begin{aligned} v\equiv 1&+\lambda^2(\lambda^2-2)+\kappa^2(\kappa^2-2)\\ &+2\sqrt{\tfrac{2}{3}}c_1\lambda\kappa(\lambda^2+\kappa^2-1)+\lambda^2\kappa^2\rho(c_1,c_2) \end{aligned} \tag{26}$$

and a symmetry-breaking part

$$v_{\mathrm{SB}}=\varepsilon\lambda \tag{27}$$

The symmetric potential has absolute minima with $v=0$ for $(\lambda,\kappa)=(1,0)$ and $(0,1)$ corresponding to the phases A and B, respectively, and a maximum at $(0,0)$, where $v=1$ (i.e., $f_p=0$), which amounts to the liquid being normal at that point. When proceeding from (0, 0) along the diagonal the energy $v$ decreases from unity and runs through a minimum at

$$\lambda = \kappa = \left(\frac{2+\sqrt{\tfrac{2}{3}}c_1}{2+4\sqrt{\tfrac{2}{3}}c_1+\rho}\right)^{1/2} \tag{28}$$

with a value

$$v = 1 - \frac{(2+\sqrt{\tfrac{2}{3}}c_1)^2}{2+4\sqrt{\tfrac{2}{3}}c_1+\rho} \tag{29}$$

We expect the path of minimal surface energy to cross the diagonal close to this point. The precise course of the path depends on the properties of the bending energy (17), which we now discuss.

Let us introduce the directional cosines of the normal to the interface with respect to the textural dreibein $\tilde{\phi}^{(1)}$, $\tilde{\phi}^{(2)}$, $l$, i.e.,

$$m_{\binom{1}{2}} = \tilde{\mathbf{\Phi}}^{\binom{1}{2}} \cdot \boldsymbol{\partial}\lambda/|\boldsymbol{\partial}\lambda|, \qquad m_3 = \mathbf{l} \cdot \boldsymbol{\partial}\lambda/|\boldsymbol{\partial}\lambda| \tag{30}$$

With the assumption of a planar interface $\partial\kappa$ has the same direction

$$\boldsymbol{\partial}\kappa \,\|\, \boldsymbol{\partial}\lambda \,\|\, \mathbf{m} \tag{31}$$

except for the sign. We shall now assume that when going from the A to the B phase the transition is monotonic. Then as $\lambda$ grows, $\kappa$ decreases and the directional vectors are oriented opposite to each other. Therefore the bending energy (17) can be written as

$$\begin{aligned} f_b/f_c^{\mathrm{B}} = {} & \tfrac{3}{5}\xi^2\{(4-2m_3^2)(\partial\lambda)^2+(10/3)(\partial\kappa)^2 \\ & -2\sqrt{\tfrac{2}{3}}[c_1+2(c_1m_1^2+c_2m_1m_2+c_3m_1m_3)]|\partial\lambda \cdot \partial\kappa|\} \end{aligned} \tag{32}$$

where we have used, at $\varepsilon = 0$,

$$\tfrac{1}{2}K\Delta_{\mathrm{A}}^2 = \tfrac{3}{2} \cdot \tfrac{1}{2}K\Delta_{\mathrm{B}}^2 = \tfrac{3}{5}\xi^2 f_c^{\mathrm{B}} \tag{33}$$

from (7) and (13). Therefore the full energy at the transition point $T_{\mathrm{AB}}$ reads, in the same reduced form as (23),

$$\begin{aligned} \tilde{f} &= (f_b+f_p)/f_c^{\mathrm{B}}+1 \\ &= \alpha\xi^2(\partial\lambda)^2+\beta\xi^2(\partial\kappa)^2(\partial\kappa)^2+\gamma\xi^2|\partial\lambda \cdot \partial\kappa| \\ &\quad \cdot+\lambda^2(\lambda^2-2)+\kappa^2(\kappa^2-2)+\sigma\lambda\kappa(\lambda^2+\kappa^2-1)+\rho\lambda^2\kappa^2+1 \end{aligned} \tag{34}$$

where we have abbreviated

$$\begin{aligned} &\alpha = \tfrac{6}{5}(2-m_3^2), \qquad \beta = 2 \\ &\gamma = -\tfrac{6}{5}\sqrt{\tfrac{2}{3}}[c_1+2m_1(c_1m_1+c_2m_2+c_3m_3)] \\ &\sigma = 2\sqrt{\tfrac{2}{3}}c_1, \qquad \rho = \rho(c_1, c_2) \end{aligned} \tag{35}$$

For this energy, we now have to find an extremal path connecting $(\lambda, \kappa) = (1, 0)$ with $(0, 1)$ for every choice of directional parameters $c_1, c_2, c_3, m_1, m_2, m_3$. The resulting surface tension

$$\sigma_{AB} = \left(\int_{-\infty}^{\infty} dx\, \tilde{f}\right) f_c^{B}$$

must be minimized with respect to those parameters. This program is far too complicated to be pursued analytically. Therefore we prefer an approximate but much simpler procedure. We first consider the straight diagonal path

$$\lambda(x) + \kappa(x) = 1 \tag{36}$$

For it the energy is simply

$$\tilde{f}(x) = A\xi^2(\partial\lambda)^2 + B\lambda^2(1-\lambda)^2 \tag{37}$$

with

$$\begin{aligned} A &= \alpha + \beta + \gamma \\ &= \tfrac{3}{5}\{22/3 - 2m_3^2 - 2\sqrt{\tfrac{2}{3}}[c_1 + 2m_1(c_1m_1 + c_2m_2 + c_3m_3)]\} \\ B &= 2 - 2\sigma + \rho \\ &= 2 - 4\sqrt{\tfrac{2}{3}}c_1 + \rho(c_1, c_2) \end{aligned} \tag{38}$$

The energy has the minimal "kink" solution

$$\lambda(x) = 1 - \kappa(x) = \tfrac{1}{2}\mathrm{th}\,[\tfrac{1}{2}(x/\xi)(B/A)^{1/2}]\} \tag{39}$$

which satisfies

$$\xi\, \partial_x\lambda(x) = (B/A)^{1/2}\lambda(1-\lambda) \tag{40}$$

and gives a surface tension

$$\begin{aligned} \sigma_{AB} &= f_c^{B} \int dx\, [A\xi^2(\partial_x\lambda)^2 + B\lambda^2(1-\lambda)^2] \\ &= \xi f_c^{B}\, 2(AB)^{1/2} \int_0^1 d\lambda\, \lambda(1-\lambda) = \tfrac{1}{3}(AB)^{1/2}\xi f_c^{B} \end{aligned} \tag{41}$$

We now determine optimal textural angles $c_1$, $c_2$, $c_3$, $m_1$, $m_2$, $m_3$ by minimizing this energy. A somewhat tedious elaboration yields the results

$$c_1 = 1, \qquad c_2 = c_2 = 0 \tag{42}$$

$$m_1 = \pm 1, \qquad m_2 = m_3 = 0 \tag{43}$$

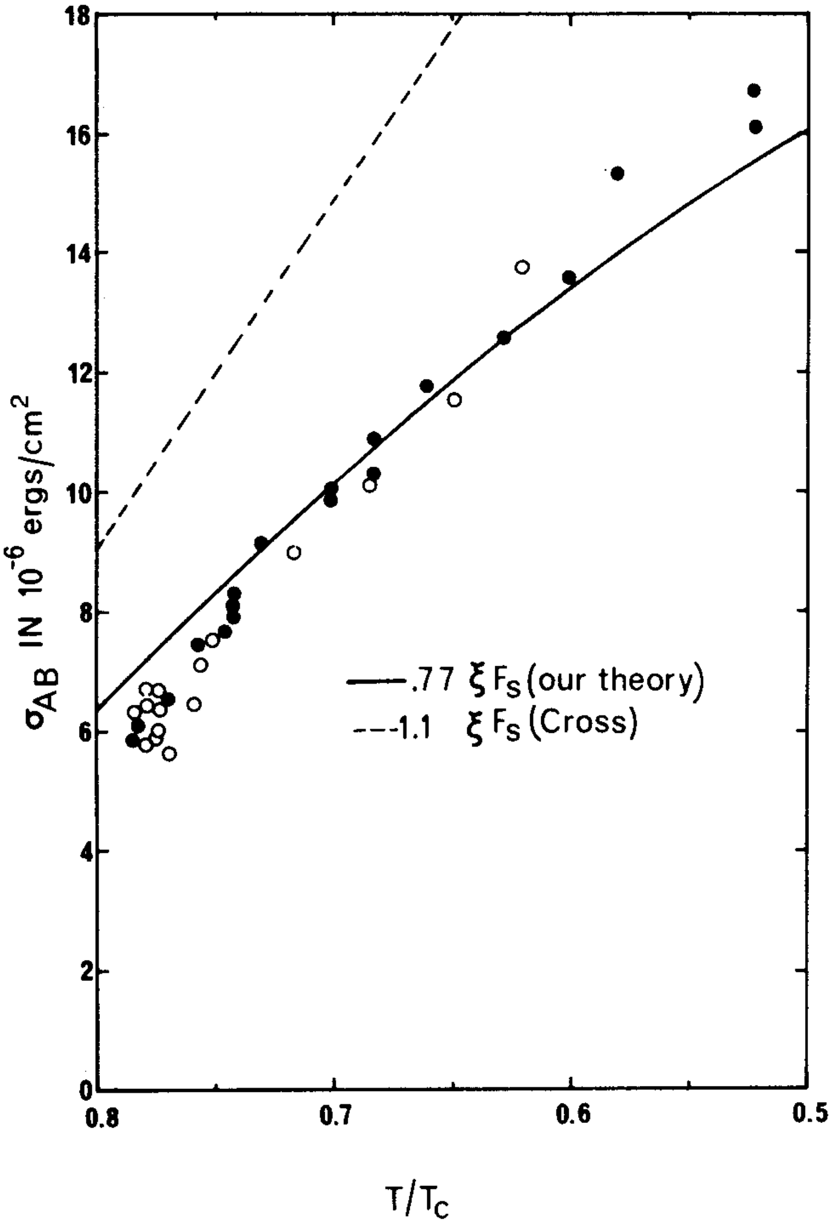

Fig. 1. The interfacial surface energy between $^3$He phase A and $^3$He phase B at melting pressure. The solid line depicts our result $\sigma_{AB} \approx 0.77 \xi f_c^B$, while the broken line represents Cross' calculation $\sigma_{AB} \approx 1.1 \xi f_c^B$. The points ○ and ● are the experimental points for $41 \pm 3$ $\mu$m and $91 \pm 3$ $\mu$m diameter holes in the grid used in the experiment by Osheroff and Cross[1].

The first line states that the vector $d^R \equiv dR$ is aligned with $\tilde{\boldsymbol{\phi}}^{(1)}$:

$$\mathbf{d}^R \parallel \tilde{\boldsymbol{\phi}}^{(1)} = \boldsymbol{\phi}^{(1)} \cos \varphi + \boldsymbol{\phi}^{(2)} \sin \varphi \tag{44}$$

while the second line fixes the orientation of $\tilde{\boldsymbol{\phi}}^{(1)}$ orthogonal to the interface and that of $\mathbf{l}$ in the interface:

$$\tilde{\boldsymbol{\phi}}^{(1)} \parallel \pm \mathbf{m}, \qquad \mathbf{l} \perp \mathbf{m} \tag{45}$$

Now we can immediately determine the sizes of $A$ and $B$:

$$A=\tfrac{3}{5}\{22/3-6\sqrt{\tfrac{2}{3}}\}\approx 1.46$$

$$\rho=(1/6\beta_{245})[2(2\beta_1+\beta_{235})+10\beta_{24}+4\beta_{235}]\approx 4.91 \tag{46}$$

$$B=2-4\sqrt{\tfrac{2}{3}}+\rho\approx 3.64$$

such that the surface tension for our straight-line connection becomes

$$\sigma_{\mathrm{AB}}\approx 0.77\xi f_c^{\mathrm{B}} \tag{47}$$

The result is plotted in Fig. 1 as a function of temperature and compared with the experimental result of Ref. 1 as well as Cross' earlier calculation.

Certainly, our solution is not truly minimal since we have chosen to proceed along the diagonal path. It is, however, easy to see that we cannot have missed the true minimal solution* by a great amount. If we keep the same textural boundary conditions (42), (43) and insert these into (28), we see that the lowest point across the potential barrier lies at

$$\lambda=\kappa\approx 0.53$$

with a value

$$v\approx 0.22 \tag{48}$$

The maximal value on our diagonal path lies only slightly higher,

$$v|_{\lambda=\kappa=1/2}\approx 0\ 227 \tag{49}$$

Figure 2 shows the potential energy. Since the optimal path is the same as the trajectory of a mass point running through the reversed potential $-v$, it is easy to imagine the smallness of the correction to our path. In fact, it can easily be seen that the straight-line connection would correspond to the exact minimum of the energy if the constants in (34) satisfied

$$\alpha=\beta, \qquad \rho=3 \tag{50}$$

with $\gamma$ and $\sigma$ being as they are under the textural conditions (42)–(43). The actual values $\alpha=12/5$, $\beta=2$, $\rho=4.91$ violate this condition only mildly.

The reader who is familiar with the difficulties in calculating strong coupling corrections may be critical toward the reliability of our results. It is gratifying to note that these do not have any appreciable dependence on which of the strong coupling corrections available in the literature are used. For example, the $\beta_i$ implied by calculations of Kuroda and Nagi[6] are

$$\beta_1=-0.53, \qquad \beta_2=1.02, \qquad \beta_3=0.97, \qquad \beta_4=0.88, \qquad \beta_5=-1.19 \tag{51}$$

where we have used the cutoff parameter $q_c/2p_{\mathrm{F}}=0.124$ instead of[6] 0.13 to

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*As far as the $\lambda$, $\kappa$ space is concerned.

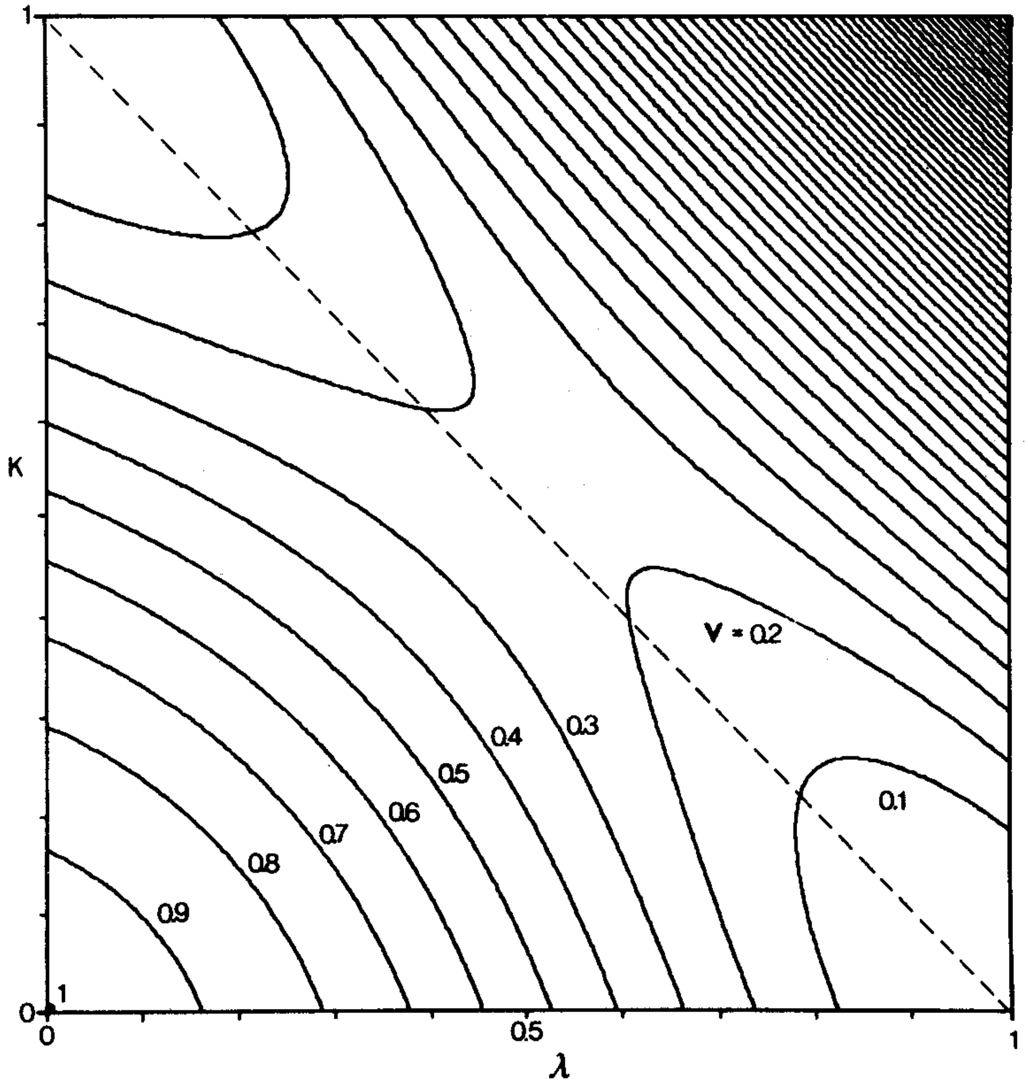

Fig. 2. The potential energy (34) in $(\lambda, \kappa)$ parameter space between A $\hat{=}$ (1, 0) and B $\hat{=}$ (0, 1) phases. Our path along the diagonal is not completely minimal but obviously is not far from it. Its surface energy is $\sigma_{AB} \approx 0.77 \xi f_c^B$.

ensure $f^A = f^B$ at $T_{AB}$. These are not appreciably different from our values (16) and yield, with boundary conditions (42) and (43),

$$\rho = 5.14, \qquad B = 3.88, \qquad \sigma_{AB} = 0.79 \xi f_c^B \tag{52}$$

instead of our earlier values 4.91, 3.64, and $0.77 \xi f_c^B$, respectively.

## 3. COMPARISON WITH CROSS' CALCULATION

Since our surface energy is so much smaller than Cross', we may ask whether our passage across the potential barrier passes through the maximum anywhere near the planar phase. In order to compare both paths we modify Cross' Ansatz for the order parameter in such a way that both his and our solution can be represented on the same two-dimensional plot. Cross chooses the following parametrization:

$$\Delta_{ai} = \frac{\Delta}{\sqrt{2}} \begin{pmatrix} \cos\chi & 0 & -i\Delta_{13} \\ 0 & \cos\theta\cos\chi & 0 \\ 0 & 0 & \sqrt{2}\sin\chi \end{pmatrix} \tag{53}$$

where $\Delta_{13} = \sin\theta\cos\chi$. This interpolates between A and the planar phase P for $\chi \equiv 0$, $\theta \in (\pi/2, 0)$ and between P and the B phase for $\theta \equiv 0$, $\chi \in (0, \chi_B)$, with $\chi_B = \sin^{-1}(1/\sqrt{3})$. The desired modification consists in multiplying the matrix element $\Delta_{13}$ by a factor $h(\chi)$:

$$\Delta_{13} \to \Delta_{13} h(\chi) = \Delta_{13} \left[ \frac{\sin(\chi_B - \chi)}{\sin(\chi_B + \chi)} \right]^{1/2} \tag{54}$$

This factor does not change the order parameter along Cross' path.

In order to see that also our linear interpolation is contained in the modified $A_{ai}$ we chose

$$\lambda = (\Delta/\Delta_A)(\cos\chi - \sqrt{2}\sin\chi), \qquad \kappa = (\Delta/\Delta_B)\sqrt{2}\sin\chi \tag{55}$$

and satisfy the textural boundary conditions by taking the order parameters for the A and B phases respectively as

$$A_{ai} = \Delta_A = \begin{pmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_{ai} = \Delta_B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{56}$$

so that the interpolation between them is

$$\Delta_{ai} = \Delta \begin{pmatrix} \cos\chi & 0 & -i(\cos\chi - \sqrt{2}\sin\chi) \\ 0 & \sqrt{2}\sin\chi & 0 \\ 0 & 0 & \sqrt{2}\sin\chi \end{pmatrix} \tag{57}$$

Now we see that along the curve

$$\cos\theta = \sqrt{2}\,\mathrm{tg}\,\chi \tag{58}$$

the linear interpolation (57) is contained in the modified Cross' Ansatz (53) and (54), for along this curve

$$\cos\chi - \sqrt{2}\sin\chi = \frac{\cos\chi - \sqrt{2}\sin\chi}{\sin\theta\cos\chi}\sin\theta\cos\chi$$

$$= \sqrt{3}\,\frac{\sin(\chi_B - \chi)}{\sin[\cos^{-1}(\sqrt{2}\,\mathrm{tg}\,\chi)]\cos\chi}\sin\theta\cos\chi \tag{59}$$

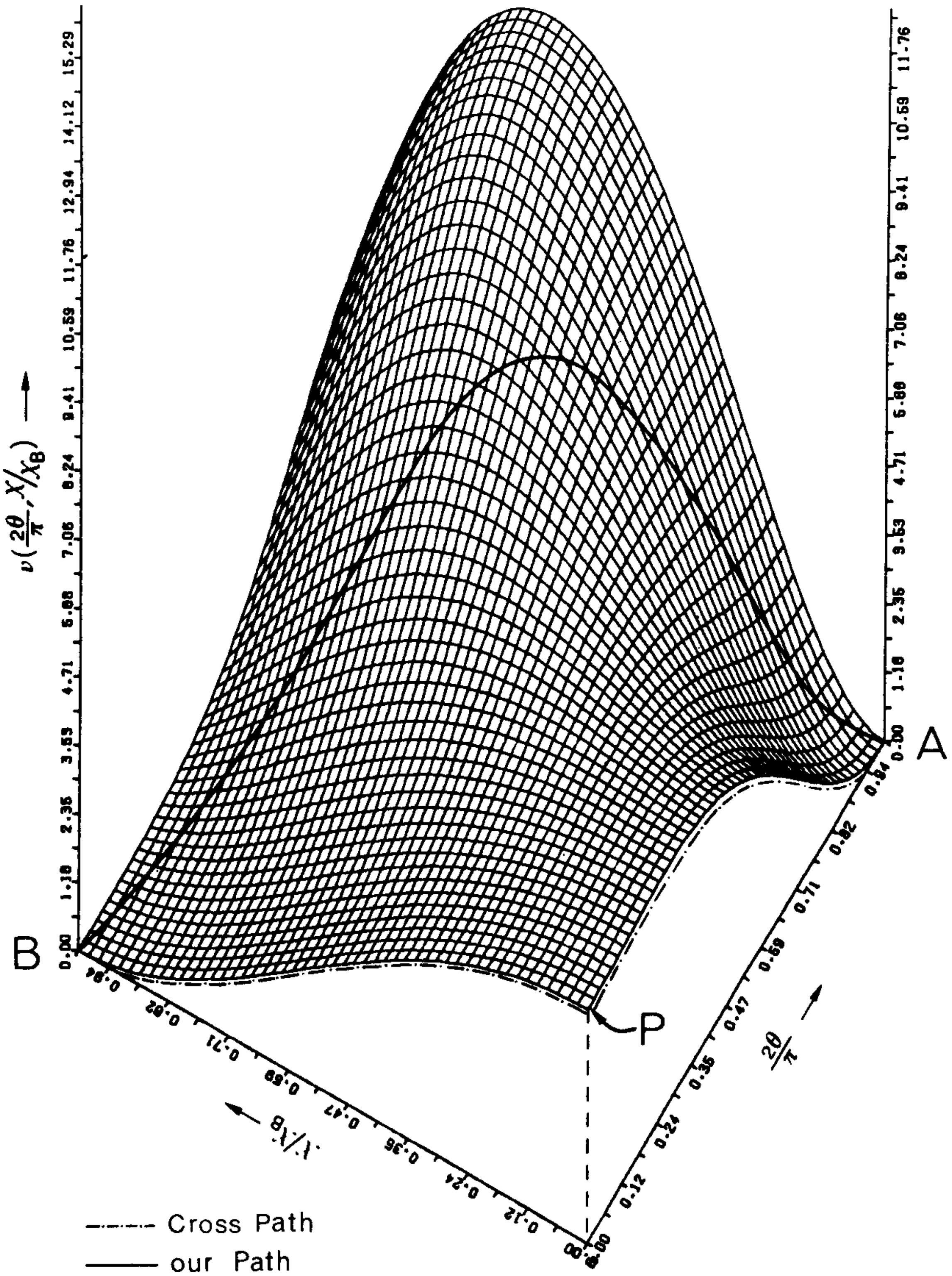

Fig. 3. The potential as a function of $2\theta/\pi \in (0, 1)$ and $\chi/\chi_B \in (0, 1)$, comparing ours with Cross' interpolation. Ours corresponds to the diagonal path in the variables shown in Fig. 2. Cross' runs along the axes and passes the planar phase at the origin. Notice how far our interpolation stays away from the planar phase. The height of the mountain is not a direct measure of the surface tension, since the bending energies have a complicated dependence on $\chi$, $\theta$ (they are simple only in the $\lambda$, $\kappa$ plane).

But

$$\sin[\cos^{-1}(\sqrt{2}\,\mathrm{tg}\,\chi)] = [\sin(\chi_B - \chi)\sin(\chi_B + \chi)]^{1/2}/(\cos\chi \sin\chi_B) \tag{60}$$

such that (53) does coincide with (54). The difference between our and Cross' paths can be now seen in Fig. 3, where we have plotted the potential as a function of $\theta$ and $\chi$ for the modified order parameters (53), (54). Cross' path runs along the axes through the planar phase at the origin, while ours following (57) does not come anywhere near it. Notice that our path climbs higher than that of Cross' and still has a lower surface tension. The reason is, of course, the strong $\theta$, $\chi$ dependence of the bending energy. Contrary to our $\lambda$, $\kappa$ path, whose bending energy takes a standard form of classical mechanics, the potential barrier in $\theta$, $\chi$ space does not permit an immediate conclusion, by inspection, as to where it can most easily be traversed.

## 4. CONCLUSION

We have seen that the straight-line interpolation between the A and B phases

$$\Delta_{ai} = \lambda(x)A_{ai} + \kappa(x)B_{ai} \tag{61}$$

with a "kink" form for $\lambda$ and $\kappa$,

$$\lambda(x) = 1 - \kappa(x) = \frac{1}{2}\left[1 - \mathrm{th}\,\frac{1}{2}\frac{x}{\xi}\left(\frac{B}{A}\right)^{1/2}\right] \tag{62}$$

leads to a surface tension $\sigma_{AB} \approx 0.77\xi f_c^B$ in excellent agreement with experiment.

The textural boundary conditions require that $(\phi^{(1)}\cos\varphi + \phi^{(2)}\sin\varphi)$ and $d_a R_{ai}(\hat{n}, \theta)$ both have to point orthogonal to the interface.

At the transition point $T = T_{AB}$ the order parameter is

$$\Delta_{ai} = \Delta_A \begin{pmatrix} 1.82 & & -i \\ & 0.81 & \\ & & 0.81 \end{pmatrix} \tag{63}$$

which is quite far from the planar phase. Certainly, the precise value of $\Delta_{ai}$ is the least dependable of our statements. Since the energy is not the most sensitive measure of the distance from the true minimal path (just as in variational calculations of a Schrödinger bound state), it could well be that the minimal path has significant deviations from ours. However, it appears highly improbable that the minimal path runs through the functional neighborhood of the planar phase with an energy even lower than that for our path.

Our result still throws no light upon the problem of how the B phase nucleates when cooling the bulk A phase. Even with our improved surface energy the radius of a critical bubble

$$r_c = 2\sigma_{AB}/|f_c^A - f_c^B| = 1.45\xi/(|f_c^A - f_c^B|/f_c^B) \tag{64}$$

with*

$$|f_c^A - f_c^B|/f_c^B \approx \varepsilon \approx 0.17(1 - T/T_{AB}) \tag{65}$$

becomes

$$r_c \approx 9\xi/(1 - T/T_{AB})$$

Even at the experimental temperature of largest supercooling $T - T_{AB} = -0.4$ mK this gives

$$r_c \approx 47\xi \tag{66}$$

The energy of this bubble is enormous,

$$\begin{aligned} E_{\text{bubble}} &= \tfrac{2}{3}\pi(2\sigma_{AB})^3/(f_c^A - f_c^B)^2 \\ &= (16/3)\pi(0.77)^3 f_c^B \xi^3/\varepsilon^2 \\ &\approx 1.8 \times 10^6\, kT \end{aligned} \tag{67}$$

such that thermally activated nucleation in the bulk is excluded. Some other nucleation mechanism will have to be found.

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*Remember that $\varepsilon$ was defined by

$$\varepsilon = (43/100)(\delta - \delta_{AB})/(1 - \tfrac{1}{5}\delta)$$

Therefore, interpolating $\delta = \delta_{AB} + (\delta_c - \delta_{AB})(T - T_{AB})/(T_c - T_{AB})$ with $\delta_{AB} = 20/43$ and $\delta_c = 0.56$, we find $\varepsilon = 0.17(1 - T/T_{AB})$.