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Decay Rate for Supercurrent in Thin Wire

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We present a new evaluation of the fluctuations triggering the decay of supercurrents. Contrary to the existing treatment available in the literature, our result emerges in a simple and closed form. This is due to the fact that, in a polar decomposition $\Delta = \rho\, e^{i\gamma}$ of the order parameter, we sum over all azimuthal paths explicitly, thereby arriving at a fluctuation determinant for the ρ variable alone which can be evaluated exactly.

1. INTRODUCTION

The calculation of the resistance of a thin superconducting wire is one of the most important applications of Feynman's path integrals to quadratic fluctuation phenomena in superconductors.[1] It is therefore disturbing that the (by now "classic") treatment of Langer, Ambegaokar, McCumber, and Halperin (LAMH) is technically quite cumbersome and gives an explicit result only for very small or close to critical currents.‡ What is needed is the evaluation of a 2×2 fluctuation determinant. LAMH proceed by calculating the infinite product of all eigenvalues of the 2×2 Schrödinger problem. We want to show that a more efficient handling of the path integral reduces the problem by one dimension and makes it soluble by the Gelfand–Yaglom method.[4]

The essential point is that by going to a polar decomposition of the order parameter

$$\Delta = \rho\, e^{i\gamma} \tag{1}$$

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‡The same technique has also been applied to the second type of superflow in ^{3}He-A,[2] resulting from a helical texture.[3]

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the azimuthal variable is cyclic and its path integral can be performed exactly.

2. GENERAL DISCUSSION

It is convenient to employ natural units, in which temperatures are measured in terms of $2f_c/k_B$, where f_c is the BCS condensation energy

$$f_c = (1/8m^2\xi_0^2)(1 - T/T_c)^2 \times \text{mass density} \tag{2}$$

with lengths taken in units of the temperature-dependent coherence length ξ,

$$\xi = \xi_0\left(1 - \frac{T}{T_c}\right)^{-1/2} = \left[\frac{7\zeta(3)}{48\pi^2}\right]^{1/2} \frac{v_F}{T_c}\left(1 - \frac{T}{T_c}\right)^{-1/2} \tag{3}$$

Then the partition function for a thin wire of length L in the regime of validity of the Ginzburg–Landau equation is given by the path integral* ($\sigma \equiv$ cross section of the wire)

$$\begin{aligned} Z[j] &= \int \rho D\rho D\gamma \exp\left\{-\frac{\sigma}{T}\int dz\,[f(\rho,\gamma) - 2j\gamma_z]\right\} \\ &= \int \rho D\rho D\gamma \exp\Bigg\{ \\ &\qquad -\frac{\sigma}{T}\int dz\left[(\rho_z)^2 - \rho^2 + \frac{1}{2}\rho^4 + \rho^2(\gamma_z)^2 - 2j\gamma_z\right]\Bigg\} \end{aligned} \tag{4}$$

Here a stationary average current

$$\langle J\rangle = \langle \rho^2(\partial_z\gamma)\rangle = j$$

is established by means of some external source $2j\gamma_z$. A quadratic completion permits us to rewrite

$$\begin{aligned} Z(j) &= \int D\rho \exp\left\{-\frac{\sigma}{T}\int_{-L/2}^{L/2} dz\left[(\rho_z)^2 - \rho^2 + \frac{1}{2}\rho^4 - \frac{j^2}{\rho^2}\right]\right\} \\ &\quad \times \int \rho D\gamma \exp\left\{-\frac{\sigma}{T}\int_{-L/2}^{L/2} dz\,\rho^2(z)\left(\gamma_z - \frac{j}{\rho^2}\right)^2\right\} \end{aligned} \tag{5}$$

Now, the important observation is that the ρ dependence of the second integral is only apparent. By performing the path integral over $D\gamma$, one

*The integral runs over all periodic functions. Its normalization on a discretized z axis with $z_n = n\varepsilon$ is

$$\rho D\rho D\gamma = \prod_n \frac{d\rho(z_n)}{(\pi\varepsilon T/\sigma)^{1/2}} \frac{\rho(z_n)\,d\gamma(z_n)}{(\pi\varepsilon T/\sigma)^{1/2}}$$

obtains the ρ-independent result:

$$Z_\gamma \equiv \int \rho(z) D\gamma(z) \exp\left[-\frac{\sigma}{T}\int_{-L/2}^{L/2} dz\, \rho^2(z)\left(\gamma_z - \frac{j}{\rho^2}\right)^2\right] \underset{L\to\infty}{\to} 1 \tag{6}$$

in the limit of a long wire such that the calculation of Z reduces to a one-dimensional problem:

$$Z[j] = \int D\rho \exp\left\{-\frac{\sigma}{T}\int_{-L/2}^{L/2} dz\left[(\rho_z)^2 - \rho^2 + \frac{1}{2}\rho^4 - \frac{j^2}{\rho^2}\right]\right\} \tag{7}$$

whose semiclassical evaluation will be shown to be extremely simple. In order to convince ourselves that (6) is true, we introduce an auxiliary space variable

$$s = \int_{-L/2}^{z} \frac{dz'}{\rho^2(z')} \tag{8}$$

which has the effect of bringing (6) to the form*

$$Z_\gamma = \int D\gamma(s) \exp\left[-\frac{\sigma}{T}\int_{-\tilde{L}/2}^{\tilde{L}/2} ds (\gamma_s - j)^2\right] \tag{9}$$

where the limits of integration are determined by

$$\tilde{L} = \int_{-L/2}^{L/2} \frac{dz}{\rho^2(z)} \tag{10}$$

Now, if γ were a usual quantum mechanical variable, the propagator of (9) would be[5]

$$\frac{1}{(2\pi i \tilde{L} T/2\sigma)^{1/2}} \exp\left[i\frac{\sigma}{T}\frac{(\gamma_b - \gamma_a - ij\tilde{L})^2}{\tilde{L}}\right] \tag{11}$$

With γ being defined on a circle, there is an additional sum over indistinguishable, periodically translated end points $\gamma_b + 2\pi m$. The statistical trace is obtained by setting $\gamma_a = \gamma_b$, integrating over one period, and dropping the factor i in front of $\tilde{L}$:

$$Z_\gamma = \left(\frac{2\pi}{\tilde{L}T/2\sigma}\right)^{1/2} \sum_m \exp\left[-\frac{\sigma}{T}\frac{(2\pi m - j\tilde{L})^2}{\tilde{L}}\right] \tag{12}$$

For a very long wire, $\tilde{L}$ grows to infinity and the sum over m can be replaced by an integral. This proves Eq. (6).

*Notice that $\rho(z)D\gamma(z) = D\gamma(s)$ since $\varepsilon = \Delta z \to \Delta s = \varepsilon\rho^2(z_n)$ in the grated form of the $\rho(z)D\gamma(z)$ measure (see footnote on p. 138).

3. CALCULATION OF Z UP TO QUADRATIC FLUCTUATIONS

Before starting it is necessary to realize that the integration over the "radial" coordinate ρ is restricted to positive values. This may be awkward when it comes to calculations, even for purely quadratic fluctuations. There is, however, a simple way of extending the integration to the full ρ axis, from $-\infty$ to $+\infty$, by the following construction: we rewrite (7) as

$$Z = \int_{\rho_a}^{\rho_b} D\rho \exp\left(-\frac{\sigma}{T}\int_{-L/2}^{L/2} dz\, g\right) - \int_{\rho_a}^{-\rho_b} D\rho \exp\left(-\frac{\sigma}{T}\int_{-L/2}^{L/2} dz\, g\right) \tag{13}$$

and integrat in each piece over all $\rho \in (-\infty, +\infty)$. This expression ensures[6] that only the paths remaining completely on the positive ρ axis contribute, all others being cancelled by a properly reflected image path of equal energy.

Now we turn directly to the evaluation of the partition function (7). At the level of the stationary-phase approximation we have to solve the equation of motion

$$\rho_{zz} = -\rho + \rho^3 + j^2/\rho^3 \tag{14}$$

There is a finite-energy solution only for j smaller than a critical value

$$j < j_c = 2/3\sqrt{3} \tag{15}$$

This condition can be ensured by parametrizing the current as*

$$j \equiv K(1-K^2) \tag{16}$$

The trivial equilibrium solution of (14) can now be written as

$$\rho = \rho_0 \equiv (1-K^2)^{1/2} \tag{17}$$

Notice that, due to (6), K is the mean value of the fluctuating γ_z at fixed $\rho \equiv \rho_0$:

$$\langle \gamma_z \rangle_{\rho\equiv\rho_0} = \langle j/\rho^2(z)\rangle_{\rho\equiv\rho_0} = K$$

The energy of the solution (17) is

$$\begin{aligned} G = \int dz\, (f - 2j\gamma_z) &= -L[\tfrac{1}{2}(1-K^2)^2 - 2jK] \equiv G_K^j \\ &= -\tfrac{1}{2}(1-K^2)(1+3K^2) \end{aligned} \tag{18}$$

*The physical current is related to this dimensionless quantity by a factor

$$(e/\xi_0 m)(1-T/T_c)^{3/2} \times \text{mass density}$$

Also, $2\gamma_z$ is the supercurrent velocity measured in terms of natural units $v_0 = 1/m\xi$.

Therefore, the partition function becomes

$$Z_K = \exp[-(\sigma/T)G_K^j] \tag{19}$$

There is no difficulty in incorporating the quadratic fluctuations around the uniform solution: they contribute the standard oscillator partition function

$$[2\,\mathrm{sh}(\omega L/2)]^{-1} \underset{L\to\infty}{\to} e^{-\omega L/2} \tag{20}$$

where $\omega = [2(1-3K^2)]^{1/2}$ measures the curvature of the potential at the minimum with f behaving as

$$f \simeq \omega^2(\rho-\rho_0)^2 \tag{21}$$

This is just the analog of the zero-point energy $\omega/2\hbar$ in the quantum mechanical case.

We are now ready to study the decay rate of the supercurrent. It is triggered by extremal excursion from the equilibrium solution (17). These are given by

$$\rho_b^2 = 1 - K^2 - \frac{\omega^2/2}{\mathrm{ch}^2(\omega z/2)} \tag{22}$$

They correspond to the critical bubbles in the description of nucleation processes in many first-order phase transitions.[7] Their contributions to the classical partition function are

$$Z_b/Z_0 = \exp[-(\sigma/F)F^b] \tag{23}$$

where F^b measures the additional energy due to the presence of a bubble

$$\begin{aligned} G_K^j + F^b &= \int_{-L/2}^{L/2} dz\, g(\rho^2) = \int_{-L/2}^{L/2} dz\left[(\rho_z)^2 - \rho^2 + \frac{1}{2}\rho^4 - \frac{j^2}{\rho^2}\right]\Bigg|_{\rho=\rho_b} \\ &= -\frac{1}{2}L(1-K^2)^2 - 2jKL + \frac{4}{3}\omega - 4j\,\mathrm{arctg}\frac{\omega}{2K} \end{aligned} \tag{24}$$

Notice that due to the Ginzberg–Landau condition $\sigma/T \gg 1$ the Boltzmann factor is very small, such that such bubbles appear very rarely.

The point is now that the center of the bubble has a reduced order parameter which allows a quadratic fluctuation of ρ to hit the origin. There, the phase becomes undefined and may change by one unit (phase slip).

Let us calculate the quadratic fluctuations for the ratio (23) [in the numerator they are centered around $\rho_b(z)$ of (22), in the denominator around $\rho_0 = 1 - K^2$]. This gives the functional integrals

$$\frac{\int D\delta \exp\left[i\int dz\, \delta(-\partial_z^2 + v(z) + \omega^2)\delta\right]}{\int D\delta \exp\left[i\int da\, \delta(-\partial_z^2 + \omega^2)\delta\right]} = \left[\frac{\det[-\partial_z^2 + v(z) + \omega^2]}{\det(-\partial_z^2 + \omega^2)}\right]^{-1/2} = \prod_n \left(\frac{\lambda_n^0}{\lambda_n^v}\right)^{1/2} \tag{25}$$

where

$$v(z) + \omega^2 \equiv \frac{1}{2}\frac{d^2}{d\rho^2}\left(-\rho^2 + \frac{\rho^4}{2} - \frac{j^2}{\rho^2}\right)\Bigg|_{\rho=\rho_b} = -1 + 3\rho_b^2 - \frac{3j^2}{\rho_b^4} \tag{26}$$

is the potential which is felt by the quadratic fluctuations in the neighborhood of the bubble. Here λ_n^v and λ_n^0 are *all* eigenvalues of the two differential operators.

This infinite ratio can be calculated most simply by means of the Gelfand–Yaglom method: let $\psi_v(z)$, $\psi_0(z)$ be solutions of the homogeneous equations

$$[-\partial_z^2 + v(z) + \omega^2]\psi_v(z) = 0 \tag{27}$$

$$(-\partial_z^2 + \omega^2)\psi_0(z) = 0 \tag{28}$$

with initial conditions

$$\psi_v(-L/2) = 0, \qquad \psi_v'(-L/2) = 1 \tag{29a}$$

$$\psi_0(-L/2) = 0, \qquad \psi_0'(-L/2) = 1 \tag{29b}$$

Then the ratio of the determinants equals the ratio of the corresponding $\psi(L/2)$ values,[4,*]

$$\frac{\det[-\partial_z^2 + v(z) + \omega^2]}{\det(-\partial_z^2 + \omega^2)} = \frac{\psi_v(L/2)}{\psi_0(L/2)} \tag{30}$$

In the present case $\psi_0(z) = (1/\omega)\,\mathrm{sh}\,\omega(z + L/2)$.

*The proof is trivial. On a grated z axis, $\det[-\partial_z^2 + v(z)]$ is proportional to

$$\psi_N = \varepsilon \det\begin{pmatrix} 2+\varepsilon^2 v_N & -1 & 0 & \cdots \\ -1 & 2+\varepsilon^2 v_{N-1} & -1 & . \\ 0 & -1 & & \end{pmatrix}$$

which can be developed as $\psi_N = (2+\varepsilon^2 v_N)\psi_{N-1} - \psi_{N-2}$. This is nothing but the grated form of $-\psi''(z) + v(z)\psi(z) = 0$. The initial conditions (29) follow from $\psi_1 = \varepsilon(2+\varepsilon^2 v_1) \to 0$, $(\psi_1 - \psi_2)/\varepsilon \to 1$.

In order to find $\psi_v(z)$ we may proceed as follows. First, the derivative of the bubble solution $\rho_b' = (d/dz)\,\rho_b$ is certainly a solution of (27) by translational invariance. Asymptotically it behaves as

$$\rho_b' \to \pm \frac{\omega^3}{(1-K^2)^{1/2}}\, e^{-\omega|z|} = \pm\alpha\, e^{-\omega|z|}, \qquad z \to \pm\infty \tag{31}$$

It may be denoted by ψ_1:

$$\psi_1(z) \equiv \rho_b' \tag{32}$$

Since it does not yet satisfy the desired initial condition (29a), it has to be combined with a second linear independent solution. It is convenient to take advantage of the asymptotic nature of both the boundary conditions (29a), (29b), and the result of (30). For this reason only the asymptotic behavior of the second solution needs to be known. But this must be exponentially increasing together with the opposite symmetry of (32), i.e.,

$$\psi_2 \to \alpha\, e^{+\omega|z|}, \qquad z \to \pm\infty \tag{33}$$

where we have chosen the prefactor to agree with (32). The proper linear combination satisfying (29a) is

$$\psi_v = (1/2\alpha m)(e^{\omega L/2}\psi_1 + e^{-\omega L/2}\psi_2) \tag{34}$$

such that we simply have

$$\psi_v(L/2) = -1/\omega \tag{35}$$

and the ratio of determinants (25) becomes[8]

$$\frac{\det[-\partial_z^2 + v(z) + \omega^2]}{\det(-\partial_z^2 + \omega^2)} = -\frac{1}{\operatorname{sh}\omega L} \to -2\, e^{-\omega L} \tag{36}$$

Actually we have committed an error in this calculation. For large length L there is a very-small-frequency mode with $\lambda_0 \sim e^{-L\omega}$ which tends to zero exponentially due to the system becoming translationally invariant for $L \to \infty$. The corresponding fluctuations carry the bubble over the whole axis $z \in (-L/2, L/2)$. They are not at all small but rather give rise to a "volume factor" L. Thus it is not sufficient to include them only to quadratic order. If one wants to deal with λ_0 correctly, one has to remove it from the ratio of eigenvalues and treat it separately. Because of its smallness one may calculate λ_0 to lowest order in perturbation theory. For this reason one writes[8]

$$\psi_{\lambda_0} \simeq \psi_v(z) + \frac{\lambda_0}{2\alpha^2 m}\int_{-L/2}^{z} dz'\,[\psi_1(z)\psi_2(z') - (z \leftrightarrow z')]\psi_v(z') \tag{37}$$

where the denominator is the Wronskian

$$\psi_1(z)\psi_2'(z)-\psi_2(z)\psi_1'(z)=2\alpha^2\omega$$

of the solutions (32) and (33). The limits of integration are chosen such as to force ψ_{λ_0} to vanish at $-L/2$. At the eigenvalue λ_0 also $\psi_{\lambda_0}(L/2)$ has to vanish. This gives the condition

$$\lambda_0=-2\alpha\left\{\int_{-L/2}^{L/2} dz'[e^{-\omega L/2}\psi_2(z')\psi(z')-e^{\omega L/2}\psi_1(z')\psi(z')]\right\}^{-1} \tag{38}$$

Inserting the decomposition (34) yields

$$\lambda_0=-4\alpha^2\omega\left[e^{-\omega L}\int_{-L/2}^{L/2} dz'\,\psi_2^2(z')-e^{\omega L}\int_{-L/2}^{L/2} dz'\,\psi_1^2(z')\right]^{-1} \tag{39}$$

Now, the second integral is simply half the bubble energy

$$\int_{-L/2}^{L/2} dz\,\psi_1^2(z)=\int_{-L/2}^{L/2} dz\,\rho_b'^2=\tfrac{1}{2}F_b \tag{40}$$

The first integral diverges as $e^{\omega L}$: thus one obtains the almost-zero frequency

$$\lambda_0=4\alpha^2\omega\, e^{-\omega L}/(F_b/2) \tag{41}$$

such that the ratio of determinants, with λ_0 removed, reads

$$\frac{\det'[-\partial_z^2+v(z)+\omega^2]}{\det(-\partial_z^2+\omega^2)}=\frac{\prod_{n\neq 0}\lambda_n^v}{\prod_n\lambda_n^0}=-\frac{F_b}{4\alpha^2\omega} \tag{42}$$

and the full ratio (25) may be written in the form

$$(-4\alpha^2\omega/F_b)^{1/2}\lambda_0^{-1/2} \tag{43}$$

Let us now properly treat the diverging factor $\lambda_0^{-1/2}$. Formally, it is the result of an expansion of the fluctuation in all eigenmodes

$$\rho=\xi_0 y_0+\sum_{n\neq 0}\xi_n y_n \tag{44}$$

and a successive integration over all normal variables ξ. The normalized zero-frequency solution is

$$y_0=\rho_b'\Big/\left(\int dz\,\rho_b'\right)^{1/2}=\rho_b'/F_b^{1/2} \tag{45}$$

An infinitesimal translation of the bubble, on the other hand, proceeds via*

$$\rho\to\rho+da\,\rho'+\cdots \tag{46}$$

*The dots denote additional pieces $\sum_{n\neq 0}\xi_n y_n'$ in (46) which do not contribute to the quadratic fluctuations since they are odd in ξ.

Therefore we can change the integration measure from ξ_0 to the position variable a, with

$$da = (F_b/2)^{-1/2} d\xi_0 \tag{47}$$

The integral which formally leads to $\lambda_0^{-1/2}$

$$\int \frac{d\xi_0}{(\pi T/\sigma)^{1/2}} \exp \frac{-\lambda_0 \xi_0^2 \sigma}{T} = \frac{1}{\lambda_0^{1/2}} \tag{48}$$

may be transformed into

$$\left(\frac{F_b}{2\pi T/\sigma}\right)^{1/2} \int da \exp \frac{-\lambda_0 \xi_0^2 \sigma}{T} \tag{49}$$

In this final version the limit $\lambda_0 \to 0$ may be taken without committing the same error as before, and results in the "volume" factor

$$\frac{1}{\lambda_0^{1/2}} \to \left(\frac{F_b}{2\pi T}\right)^{1/2} L \,. \tag{50}$$

The final result for (23), therefore, is

$$\frac{Z_b}{Z_0} = \left(\frac{F_b \sigma}{2\pi T}\right)^{1/2} L \left(\frac{-4\alpha^2 \omega}{F_b}\right)^{1/2} e^{-F_b \sigma / T} \tag{51}$$

The minus sign is the signal for the presence of a mode whose frequency is negative. That such a mode must be present follows from the fact that ρ_b' is an antisymmetric wave function of *zero* frequency.[7] There has to be at least one lower eigenvalue. It corresponds to an instability of the bubble under phase slips: once the phase slip takes place the bubble disappears in favor of a uniform solution, with the current reduced or increased by one unit of flux. From the discussion of nucleation rates in Ref. 7 we know that the formal imaginary result (51) actually lacks a factor 1/2 since the analytic continuation of the functional integral to negative frequencies requires the selection of a branch along the upper or lower functional imaginary axis. Only the upper (say) gives the rate for decay. The opposite complex conjugate direction is responsible for the return of the system from the saddle point to the original state in our situation; this corresponds to the current-decreasing phase slips. Thus we can write for only the current-decreasing part

$$\frac{Z_b^{\text{decr}}}{Z_0} = \frac{i}{2} L \frac{\alpha \sqrt{\omega}}{(2\pi T/\sigma)^{1/2}} e^{-F_b \sigma / T} \tag{52}$$

It is worthwhile stressing that in this formula we have expressed the result of all quadratic fluctuations in Eq. (4) in terms of the two parameters ω and α which are trivial to read off the asymptotic form of the critical bubble solution $[\rho_b'(z) \to \alpha\, e^{-\omega |z|}]$.

Certainly the one-bubble result (52) is only the first term in an expansion of the partition function into many-bubble contributions. The total sum reads

$$Z = Z_0 \exp(Z_v^{\text{decr}}/Z_0) = Z_0 \exp[i(\sigma/T)\Gamma \exp(-F_b\sigma/T)] \tag{53}$$

Explicitly, Γ is given by

$$\begin{aligned}\Gamma &= \frac{LT/\sigma}{(2\pi T/\sigma)^{1/2}}\alpha\sqrt{\omega} = \frac{LT/\sigma}{(2\pi T/\sigma)^{1/2}}\frac{\omega^{7/2}}{(1-K^2)^{1/2}} \\ &= \frac{L}{(\pi T/\sigma)^{1/2}}2^{5/4}\frac{(1-3K^2)^{7/4}}{(1-K^2)^{1/2}}\frac{T}{\sigma}\end{aligned} \tag{54}$$

For $K \simeq 0$ and $K \sim 1/\sqrt{3}$ this agrees precisely with the result of McCumber and Halperin,[1] who obtain for $K \simeq 0$

$$\begin{aligned}\Gamma &= \frac{1}{2}\left(\frac{\frac{1}{3}\omega^3}{\pi T/\sigma}\right)^{1/2}\Bigg|_{K=0}(24)^{1/2}(1-\tfrac{10}{4}K^2+\cdots) \\ &= \frac{LT/\sigma}{2}\left[\frac{\frac{8}{3}(1/2\sqrt{2})}{\pi T/\sigma}\right]^{1/2}(24)^{1/2}(1-\tfrac{19}{4}K^2+\cdots)\end{aligned} \tag{55}$$

and the same multiplied with a factor $\sqrt{\frac{2}{3}}\,(1-3K^2)^{7/4}$ for $K \to 1/\sqrt{3}$. Here $(24)^{1/2}$ is the product of all frequency ratios, and $(\frac{1}{3}\omega^3/\pi T/\sigma)^{1/2}$ is the factor from the translational mode.

4. OBSERVATIONAL CONSEQUENCES

The imaginary part of the energy Γ gives rise to a rate of phase slip which was derived in Ref. 9 by using the Fokker–Planck equation as

$$\text{Rate} = \frac{|\lambda_{-1}|}{\tau}\frac{1}{\pi T/\sigma}\Gamma\, e^{-F_a\sigma/T} \tag{56}$$

where λ_{-1} is the negative frequency mode via which the decay occurs and τ is the relaxation time for the movement of the corresponding normal coordinate:

$$\tau\partial\xi_{-1}/\partial t = |\lambda_{-1}|\xi_{-1} \tag{57}$$

The value of λ_{-1} can be taken from McCumber and Halperin[1]:

$$\lambda_{-1} = \tfrac{1}{2}(1+K^2) - [(1+K^2)^2 + \tfrac{3}{2}\omega^2]^{1/2} \tag{58}$$

Therefore the rate of phase slip becomes

$$\text{Rate} = \frac{1}{2\pi\tau}\left[\frac{1}{(\pi T/\sigma)^{1/2}} 2^{5/4}\frac{(1-3K^2)^{7/4}}{(1-K^2)^{1/2}}\right]$$
$$\times\{(1+K^2)-[(1+K^2)^2+3(1-3K^2)^2]^{1/2}\}\, e^{-F_b\sigma/T} \tag{59}$$

It must be stressed that λ_{-1} is the negative eigenvalue in the *original* full 2×2 fluctuation problem, i.e., before the path integral over $D\gamma$ is performed. The negative eigenvalue of the final, pure ρ, determinant is completely unrelated to this λ_{-1}, since one degree of freedom, which is not an eigenmode, has been eliminated. In order to appreciate this remark, consider the example of a trivial fluctuation integral with eigenvalues $+1$ and -1:

$$\oint \frac{dx}{\sqrt{\pi}}\frac{dy}{\sqrt{\pi}}\exp[-(x^2-y^2)] \cong \frac{1}{2}\frac{1}{\sqrt{-1}} \tag{60}$$

By a variable change

$$x=(\cos\theta)\zeta+(\sin\theta)\eta, \qquad y=-(\sin\theta)\zeta+(\cos\theta)\eta \tag{61}$$

this becomes

$$\int \frac{d\zeta}{\sqrt{\pi}}\frac{dy}{\sqrt{\pi}}\exp\left\{-(\cos 2\theta)[\zeta-(\text{tg}\,2\theta)\eta]^2+\frac{1}{\cos 2\theta}\eta^2\right\} \tag{62}$$

Now if the integral over ζ is done, this leaves

$$(\cos 2\theta)^{-1/2}\int \pi^{-1/2}\,d\eta\,\exp(\eta^2/\cos 2\theta) \approx \tfrac{1}{2}(\cos 2\theta)^{-1/2}(-\cos 2\theta)^{1/2} \tag{63}$$

and the eigenvalue of the remaining variable η can be anything between 1 and ∞. In fact, the negative eigenvalue of our fluctuation problem

$$[-\partial_z^2+v(z)+\omega^2]\psi=\lambda\psi \tag{64}$$

diverges as $-1/K$ for $K\to 0$, and vanishes as $-\frac{5}{4}\omega^2$ for $K\to 1/\sqrt{3}$. The true negative frequency λ_{-1}, on the other hand, moves from $-1/2$ at $K=0$ to $-(9/64)\omega^4$ for $K\to 1/\sqrt{3}$.

In order to extract an observable quantity from the result (59) it is convenient to consider an ensemble of fluctuating uniform currents j_{fl} in some neighborhood of the external source value j. The exponent of the partition function (42) can then be split according to

$$\int dz\,(f-2j\gamma_z)=\int dz\,(f-2j_{\text{fl}}\gamma_z)-(j-j_{\text{fl}})2\int dz\,\gamma_z \tag{65}$$

The first piece can be treated as in the last section, finding the two kinds of extremal contributions: those with uniform order parameter

$$\rho(z) \equiv \rho_0 = (1 - K^2), \qquad \gamma_0(z) = Kz \tag{66}$$

and the bubble solutions

$$\rho(z) = \rho_b(z),$$
$$\gamma_b(z) = j_{\text{fl}} \int_{-L/2}^{z} dz \, \frac{1}{\rho_b^2(z)} = Kz + \text{arctg}\left(\frac{\omega}{2K} \text{th} \frac{\omega}{K} z\right) \tag{67}$$

The situation can be described most clearly if we imagine the corresponding currents

$$j_{\text{fl}} = K(1 - K^2) \tag{68}$$

to run through a closed wire of length L. Then the values of K are quantized as

$$\int_{-L/2}^{L/2} dz \; \partial_z \gamma_0(z) = K_n L = 2\pi n$$
$$\int_{-L/2}^{L/2} \partial_z \gamma_b(z) = K_n^b L + 2 \text{ arctg} \frac{\omega}{2K} \equiv K_n^b L + 2\delta_\infty \equiv 2\pi n \tag{69}$$

Notice that K_n^b is slightly lower than K_n which is necessary in order to accommodate the narrower winding of $\gamma_b(z)$ in the bubble region.

As far as the decay rate is concerned, it is important to realize that each uniform state K_n has two neighboring bubble states: K_n^b and K_{n+1}^b. The first leads to a decay of the state K_n into the next lower state K_{n-1}. The second, however, is responsible for the decay of the next higher state K_{n+1} into K_n itself. Therefore it corresponds to a current-increasing transition. Looking at (65), we see that the free energy of the first bubble is [compare (24)]

$$G_K^{j_{\text{fl}}} + F_b = G_K^{j_{\text{fl}}} + \tfrac{4}{3}\omega - 4 j_{\text{fl}} \delta_\infty \tag{70}$$

Its decay rate is therefore precisely given by formula (59).

The current-increasing bubble, on the other hand, is accompanied by a change of $\int_{-L/2}^{L/2} dz \; \gamma_z(x)$ by 2π. Therefore its energy has the different value

$$G_K^{j_{\text{fl}}} + F_b - (j - j_{\text{fl}}) 4\pi \tag{71}$$

Its rate follows formula (59) but with an additional Boltzmann factor

$$e^{-(j_{\text{fl}} - j) 4\pi\sigma / T} \tag{72}$$

And with a negative sign accounting for its opposite direction.

Hence the total decay rate of the fluctuating current j_{fl} is given by formula (59) with the factor*

$$(1-e^{-(j_{fl}-j)4\pi\sigma/T}) \tag{73}$$

If the current j_{fl} is set up above j, it decreases, phase slip by phase slip,* until equilibrium is reached at $j_{fl}=j$.

For applications it is useful to return from our reduced variables to physical quantities: the physical current density reads

$$\begin{aligned} J^{\text{phys}} &= \frac{e\hbar}{m^2\xi_0}(\text{mass density})\left(1-\frac{T}{T_c}\right)^{3/2} j_{fl} \\ &= \frac{4e}{\hbar}\xi(2f_c)j_{fl} = \frac{4\pi}{\Phi_0/c}\xi(2f_c)j_{fl} \end{aligned} \tag{74}$$

where

$$\Phi_0 \equiv \pi c\hbar/e \tag{75}$$

is the fundamental unit of flux. Hence, the total physical current through the wire

$$I = J^{\text{phys}}\sigma\xi^2 \tag{76}$$

satisfies

$$I\Phi_0/c = 4\pi j_{fl}(2f_c\xi^3)\sigma \tag{77}$$

where $2f_c\xi^3$ was our unit energy. Thus in the absence of an external source, the Boltzmann factor in (59) becomes

$$\begin{aligned} &\exp\left(-\frac{2f_c\xi^3\sigma}{K_BT}\frac{4}{3}\omega\right)\exp\left(\frac{1}{K_BT}I\frac{\Phi_0}{c}\frac{1}{4\pi}\delta_\infty\right) \\ &\times\left[1-\exp\left(-I\frac{\Phi_0}{c}\frac{1}{k_BT}\right)\right] \end{aligned} \tag{78}$$

where all quantities are now in arbitrary units.

The net rate of phase slips can directly be seen experimentally since by Josephson's relation

$$\partial\gamma/\partial t = (2e/\hbar)\,\nabla V \tag{79}$$

a rate of phase slips equals $2e/\hbar$ times the voltage V across the wire.

$$\frac{1}{2\pi}\frac{\Delta\int dz\,\gamma}{\Delta t} = \frac{1}{2\pi}\frac{2e}{\hbar}V \tag{80}$$

*For a discussion of the topological aspects of the decay see Kleinert.[10]

5. CONCLUSION

If angular variables can be treated exactly, it is advisable to do so. The resulting reduction of functional dimensionality may lead to drastic simplification of the fluctuation problem.

This problem has been shown to become trivial in one dimension: the full fluctuation determinant is derived from only the *asymptotic behavior of the bubble solution.*

Certainly, the method presented here is very general and can be applied to many quasi-one-dimensional systems (e.g., Ref. 2). It will be interesting to see what corresponding result can be obtained for different dimensionalities.

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