FIELD THEORY OF COLLECTIVE EXCITATIONS III
CONDENSATION OF FOUR-PARTICLE CLUSTERS

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For many fermions with arbitrary two-body forces we investigate the effective action $\Gamma[\rho, \alpha]$ where ρ is the density matrix including pair correlations and α is the four-particle vertex. The physical state including all radiative corrections is obtained by extremizing $\Gamma[\rho, \alpha]$ with respect to ρ and α. Just as $\rho \neq 0$ corresponds to non-zero density and condensation of pairs, $\alpha \neq 0$ signals the presence of a condensate of four-particle clusters. A simple loop expansion is given for $\Gamma[\rho, \alpha]$. To lowest order, the extremum amounts to the Hartree–Fock–Bogoliubov equations. The next two steps bring in the four-particle condensate.

Effective actions Γ seem to provide the ideal tool for deriving non-perturbative properties of quantum systems in terms of classical variables. Well-known diagrammatic resummation techniques such as the Hartree–Fock–Bogoliubov self-consistency equation follow from extremizing the lowest approximation to Γ, and higher corrections can be obtained from simple diagrammatic rules. Large-amplitude collective excitations and tunneling phenomena become equally accessible via extremal principles, once with solutions periodic in time and once connecting different time-independent extrema on a path along the imaginary time axis [1]. The extremal value itself of the effective action either serves to quantize the orbit or determines the barrier penetration amplitude.

For fermionic systems, the basic effective action is a functional $\Gamma[\rho]$ of the 2 × 2 matrix ρ which collects density and pair correlations $\langle T\psi\psi^+\rangle$ and $\langle T\psi\psi\rangle$. It is defined as the Legendre transform of the generating functional

$$W[K] = -\mathrm{i}\log Z[K] = -\mathrm{i}\log \int \mathcal{D}\varphi \exp(\mathrm{i}\mathcal{A}[\varphi] + \mathrm{i}\varphi K\varphi/2), \tag{1}$$

where

$$\varphi_a \equiv \begin{pmatrix}\varphi_{\alpha\uparrow}\\ \varphi_{\alpha\downarrow}\end{pmatrix} \equiv \begin{pmatrix}\psi_\alpha\\ \psi^+_\alpha\end{pmatrix}$$

is the doubled notation for the fermion field, K_{ab} is an auxiliary external source, and all orbital and time indices have been suppressed. The action is written as

$$\mathcal{A} = \tfrac{1}{2}\varphi \mathrm{i}G_0^{-1}\varphi - (1/4!)\,V\varphi\varphi\varphi\varphi, \tag{2}$$

where

$$\mathrm{i}G_0^{-1} = \begin{pmatrix}0 & \mathrm{i}\partial_t + (\epsilon-\mu)\\ \mathrm{i}\partial_t - (\epsilon-\mu) & 0\end{pmatrix} \equiv \begin{pmatrix}0 & \mathrm{i}\partial_t + \xi\\ \mathrm{i}\partial_t - \xi & 0\end{pmatrix} \tag{3}$$

is the kinetic term in a somewhat unconventional form which, however, will be useful for our formalism. The potential V has upper and lower entries for each of its four doubled indices, is completely antisymmetric, and may be normalized to the standard interaction $-\frac{1}{2}v_{\alpha\beta\gamma\delta}\psi^+_\alpha\psi_\beta\psi^+_\gamma\psi_\delta$ by choosing $V_{\alpha\downarrow\beta\uparrow\gamma\downarrow\delta\uparrow} = 2v_{\alpha\beta\gamma\delta}$.

The effective action $\Gamma[\rho]$ is given by the Legendre transform

$$\Gamma[\rho] \equiv W[K] - \tfrac{1}{2}\rho_{ab}K_{ab}, \tag{4}$$

where

$$\tfrac{1}{2}\rho_{ab} = \delta W/\delta K_{ab} = Z^{-1}\int \mathcal{D}\varphi\, \varphi_a\varphi_b \exp(\mathrm{i}\mathcal{A} + \mathrm{i}\varphi K\varphi/2) \tag{5}$$

is, by definition, the density matrix of the system,

$$\rho_{ab} \equiv \langle \varphi_a(t) \varphi_b(t') \rangle . \tag{6}$$

By construction, $\Gamma[\rho]$ is extremal for physical ρ configurations at no external source, as can be checked by differentiating (4):

$$\delta \Gamma[\rho] / \delta \rho_{ab} = -\tfrac{1}{2} K_{ab} = 0 . \tag{7}$$

This is why it is called the effective action. The simple rule for constructing $\Gamma[\rho]$ was given in ref. [1] as follows: Write $\Gamma[\rho]$ as a sum of free and interacting parts:

$$\Gamma[\rho] = \Gamma^{(0)}[\rho] + \Gamma^{(1)}[\rho] + \Gamma^{\text{int}}[\rho] = \tfrac{1}{2} \mathrm{i} G_{0ab}^{-1} \rho_{ab} - \tfrac{1}{2} \mathrm{i} \operatorname{tr} \log \rho^{-1} + \sum_{n=2}^{\infty} \Gamma^{(n)}[\rho] . \tag{8}$$

Then the terms in $\Gamma^{(n)}[\rho]$ consist precisely of all two-particle irreducible (TPI) vacuum graphs in which a line stands for the density matrix ρ and a vertex for the potential V. The label n organizes the expansion according to the number of loops. The diagrams were pictured in ref. [1] together with their multiplicities. The first four terms may be written in a compact fashion as

$$\sum_{n=2}^{\infty} \Gamma^{(n)}[\rho] = -\tfrac{1}{8} V \rho\rho + \tfrac{1}{48} \mathrm{i} (V \rho\rho)^2 + \tfrac{1}{48} (V \rho\rho)^3 - \frac{5}{8 \cdot 16} \mathrm{i} (V \rho\rho)^4 - \dots , \tag{9}$$

where up to V^3 there is only one, for V^4 there are two different index contractions which are TPI. These and all higher terms follow from the two functional equations

$$\Gamma^{\text{int}}_{\rho}[\rho] \rho = -\tfrac{1}{4} V \rho\rho + \tfrac{1}{12} \mathrm{i} V \rho^4 \alpha , \tag{10}$$

$$\alpha = -4 \Gamma^{\text{int}}_{\rho\rho} (1 - 2\mathrm{i} \rho\rho \Gamma^{\text{int}}_{\rho\rho})^{-1} = -4 \Gamma^{\text{int}}_{\rho\rho} + 2\mathrm{i} \Gamma^{\text{int}}_{\rho\rho} \rho\rho\alpha , \tag{11}$$

where α is the complete four-particle vertex function. See again ref. [1] for the index contractions. Notice that expansion (9) is non-perturbative since ρ refers to the fully interacting Green's functions.

Extremization of $\Gamma[\rho]$ gives the generalization of the Hartree–Fock–Bogoliubov equation correct to all orders in the coupling:

$$\rho = \mathrm{i} [\mathrm{i} G_0^{-1} + 2 \Gamma^{\text{int}}_{\rho}]^{-1} , \tag{12}$$

whose approximate solutions may be reinserted back into $\Gamma[\rho]$ for quantization or barrier penetration factors.

The particular strength of eq. (12) lies in the possibility of accounting for Bose condensates of pairs of fermions (Cooper pairs). In some many-fermion systems such as atomic nuclei, however, four-particle clusters play an important role in the observed phenomena. There is an obvious need for an equation analogous to (12) which allows for the formation of a condensate of these clusters. This can indeed be found by a straightforward extension of the technique of Legendre transformation. For this we insert into the action (2) a coupling to a four-fermion source γ by substituting $V \to V + \gamma$. Then we perform a further Legendre transformation

$$\Gamma[\rho, \alpha] \equiv \Gamma[\rho] - \Gamma_{\gamma}[\rho] \gamma . \tag{13}$$

As our new variable α we choose precisely the four-particle vertex function which occurred in the intermediate functional equations (10), (11). This can be extracted from $\Gamma_{\gamma}[\rho]$ by separating out disconnected parts and amputating external legs,

$$\Gamma_{\gamma}[\rho] = \tfrac{1}{4} \mathrm{i} \alpha \rho\rho\rho\rho - \tfrac{3}{4} \rho\rho . \tag{14}$$

Inserting (9) we find

$$\Gamma[\rho, \alpha] = \Gamma^{(0)}[\rho] + \Gamma^{(1)}[\rho] + \Gamma^{\text{int}}[\rho, \alpha] = \Gamma^{(0)}[\rho] + \Gamma^{(1)}[\rho] - \tfrac{1}{8} V \rho\rho + \tfrac{1}{48} \mathrm{i} (2\alpha V - \alpha^2) \rho^4 + \tfrac{1}{48} \alpha^3 \rho^6 + \frac{1}{8 \cdot 16} \mathrm{i} \alpha^4 \rho^8 - \dots . \tag{15}$$

The terms $\Gamma^{(n)}[\rho, \alpha]$ can again be pictured graphically (see fig. 1) with lines representing ρ but, contrary to the previous case, vertices stand for the fully interacting vertex function α rather than the potential V. Up to $n = 4$, the diagrams are the same as before. For $n \geqslant 5$, their number is considerably reduced.

This new functional is truly an effective action since from (7), (14) its derivatives satisfy

$$\Gamma_{\rho}[\rho, \alpha] = -K/2 - (\mathrm{i}/3!) \alpha \rho^3 \gamma + \tfrac{1}{4} \rho \gamma , \tag{16}$$

$$\Gamma_{\alpha}[\rho, \alpha] = -(\mathrm{i}/4!) \rho^4 \gamma , \tag{17}$$

such that Γ is extremal for physical ρ, α configurations at no external source. Actually, eq. (17) represents the

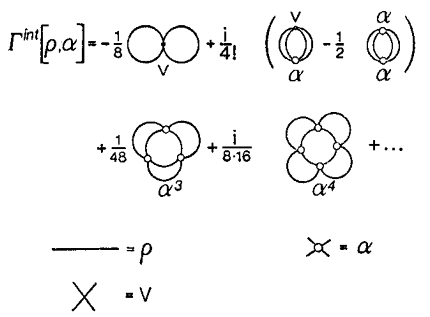

Fig. 1. The loop expansion of the interacting part of the higher effective action whose extrema determine two- and four-particle correlations. Lines stand for the full density matrix ρ, small blobs for the interaction vertex α, and simple vertices for the potential V.

fastest access to the explicit expansion (15) since we may use (9) and (14) to find the functional $\alpha[\rho, \gamma]$, invert this to obtain $\gamma = \gamma[\alpha, \rho]$, and integrate the result over α. This is also what we have done [2].

Extremizing the higher effective action (15) gives the equations of motion

$$\rho = \mathrm{i}[\mathrm{i}G_0^{-1} - \tfrac{1}{2}V\rho + \tfrac{1}{6}\mathrm{i}\alpha G^3 V]^{-1}, \qquad (18)$$

$$V = V[\rho, \alpha] = \alpha + \tfrac{3}{2}\mathrm{i}\alpha\rho^2\alpha - \tfrac{3}{4}\alpha^3\rho^4 + \dots . \qquad (19)$$

Notice that the infinitely many terms which are present at first in eq. (18) can all be summed by using (19) since all terms after the fourth in the expansion (15) depend on ρ and α only in the combination $\alpha\rho^2$!

Eq. (19) does represent the desired self-consistency condition which allows for the formation of a condensate of four-particle clusters. We see by inspection that the extremal α configuration does not necessarily have to be a perturbatively improved version of the original potential V (even though this possibility is certainly included). There may be additional parts in α corresponding to a correlation of four particles $\langle\psi\psi\psi\psi\rangle$ (not just $\langle\psi^+\psi\,\psi^+\psi\rangle$). This happens if the homogeneous part of (19) satisfies

$$V[\rho, \alpha]\big|_{\psi^4 \mathrm{part}} = 0, \qquad (20)$$

if also $\alpha|_{\psi^4 \mathrm{part}} \neq 0$ for this projection. Obviously, this condition may be interpreted as signaling the presence of a condensate of four-particle clusters in the ground state just as $\langle\psi^2\rangle$ does with respect to Cooper pairs. Such a "spontaneously generated" α vertex contributes a completely new piece to the denominator of of (18).

Eqs. (18), (19) permit a deeper investigation of many collective phenomena. In particular they lead to a natural extension of Landau's theory of Fermi liquids in which not only the particle densities but also the vertices show dynamical excitation [2]. More details will be published elsewhere [3].

References

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