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Calculation of Relativistic Transition Probabilities and Form Factors from Noncompact Groups*

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Relativistic transition probabilities and form factors have been evaluated in closed form (in terms of hypergeometric functions) for a model of fundamental particles constructed on the unitary irreducible representation space of a noncompact dynamical group. Majorana-type equations have been used to project out irreducible subspaces of the Poincaré group and to specify the transformation properties of the electromagnetic current. The magnetic moment of the spin-$\frac{1}{2}$ ground state in the simplest representation is $-\frac{1}{2}$, and one obtains the observed symmetry $G_E(t)=G_M(t)/\mu$.

I. INTRODUCTION

IN a previous study[1] we have investigated the problem of transition probabilities based on noncompact dynamical groups. The purpose of this paper is to calculate for a relativistic model of elementary particles constructed on the space of unitary irreducible representations of a noncompact dynamical group all transition probabilities and form factors.

A problem vigorously pursued in recent years concerned itself with an algebraic description of relativistic mass spectra of a system with internal degrees of freedom and of its interactions with external fields. More specifically, the algebraic structure must contain the Poincaré group $\mathfrak{P}$, such that the quantity

$$P_\mu P^\mu = m^2$$

has a discrete spectrum. Moreover, the levels at rest, $P\mu$: $(m,0,0,0)$, belong to an irreducible representation of a noncompact dynamical group G, and represent, physically, the excited states of the relativistic quantum system. It is clear that we have to do here with a reducible representation of the Poincaré group containing a countably infinite number of irreducible representations of different masses and spins as determined by the noncompact group G.[2]

We use, in this and following papers, Majorana-type equations[3] involving infinite-dimensional unitary representations of the group G in order to determine the transformation properties of electromagnetic interactions. The transition probabilities are then calculated purely group theoretically within the framework of a larger group $\mathfrak{G}$ containing G, as in the nonrelativistic case.[1]

Majorana-type equations have recently been used by Nambu[4] to calculate mass spectra and form factors. Nambu has calculated one form factor in the case of nonunitary representations of G and finds it to be unphysical. For the unitary representations he has only given a mass formula. We present here the calculation of form factors and transition probabilities based on infinite-dimensional unitary representations and the result is physical. We remark that the electric part of the form factor has also been calculated from the semidirect product $\mathfrak{P}\otimes SL(6,C)$ by a different method.[5,6]

In Sec. II we give the algebraic content of the solution, how the irreducible representation of G is extended to a reducible representations of $\mathfrak{P}$ and how the irreducible parts of the latter are picked up, and finally, the most general mass spectra allowed by the formalism. The calculation of form factors and transition form factors is given in Sec. III and interpreted in Sec. IV. The present paper deals with the simplest triangular representations of the noncompact group. More general representations and groups are considered in the papers that will follow.

II. ALGEBRAIC STRUCTURE OF COMBINED POINCARÉ GROUP AND DYNAMICAL GROUP

Let α be the indices, collectively, labeling the irreducible unitary representations of a noncompact dynamical group G, containing the homogeneous Lorentz group as a subgroup. Let p be the continuous labels of the translation generators, that is the momenta. The states $|\alpha;\, p\rangle$ represent then the Poincaré group $\mathfrak{P}$ in a reducible way, because the states of an irreducible representation of $\mathfrak{P}$ are labeled by p and by the labels of the little groups $O(3)$ or $O(2,1)$. The indices α contain those of the little group a great many times. In fact, the

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[1] A. O. Barut, second preceding paper, Phys. Rev. **156**, 1538 (1967), A. O. Barut and H. Kleinert, preceding paper, *ibid.* **156**, 1541 (1967).

[2] The Hilbert space is the direct sum $\otimes\sum\mathcal{H}[m,s]$, where $\mathcal{H}[m,s]$ are the Hilbert spaces of the irreducible representation of the Poincaré group and the range of the sum is determined by the representation of the dynamical group G. See A. O. Barut, in *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), and in *Non-Compact Groups in Particle Physics*, edited by Y. Chow (W. A. Benjamin, Inc., New York, 1966).

[3] E. Majorana, Nuovo Cimento **9**, 335 (1932); I. M. Gel'fand and A. M. Yaglom, Zh. Eksperim. i. Teor. Fiz. **18**, 703 (1948).

[4] Y. Nambu, Suppl. Theoret. Phys. (Kyoto), Progr. **37** & **38**, 368 (1966).

[5] G. Cocho, C. Fronsdal, Harun Ar-Rashid, and R. White, Phys. Rev. Letters **17**, 275 (1966).

[6] H. Leutwyler, Phys. Rev. Letters **17**, 156 (1966).

reduction of G with respect of $O(3)$ [or $O(2,1)$] gives also the possible spin values in the theory.

Because the two invariants of the Poincaré group, $\mathfrak{P}^2$ and W^2, define mass and spin, we can single out particular values of mass and spin from the Hilbert space $|\alpha; p\rangle$ by the following Poincaré invariant projection operations

$$(p^2-m^2)|\alpha; p\rangle=0, \qquad (2.1)$$
$$[W^2-m^2s(s+1)]|\alpha; p\rangle=0,$$

whereas a more general equation of the type

$$(p^2+\lambda W^2-K)|\alpha; p\rangle=0, \qquad (2.2)$$

fixes a relation between mass and spin.

These equations are however only *kinematical*, because p^2 and W^2 being invariant scalar, they do not tell us the transformation properties of the states (i.e., of indices α) under pure Lorentz transformations (i.e., boost operations). We can write dynamical equations by passing to linear equations which mix the states $|\alpha; p\rangle$ in a particular way and specify how the states are to be boosted. The Majorana equation is of this type

$$(\Gamma^\mu P_\mu-m_0)|\alpha; p\rangle=0, \qquad (2.3)$$

where Γ^μ is a vector operator operating in the infinite dimensional representation of G. More generally, we can write

$$(\Gamma^\mu P_\mu+\lambda\Gamma'^\mu\epsilon_{\mu\nu\lambda\rho}P^\nu L^{\lambda\rho}-\mu)|\alpha; p\rangle=0. \qquad (2.4)$$

We have in fact a generalization of the Majorana equation insofar as we can take in (2.3) and (2.4) m_0, λ, and μ to be an arbitrary function of the invariants m^2 and $m^2S(S+1)$ subject to restrictions imposed by gauge invariance (see Sec. V). This freedom physically means that a whole class of systems with different mass spectra have states which are boosted exactly in the same way under pure Lorentz transformations. This has to be so, because we only specify by G the possible spins of the system and by (2.3) the external motion of the system. This leaves still a lot of freedom of internal dynamics compatible with the same external motion.

The main advantage of the linear equation of the type (2.3) is that it specifies for us the transformation property of the electromagnetic interactions. This is done under the *assumption* that the minimal electromagnetic interactions are obtained by the usual recipe of replacement

$$p_\mu \to p_\mu-eA_\mu, \qquad (2.5)$$

where e is the "charge matrix" (see Sec. V). We shall see in the next section, that (2.5) is the equivalent of specifying the group element in $\mathfrak{G}$ that describes the electromagnetic transitions.

We take in this first paper $\mathfrak{G}$ to be the simplest possible case, namely the group $SL(2,C)$, the covering group of the homogeneous Lorentz group itself, with generators L_i and M_i and consider a triangular representation with one vansihing Casimir operator

$$Q_1=L\cdot M=0. \qquad (2.6)$$

This representation can be obtained by writing the generators in terms of the boson creation and annihilation operators in the following form

$$L^i=a^\dagger\sigma^i a, \quad a=\begin{pmatrix}a_1\\a_2\end{pmatrix},$$
$$M^i=\tfrac{1}{2}i(a^\dagger\sigma^i Ca^\dagger+aC\sigma^i a), \qquad (2.7)$$
$$C=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

These operators act on the normalized states

$$|S,S_z\rangle=[(S-S_z)!(S+S_z)!]^{-1/2}a_1{}^{\dagger S+S_z}a_2{}^{\dagger S-S_z}|0\rangle. \quad (2.8)$$

With (2.7), Eq. (2.6) is automatically satisfied, and the second-order Casimir operator can be evaluated giving

$$Q_2=L^2-M^2=(a^\dagger aa^\dagger a+2a^\dagger a) -(a^\dagger aa^\dagger a+2a^\dagger a+3)=-3. \quad (2.9)$$

The unitary representation in question, Eq. (2.7), contains all spins

$$S=0,\ 1,\ 2,\ 3,\cdots. \qquad (2.10)$$

The representation with the lowest spin $S_0=\frac{1}{2}$ is obtained by acting the operators (2.7) on the lowest state $(\alpha^\dagger|0\rangle)$. Other unitary representations to be considered in the following papers contains all spins S_0, S_0+1, $S_0+2,\cdots$, starting with a lowest spin $S_0=1, \frac{3}{2}, 2, \cdots$. The vector Γ^μ in (2.3) is uniquely determined in our case to be

$$\Gamma^0=a^\dagger a+1, \quad \Gamma^i=-\tfrac{1}{2}(a^\dagger\sigma^i Ca^\dagger-aC\sigma^i a). \quad (2.11)$$

We observe that the quantities $\mathbf{L}$, $\mathbf{M}$, and Γ^μ form the Lie algebra of a larger group: If we identify

$$\tfrac{1}{2}L^i\to(L_{12},L_{13},L_{23}),$$
$$\tfrac{1}{2}M^i\to(L_{14},L_{24},L_{34}), \qquad (2.12)$$
$$\tfrac{1}{2}\Gamma^i\to(L_{15},L_{25},L_{35}),$$
$$\tfrac{1}{2}\Gamma^0\to L_{45},$$

we have the commutation relations of a five-dimensional algebra

$$[L_{ab},L_{cd}]=-i(-g_{bc}L_{ad}-g_{ad}L_{bc}+g_{ac}L_{bd}+g_{bd}L_{ac}),$$
$$a,b,c,d=1,2,3,4,5, \qquad (2.13)$$
$$g_{11}=g_{22}=g_{33}=+1, \quad g_{44}=g_{55}=-1.$$

This situation is analogous to that of the nonrelativistic hydrogen atom where the transition operators and the dynamical group $O(4,1)$ generate a larger group $\mathfrak{G}$ which is $O(4,2)$.[1] The irreducible representation of $\mathfrak{G}$ remains also irreducible for the smaller group G.

The equation $(\Gamma^\mu p_\mu-m_0)|\alpha; p\rangle=0$ generates a whole set of representations of the Poincaré group. To see this we go to the rest frame of the system where

$$(\Gamma^0 p_0 - m_0)|\alpha;\,0\rangle = 0. \tag{2.14}$$

Because the eigenvalues of Γ^0 are

$$\Gamma^0 : 2S+1\,, \tag{2.15}$$

the spectrum is calculated from the equation

$$(2S+1)m = m_0(m^2, m^2 S(S+1))\,; \tag{2.16}$$

in particular the Majorana case[3] corresponds to a constant m_0. It will be shown that the form factors are essentially independent of the choice of m_0; they will depend on it only through the invariant momentum transfer t.

III. FORM FACTORS

We shall evaluate the electromagnetic vertex under the assumption (2.5) for the electromagnetic coupling. Thus we have to evaluate (Fig. 1)

$$\langle\alpha',p'|J^\mu(0)|\alpha,p\rangle = \langle\alpha',p'|\Gamma^\mu|\alpha,p\rangle. \tag{3.1}$$

In the case $G=SL(2,c)$, we label the states by spin S and its component S_z; then

$$|\alpha;\,p\rangle = |S,S_z;\,p\rangle = \exp(+i\tfrac{1}{2}\boldsymbol{\xi}\cdot\mathbf{M})|S,S_z\rangle\,, \tag{3.2}$$
$$\tanh\xi = v = p/E,\quad \xi=\sqrt{(\xi^2)}\,,$$

where $|S,S_z\rangle$ are pure $SL(2,c)$ states and the exponential factor is the boost operation to momentum p.

Without loss of generality we can go to the rest frame of the particle p'. We have then to evaluate

$$\langle S',S'_z|\Gamma^\mu \exp(+\tfrac{1}{2}i\boldsymbol{\xi}\cdot\mathbf{M})|S,S_z\rangle. \tag{3.3}$$

Let us take first a boost operation in the direction of positive z axis, i.e.,

$$e^{+i\frac{1}{2}\xi_3 M_3}. \tag{3.3$'$}$$

The evaluation of (3.3) can be made quite easy by the following observation: The operator

$$M_3 = \tfrac{1}{2}i(a^\dagger\sigma^1 a^\dagger - a\sigma^1 a) = i(X_3{}^\dagger - X_3{}^-) = 2K_1 \tag{3.4}$$

forms together with

$$\Gamma^0 = a^\dagger a + 1 = 2K_3 \tag{3.5}$$

and

$$-\Gamma^3 = \tfrac{1}{2}(a^\dagger\sigma^1 a^\dagger + a\sigma^1 a) = (X_3{}^+ + X_3{}^-) = 2K_2 \tag{3.6}$$

a Lie subalgebra of $O(2.1)$ of our big algebra $\mathfrak{G}$: $O(3,2)$.

$$K^+ = iX^+ = \tfrac{1}{2}i(a^\dagger\sigma^1 a^\dagger),\quad K^- = -iX^- = -\tfrac{1}{2}i(a\sigma^1 a). \tag{3.7}$$

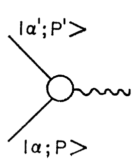

FIG. 1. The electromagnetic vertex with composite particles formed according to the dynamical group.

We find the commutation relations

$$[K^+,K^-] = -2K_3,\quad [K_3,K^\pm] = \pm K^\pm. \tag{3.8}$$

This particular $O(2,1)$, which we call the *transition group* T [in terms of L_{ab} of Eq. (2.13) this is the $O(2,1)$ generated by L_{34}, L_{35}, L_{45}], commutes with L_{12}; hence it changes the values of S by one unit without changing S_z (Fig. 2[7]).

The representation of $G=O(3,1)$ that we are considering reduces into a number of discrete $\mathfrak{D}^+$ representations with respect to T:$O(2,1)$. For $S_z=0$ we have the representation $\mathfrak{D}^+(\phi=-\frac{1}{2})$; for an arbitrary S_z, the representation is $\mathfrak{D}^+(\phi=-\frac{1}{2}-S_z)$ (see Fig. 2). We can see that also by calculating the Casimir operator of T

$$Q = J_1{}^2 + J_2{}^2 - J_3{}^2 = (X^+X^- + K_3) - K_3{}^2. \tag{3.9}$$

We have

$$Q|(S=0,\,S_z=0)\rangle = \phi(\phi+1)|0,0\rangle = -\tfrac{3}{4}|0,0\rangle\,, \tag{3.10}$$

and similarly for the other states.

In Eq. (3.3) we need thus the matrix element of a *finite* $O(2,1)$ rotation. Notice that the calculation of transition probabilities involve group elements rather than infinitesimal generators! The matrix element in question is

$$e^{\frac{1}{2}i\xi M_3}|S,S_z\rangle = V_{s's}{}^{\phi}(a)|S',S_z\rangle. \tag{3.11}$$

Here we can get the finite group rotation a from the two-dimensional fundamental representation for which $M_3=+i\sigma_1$, hence

$$a = e^{\frac{1}{2}i\xi(i\sigma_1)} = e^{-\frac{1}{2}\xi\sigma_1} = \cosh\tfrac{1}{2}\xi - \sigma_1\sinh\tfrac{1}{2}\xi. \tag{3.12}$$

If we compare this with the spinor group

$$W=\begin{pmatrix}\alpha & \beta\\ +\bar\beta & \bar\alpha\end{pmatrix},\quad W^{-1}=\begin{pmatrix}\bar\alpha & \beta\\ -\bar\beta & \alpha\end{pmatrix}, \tag{3.13}$$

we identify

$$\alpha=\cosh\tfrac{1}{2}\xi,\quad \beta=-\sinh\tfrac{1}{2}\xi. \tag{3.14}$$

The matrix elements $V_{s's}{}^{\phi}(a)$ of $\mathfrak{D}^+(\phi)$ have the following closed form[8] (with the identification $\phi=-\frac{1}{2}-S_z$, $m=\frac{1}{2}+S$, Ref. 8b).

$$V_{s's}{}^{\phi}(a) = \theta_{s's}(\phi)\bar\alpha^{(S+S'+1)}\beta^{S'-S}$$
$$\times F(S_z-S,\,-S_z-S,\,1+S'-S;\,-\beta\bar\beta)\,,$$

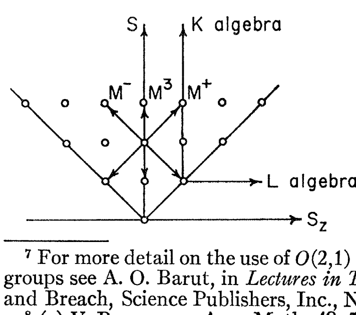

FIG. 2. Weight diagram for the triangular representation of $SL(2,C)$ and the transition subalgebra K.

[7] For more detail on the use of $O(2,1)$ subalgebras of noncompact groups see A. O. Barut, in *Lectures in Theoretical Physics* (Gordon and Breach, Science Publishers, Inc., New York, 1967), Vol. 9b.

[8] (a) V. Bargmann, Ann. Math. **48**, 568 (1947); (b) A. O. Barut and C. Fronsdal, Proc. Roy. Soc. (London) **A287**, 532 (1965).

where

$$\theta_{S'S}{}^{S_z}=\frac{1}{(S'-S)!}\left[\frac{\Gamma(S'+1-S_z)\Gamma(S'+1+S_z)}{\Gamma(S+1-S_z)\Gamma(S+1+S_z)}\right]^{1/2}. \quad (3.15)$$

The hypergeometric function is a polynomial [since S_1, S_3 are integers] with the highest power $(S-S_3)$.

For $S'<S$, the matrix element is $(V^{-1})^{T*}$, that is, one has to interchange in (3.15):

$$S\leftrightarrow S',\quad \alpha\leftrightarrow\alpha,\quad \beta\rightarrow-\bar{\beta}. \quad (3.16)$$

We have thus

$$\begin{aligned}\langle\alpha'p'|J^\mu(0)|\alpha p\rangle&\rightarrow\langle S',S_z'|\Gamma^\mu|S'',S_z\rangle V_{S''S}{}^{\phi}(a)\\ &=\mathfrak{F}_{S'S_z'SS_z}{}^{(\mu)}=\langle S',S_z'|\Gamma^\mu|S'',S_z\rangle\theta_{S''S}{}^{S_z}\\ &\quad\times(\cosh\tfrac{1}{2}\xi)^{-(S+S''+1)}(-\sinh\tfrac{1}{2}\xi)^{S''-S}\\ &\quad\times\sum_{n=0}^{S-S_z}\frac{(S_z-S)_n(-S_z-S)_n}{(1+S''-S)_n}[-\sinh^2(\tfrac{1}{2}\xi)]^n, \quad (3.17)\end{aligned}$$

where $(S_z-S)_n=\Gamma(S_z-S+n)/\Gamma(S_z-S)$, etc.

It remains now to evaluate the matrix elements

$$\langle S',S_z'|\Gamma^\mu|S,S_z\rangle. \quad (3.18)$$

But this is relatively simple, because the Γ's at most connect the neighboring S values. We have

$$\langle S',S_z'|\Gamma^0|S,S_z\rangle=(2S+1)\delta_{S'S}\delta_{S_z'S_z}, \quad (3.19)$$

$$\langle S',S_z'|\Gamma^3|S,S_z\rangle=-(S_>{}^2-S_z{}^2)^{1/2}\times(\delta_{S',S+1}+\delta_{S',S-1})\delta_{S_z'S_z}, \quad (3.20)$$

where $S_>=\max(S',S)$. For the evaluation of the matrix elements of Γ^1 and Γ^2 note that

$$|S,S_z\rangle=[(S+S_z)!(S-S_z)!]^{1/2}a_1{}^{\dagger S+S_z}a_2{}^{\dagger S-S_z}|0\rangle. \quad (3.21)$$

Hence one has to evaluate matrix elements of the type

$$-\tfrac{1}{2}\langle 0|a_2{}^{m'}a_1{}^{n'}\{a^\dagger\sigma^iCa^\dagger-aC\sigma^ia\}a_1{}^{\dagger n}a_2{}^{\dagger m}|0\rangle.$$

One obtains

$$\begin{aligned}\langle S',S_z'|\Gamma^1|S,S_z\rangle&=\tfrac{1}{2}[(S'+S_z+1)(S'+S_z)]^{1/2}\\&\quad\times(\delta_{S',S+1}\delta_{S_z',S_z+1}+\delta_{S,'S-1}\delta_{S_z',S_z-1})\\&\quad-\tfrac{1}{2}[(S'-S_z+1)(S'-S_z)]^{1/2}\\&\quad\times(\delta_{S',S+1}\delta_{S_z',S_z-1}+\delta_{S',S-1}\delta_{S_z',S_z+1}), \quad (3.22)\\ \langle S',S_z'|\Gamma^2|S,S_z\rangle&=\tfrac{1}{2}i[(S'+S_z+1)(S'+S_z)]^{1/2}\\&\quad\times(\delta_{S',S+1}\delta_{S_z',S_z+1}-\delta_{S',S-1}\delta_{S_z',S_z-1})\\&\quad-\tfrac{1}{2}[(S'-S_z+1)(S'-S_z)]^{1/2}\\&\quad\times(\delta_{S',S+1}\delta_{S_z',S_z-1}+\delta_{S',S-1}\delta_{S_z',S_z+1}).\end{aligned}$$

With (3.17) and (3.19)–(3.22) the calculation of the transition form factor is completed. To simplify the discussion, we shall go into the center-of-mass frame of the process

$$|\alpha'p'\rangle\rightarrow|\alpha p\rangle+\gamma$$

and let **p** point in the z direction. We then have to consider only the matrix element of Γ^1 and Γ^2 because of transversality of the photon. The general process is obtained from this configuration by interpreting S_z as the velicity index in the center-of-mass frame.

IV. DISCUSSION AND INTERPRETATION OF THE RESULTS

We express the result in terms of the invariant momentum transfer t which is related to the $\gamma=E/m$ of particle $|\alpha p\rangle$ in center-of-mass frame by

$$t=(m-m')^2-2mm'(\gamma-1), \quad (4.1)$$

while

$$\tanh\xi=(\gamma^2-1)^{1/2}/\gamma,\quad \sinh\xi=(\gamma^2-1)^{1/2},\quad \cosh\xi=\gamma. \quad (4.2)$$

We first consider transitions from the ground state $S=0$, $S_z=0$. Then

$$\begin{aligned}V_{S''0}{}^{S_z=0}(\xi)&=\theta_{S''0}(\cosh\tfrac{1}{2}\xi)^{-S''+1}(-\sinh\tfrac{1}{2}\xi)^{S''}F_{S'0}{}^0,\\ \theta_{S''0}&=+1,\\ F_{S''0}{}^0&=1,\end{aligned}$$

hence

$$V_{S''0}{}^0(\xi)=(-\tanh\tfrac{1}{2}\xi)^{S''}\frac{1}{\cosh\frac{1}{2}\xi}. \quad (4.3)$$

Thus,

$$\mathfrak{F}_{S'S_z'00}{}^\mu=\sum_{S''}V_{S''0}(a)\langle S'S_z'|\Gamma^\mu|S''0\rangle, \quad (4.4)$$

$$\mathfrak{F}_{S'S_z'00}{}^0=(2S'+1)V_{S'0}(a)\delta_{S_z'0}, \quad (4.5)$$

$$\begin{aligned}\mathfrak{F}_{S'S_z'00}{}^3&=-\sum_{S''S''_z}V_{S''0}(a)(S''^2-S_z''^2)^{1/2}\\&\quad\times[\delta_{S',S''+1}+\delta_{S',S''-1}]\delta_{S_z'S_z''}\\&=-[(S'-1)^2-S_z'^2]^{1/2}V_{S'-1,0}(a)\\&\quad-[(S'+1)^2-S_z'^2]^{1/2}V_{S'+1,0}(a). \quad (4.6)\end{aligned}$$

In the center-of-mass coordinate system that we use, the factors $\langle S',S_z'|\Gamma^{1,2}|S'',S_z\rangle$ vanish.

In particular, the form factor of the ground state itself is given by

$$\begin{aligned}\mathfrak{F}_{0000}{}^3&=-V_{1,0}(a)\\&=\frac{\tanh\frac{1}{2}\xi}{\cosh\frac{1}{2}\xi}=\sqrt{2}\frac{\sinh\xi}{(1+\cosh\xi)^{3/2}}\\&=\sqrt{2}\frac{(\gamma^2-1)^{1/2}}{(1+\gamma)^{3/2}}=\sqrt{2}\frac{(\gamma-1)^{1/2}}{\gamma+1}, \quad (4.7)\end{aligned}$$

or, in terms of the invariant t,

$$\mathfrak{F}_{0000}{}^3=\sqrt{2}\frac{(-t/2m^2)^{1/2}}{2-t/2m^2}\,. \tag{4.8}$$

On the other hand, the scalar form factor $G(t)$ is related to $\mathfrak{F}^\mu$ by

$$\mathfrak{F}^\mu=G(t)\frac{(p+p')^\mu}{2m}\,.$$

In our frame of reference $(p+p')=(E+m,\ 0,\ 0,\ (E^2-m^2)^{1/2})$ and we obtain

$$G(t)=\frac{2}{\gamma+1}\mathfrak{F}^{(0)}=\frac{2}{(\gamma^2-1)^{1/2}}\mathfrak{F}^{(3)}\,.$$

Both components $\mathfrak{F}^{(0)}$ and $\mathfrak{F}^{(3)}$ give the same $G(t)$ as of course they should and we get finally

$$G(t)=\frac{2\sqrt{2}}{(\gamma+1)^{3/2}}=\frac{2\sqrt{2}}{(2-t/2m^2)^{3/2}}$$

and

$$G(0)=1\,.$$

Spin-$\frac{1}{2}$—Form Factor

We consider the representation with the lowest spin $\frac{1}{2}$ and calculate the form factor of this ground state:

$$\mathfrak{F}^\mu=\langle\tfrac{1}{2}\tfrac{1}{2}|J^\mu(0)|\tfrac{1}{2}\tfrac{1}{2}\rangle=\sum_{S'}\langle\tfrac{1}{2}S_z'|\Gamma^\mu|S'\tfrac{1}{2}\rangle V_{S',1/2}{}^{(\phi=-1)}\,.$$

From (3.17), because Γ^0 is diagonal,

$$\mathfrak{F}^{(0)}=\langle\tfrac{1}{2}\tfrac{1}{2}|\Gamma^0|\tfrac{1}{2}\tfrac{1}{2}\rangle V_{1/2,1/2}{}^{-1}=2\frac{1}{(\cosh\frac{1}{2}\xi)^2}=\frac{4}{\gamma+1}.$$

For $\mathfrak{F}^3$ we need

$$V_{3/2,1/2}{}^{-1}=-\sqrt{2}\frac{\sinh\frac{1}{2}\xi}{(\cosh\frac{1}{2}\xi)^3}=-2\sqrt{2}\frac{(\gamma-1)^{1/2}}{(\gamma+1)^{3/2}}\,.$$

Hence,

$$\mathfrak{F}^{(3)}=4(\gamma^2-1)^{1/2}/(\gamma+1)^2.$$

On the other hand, usual form factors of a spin-$\frac{1}{2}$ particle are defined by, in our normalization $\bar{u}u=1$,

$$\langle p'|J^\mu(0)|p\rangle=\bar{u}(p')[F_1(t)\gamma^\mu+iK'F_2(t)\sigma^{\mu\nu}q_\nu]u(p)\,,$$

where $K'=K/2m$, and

$$q=p'-p,\quad t=q^2,\quad \sigma_{\mu\nu}=\frac{i}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu);$$

$$u(p)=\frac{1}{\sqrt{2}}\begin{pmatrix}(p\cdot\sigma/m)^{1/2}\\(p\cdot\bar{\sigma}/m)^{1/2}\end{pmatrix},\quad F_1(0)=F_2(0)=1,$$

$K=$ anomalous magnetic moment.

In our frame: $q=(E-m,\ 0,\ 0,\ (E^2-m^2)^{1/2})$, $t=-2m^2\times(\gamma-1)$, $\gamma=E/m$. Then

$$\begin{aligned}\langle p'|J^0(0)|p\rangle&=[\bar{u}(p')\gamma^0u(p)F_1+iK'F_2\bar{u}(p')\sigma^{03}q_3u(p)]\\&=\frac{1}{\sqrt{2}}[F_1(\gamma+1)^{1/2}+K'F_2(\gamma-1)(\gamma+1)^{1/2}]\\&\equiv\left(\frac{\gamma+1}{2}\right)^{1/2}G_E,\end{aligned}$$

and similarly,

$$\begin{aligned}\langle p'|J^3(0)|p\rangle&=[\bar{u}(p')\gamma^3u(p)F_1+ik'F_2u^+(p')\gamma^3u(p)q_0]\\&=\frac{1}{\sqrt{2}}[F_1(\gamma-1)^{1/2}+K'F_2(\gamma-1)(\gamma-1)^{1/2}]\\&\equiv\left(\frac{\gamma-1}{2}\right)^{1/2}G_E.\end{aligned}$$

Notice that

$$\langle p'|J^0|p\rangle=\langle p'|J^3|p\rangle\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}=+\frac{q^3}{q^0}J^3,$$

as it should be by current conservation.

In order to separate F_1 and F_2 we need an independent equation. Consider $J^1(0)$ and the matrix element

$$\langle\tfrac{1}{2},-\tfrac{1}{2}|J^1(0)|\tfrac{1}{2},\tfrac{1}{2}\rangle=\langle\tfrac{1}{2},-\tfrac{1}{2}|\Gamma^1|S',\tfrac{1}{2}\rangle V_{S',1/2}{}^{-1}.$$

From (3.22) we see that only

$$V_{3/2,1/2}{}^{-1}=-\sqrt{2}\frac{\sinh\frac{1}{2}\xi}{(\cosh\frac{1}{2}\xi)^3}=-2\sqrt{2}\frac{(\gamma-1)^{1/2}}{(\gamma+1)^{3/2}}$$

contributes and

$$\langle\tfrac{1}{2},-\tfrac{1}{2}|\Gamma^1|\tfrac{3}{2},\tfrac{1}{2}\rangle=+\sqrt{2}/2.$$

Hence

$$\langle\tfrac{1}{2},-\tfrac{1}{2}|J^1(0)|\tfrac{1}{2},\tfrac{1}{2}\rangle=-2\frac{(\gamma-1)^{1/2}}{(\gamma+1)^{3/2}}.$$

On the other hand

$$\begin{aligned}\langle p'|J^1(0)|p\rangle&=[\bar{u}(p')\gamma^1u(p)F_1+iK'F_2\bar{u}(p')\sigma^{1\nu}q_\nu u(p)]\\&=\frac{1}{\sqrt{2}}[(\gamma-1)^{1/2}F_1+2K'F_2(\gamma-1)^{1/2}]\\&\equiv\left(\frac{\gamma-1}{2}\right)^{1/2}G_M.\end{aligned}$$

From $J^1(0)$ and $J^0(0)$ we can now evaluate F_1 and F_2. To obtain also the correct absolute values of form factors, we have to normalize Γ^μ in the basic equation (2.3). The zeroth component Γ^0 is a measure for the charge of the system. From (3.19) we see that the matrix element of Γ^0 is proportional to $(2S+1)$. Thus we have to divide Γ^μ with the Poincaré scalar $(2S+1)$. In our case this means dividing F_1, F_2 by 2. Thus,

$$G_E=F_1+\frac{t}{4m^2}KF_2=2\sqrt{2}\frac{1}{(\gamma+1)^{3/2}},$$

$$G_M(t)=F_1(t)+KF_2(t),$$

$$G_M=-2\sqrt{2}\frac{1}{(\gamma+1)^{3/2}}=-\tfrac{1}{2}G_E.$$

Indeed we find, as we should, that

$$F_1(0)=1 \quad \text{or} \quad G_E(0)=1,$$

and $F_2(0)=1$ gives an anomalous magnetic moment $K=-\frac{3}{2}$ Bohr magneton, or

$$G_M(0)=-\tfrac{1}{2}.$$

The decrease of the form factors in this simple model is much slower than the experimental form factors of nucleons (Fig. 3), as this model does not contain enough states to be completely realistic.

V. GAUGE INVARIANCE

Because we have allowed an arbitrary invariant mass spectrum to be calculated from Eq. (2.16), the gauge problem in the electromagnetic minimal coupling (2.5) is not trivial, as can be seen by the equation

$$q^\mu\langle 2|\Gamma_\mu|1\rangle=[K(m_2^2)-K(m_1^2)]\langle 2,p|1p'\rangle\neq 0.$$

Therefore, to guarantee the gauge invariance we modify the equation multiplying Γ^μ with an operator A,

$$(A\Gamma^\mu P_\mu+K_0)|\alpha;p\rangle=0,$$

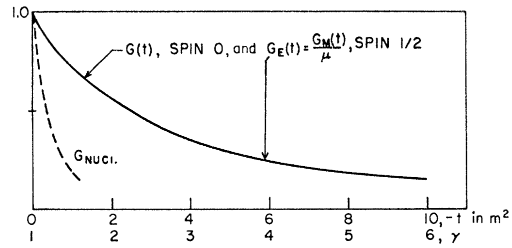

FIG. 3. Form factors for the spin-zero and spin-$\frac{1}{2}$ ground states (without dynamical corrections) for the two irreducible representations $J_0=0$ and $J_0=\frac{1}{2}$, respectively.

so that K_0 is now a constant. This equation gives again the same mass spectrum (2.16). Now if we introduce the electromagnetic coupling, we have to see that the total charge of each state with different spin and mass is the same. This is achieved by taking e in (2.5) as a matrix $\mathcal{E}$ so that

$$\langle|\mathcal{E}A\Gamma^\mu|\rangle=\text{constant},$$

for all the states of the composite system.

VI. CONCLUSIONS

We have presented an exactly soluble model of a composite quantum-mechanical system whose electromagnetic interactions can also be completely calculated in a relativistic way; that is, without making any dipole or multipole approximations. The model is the simplest one compatible with relativistic invariance and uses a generalization of the Majorana equations. The method can, however, be generalized to more complicated systems.