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Condensation of four-particle clusters: a soluble model

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Abstract. We illustrate the gap type of equation for four-particle vertices discussed recently by means of a simple soluble model involving N (where $N\to\infty$) spin-$\frac{1}{2}$ fermions $\psi_a(a=1,\ldots,N)$ in $2+\varepsilon$ space–time dimensions with the interaction $-(1/2N)g_0|\psi^{\mathrm{T}}C\psi|^2$ in the limit $N\to\infty$. The model exhibits a condensate of four-particle clusters if the renormalised coupling constant g exceeds some critical value g_c.

In recent papers (Kleinert 1981a, b, 1982) we pointed out the use of higher effective actions $\Gamma(G,\alpha)$ for the understanding of cluster properties in many-body systems. The symbol G collects the full particle density and pair correlation function in a 2×2 matrix and α denotes the exact four-particle vertex. The ground state is found by extremising Γ, which leads to the equations (Kleinert 1981b)

$$G=\mathrm{i}(\mathrm{i}G_0^{-1}-\tfrac{1}{2}VG+\tfrac{1}{6}\mathrm{i}VG^2\alpha)^{-1} \tag{1}$$

$$V=V(G,\alpha)=\alpha+\tfrac{3}{2}\mathrm{i}\alpha^2G^2-\tfrac{3}{4}\alpha^3G^4+\ldots. \tag{2}$$

The first equation is the standard Hartree–Fock–Bogoljubov gap equation. Previously (Kleinert 1981b) we argued that the second equation may be considered, by analogy with (1), as a gap-type equation for the vertex function. There may be non-perturbative solutions with α being non-zero in channels in which the original potential V vanishes. Equation (2) will be the key for understanding the formation of four-particle clusters in atomic nuclei.

It is the purpose of this paper to illustrate the properties of equation (2) by exhibiting a simple exactly soluble model in which such solutions exist. It consists of N relativistic fermion fields ψ_a in $2+\varepsilon$ space–time dimensions with the Lagrangian†‡

$$\mathscr{L}(x)=\bar\psi_a(x)\mathrm{i}\not\partial\psi_a(x)+(g_0/2N)\bar\psi_a(x)C\bar\psi_a^{\mathrm{T}}(x)\psi_b^{\mathrm{T}}(x)C\psi_b(x) \tag{3}$$

in the limit $N\to\infty$. This limit has the great technical advantage that there are so many different particles that exchange forces are irrelevant, i.e. there are no Fock terms in equation (1). Moreover, it can easily be seen that the term $VG^2\alpha$ in (1) is suppressed

† C is the charge conjugation matrix: $C=\gamma^1=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ with $\gamma^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Otherwise our notation follows Bjorken and Drell (1964).

‡ This is a modified Fierz-transformed version of a model studied extensively in the literature (Vaks and Larkins 1961, Gross and Neveu 1974, Abarbanel 1977, Muta 1978; also Kleinert 1978a). Moreover, being in two space–time dimensions, it is equivalent by a change of field variables to a model of different appearance first used by Nambu and Jona-Lasino (1961a, b) in order to explain pions as Nambu–Goldstone bosons.

compared with VG by a factor N^{-1} and thus can be neglected. In the expansion (2) only a single channel survives and the infinite number of terms can be summed to give

$$V(G,\alpha)=\alpha/(1-\tfrac{1}{2}\mathrm{i}\alpha G^2). \tag{4}$$

We shall see that this equation does indeed have a solution $\alpha\neq 0$ in the four-particle ψ^4 channel where there is no coupling in (3) to any order in perturbation theory if the renormalised coupling constant g is larger than some critical value g_c which is proportional to ε.

In order to see all this consider the generating functional

$$Z(\eta,\bar\eta)=\int D\psi D\bar\psi \exp\left(\mathrm{i}\int \mathrm{d}x\,(\mathscr{L}+\eta\psi+\bar\psi\eta)\right) \tag{5}$$

and use Stratonovic's trick (Stratonovic 1958, Kleinert 1977, 1978b, 1979) to introduce a collective pair field $\Delta(x)$ in analogy to superconductors

$$Z(j)=\int D\varphi D\Delta \exp\left(\mathrm{i}\int \mathrm{d}x\,[\tfrac{1}{2}\varphi_a^{\mathrm{T}}\mathrm{i}G_\Delta^{-1}\varphi_a-(N/2g_0)|\Delta|^2]\right)$$
$$\times\exp\left(\tfrac{1}{2}\mathrm{i}\int \mathrm{d}x\,\mathrm{d}y\, j_a^{\mathrm{T}}(x)G_\Delta(x,y)j_a(y)\right) \tag{6}$$

where we have used a doubled notation for fields and sources

$$\varphi_a\equiv\begin{pmatrix}\psi_a\\ C\bar\psi_a^{\mathrm{T}}\end{pmatrix}\qquad j_a\equiv\begin{pmatrix}\bar\eta_a^{\mathrm{T}}\\ C^{-1}\eta\end{pmatrix}$$

and abbreviated

$$\mathrm{i}G_\Delta^{-1}\equiv\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}\begin{pmatrix}-\Delta(x) & \mathrm{i}\partial\!\!\!/\\ \mathrm{i}\partial\!\!\!/ & -\Delta^+(x)\end{pmatrix}. \tag{7}$$

The integral over fermions can be executed with the result

$$Z_1(j)=\mathrm{e}^{\mathrm{i}W(j)}$$
$$=\int D\Delta \exp\left(\mathrm{i}N\mathscr{A}_{\mathrm{coll}}(\Delta)+\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, j_a^{\mathrm{T}}(x)G_\Delta(x,y)j_a(y)\right) \tag{8}$$

where the collective action

$$\mathscr{A}_{\mathrm{coll}}(\Delta)=-(1/g_0)|\Delta|^2-\tfrac{1}{2}\mathrm{i}\,\mathrm{Tr}\,\mathrm{lg}(\mathrm{i}G_\Delta^{-1}) \tag{9}$$

is the same for all N particles such that we have divided out the factor N. In the limit $N\to\infty$, the field Δ is squeezed into the extremum, say $\Delta(j)$, of the exponent and ceases to fluctuate. Thus the generating functional of the connected Green function $W(j)$ is known up to corrections $1/N$ as

$$N^{-1}W(j)=-\int \mathrm{d}x\,(1/g_0)|\Delta(x)|^2-\tfrac{1}{2}\mathrm{i}\,\mathrm{Tr}\,\mathrm{lg}(\mathrm{i}G_\Delta^{-1})+(\mathrm{i}/N)\int \mathrm{d}x\,\mathrm{d}y\, j_a^{\mathrm{T}}(x)G(x,y)j_a(y). \tag{10}$$

We may go over to the effective action by performing the Legendre transform

$$\Gamma(\Phi)=W(j)-\int \mathrm{d}x\,(\delta W(j)/\delta j_a(x))^{\mathrm{T}}j_a(x) \tag{11}$$

with $\delta W/\delta j_a \equiv \Phi_a(x)$ being the expectation value of the Fermi fields $\Phi_a \equiv \langle \varphi_a \rangle_j$ in the presence of the source j. Using (10) this may be written in the form

$$N^{-1}\Gamma(\Phi, \Delta) = -\int \mathrm{d}x\,(1/2g)|\Delta(x)|^2 - \tfrac{1}{2}\mathrm{i}\,\mathrm{Tr}\,\lg(\mathrm{i}G_\Delta^{-1}) + N^{-1}\int \mathrm{d}x\,\Phi_a(x)\,\mathrm{i}G_\Delta^{-1}\Phi_a(x) \qquad (12)$$

where it is understood that Δ has to be taken at the extremum of $\Gamma(\Phi, \Delta)$, $\Delta \equiv \Delta(\Phi)$. Expanding in powers of Φ gives all one-particle irreducible Green functions of the theory, in particular $\alpha = -\delta^4\Gamma/\delta\Phi\delta\Phi\delta\Phi\delta\Phi$, which are the vertex functions to be studied here. Equivalently, one may view Γ as an effective potential of both variables, Φ_a and $\Delta' \equiv \Delta - \Delta(\Phi)$, and obtain all Green functions by joining together all tree graphs involving Φ and Δ' lines. Going back to the undoubled fields, ψ, ψ^+, but now without fluctuations, which we indicate by capital letters, Ψ, Ψ^+, we may write (12) more explicitly as

$$\begin{aligned} N^{-1}\Gamma(\Psi, \bar\Psi, \Delta) = N^{-1}\int \mathrm{d}x\,[\bar\Psi_a\,\mathrm{i}\partial\!\!\!/\Psi_a - \tfrac{1}{2}(\bar\Psi C\bar\Psi_a^{\mathrm{T}}\Delta + \Delta^+\Psi_a^{\mathrm{T}}C\Psi_a)] \\ -\int \mathrm{d}x\,(1/2g_0)|\Delta|^2 - \tfrac{1}{2}\,\mathrm{i}\,\mathrm{Tr}\,\lg(\mathrm{i}\,G_\Delta^{-1}) \end{aligned} \qquad (13)$$

such that there are the non-local vertices

$$\tfrac{1}{4}N^{-1}[\alpha_{11}(\Psi_a^{\mathrm{T}}C\Psi_a)^2 + (\alpha_{12}\bar\Psi_a C\bar\Psi_a^{\mathrm{T}}\Psi_b^{\mathrm{T}}C\Psi_b + \mathrm{cc} + \alpha_{22}(\bar\Psi_a C\bar\Psi_a^{\mathrm{T}})^2] \qquad (14)$$

which follow from the Δ' propagators

$$\begin{aligned} \alpha_{11} &= \tfrac{1}{2}\,\mathrm{i}N\overline{\Delta'^+\Delta'^+} = \alpha_{22}^+ \\ \alpha_{12} &= \tfrac{1}{2}\,\mathrm{i}N\overline{\Delta'\Delta'^+} = \alpha_{21}^+. \end{aligned} \qquad (15)$$

In perturbation theory $\alpha_{12} = g_0 + \mathrm{O}(g_0^2)$, $\alpha_{11} \equiv \alpha_{22} \equiv 0$. Here, however, we shall find non-vanishing solutions for the four-particle vertices α_{11}, α_{22} which signal the presence of an α-particle-like condensate.

In order to calculate α we first determine the ground state by extremising (12) or (13):

$$\mathrm{i}G_{\Delta^0}^{-1}\Phi_a = 0 \qquad (16)$$

$$\Delta^0/g_0 = \tfrac{1}{2}\,\mathrm{Tr}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}G_{\Delta^0}\right) + (\Psi_a^{\mathrm{T}}C\Psi_a + \mathrm{cc}) \qquad (17)$$

where we have used the global gauge invariance under $\Psi \to \mathrm{e}^{\mathrm{i}\alpha}\Psi$, $\Delta \to \mathrm{e}^{2\mathrm{i}\alpha}\Delta$, to choose Δ^0 real. The first equation implies† $\Phi_a = 0$, the second is this model's version of equation (1) with $\Delta^0 = \tfrac{1}{2}VG$. It may be brought to Euclidean form to read

$$\begin{aligned} \frac{1}{g_0} &= 2^{D/2}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\,\frac{1}{p^2 + M^2} \\ &= \tfrac{1}{2}D\mu^\varepsilon b_\varepsilon (M/\mu)^\varepsilon \end{aligned} \qquad (18)$$

where

$$b_\varepsilon \equiv (2/D)(2\pi)^{-D/2}\Gamma(1 - \tfrac{1}{2}D) \sim -(1/\pi\varepsilon) + \mathrm{O}(\varepsilon^0) \qquad M \equiv \sqrt{(\Delta^0)^2} \qquad (19)$$

and μ is some arbitrary mass parameter.

† Fermi fields cannot have a ground-state expectation value.

There is a solution for every negative value of the bare coupling g_0. In two space–time dimensions, as $\varepsilon \to 0$, we may introduce a renormalised coupling

$$\frac{1}{g_0 \mu^{\varepsilon}} = \frac{1}{g} + b_{\varepsilon} \tag{20}$$

such that (18) becomes

$$1/g = b_{\varepsilon}[\tfrac{1}{2}D(M/\mu)^{\varepsilon} - 1] \tag{21}$$

which is finite in the limit $\varepsilon \to 0$. As is obvious from (19), a negative g_0 amounts to g being larger than some critical ε-dependent value $g_c = -b_{\varepsilon}^{-1} \geqslant 0$. For $\varepsilon = 0$ this means all positive values. We may now calculate the propagators of Δ' by expanding Γ in $\Delta' \equiv \Delta - \Delta^0$ up to quadratic order as

$$N^{-1}\Gamma = -\tfrac{1}{2}[(g_0^{-1} + A)|\Delta'|^2 + \tfrac{1}{2}B(\Delta'^2 + \Delta'^{+2})] \tag{22}$$

and find

$$\begin{aligned} \alpha &= \begin{pmatrix} B & g_0^{-1} + A \\ g_0^{-1} + A & B \end{pmatrix}^{-1} \\ &= \frac{1}{(g_0^{-1} + A)^2 - B^2} \begin{pmatrix} -B & g_0^{-1} + A \\ g_0^{-1} + A & -B \end{pmatrix} \end{aligned} \tag{23}$$

where A and B are the following expressions in Euclidean form ($z \equiv q^2/M^2$):

$$\begin{aligned} A(z) &= -\tfrac{1}{2}Db_{\varepsilon}M^{\varepsilon}[(1+\varepsilon)J_1(z) - \tfrac{1}{2}\varepsilon J_2(z)] \\ B(z) &= -\tfrac{1}{2}Db_{\varepsilon}M^{\varepsilon}\tfrac{1}{2}\varepsilon J_2(z) \end{aligned} \tag{24}$$

with the integrals

$$J_{\binom{1}{2}}(z) = \int_0^1 \mathrm{d}x\, [zx(1-x) + 1]^{\frac{1}{2}D - \binom{1}{2}}. \tag{25}$$

We verify the Nambu–Goldstone theorem by noticing the $q^2 = 0$ pole in the propagator

$$\mathrm{Im}\, \overline{\Delta' \,\mathrm{Im}}\, \Delta' = -\frac{\mathrm{i}}{N}\left(\frac{1}{g_0} + A - B\right)^{-1} \sim -\frac{2}{D\varepsilon b_{\varepsilon} N} 4M^2 \frac{\mathrm{i}}{q^2}. \tag{26}$$

For $\varepsilon \to 0$ there are the finite limits

$$\frac{1}{g_0} + A(z) \underset{\varepsilon=0}{=} \frac{1}{2\pi}\,\frac{z+2}{z+4}\,J(z) \qquad B(z) \underset{\varepsilon=0}{=} \frac{1}{2\pi}\,\frac{2}{z+4}\,J(z) \tag{27}$$

with

$$J(z) = \int_0^1 \mathrm{d}x\, \lg[zx(1-x) + 1] + 2 = 2\theta \coth\theta \tag{28}$$

and

$$\theta = \tanh^{-1}[z/(z+4)]^{1/2}.$$

Now we see that equation (23) may be inverted to render this model's version of the general equation (2) which here has the specific form

$$g_0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\alpha\left[1-\begin{pmatrix}B & A\\ A & B\end{pmatrix}\alpha\right]^{-1}. \tag{29}$$

Moreover, this illustrates precisely what we announced in the beginning: the initial potential is a matrix with vanishing entries in the four-particle channels. The matrix of the vertices α, on the other hand, does contain such entries. They are completely non-perturbative functions of g, as can be seen by solving (21) for M as a function of g and inserting this into (24). In two dimensions, $\varepsilon=0$ and (21) yields

$$M=\mu\exp[-\tfrac{1}{2}-(\pi/g)] \tag{30}$$

such that

$$z=\frac{q^2}{\mu^2}\exp[(2\pi/g)+1] \tag{31}$$

and the four-particle vertex becomes

$$\alpha_{11}=-4\pi/zJ(z), \tag{32}$$

displaying non-analytic behaviour for $g\to 0$ at fixed q^2/μ^2, i.e.,

$$\alpha_{11}\sim-\frac{2g}{q^2/\mu^2}\exp[-(2\pi/g)+1]. \tag{33}$$

Thus equation (29) (which is this model's version of equation (21)) is indeed the counterpart of the gap equation (17) (which is this model's version of equation (1)). It renders non-perturbative ground-state values for the product of four field operators $\langle\psi\psi\psi\psi\rangle$. This may be interpreted as a signal for the presence of a condensate of alpha-like clusters in the same way as $\langle\psi\psi\rangle\neq 0$ signals Cooper pairs.

It should be noted that, in the present case, the four-particle condensate arises via the same dynamical mechanism as the Cooper pairs. It would be instructive to find different models where either condensate appears separately. As a related problem one may wonder whether there are theories in which not only certain matrix elements of α but the whole interaction can be spontaneously generated, i.e. where $V=V(G,\alpha)\equiv 0$ has a non-trivial vertex solution α. If this happened in one of the presently popular grand unified theories of weak, electromagnetic and strong interactions it could lead to a determination of all fundamental parameters (for example, the fine structure constant) with the theory being completely specified in terms of a single mass scale.

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