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Higher Effective Actions for Bose Systems

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Abstract

We study the generalization of the usual effective action $\Gamma[\Phi]$ of Bose systems to an explicit functional $\Gamma[\Phi, G, \alpha_2, \alpha_4]$ of field Φ, Green's function G, and three- and four-point vertex functions α_3, α_4. The equations of motion following by extremization with respect to Φ, G, α_3, α_4 provide for non-linear integral equations whose solution can account for a wide variety of non-perturbative effects: Condensation of particles, pairs, and three and four-particle clusters. There can be spontaneous generation of mass as well as of interaction.

I. Introduction

Phase transitions are the most dramatic manifestation of non-perturbative effects in physical systems. It is well known that the loop expansion of the effective action $\Gamma[\Phi]$ provides a powerful tool for a discussion of the ensuing phenomena [*1*]. The effective action $\Gamma[\Phi]$ is the Legendre transform of the generating functional $W[j]$ of connected Green's functions. Its great technical advantage lies in the fact that the explicit calculation of the loop expansion involves expressions which are very similar to those occurring in perturbation theory except that the propagators depend again on the field quantities [*2*]. This causes a strong non-linearity of the resulting equations of motion and corresponds to an infinite string of normal Feynman graphs being summed up in a single term.

It is also known [*3*] that the understanding of non perturbative effects can be driven further by continuing the technique of Legendre transforms to include also the fully interacting two point function G as an explicit variable into $\Gamma[\Phi]$, forming $\Gamma[\Phi, G]$. Then the equations of motion allow not only for a condensate of the initial bosons but also of bound-state pairs of these.

This represents a decisive progress since it can easily be seen that there are certain models of the spherical type with infinitely many components which at the two-loop level of this effective action are solved *exactly*. Since most physical systems have only a finite number of components it must be expected that an extension of the technique of Legendre transform can bring a further improvement of the theoretical understanding of fluctuation phenomena. It is the purpose of this paper to present such an extension in detail. Some results have already been published elsewhere, in particular for the simpler case of Fermi systems where there are many applications in nuclear physics.

The essential results consist in non-linear gap type of equations also for coupling constants such that there can be spontaneous generation not only of mass but also of interaction [*4*].

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II. The Generating Functional

2.1. General properties

We shall address ourselves to a general theory of interacting boson fields whose action reads

$$A[\varphi] = A^0[\varphi] + A^{\text{int}}[\varphi] = \frac{1}{2}\,\varphi i G_0^{-1}\varphi - \frac{1}{3!}\,V_3\varphi\varphi\varphi - \frac{1}{4!}\,V_4\varphi\varphi\varphi\varphi. \tag{1}$$

The field φ has space time and internal degrees of freedom which may all be collected in a single index x which, in turn, is suppressed in the notation (1). The potentials V_3 and V_4 may depend on three and four x variables, respectively. The matrix iG_0^{-1} contains the kinetic part. For a scalar relativistic field has of the form

$$iG_0^{-1} = -\Box - m^2 \tag{2}$$

where m^2 is a mass matrix.
In the notation (1) the fields are assumed to be real. If this is not the case we find it convenient to keep the same notation which is always possible if we combine the field, say ψ, and its complex conjugate ψ^+ to a doubled object

$$\varphi = \begin{pmatrix}\varphi_\uparrow\\ \varphi_\downarrow\end{pmatrix} = \begin{pmatrix}\psi\\ \psi^+\end{pmatrix}. \tag{3}$$

The index $\uparrow$ and $\downarrow$ vor the doubling may again be absorbed into the variable x. Thus we remain with a simple looking action of the φ^3, φ^4 type whose main complication consists in the potentials V_3, V_4 being tensors of rank three and four in the index x. The field either is real or, in the case of complex initial fields, quasi-real in the sense that φ is similar to φ^+ via

$$\varphi = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\varphi^+. \tag{4}$$

The standard non-relativistic many-boson problem is contained in the latter formulation in which case the kinetic part of the action reads

$$iG_0^{-1} = \begin{pmatrix}0 & \mp i\,\partial_t \mp \varepsilon\\ i\partial_t - \varepsilon & 0\end{pmatrix} \tag{5}$$

where ε is the matrix of single-particle energies. The potential V_3 vanishes and the potential V_4 may be related to the standard form of the interaction (α = space time $\otimes$ internal index)

$$-\frac{1}{2}\,v_{\alpha\beta\gamma\delta}\psi_\alpha{}^+\psi_\beta\psi_\gamma{}^+\psi_\delta \tag{6}$$

by simply choosing V_4 with the doubled indices $x = \alpha\uparrow$ or $\alpha\downarrow$ as

$$V_{4\,\alpha\downarrow\beta\uparrow\gamma\downarrow\delta\uparrow} = 2v_{\alpha\beta\gamma\delta} = \pm V_{4\,\beta\uparrow\alpha\downarrow\gamma\downarrow\delta\uparrow} = \cdots. \tag{7}$$

and determining the others from symmetry in all four doubled indices. Moreover due to (6), V_4 has to vanish if the number of up and down entries is not equal such that particle number is conserved.
Actually, fermionic systems may be described by the same formalism even though the situation there is much simpler and has been discussed elsewhere [4]. We shall, however,

find it useful to keep the fermionic alternative in the picture, for comparison. It usually amounts to a second alternative as fas as signs are concerned which will be marked below the boson sign. This explains the minus sign in (7) and the lower signs in (5) since, for fermions, V_4 and $\varphi\varphi$ are antisymmetric in the doubled indices. The full quantum field theory is contained in the set of all n-particle Green's functions involving the fully interacting Heisenberg fields

$$G^{(n)}(x_1 \cdots x_n) = \langle T\varphi(x_1) \cdot \cdots \cdot \varphi(x_n)\rangle. \tag{8}$$

The symbol $\langle\ \rangle$ denotes the ground state expectation value. Alternatively we may deal with the generating functional[2])

$$Z[j] = \langle Te^{ij\varphi}\rangle \tag{9}$$

from which all $G^{(n)}$'s can be obtained by functional differentiation with respect to the auxiliary external source j:

$$G^{(n)}(x_1 \ldots x_n) = Z[j]^{-1}\frac{\delta}{i\delta j(x_1)} \cdot \cdots \cdot \frac{\delta}{i\delta j(x_n)} Z[j] \tag{10}$$

if one sets $j = 0$ at the end. For technical reasons it will be convenient to define the objects $G^{(n)}$ via equ. (10) also for $j \neq 0$ and call them the n-point functions in the presence of the source j.

For fermions we use the same rules but have to agree that the source is an anticommuting object

$$j(x)\, j(y) = -j(y)\, j(x) \tag{11}$$

in order to cope with the antisymmetry of $G^{(n)}(x_1 \cdots x_n)$ under exchange of two of its variables.

The set of all Green's functions generated by $Z[j]$ is not very economic to deal with since they are, in general, disconnected. Therefore it is preferable to work with the logarithm of $Z[j]$:

$$W[j] = -i \log Z[j]. \tag{12}$$

This is known to contain only connected diagrams, a property which will emerge later as a result of explicit calculations.

The generating functional (12) may be calculated in the interaction picture of quantum field theory where it is shown that

$$Z[j] \equiv \mathcal{N}\langle 0|\ e^{iA^{\mathrm{int}}[\varphi_0]+ij\varphi_0}\ |0\rangle. \tag{13}$$

The normalization factor $\mathcal{N}$ collects the infinite phase aquired by the vacuum through the interaction

$$\mathcal{N} = \langle 0|\ e^{iA^{\mathrm{int}}[\varphi_0]}\ |0\rangle^{-1}. \tag{14}$$

Since the most important property (10) of $Z[j]$ is independent of $\mathcal{N}$ this factor will from now on be omitted in the definition (13). The field φ_0 follows the free field equations and has the propagator

$$G_0(x_1x_2) = \langle 0|\ T\varphi_0(x_1)\ \varphi_0(x_2)\ |0\rangle \equiv \overbrace{\varphi_0(x_1)\ \varphi_0(x_2)} \tag{15}$$

with $|0\rangle$ being the free field vacuum.

The calculation of $Z[j]$ proceeds as follows: Since the operator φ_0 may be generated by $(1/i)\ \delta/\delta j$ acting on the exponent we may write $A^{\mathrm{int}}[\varphi_0] = A^{\mathrm{int}}[(1/i)\ \delta/\delta j]$ such that the interaction is no longer an operator and can be removed from the vacuum expectation

[2]) In accordance with (1), $j\varphi$ means $\sum_x j(x)\ \varphi(x)$.

value obtaining

$$Z[j] = e^{iA^{\text{int}}[(1/i)(\delta/\delta j)]} Z^0[j]. \tag{16}$$

Here we have introduced $Z^0[j]$ as the generating functional of all free Green's functions. This functional can be calculated very simply. For this one uses the free equations of motion

$$iG_0^{-1}\varphi_0 = 0 \tag{17}$$

and the canonical commutation rule to derive [1]

$$iG_0^{-1} \frac{1}{i} Z_j^0[j] + jZ^0[j] = 0 \tag{18}$$

where a subscript denotes functional differentiation, i.e. $Z_j \equiv \delta/\delta j\, Z$. This equation is, of course, just a reflection of the operator equation of motion in the presence of the source j

$$iG_0^{-1}\varphi + j = 0. \tag{19}$$

The equation can be integrated[3]) to obtain

$$Z^0[j] = e^{-1/2jG_0 j} \tag{20}$$

if the free overall factor is adjusted to make $Z^0[0] = 1$; just as in (9).
The generating functional (20) can be expanded in powers of j and exhibits the general free n-point functions as a sum of products of two-point functions in accordance with Wick's theorem: For n fields $\varphi_0(x_1) \ldots \varphi_0(x_n)$ in $G^{(n)}$, there are $(n-1)!!$ pair contractions, each amounting to a Green's function $G(x_i x_k)$ for the corresponding pair (ik). The graphical representation of this expansion amounts to a sum of combinations of disconnected lines each of which represents G_0 which is the only connected subdiagram.
Contrary to this, the exponent

$$W^0[j] = -i \log Z^0[j] = \frac{i}{2} jG_0 j \tag{21}$$

contains the only connected graph G_0 of the free theory. If the subscript c is meant to collect the connected subset of all graphs, this amounts to the trivial statement

$$G_c^{(n)} = \delta_{n,2} G_0. \tag{22}$$

2.2. Differential equations

The standard perturbation series for $Z[j]$ can be obtained by expanding (16) in powers of A^{int} and executing the functional differentiations.
Alternatively, we may combine (16) with (18) and derive an analogous equation for the interacting generating functional

$$\begin{aligned} &iG_0^{-1} \frac{1}{i} Z_j[i] - [j,\, e^{iA^{\text{int}}[(1/i)(\delta/\delta j)]}]\, Z^0[j] + jZ[j] \\ &= iG_0^{-1} \frac{1}{i} Z_j[j] + A_\varphi^{\text{int}} \left[\frac{1}{i} \frac{\delta}{\delta j}\right] Z[j] + jZ[j] = 0. \end{aligned} \tag{23}$$

[3]) This is unique only in Euclidean space or, alternatively, if G_0 has a proper $i\varepsilon$ prescription.

Inserting (1) this can be written more explicitly as

$$iG_0^{-1}\frac{1}{i}Z_j - \frac{1}{2!}V_3\frac{1}{i^3}Z_{jjj} - \frac{1}{3!}V_4\frac{1}{i^4}Z_{jjjj} + jZ = 0. \tag{24}$$

For the generating functional $W[j]$, this amounts to

$$iG_0^{-1}W_j - \frac{1}{2!}V_3\frac{1}{i^2}(iW_{jj} - W_j^2) - \frac{1}{3!}V_4\frac{1}{i^3}(iW_{jjj} - 3W_{jj}W_j - iW_j^3) + j = 0. \tag{25}$$

Multiplying with $-iG_0$ gives

$$W_j = -iG_0\frac{1}{2!}V_3\frac{1}{i^2}(iW_{jj} - W_j^2) - iG_0\frac{1}{3!}V_4\frac{1}{i^3}(iW_{jjj} - 3W_{jj}W_j - iW_j^3) + iG_0 j. \tag{26}$$

We now introduce the interacting connected n-points functions $G_c^{(n)}(x_1 \dots x_n)$ in the presence of the source j as the functional derivatives of $W[j]$

$$G_c^{(n)}(x_1 \dots x_n) \equiv \frac{1}{i^n}\frac{\delta}{\delta j(x_1)} \cdot \dots \cdot \frac{\delta}{\delta j(x_n)} iW[j] \tag{27}$$

and find for them the equation

$$G_c^{(1)} = -iG_0\frac{1}{2!}V_3(G_c^{(2)} + G_c^{(1)}G_c^{(1)}) - iG_0\frac{1}{3!}V_4(G_c^{(3)} + 3G_c^{(2)}G_c^{(1)} + G_c^{(1)3}) + iG_0 j. \tag{28}$$

Observing that

$$G_c^{(n)} = \frac{1}{i}\frac{\delta}{\delta j}G_c^{(n-1)} = -iG_c^{(n-1)}{}_j \tag{29}$$

this can also be written as a differential equation for $G_c^{(1)}$

$$G_c^{(1)} = -G_0\frac{1}{2!}(G_c^{(1)}{}_j + iG_c^{(1)2}) + G_0\frac{1}{3!}(iG_c^{(1)}{}_{jj} - 3G_c^{(1)}{}_jG_c^{(1)} - iG_c^{(1)3}) + iG_0 j. \tag{30}$$

2.3. Perturbation solution

This equation can be solved iteratively as shown in Fig. 1. Once $G_0^{(1)}$ is known, the generating functional $W[j]$ follows simply by integrating functionally in j:

$$W[j] = \int Djj\, W_j[j] = \int Dj\, G_c^{(1)} \tag{31}$$

which amounts to multiplying each term by j contracting the index x and dividing every power j^n by n. This can best be done graphically as shown in Fig. 1.

Notice that this procedure leaves open the additive constant $W[0]$. It has to be calculated using the perturbation expansion (16) for $Z[0]$. This results in the sum of all vacuum diagrams which are those composed of vertices V_3, V_4 and propagators G_0 without any external line. Taking the logarithm of these, $W[0]$ collects only the connected subset of these diagrams. This is illustrated in Fig. 2.

We can now convince ourselves by inspection that the connected Green's functions defined as the derivatives (27) of $W[j]$ are indeed connected in the naive sence of the word as far as their graphical representation is concerned.

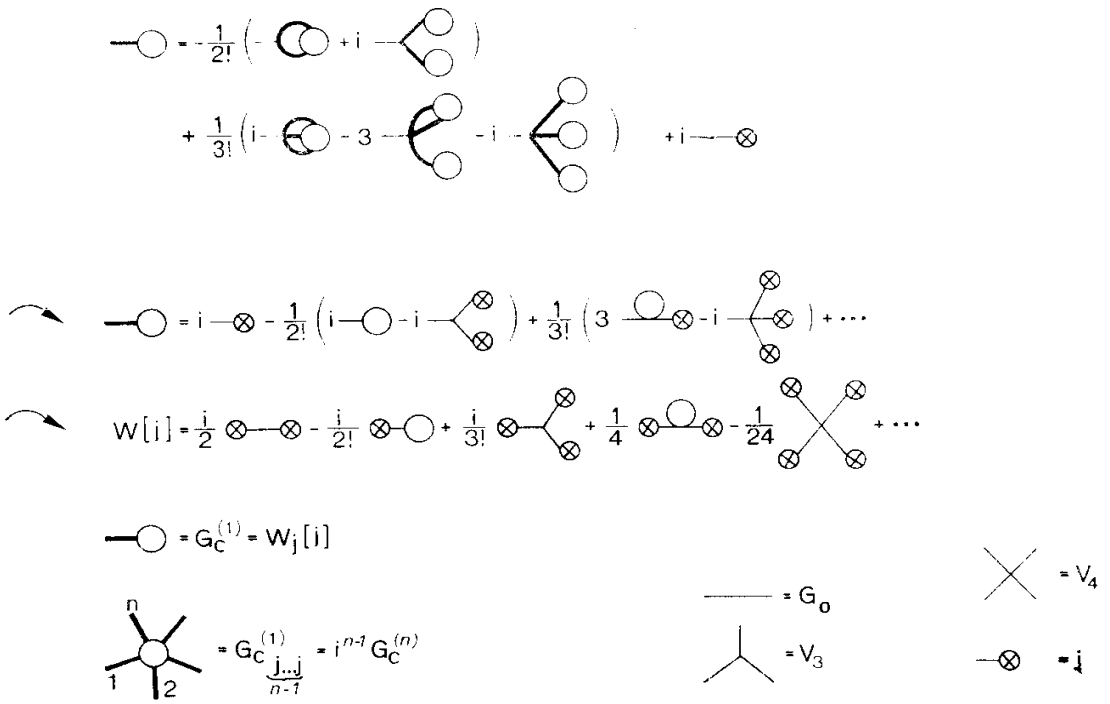

Fig. 1. The differential equation for $\Phi = G^{(1)} = W_i[j]$ in the presence of an external source and its recursive solution. Fat legs correspond to derivatives with respect to the source j. The generating functional $W[j]$ of connected Green's functions is obtained by functional integration with respect to j which amounts to closing the free line by a j source and dividing each term j^n by n.

Using the exponential relation between W and Z we can derive the relations between the n-point functions $G^{(1)}$ and the connected ones $G_c^{(n)}$ as

$$G^{(1)} = G_c^{(1)} \tag{32}$$

$$G^{(2)} = G_c^{(2)} = G_c^{(1)} G_c^{(1)}$$

$$G^{(3)} = G_c^{(3)} + (G_c^{(2)} G_c^{(1)} + 2 \text{ permutations}) + G_c^{(1)3} \tag{33}$$

$$\begin{aligned} G^{(4)} = G_c^{(4)} &+ (G_c^{(3)} G_c^{(1)} + 3 \text{ permutations}) \\ &+ (G_c^{(2)} G_c^{(2)} + 2 \text{ permutations}) \\ &+ (G_c^{(2)} G_c^{(1)} G_c^{(1)} + 5 \text{ permutations}) + G_c^{(1)4} \end{aligned} \tag{34}$$

$$\vdots$$

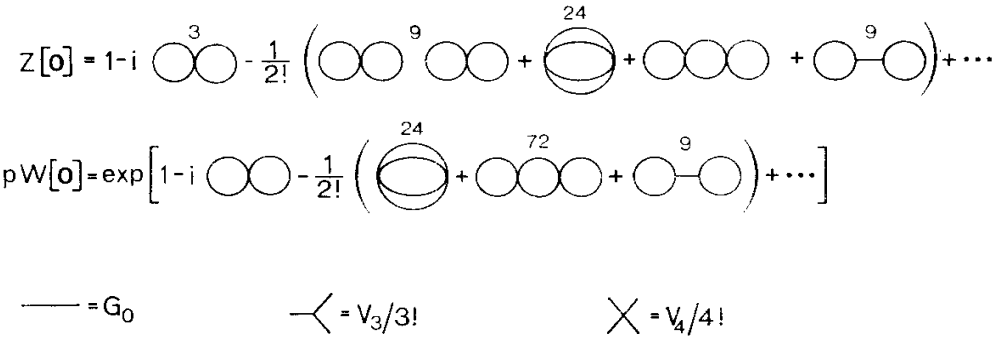

Fig. 2. The vacuum graphs contained in $Z[0]$ are an exponential of the connected graphs in $iW[0]$

These can be inverted to give

$$G_c^{(1)} = G^{(1)}$$

$$G_c^{(2)} = G^{(2)} - G^{(1)}G^{(1)} \tag{35}$$

$$G_c^{(3)} = G^{(3)} - 3G^{(2)}G^{(1)} + 2G^{(1)3} \tag{36}$$

$$G_c^{(4)} = G^{(4)} - 4G^{(3)}G^{(1)} + 12G^{(2)}G^{(1)2} - 3G^{(2)2} - 6G^{(1)4} \tag{37}$$

$$\vdots$$

where the numbers indicate how many permutations have to be formed of the product of Green's functions behind them. In the future we shall always use this abbreviated notation.
For fermion Green's functions the connectedness structure is the same if the indices are ordered such that they are even permutations of each other, otherwise there is a minus sign.

III. The Effective Action

3.1. General properties

Since the beginning of modern theoretical physics, variational principles have played an important role in the formulation of equations of motion. In classical mechanics and electrodynamics, particle orbits and field configurations are obtained by extremizing the action with respect to variations in the path. It is gratifying to realize that similar principles can also be found for the observable quantities of quantum field theory. There exists a whole sequence of functionals $\Gamma[G_c^{(1)}, G_c^{(2)}, G_c^{(3)}, \ldots]$ whose extrema determine the set of connected Green's functions contained in the list of arguments for arbitrary many-body systems [*1*, *2*, *3*].
In this section we shall consider only $\Gamma[G_c^{(1)}]$. This functional is introduced as the Legendre transform of the generating functional $W[j]$

$$\Gamma[G_c^{(1)}] \equiv W[j] - W_j[j]\, j \tag{38}$$

where by definition of $W[j]$

$$G_c^{(1)}(x) \equiv W_{j(x)}[j] \tag{39}$$

is the expectation value of the field operator in the presence of the external source j

$$G_c^{(1)}(x) \equiv W_{j(x)}[j] = \langle \varphi(x) \rangle_j \equiv \Phi(x). \tag{40}$$

Conventionally, the capital letter $\Phi(x)$ is used for this field expectation and we shall do the same. The physical situation corresponds to zero source j in which case $\Phi \equiv \Phi_0$ determines the expectation in the ground state. This is an important physical quantity since it informs us about the presence or absence of a condensate of the bosons φ in the ground state.
For fermions, Φ_0 always vanishes since there can be no state which is a superposition of an even and odd number of fermions. This is why fermion systems are simpler and have been treated separately before [*4*].
The important property of this new functional consists in the following: If $\Gamma[\Phi]$[4]) is differentiated with respect to Φ, we obtain

$$\Gamma_{\Phi(x)}[\Phi] = (W_j - \Phi)\, j_{\Phi(x)} - j(x) = -j(x). \tag{41}$$

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[4]) From now on, we shall always use the variable Φ for $G_c^{(1)}$.

Since the physical situation corresponds to $j = 0$ this means that $\Gamma[\Phi]$ is extremal on physical field configurations Φ. Thus the functional $\Gamma[\Phi]$ plays the same role in quantum physics with respect to the field expectation as classical actions do with respect to particle orbits. This is why $\Gamma[\Phi]$ is called the effective action of the system.

3.2. Differential equations for the effective action $\Gamma[\Phi]$

It is quite simple to determine $\Gamma[\Phi]$ via differential equations which are a direct consequence of equs (29) for $W[j]$. For this we go to the variable Φ by using $W_j = \Phi$. Differentiating this gives

$$W_{jj} = \Phi_j = (j_\Phi)^{-1} = -\Gamma^{-1}_{\Phi\Phi} \tag{42}$$

where the inverse is understood in the matrix sense. But $-iW_{jj}$ is the connected Green's function such that we recognize and important property of $\Gamma[\Phi]$:

$$\Gamma_{\Phi\Phi}[\Phi] = iG_c^{(2)-1} = iG^{-1} \tag{43}$$

i.e. the second derivative of the effective action determines directly the connected two-point function of the system. Since $G_c^{(2)}$ will occur quite often in what follows we shall denote it simply by G. Differentiating (43) once more gives

$$W_{jjj} = -\Gamma^{-1}_{\Phi\Phi} \times \Gamma^{-1}_{\Phi\Phi} \times \Gamma^{-1}_{\Phi\Phi} \cdot \Gamma_{\Phi\Phi\Phi}\,. \tag{44}$$

Since W_{jjj} is the connected three-point function $-G_c^{(3)}$ this amounts to the relation

$$i\Gamma_{\Phi\Phi\Phi} = G^{-1}G^{-1}G^{-1}G_c^{(3)}\,. \tag{45}$$

We have omitted indices which are such that each index in $G_c^{(3)}$ is contracted with one of the indices in G. Equ. (45) has a simple graphical meaning: The Green's function $G_c^{(3)}$ collects all connected Feynman graphs with three external legs. The matrix multiplication by G^{-1} on each index amounts to amputating the three external legs. The remaining amputated connected three point function is called three-point vertex function. Thus the third functional derivative of the effective action $\Gamma[\Phi]$ is the three-point vertex function.

We now use (39), (42), (44) to rewrite the differential equations for $W[j]$ as an equation for $\Gamma[\Phi]$

$$iG_0^{-1}\Phi - \frac{1}{2!}\,V_3(i\Gamma^{-1}_{\Phi\Phi} + \Phi^2) - \frac{1}{3!}\,V_4(G^{-3}_{\Phi\Phi}\Gamma_{\Phi\Phi\Phi} + 3i\Gamma^{-1}_{\Phi\Phi}\Phi + \Phi^3) - \Gamma_\Phi = 0\,. \tag{46}$$

This can be solved by iteration. For $V_3 = V_4 = 0$ we find the free effective action

$$\Gamma^0[\Phi] = \frac{1}{2}\,\Phi iG_0^{-1}\Phi \tag{47}$$

which is precisely the same as the free action $A^0[\varphi]$ except that the field operator φ is replaced by the expectation value Φ. Before proceeding with the iteration it is convenient to use Γ^0 and separate off the interacting part of the effective action as

$$\Gamma[\Phi] = \Gamma^0[\Phi] + \Gamma^{\text{int}}[\Phi] \tag{48}$$

such that

$$\Gamma_{\Phi\Phi} = iG_0^{-1} + \Gamma^{\text{int}}_{\Phi\Phi} \tag{49}$$

$$\Gamma_{\Phi\Phi\Phi} = \Gamma^{\text{int}}_{\Phi\Phi\Phi} \tag{50}$$

and (46) becomes

$$\Gamma_{\Phi}^{\text{int}} = -\frac{1}{2!} V_3 \Phi^2 - \frac{1}{3!} V_4 \Phi^3 - \frac{1}{2!} (V_3 + V_4 \Phi) G - \frac{1}{3!} V_4 i G^3 \Gamma_{\Phi\Phi\Phi}^{\text{int}} \tag{51}$$

where the Green's function is given by (43) and (49) as

$$G = G_0 (1 - i G_0 \Gamma_{\Phi\Phi}^{\text{int}})^{-1}. \tag{52}$$

Expanding the denominator and recollecting terms this can also be written as an integral equation

$$G = G_0 + i G_0 \Gamma_{\Phi\Phi}^{\text{int}} G. \tag{53}$$

At this point it is useful to introduce a quantity Σ called the self-energy which is defined by

$$G \equiv i[i G_0^{-1} - \Sigma]^{-1}, \qquad \Sigma \equiv i G_0^{-1} - i G^{-1}. \tag{54}$$

Comparing with (53) we many identify

$$\Sigma = -\Gamma_{\Phi\Phi}^{\text{int}}. \tag{55}$$

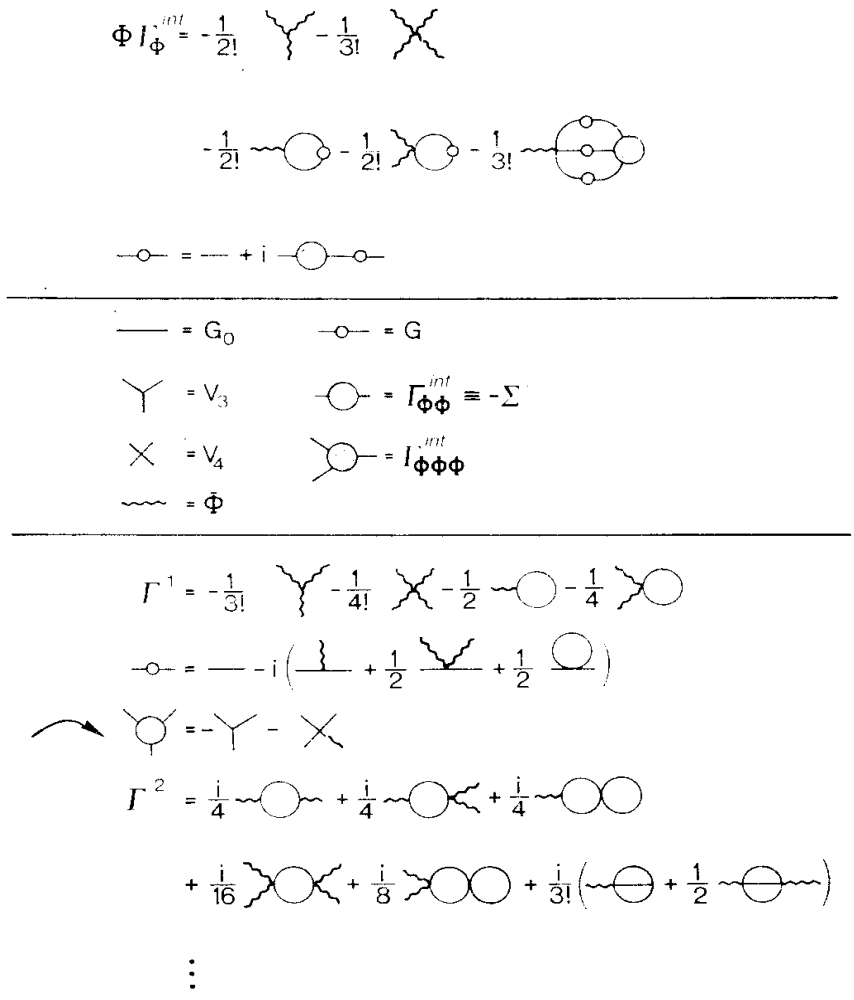

Fig. 3. The integral equation for $\Gamma^{\text{int}}[\Phi]$ and its graphical iterative solution up to second order in the potentials

Equations (51), (52) will be solved iteratively. Before starting it is useful to observe that there exists a trivial relation between $\Gamma[\Phi]$ and $\Phi\Gamma_\Phi^{\rm int}$ which merely consists in a factor $1/n$ in front of every power Φ^n. Thus it is not necessary to alculate $\Gamma_\Phi^{\rm int}$ but we may multiply (51) by $\Phi(x)$ and contract the indices x. The resulting equations are displayed graphically in Fig. 3.

3.3. Iterative Solution for the Effective Interaction

We can now proceed by iterating (51), (52). Neglecting on the right-hand sides $\Gamma^{\rm int}$, we find the first order correction

$$\Gamma^1[\Phi] = -\frac{1}{3!} V_3\Phi^3 - \frac{1}{4!} V_4\Phi^4 - \frac{1}{2} V_3 G_0\Phi - \frac{1}{4} V_4 G_0\Phi^2. \tag{56}$$

Two pieces in this are simply the interacting parts of the original action $A^{\rm int}[\Phi]$ with φ replaced by Φ. The other two pieces are new. They correspond to partial contractions in the interacting part of the action and setting $\varphi = \Phi$ in the remaining uncontracted fields

$$-\frac{1}{3!} V_3\varphi^3 \to -\frac{1}{3!} V_3 3\overbrace{\varphi\varphi}\Phi \tag{57}$$

$$-\frac{1}{4!} V_4\varphi^4 \to -\frac{1}{4!} V_4 \frac{4\cdot 3}{2} \overbrace{\varphi\varphi}\Phi\Phi. \tag{58}$$

The iteration can be continued be in a graphical way and this has been done up to the next order in Fig. 2.

If we imagine proceeding to arbitrary order we discover an important topological characteristic of the graphical expansion: When cutting a single line G_0, none of the graphs decomposes into disconnected pieces. Such diagrams are called one-particle irreducible (OPI).

It is easy to recognize the general rule according to which one can find all higher order contributions: One simply writes down the action in which the field φ is translated by the expectation value $\varphi \equiv \Phi + \varphi'$:

$$\begin{aligned} A_{\rm aux}[\varphi'] &\equiv A[\Phi + \varphi'] = A[\Phi] + A^0_{\rm aux}[\varphi'] + A^{\rm int}_{\rm aux}[\varphi'] \\ &= A[\Phi] + \frac{1}{2}\varphi' i G_0^{-1}\varphi' + \Big[-\frac{1}{2} V_3\Phi^2\varphi' - \frac{1}{3!} V_4\Phi^3\varphi' \\ &\quad - \frac{1}{2} V_3\Phi\varphi'^2 - \frac{1}{4} V_4\Phi^2\varphi'^2 - \frac{1}{3!}(V_3 + V_4\Phi)\varphi'^3 - \frac{1}{4!} V_4\varphi'^4\Big] \end{aligned} \tag{59}$$

and calculates the one-particle irreducible subset of all vacuum diagrams arising from the different interaction terms using G_0 as the propagator of the shifted field φ'. There is, however, one exception to this rule: We must leave out all one-particle reducible diagrams. In particular, the two terms in (59) linear in φ' may be dropped. This, of course, reflects the fact that by definition, the field φ' has a vanishing expectation $\langle\varphi'\rangle = \langle\varphi\rangle - \langle\Phi\rangle = 0$. Thus the sum of all graphs connected to a single φ' leg must disappear since these would ceratainly be one-particle reducible.

3.4. Non-perturbation solution

When solving the differential equations (51), (52) iteratively we found empirically that the auxiliary action $A_{\text{aux}}[\varphi']$ supplies precisely the correct Feynman rules. The graphical expansion was, however, somewhat unconventional. There are two terms in $A^{\text{int}}_{\text{aux}}[\varphi']$

$$-\frac{1}{2} V_3 \Phi \varphi'^2 - \frac{1}{4} V_4 \Phi^2 \varphi'^2 \tag{60}$$

which have been treated like an interaction even though they are quadratic in the field φ'. Therefore in the usual procedure of splitting an action intofree and interacting part $A[\varphi'] = A^0[\varphi'] + A^{\text{int}}[\varphi']$ these terms would have been considered as part of the free action $A^0[\varphi']$. In this case we would have arrived at a free propagator

$$G^{\Phi} = i\left[iG_0^{-1} - V_3\Phi - \frac{1}{2} V_4 \Phi^2\right]^{-1} \tag{61}$$

for the φ' field rather than G_0 and would be left only with the interaction

$$A^{\text{int}}[\varphi'] = -\frac{1}{3!}(V_3 + V_4\Phi)\,\varphi'^3 - \frac{1}{4!} V_4 \varphi'^4. \tag{62}$$

Now $\Gamma[\Phi] - A[\Phi]$ would collect all vacuum diagrams involving G^{Φ} and these two vertices. By comparing this graphical expansion with the previous one we find that they are the same, order by order, if the denominators in the propagators are expanded in powers of V_3 and V_4. There is only one infinite family of graphs which cannot be obtained by the new rules and which has to be included separately: It consists the single loop diagrams with an arbitrary number of Φ lines attached to them (see Fig. 3). It can be verified by looking at the iteration that these are of the form

$$-\left(\frac{i}{2}\right) \operatorname{tr} \sum_{n=0}^{\infty} \left[iG_0\left(V_3\Phi + \frac{1}{2} V_4 \Phi\Phi\right)\right]^n \Big/ n. \tag{63}$$

They can be collected in a single expression

$$\frac{i}{2} \operatorname{tr} \log\,(iG^{\Phi-1}). \tag{64}$$

Thus the effective action may be expanded alternatively in the form

$$\Gamma[\Phi] = A[\Phi] + \frac{i}{2} \operatorname{tr} \log\,(iG^{\Phi-1}) + \sum_{\substack{\text{OPI vacuum} \\ \text{graphs}}} (G^{\Phi},\ V_3 + V_4\Phi,\ V_4) \tag{65}$$

where the sum covers all OPI diagrams with propagators (61) and vertices (62).
This alternative way of calculating the effective action within the shifted field theory puts us in a position to learn what is the graphical content of $\Gamma[\Phi]$ at vanishing effective field $\Phi = \langle\varphi\rangle = 0$. This quantity was inaccessible in the previous iteration scheme since $\Gamma[\Phi]$ was obtained from a differential equation which gave no information on $\Gamma[0]$. What the iteration did disclose was the one-particle irreducibility of the graphs. By re-expressing the action in the form (59) in terms of the fluctuating part φ' of the field, these graphs turned into one-particle irreducible vacuum graphs involving lines G^{Φ} and vertices V_4, $V_3 + V_4\Phi$. But in this latter formulation we may simply set $\Phi = 0$ and see that $\Gamma[0]$

still collects all one-particle irreducible vacuum graphs but now constructed from the original propagators G_0 and vertices V_4, V_3, i.e.

$$\Gamma[0] = \sum_{\substack{\text{OPI vacuum} \\ \text{graphs}}} (G_0, V_3, V_4) + \text{const.} \tag{66}$$

This is to be contrasted with $\Gamma[\Phi_0]$ where Φ_0 is the field at $j = 0$, which coincides with $W[0]$ due to

$$W[0] = \Gamma[\Phi_0] - 0 \cdot \Phi_0 \tag{67}$$

and therefore collects *all* connected vacuum graphs involving G_0 and V_4, V_3, i.e.

$$\Gamma[\Phi_0] = \sum_{\substack{\text{all connected} \\ \text{vacuum graphs}}} (G_0, V_3, V_4). \tag{68}$$

It may be worthwhile seeing this result once more from the point of view of the generating functional $W[j]$. Let j_0 be the current which would force the field expectation Φ to vanish. Then there is the Legendre transform converse to (67)

$$\Gamma(0) = W[j_0] - j_0 \cdot 0 \tag{69}$$

such that we see that $\Gamma[0]$ coincides with $W[j_0]$. Thus while $W[0]$ collects all vacuum graphs, $W[j_0]$ picks out only the one-particle irreducible subset among these. Since $W[j_0]$ can be expanded in terms of connected Green's functions as

$$W[j_0] = \sum_{n=0}^{\infty} \frac{i^n}{n!} G_0^{(n)}(x_1 \cdots x_n)\, j_0(x_1) \cdots j_0(x_n) \tag{70}$$

this statement amounts to the fact that j_0 is just the correct value capable of subtracting from $W[0] = G^{(0)}(x_1 \ldots x_n)$ precisely all one-particle reducible vacuum diagrams. In order to appreciate this statement it should be remembered, that due to (48),

$$j_0 = -\Gamma_\phi[0] = -\Gamma_\Phi^{\text{int}}[0] \tag{71}$$

collects all one-particle irreducible vacuum diagrams linked up to a single φ leg. Thus, except for $W[0] = G_0^{(0)}$, each term in (70) with $n \geq 1$ is certainly one-particle reducible and according to what we have just learned about $W[j_0]$ being OPI, these must be all of them!

3.5. Phase transitions

The new non-perturbative expansion of $\Gamma[\Phi]$ has decisive advantages over the original perturbative one. The most important of these lies in the possibility of describing phase transitions. This can best be illustrated by looking at the simplest case of a relativistic local $g\varphi^4$ theory. Perturbatively, it can be calculated only as long as its mass^2 is positive. All loop integrals in the Wick rotated form contain denominators of the form $(p^2 + m^2)^{-1}$ and there are no problems in evaluating them at small momenta. In many systems, however, which may be described by φ^4 theory, the mass term is strongly temperature dependent and changes sign at some critical temperature T_c. Below T_c, the Feynman integrals are no longer defined. The effective action, however, can still be calculated if one uses the non-perturbative graphical rules derived in the last section. First of all we observe that without any vacuum diagrams we had as lowest approximation

$$\Gamma[\Phi] = A[\Phi]. \tag{72}$$

For $m^2 < 0$ we see that there are two values of Φ_0 for which $\Gamma[\Phi]$ is extremal namely

$$\Phi = \Phi_0 = 0 \tag{73}$$

$$\Phi = \Phi_0 = -\sqrt{\frac{m^2}{g/3!}}. \tag{74}$$

Since both are time independent the effective action may be used to calculate the energy of these states as

$$E_0 = -\frac{1}{T}\Gamma[\Phi_0]. \tag{75}$$

Conventionally one defines an effective potential from the action density at arbitrary constant fields Φ

$$V(\Phi) \equiv -\frac{1}{\text{Volume} \cdot T}\Gamma[\Phi]\Big|_{\Phi=\text{const.}} \tag{76}$$

such that $E_0 = \text{Volume} \cdot V[\Phi_0]$. Then one notices that the energy corresponding to the extremum (73) vanishes while that of (74) is negative

$$E_0 = -\frac{m^4}{4}\frac{3!}{g}. \tag{77}$$

Thus (74) will correspond to the physical ground state. The nonzero value (74)

$$\langle\varphi\rangle = \Phi_0 \neq 0 \tag{78}$$

is interpreted as a signal for the condensation of φ particles in the ground state and (77) is called the condensation energy. Inserting the typical temperature dependence of the mass term close to the critical temperature

$$m^2 \approx \mu^2\left(\frac{T}{T_c} - 1\right) \tag{79}$$

we see that there is a phase transition of second order: the energy which vanishes for $T > T_c$ goes smoothly over into the condensation energy energy (77). The specific heat, however, $c = -T(d^2E/dT^2)$, has a jump at T_c.

The important point is now that below T_c the Feynman graphs involving G_0 can no longer be calculated but the new ones with propagators G^Φ can. They have the form

$$(p^2 + M^2)^{-1} \tag{80}$$

where the effective mass term is a functional of Φ. Now, at the ground state value Φ_0, this mass term is

$$M^2 = m^2 + \frac{1}{2}g\Phi_0^2 = -m^2 > 0 \tag{81}$$

such that there is a whole neighbourhood of the ground state value for which the new Feynman integrals are well defined.

In general, if the potential has a stable minimum at Φ_0, the effective masses of the modes described by the propagator G^Φ will be of two types: either they are positive or they vanish for $\delta\Phi$ along directions of symmetry transformations. The latter are the Nambu Goldstone modes. In either case, there are no negative mass squares and the Feynman graphs can be calculated.

3.6. Reconstruction of all Green's functions from effective action

The effective action was introduced in order to make extremal principles available for the discussion of Green's functions. Up to now, the information contained in the extremum is rather scarce since it deals only with the vacuum expectation of the quantum field. It must be pointed out that even though $\Gamma[\Phi]$ involves only this one-point function explicitly, it nevertheless contains information about all Green's functions, except that this has to be extracted from the functional derivatives at the extremum. Only later shall we develop effective action in which also higher Green's functions can be obtained from the extremum itself. Let us here show how all Green's functions can be reconstructed once the effective action is known [1].

When deriving the differential equations for $\Gamma[\Phi]$ we have already noticed that the second functional derivative is the inverse of the connected two-point function and that the third derivative is the amputated connected three point function. In order to see how the higher Green's functions can be obtained from $\Gamma[\Phi]$ we differentiate (44) once more with respect to j and find

$$W_{j_4j_3j_2j_1} = W_{j_4j_3j_3'}W_{j_2j_2'}W_{j_1j_1'}\Gamma_{\Phi_{3'}\Phi_{2'}\Phi_{1'}} + W_{j_3j_3'}W_{j_1j_2j_2'}W_{j_1j_1'}\Gamma_{\Phi_{2'}\Phi_{2'}\Phi_{1'}} + W_{j_3j_3'}W_{j_2j_2'}W_{j_1j_1'}\Gamma_{\Phi_{3'}\Phi_{2'}\Phi_{1'}} + W_{j_4j_4'}W_{j_3j_3'}W_{j_2j_2'}W_{j_1j_1'}\Gamma_{\Phi_{4'}\Phi_{3'}\Phi_{2'}\Phi_{1'}}, \tag{82}$$

where the indices x_i have been replaced shortly by i itself. Using equ. (44) once more, this can be rewritten as

$$W_{j_4j_3j_2j_1} = \{W_{j_4j_4'}W_{j_3j_3'}\Gamma_{\Phi_{4'}\Phi_{3'}\Phi_5}W_{j_5j_5'}\Gamma_{\Phi_{5'}\Phi_{2'}\Phi_{1'}}W_{j_2'j_2}W_{j_1'j_1} + (2 \text{ permutations of (12) (34)})\} + W_{j_4j_4'}W_{j_3j_3'}W_{j_2j_2'}W_{j_1j_1'}\Gamma_{\Phi_{4'}\Phi_{3'}\Phi_{2'}\Phi_{1'}} \tag{83}$$

which amounts to the following relation for the connected Green's functions

$$G^{(4)}_{c\,4321} = \{G_{4'1'}G_{33'}iG\Gamma_{\Phi_{4'}\Phi_{3'}\Phi_5}G_{55}iG\Gamma_{\Phi_{5'}\Phi_{2'}\Phi_{1'}}G_{2'2}G_{1'1} + (2 \text{ permutations of (12) (34)})\} + G_{44'}G_{33'}G_{22}G_{11'}iG\Gamma_{\Phi_{4'}\Phi_{3'}\Phi_{2'}\Phi_{1'}}. \tag{84}$$

The result is displayed graphically in Fig. 4. Thus given the effective action, the four-point function is found simply by calculating the derivatives of $\Gamma[\Phi]$ up to the forth order and constructing all diagrams involving the legs G and vertices $\Gamma_{\Phi\ldots\ldots\Phi}$ but without forming any loops. Because of their physical appearance these are called tree diagrams. Notice that all loop integrals of the original Feynman diagrams are contained in the propagators G and the higher vertex functions $\Gamma_{\Phi\ldots\ldots\Phi}[\Phi]$. We now understand why in the calculation of $\Gamma[\Phi]$ there appeared only OPI graphs: All one-particle reducible contributions to the higher connected Green's functions come from the tree like composition of the OPI graphs contained in G and the vertex functions. By cutting one branch, the tree decomposes.

The reconstruction formulas for higher connected Green's functions $G_c^{[n]}$ are best derived in a graphical way. If we differentiate equ. (83) further with respect to j this amounts to adding one subscript j to each W_{jj} which, according to (44) can be represented as changing a propagator G into a three-point vertex function with three propagators sprouting from it. The derivatives on the vertex functions, on the other hand, adds one more Green's function leg to it via the chain rule. The procedure is illustrated in Fig. 4 up to the five-point function.

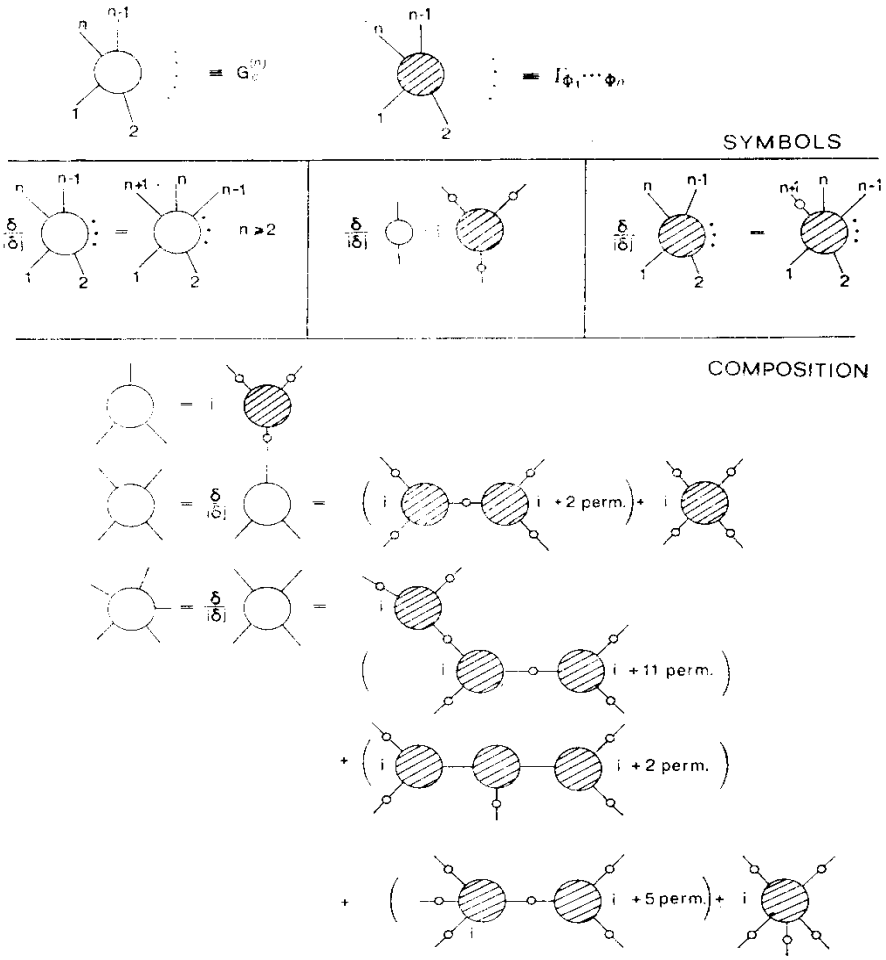

Fig. 4. The rules of composing the higher connected Green's functions $G^{(n)}$ from tree diagrams involving vertex functions $\Gamma_{\Phi_1 \cdots \Phi_n}$ and connected propagators G. The higher $G^{(n)}$'s are obtained by differentiation $(1/i)(\delta/\delta j)$ at each bubble element of the graphs using the rules stated in the middle of the figure

IV. Fluctuations and Loop Expansion

4.1. Path integrals

In the last chapter we have derived a non-perturbative expansion for the effective action $\Gamma[\Phi]$ and seen that this permits a passage of the theory from positive mass^2 to negative mass^2 and the ensuing phase transition. When stating the graphical rules of using the propagators G^{Φ} and vertices (62) there was one point which remained unclear: What is the proper parameter of smallness which organizes this graphical expansion? The potential strength can no longer be the relevant candidates since one Feynman graph involving G sums infinitely many V_3 and V_4 correction at a time. We shall now see that the natural systematics of the expansion is provided by the size of fluctuations in the quantum field.

In the usual formulation of quantum field theory, by which we derived the previous formulas, fluctuations are automatically taken care of by the use of field operators. The product of two operators

$$\sum_{n'} O^1_{nn'} O^2_{n'n''} = (O^1 O^2)_{nn''}$$

represents the sum of all "fluctuating indices" n' which are eigenvalues of some observables. When writing the time displacement operator in a product form factorized into many

infinitesimal pieces

$$\langle q_f|\, e^{-(i/\hbar)HT}\, |q_i\rangle = \lim_{N\to\infty} \langle q_f|\, (e^{-(i/\hbar)HT/N})^N\, |q_i\rangle \tag{85}$$

and inserting a complete set of intermediate states in the Schrödinger basis

$$\int dq_n\, |q_n\rangle\, \langle q_n| = 1 \tag{86}$$

between the n^{th} and $(n+1)^{st}$ factor, the operator becomes the product of infinitely many integrals

$$\langle q_f|\, e^{-(i/\hbar)HT}\, |q_i\rangle = \prod_{n=1}^{N-1} \int dq_n \langle q_f|\, e^{-(i/\hbar)HT/N}\, |q_{N-1}\rangle\, \langle q_{N-1}|\, e^{-(i/\hbar)HT/N}|\, q_{N-2}\rangle \cdots \langle q_1|\, e^{-(i/\hbar)HT/N}\, |q_i\rangle \tag{87}$$

one at every time $t_n = nT/N$. In the limit $N \to \infty$ this may be visualized as the sum over all possible zig-zag paths leading from the initial to the final Lagrange parameter q. Looking at the exponent closer we see that for a single mass point in an external potential it is

$$\langle q_{n+1}|\, e^{-(i/\hbar)HT/N}\, |q_n\rangle = \frac{1}{\sqrt{2\pi i/N}} \exp\left\{\frac{i}{2\hbar} N(q_{n+1} - q_n)^2 - \frac{i}{\hbar N} V(q_n)\right\}. \tag{88}$$

The infinite product can therefore be collected to

$$\frac{1}{\sqrt{2\pi i/N}^N} \exp\left\{\frac{i}{\hbar} \sum_{n=1}^{N-1} \left\{\frac{1}{2} N(q_{n+1} - q_n)^2 - \frac{1}{N} V(q_n)\right\}\right\} \tag{89}$$

such that for $N \to \infty$ the combined exponent is simply the action of the problem. Therefore there exists the following representation for the time displacement amplitude [5]

$$\langle q_f|\, e^{-(i/\hbar)HT}\, |q_i\rangle = \mathcal{N} \prod_{n=1}^{N-1} \int dq_n \Big/ \sqrt{\varepsilon}\, \exp\left\{\frac{i}{\hbar} A[q]\right\} \tag{90}$$

where

$$\mathcal{N} = (2\pi i\hbar)^{-N/2};\ \varepsilon \equiv 1/N \tag{91}$$

is the appropriate normalization factor. The infinite product of integrals is called path integral and denoted by

$$\mathcal{N} \prod_{n=1}^{N-1} \int dq_n \Big/ \sqrt{\varepsilon} \to \int Dq. \tag{92}$$

Notice that the path integral representation (90) contains all quantum information on the system without the use of any operators. The price to pay for this advantage is that all quantum fluctuations have to be summed explicitly. At first sight this seems an awful task and in a way it is: Of all quantum mechanical problems which in Schrödinger theory are simply solved by diagonalizing the Hamiltonian differential operator, it was for a long time only the harmonic oscillator for which the sum over fluctuating paths could be performed [5]. The other standard physical system, the hydrogen atom, was solved only very recently by really executing the path integral [6].

While it is very hard to sum over all fluctuating paths there is one important advantage of the path integral formulation if the physical system is such that fluctuations are small in some sense. Then it provides a convenient tool for an expansion of observables in powers of this parameter of smallness. From formula (90) it is obvious that the size of fluctuations is controlled by Planck's fundamental constant $\hbar$. In fact, in the classical

limit $\hbar \to 0$ there can be no fluctuations. In formula (90) we see that for $\hbar \to 0$ the exponential oscillates rapidly for the slightest changes of path. In the limit $\hbar \to 0$ only those paths can contribute where these oscillations are minimized which happens at the extremum of the action [5]

$$\left.\frac{\delta A[q]}{\delta q}\right|_{q(t)=q_{\mathrm{cl}}(t)} = 0. \tag{93}$$

This is, of course, just the extension of the standard saddle point theorem to path integrals. The extremal path is the solution of the classical equations of motion and this explains the subscript on the variable $q_{\mathrm{cl}}(t)$.
Since $\hbar$ is finite there are contributions to the path integral for $q(t)$ deviating from the classical path. If we write

$$q(t) = q_{\mathrm{cl}}(t) + \delta q(t) \tag{94}$$

we may expand the action into classical and fluctuating parts

$$A[q] = A[q_{\mathrm{cl}}] + \frac{1}{2}\,\delta q A_{qq}[q_{\mathrm{cl}}]\,\delta q + \cdots. \tag{95}$$

The linear term is absent due to the extremality property of $q_{\mathrm{cl}}(t)$. The quadratic piece contributes a path integral

$$\int D\delta q \exp\left\{\frac{i}{\hbar}\,\frac{1}{2}\,\delta q A_{qq}[q_{\mathrm{cl}}]\,\delta q\right\}. \tag{96}$$

When decomposing this into the infinite product of integrals according to (92) we see that in each factor $\int d\delta q_n$ there is a significant contribution only as long as δq_n is of the order of

$$\sqrt{\hbar}\, A_{qq}[q_{\mathrm{cl}}]_{n,n}. \tag{97}$$

Thus the quantum fluctuations remain limited by this quantity [5]. The important point to realize is that there are many physical systems in which the interaction is quite large but nevertheless the quantity (97) remains rather small, i.e. strongly interacting systems may carry quasiclassical properties. This holds at least within certain limited ranges of energy and momentum. Examples are large nuclei and superfluid He (both ^{3}He and ^{4}He) where low-energy and long-wave length properties are governed by hydrodynamic laws [7].

4.2. Path integral representation of quantum field theory

The quantum mechanical discussion of the last section can be carried over to quantum field theory in a very simple way. Let us begin with the free theory and consider the Fourier transform of the generating functional $Z^0[j]$

$$\tilde{Z}^0[\varphi] \equiv \int Dj e^{-ij\varphi} Z^0[j]. \tag{98}$$

The path integral over the source is defined in analogy with (92) as

$$\int Dj = C \prod_i \int dj(x_i)/\sqrt{\varepsilon} \tag{99}$$

where now the product runs over all doubled space time indices. Here C is a normalization factor which will be adjusted later. The inverse of (98) reads

$$Z^0[j] = \int D\varphi e^{ij\varphi} \tilde{Z}^0[\varphi] \tag{100}$$

where the path integral over $D\varphi$ is defined with some other normalization factor $\tilde{C}$. Taking the special case $Z[j] \equiv 1$ the Fourier transform becomes an infinite product of δ functions called a δ functional

$$\tilde{Z}[\varphi] = C \prod_i 2\pi\delta\big(\varphi(x_i)\big). \tag{101}$$

The inverse Fourier transform gives

$$Z[j] = \tilde{C}C \prod_i (2\pi\varepsilon). \tag{102}$$

Hence we find the condition

$$C\tilde{C} = \prod_i (2\pi\varepsilon)^{-1}. \tag{103}$$

Let us now use the explicit form of $Z^0[j]$ to actually calculate the Fourier transform.

$$\tilde{Z}^0[\varphi] = \int Dj \exp\left\{-\frac{1}{2}\, jG_0 j - ij\varphi\right\}. \tag{104}$$

Since each integral over $dj(x_i)$ runs from minus to plus infinity, we may change in every from j to $j' = j - iG_0^{-1}\varphi$ and rewrite

$$\tilde{Z}^0[\varphi] = \int Dj' \exp\left\{-\frac{1}{2}\, j'G_0 j'\right\} \exp\left\{\frac{i}{2}\, \varphi i G_0^{-1}\varphi\right\}. \tag{105}$$

The φ part does not depend on j' and can be taken out of the integral such that we remain with

$$\tilde{Z}^0[\varphi] = \mathcal{N} \exp\left\{\frac{i}{2}\, \varphi i G_0^{-i}\varphi\right\} \tag{106}$$

where

$$\mathcal{N} = \int Dj \exp\left\{-\frac{1}{2}\, jG_0 j\right\} \tag{107}$$

is a constant which does not depend on φ. It can easily be calculated by using the decomposition (94) as

$$\mathcal{N} = C \prod_k \int dj(x_k) \exp\left\{-\frac{i}{2}\, j_k (G_0/i)_{kl}\, j_l\right\}. \tag{108}$$

For this we merely have to bring the symmetric matrix G_0/i to diagonal form

$$(G_0/i)_{kl} \to (G_0{}^d/i)_k\, \delta_{kl} \tag{109}$$

via some rotation

$$j \to j^d = Rj \tag{110}$$

and may factorize

$$\mathcal{N} = C \prod_k \int dj^d(x_k) \exp\left\{-\frac{i}{2}\, j_k{}^{d2} (G_0{}^d/i)_k\right\}. \tag{111}$$

The measure of integration does not change since the Jacobian of the rotation is unity. Now each integral is of the Fresnel type and can be performed with the result

$$\mathcal{N} = C \prod_k \sqrt{-2\pi i}\, 1/\sqrt{(G_0{}^d/i)_k}. \tag{112}$$

The product of eigenvalues can be rewritten in a basis independent way as

$$\prod_k 1/\sqrt{(G_0{}^d/i)_k} = (\det G_0/i)^{-1/2} = (\det iG_0{}^{-1})^{1/2} \tag{113}$$

or also as

$$\mathcal{N} = \exp\left\{\frac{1}{2}\operatorname{tr}\log iG_0{}^{-1}\right\}. \tag{114}$$

The infinite factor has disappeared by choosing the constant C as

$$C = \prod_k \sqrt{-2\pi i\varepsilon}^{-1} \tag{115}$$

which we shall always do from now on.
Given this result for $\tilde{Z}^0[\varphi]$ the inverse Fourier transformation reads

$$Z^0[j] = \mathcal{N}\int Dj \exp\left\{\frac{i}{2}\varphi iG_0{}^{-1}\varphi + ij\varphi\right\}. \tag{116}$$

But here we realize that the exponent is simply the free action written in terms of the c-number field variable φ such that (116) is the path integral representation of the free generating functional $Z^0[j]$ in complete analogy with the quantum mechanical formula (90).
There is no problem in including interactions if we use formula (19). Performing explicitly the differentiations with respect to the currents we obtain

$$Z[j] = \mathcal{N}\int D\varphi \exp\left\{\frac{i}{2}\varphi iG_0{}^{-1}\varphi + iA^{\text{int}}[\varphi] + ij\varphi\right\}. \tag{117}$$

The exponent is the full action of the system written in terms of fluctuating field variables. In this way any interacting field theory is brought to the path integral form.
It is useful to notice that the normalization factor $\mathcal{N}$ can also be expressed in terms of a path integral over the field φ rather than j as

$$\mathcal{N}^{-1} = \int D\varphi \exp\left\{\frac{i}{2}\varphi iG_0{}^{-1}\varphi\right\} \tag{118}$$

since the right hand side gives by the same argument as in (108)

$$\mathcal{N}^{-1} = \tilde{C}\prod_k \sqrt{2\pi i\varepsilon}\,\sqrt{iG_0{}^{d\,-1}}^{-1} \tag{119}$$

which together with (103) agrees with (114). Thus we may write

$$Z[j] = \int D\varphi e^{(i/\hbar)A[\varphi]+(i/\hbar)j\varphi}/\int D\varphi e^{(i/\hbar)A^0[\varphi]} \tag{120}$$

which looks very similar to partition functions in statistical mechanics. In this final formula we have reintroduced the Planck constant $\hbar$ which had been set equal to one in the previous discussion.
For fermion systems, no such path integral representation really exists. There is, however, an algebraic formalism which can be set up for the anticommuting sources j and which may be considered as the generalization of path integrals to Fermi fields.
The only requirement is that it reproduces the correct quantum field theoretic results known from the operator formalism. It can be shown that all one has to do is define the

path integral for $\tilde{Z}^0[j]$ to be the same (106), (108), (111) except that the inverse of the diagonal elements appears on the right-hand side. For the normalization factor $\mathcal{N}$ this amounts to the exponent changing its sign.

$$\mathcal{N} \to \mathcal{N}_{\text{fermi}} = \exp\left\{-\frac{1}{2}\,\text{tr}\,\log iG_0^{-1}\right\}. \tag{121}$$

We have seen in the last chapter that such an expression is related to one loop integrals (see (70)) which now for fermions have the opposite sign of that for bosons. But this is precisely what is required for ordinary Feynman graphs.
Once (116) has been defined for fermions, also the interactions can be brought in just as in the boson case except that the differentiations in the sources are anticommuting variables. Therefore we have the same formal expression (117) for the interacting partition function except that the normalization factor is inverted. The analogy can be made perfect by expressing again $\mathcal{N}_{\text{fermi}}$ in terms of a path integral over φ. Then formula (120) is seen to hold for both, bosons and fermions.[5])

4.3. Fluctuation expansion of effective action

Let us now use the path integral representation (120) to expand the effective action according to the size of fluctuations. To lowest order in $\hbar$, the exponent is governed by the extremum which satisfies the condition

$$A_\varphi[\varphi] + j|_{\varphi=\varphi_{\text{cl}}} = 0. \tag{122}$$

The solution of this is the classical field configuration which depends functionally on the choice of the external source

$$\varphi_{\text{cl}} = \varphi_{\text{cl}}[j]. \tag{123}$$

Reinserting this into the exponent of (120) we find for the generating functional

$$W[j] = A\big[\varphi_{\text{cl}}[j]\big] + j\varphi_{\text{cl}}[j] + \text{const.} \tag{124}$$

From this it is easy to find the effective action. The field expectation coincides with the classical field:

$$\Phi = W_j = (A_\varphi + j)|_{\varphi=\varphi_{\text{cl}}[j]}\,\varphi_{\text{cl}\,j} + \varphi_{\text{cl}}[j]. \tag{125}$$

The first two terms cancel because of (122) such that the effective action becomes at this classical level

$$\Gamma^{(0)}[\Phi] = A[\Phi] + \text{const.} \tag{126}$$

i.e. the original action with the field operator replaced by the field expectation.
Let us now include fluctuations. According to the estimate (97) they will be limited to be of the order of $\sqrt{\hbar}$ such that we may expand

$$\varphi(t) = \varphi_{\text{cl}} + \sqrt{\hbar}\,\varphi_1. \tag{127}$$

If we keep only quadratic fluctuations, the exponent reads

$$A[\varphi_{\text{cl}}] + j\varphi_{\text{cl}} + \frac{\hbar}{2}\,\varphi_1 A_{\varphi\varphi}[\varphi_{\text{cl}}]\,\varphi_1. \tag{128}$$

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[5]) For more details see [7] and references therein.

Thus we may rewrite the generating functional as a path integral over the fluctuations

$$Z[j] = e^{(i/\hbar)A[\varphi_{\rm cl}]+j\varphi_{\rm cl}} \frac{\int D\varphi_1 e^{i\varphi_1 A_{\varphi\varphi}[\varphi_{\rm cl}]\varphi_1}}{\int D\varphi e^{i\varphi i G_0^{-1}\varphi}} \tag{129}$$

we may now use formulas (107), (114) to evaluate the quadratic functional integrals in numerator and denominator

$$\begin{aligned} Z[j] &= e^{(i/\hbar)W[j]} \\ &= \exp\left\{\frac{i}{\hbar}(A[\varphi_{\rm cl}] + j\varphi_{\rm cl}) + \frac{i}{2}\,\mathrm{tr}\,\log A_{\varphi\varphi}[\varphi_{\rm cl}] - \frac{i}{2}\,\mathrm{tr}\,\log iG_0^{-1}\right\}. \end{aligned} \tag{130}$$

For fermions, the last two terms would have the opposite sign. We may now use this result and calculate the lowest fluctuation correction to the effective action (124): the field expectation is

$$\Phi = W_j = (A_\varphi[\varphi] + j)\,\varphi_j|_{\varphi_{\rm cl}} + \varphi_{\rm cl} + \frac{i\hbar}{2} A_{\varphi\varphi}[\varphi_{\rm cl}]^{-1} A_{\varphi\varphi\varphi}[\varphi_{\rm cl}]\,\varphi_{{\rm cl}\,j} + O(\hbar^2). \tag{131}$$

The first two terms have cancelled again because of (122). Inserting this into (128) we find the effective action

$$\Gamma_{[\Phi]} = \Gamma^{(0)}_{[\Phi]} + \Gamma^{(1)}_{[\Phi]} + O(\hbar^2) = A[\Phi] + \frac{i}{2}\,\hbar\,\mathrm{tr}\,\log A_{\varphi\varphi}[\Phi] + O(\hbar^2). \tag{132}$$

We now see that up to this level the effective action agrees precisely with the first two terms of the non-perturbative expansion (65).

We shall now convince ourselves that by further expanding the effective action in powers of the fluctuation size $\sqrt{\hbar}$ we obtain the prescription (65) also for all higher terms with the additional advantage that by these powers they receive a definite organization via a quantity that in many physical systems can be considered as small:

In the general case, the field expectation will no longer lie at the extremum. Let us expand the field according to fluctuations around Φ rather than $\varphi_{\rm cl}$ and set

$$\varphi = \Phi + \varphi'. \tag{133}$$

Then the fluctuation field φ' has the property that its field expectation vanishes

$$\langle\varphi'\rangle = 0. \tag{134}$$

If we substitute (133) in formula (120) and use our first order results (132) we may rewrite the generating functional as

$$\begin{aligned} Z[j] &= e^{(i/\hbar)W[j]} \\ &= \exp\left\{\frac{i}{\hbar}(A[\Phi] + j\Phi) + \frac{i}{2}\,\mathrm{tr}\,\log A_{\varphi\varphi}[\Phi] - \frac{i}{2}\,\mathrm{tr}\,\log iG_0\right\} \\ &\quad \times \int D\varphi' e^{iA_{\rm fl}[\varphi']} \Big/ \int D\varphi' e^{iA^0_{\rm fl}[\varphi']} \end{aligned} \tag{135}$$

where

$$\begin{aligned} A^0_{\rm fl}[\varphi'] &\equiv \frac{1}{2}\,\varphi' A_{\varphi\varphi}[\Phi]\,\varphi' \\ A_{\rm fl}[\varphi'] &\equiv A^0_{\rm fl}[\varphi'] + A^{\rm int}_{\rm fl}[\varphi'] \equiv A[\Phi + \varphi'] + j\varphi' - A[\Phi] \end{aligned} \tag{136}$$

is the fluctuating part of the action. Notice that this does have terms linear in φ' since contrary to the low order discussion, $\Phi + \varphi_{\rm cl}$ does not solve the equation of motion (122) and therefore

$$j \neq A_\varphi[\Phi]. \tag{137}$$

We now recognize in the fluctuating action our previous auxiliary action (59) which was derived there heuristically in order to reproduce the correct graphical expansion for the effective action. In fact, the graphical rules are now an automatic consequence of seeing the fluctuation factor

$$Z_{\rm fl}[j] = \int D\varphi' \exp\{iA^0_{\rm fl}[\varphi'] + A^{\rm int}_{\rm fl}[\varphi'] + j\varphi'\}\left[\int D\varphi' \exp\{iA^0_{\rm fl}[\varphi']\}\right]^{-1} \tag{138}$$

as a generating functional of its own. In it, φ' plays the role of the fundamental quantum field, the quadratic piece

$$A^0_{\rm fl}[\varphi'] = \frac{1}{2}\varphi'\left(iG_0^{-1} - V_3\Phi - \frac{1}{2}V_4\Phi\Phi\right)\varphi' \tag{139}$$

supplies the propagator

$$G^\Phi = i\hbar\left[iG_0^{-1} - V_3\Phi - \frac{1}{2}V_4\Phi\Phi\right]^{-1} \tag{140}$$

and the remaining terms give the interactions which depend functionally on the particular configuration of the field expectation Φ. There is again a generating functional of connected Green's functions

$$W_{\rm fl}[j] = -i\log Z_{\rm fl}[j] \tag{141}$$

and it may be used to define the effective action $\Gamma_{\rm fl}[\Phi']$ of the fluctuating part of the theory where Φ' is the expectation of φ', i.e.

$$\Phi' \equiv \langle\varphi'\rangle, \qquad W_{\rm fl}[j] = \Gamma_{\rm fl}[\Phi'] + j\Phi'. \tag{142}$$

The important point is now that we chose Φ as the center of fluctuations such that the $\Gamma_{\rm fl}[\Phi']$ has to be evaluated at $\Phi' = 0$. This imposes a restriction upon the external source j which has to be a functional of Φ such that $\Phi' = 0$. But this means that $j = j_0[\Phi]$ plays exactly the same role with respect to the fluctuating part of the theory as was discussed in the context of equ. (65) for the full theory. Hence we can conclude that only one-particle irreducible vacuum diagrams are contained in $\Gamma_{\rm fl}[0]$. Thus we have found the same graphical rules as before in the non-perturbative discussion of Sect. 2.4.
Let us now see how the parameter of smallness of fluctuations $\hbar$ appears in this graphical expansion. Every propagator G^Φ carries an explicit factor $\hbar$, every vertex $V_3 + V_4\Phi$ or V_4 is accompanied by an inverse factor $\hbar^{-1}$, and finally there is one overall factor $\hbar$ in the definition of $\Gamma_{\rm fl}[\Phi]$. In a vacuum graph, all lines emerging from the vertices have to be connected with each other. Thus, if there are n_3 vertices $V_3 + V_4\Phi$ and n_4 vertices V_4 the total power in $\hbar$ is

$$L \equiv (3n_3 + 4n_4)/2 - n_3 - n_4 + 1 = n_3/2 + n_4 + 1. \tag{143}$$

This is the new parameter which organizes the non-perturbative expansion. The expansion can be truncated after a finite number of terms if the system happens to possess quasiclassical properties.
The determination of the order of each graph is simplified by realizing that it coincides with a simple topological property: it is the number of independent loop integrals in the

Feynman diagram. For, the total number of internal lines is

$$(3n_3 + 4n_4)/2$$

which have to be integrated. But of these, $n_3 + n_4 - 1$ are trivial due to an energy momentum δ-function at each vertex. The δ-function at the last vertex guarantees total energy momentum conservation and is usually factored out of the Feynman integral. Therefore, the power L coincides with the number of loops and this is why the non-perturbative expansion is often referred to as loop expansion. Since its speed of convergence is determined by the degree of quantum fluctuations, it is also known as quasiclassical expansion.

V. The Effective Action of the Second Type

5.1. Two particle source

Up to now, we have considered the effective action $\Gamma[G_c^{(1)}] \equiv \Gamma[\Phi]$ of one argument. The extremal principle is not very powerful since it determines only the field expectation Φ_0 at $j = 0$. Even though the complete theory can be reconstructed from $\Gamma[\Phi]$, (see Sect. 2.5) the information on the higher Green's functions is not contained in the extremum itself but in the functional derivatives at the extremum. It would be preferable to extend the list of arguments of $\Gamma[\Phi]$ to comparise also these higher Green's functions into an extremal principle. The basic reason is the following: For any realistic theory, $\Gamma[\Phi]$ may never be known exactly. Only some truncated graphical expansion is accessible to calculations. As in any variational method, if we increase the list of parameters which may directly be varied we expect to find a better approximation to the true solution. This will be the case also here.

Let us therefore proceed by one step and search for an effective action

$$\Gamma[G_c^{(1)}, G_c^{(2)}] \equiv \Gamma[\Phi, G] \tag{144}$$

which is extremal for both the connected one- and two-point functions Φ and G.

At this place it should be pointed out that in non-relativistic many-body systems, the two point function does not only describe the usual particle propagators $\langle T\psi\psi^+\rangle$. Due to our doubled field notation the diagonal elements consist of the so called anomalous Green's functions $\langle T\psi\psi\rangle$, $\langle T\psi^+\psi^+\rangle$ which account for pair correlations. These are absent in perturbative calculations at any finite order. It will be the virtue of the extended effective action to allow for the presence of these anomalous Green's functions. For many physical systems this is essential for a correct description of the observed phenomena. The best example is the superconductor where the fundamental particles themselves cannot form a condensate and therefore $\Phi = 0$, but pairs of particles, called Cooper pairs, can, and such a condensate is signalized by the non-vanishing of pair correlation functions in the ground state. In the example these are local and observable as a gap in the single particle energy spectrum since it is related to the breakup energy of the pairs.

The extended effective action $\Gamma[\Phi, G]$ can be found in complete analogy with $\Gamma[\Phi]$. For this we introduce an external bilocal source

$$A^{\text{source}} = \frac{1}{2}\varphi K\varphi \tag{145}$$

into the action and consider the generating functional of two arguments

$$Z[j, K] \equiv e^{iW[j,K]} = \langle Te^{ij\varphi+(i/2)\varphi K\varphi}\rangle. \tag{146}$$

The new source K is not really an independent variable. In fact, the functional derivatives are related by

$$2Z_K = -iZ_{jj}. \tag{147}$$

For $W[j, K]$ this amounts to

$$2iW_K = W_{jj} + iW_j^2 \tag{148}$$

such that

$$2iW_{Kj} = W_{jjj} + 2iW_{jj}W_j \tag{149}$$

and

$$W_{jjj} + 3iW_jW_{jj} - W_j^3 = 2i(W_{Kj} + iW_KW_j). \tag{150}$$

We may now derive differential equations for $W[j, K]$ just as we did previously for $W[j]$. We can save labor by noting that adding the source term (145) leaves the equations of motion (24) the same as before except that the matrix iG_0^{-1} in the free part of the action is replaced by $iG_0^{-1} + K$ such that

$$G^K = i[iG_0^{-1} + K]^{-1} \tag{151}$$

appears in place of the free propagator G_0. Thus $W[j, K]$ satisfies the same equ. (26) as $W[j]$ with G^K instead of G_0 and we may use (149) to rewrite (26) as

$$i(G^K)^{-1}W_j - V_3W_K + \frac{i}{3}V_4(W_{Kj} + iW_KW_j) + j = 0 \tag{152}$$

which may be solved in conjunction with the constraint (149). We shall not do so but employ this equation a little later on for the purpose of deriving a differential equation for the effective action of the second type.

5.2. The combined Legendre transform

The advantage of using the bilocal source lies in the fact that we can now, in a straight-forward extension of the previous procedure, introduce a Legendre transform of $W[j, K]$ with respect to both source variables

$$\Gamma[\Phi, G] = W[j, K] - W_jj - W_KK. \tag{153}$$

The field expectation Φ is now defined in the presence of both sources j and K[6])

$$\Phi \equiv \langle\varphi\rangle_{j,K} = W_j[j, K]. \tag{154}$$

The same thing holds for the connected two point function

$$G \equiv G_c^{(2)} = \langle T\varphi\varphi\rangle_{j,K} - \langle\varphi\rangle_{j,K}\langle\varphi\rangle_{j,K} = \frac{1}{i}W_{jj}[j, K]$$

which is now obtained, from (148), as

$$\begin{aligned} G = G_c^{(2)} &= \langle T\varphi\varphi\rangle_{j,K} - \langle\varphi\rangle_{j,K}\langle\varphi\rangle_{j,K} = \frac{1}{i}W_{jj}[j, K] \\ &= 2W_K - W_j^2 = 2W_K - \Phi^2 \end{aligned} \tag{155}$$

or

$$W_K = \frac{1}{2}(G + \Phi^2) = \frac{1}{2}\langle T\varphi\varphi\rangle_{j,K}. \tag{156}$$

---

[6]) The subscripts j, k on $\langle\varphi\rangle$ and $\langle T\varphi\varphi\rangle$ record functional dependencies, not derivatives!

Writing

$$\Gamma[\Phi, G] = W[j, K] - \Phi j - \frac{1}{2}(G + \Phi^2) K \tag{157}$$

we see that $\Gamma[\Phi, G]$ has, by construction, the following derivatives

$$\Gamma_\Phi[\Phi, G] = -j - K\Phi \tag{158}$$

$$\Gamma_G[\Phi, G] = -\frac{1}{2} K. \tag{159}$$

Since the physical situation corresponds to the absence of external sources, $\Gamma[\Phi, G]$ is extremal in both, field expectation Φ and connected Green's function G, and will therefore be called the effective action of the second type.
Notice that $\Gamma[\Phi, G]$ can also be obtained sequentially by finding the first effective action

$$\Gamma^K[\Phi] = W[j, K] - W_j j \tag{160}$$

$$\Phi \equiv W_j[j, K] \tag{161}$$

albeit in the presence of the external bilocal source K, and then taking the Legendre transform of $\Gamma^K[\Phi]$ with respect to K

$$\Gamma[\Phi, G] \equiv \Gamma^K[\Phi] - \Gamma^K{}_K[\Phi] K \tag{162}$$

$$\frac{1}{2}(\Phi^2 + G) = \Gamma^K{}_K[\Phi]. \tag{163}$$

Since

$$\begin{aligned} \Gamma^K{}_K[\Phi] &= (W_j[j, K] - \Phi) j_K + W_K[j, K] \\ &= W_K[j, K] \end{aligned} \tag{164}$$

this reduces again to (153).
A remark is in order concerning the treatment of macrocanonical ensembles. There, an average particle number is ensured by an external chemical potential μ. In our doubled field notation this corresponds to a constant source

$$A^{\text{c.p.}} = \mu\psi^+\psi = \frac{1}{2}\varphi K^{\text{c.p.}}\varphi \tag{165}$$

where

$$K^{\text{c.p.}} = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} \tag{166}$$

ist the 2×2 matrix for the chemical potential. There are two options of including this into the formalism: Either we absorb $A^{\text{c.p.}}$ into A^0 and replace the propagator G_0 by $i[iG_0^{-1} + K^{\text{c.p.}}]^{-1}$ in all formulas. Or we leave G_0 as it is but set the external source K equal to $K^{\text{c.p.}}$ at the end, rather than zero. In the latter case, $\Gamma[\Phi, G]$ is not really extremal with respect to variations in Φ and G (see (158), (159)) but satisfies

$$\Gamma_\Phi[\Phi, G] = -K^{\text{c.p.}}\Phi \tag{167}$$

$$\Gamma_G[\Phi, G] = -\frac{1}{2} K^{\text{c.p.}}. \tag{168}$$

There ist, however, a simple modification which is properly extremal

$$\Gamma^{\text{g.c.}}[\Phi, G] = \Gamma[\Phi, G] + \frac{1}{2} K^{\text{c.p.}}(\Phi^2 + G) \tag{169}$$

as can be verified by differentiation. Of course, the connection between $\Gamma[\Phi, G]$ and $\Gamma^{\text{g.c.}}[\Phi, G]$ corresponds precisely to going from the Hamiltonian H to the grand canonical energy $H - \mu N$.

5.3. Calculation of $\Gamma[\Phi, G]$

Let us now calculate the second effective action $\Gamma[\Phi, G]$. For this we use the differential equation (152) and insert

$$W_j = \Phi, \qquad W_K = \frac{1}{2}(G + \Phi^2), \qquad j = -\Gamma_\Phi - K\Phi \qquad K = -2\Gamma_G \tag{170}$$

to find

$$iG_0^{-1}\Phi - \frac{1}{2!} V_3(G + \Phi^2) + \frac{i}{3} V_4 W_{Kj} - \frac{1}{3!} V_4(G + \Phi^2)\Phi - \Gamma_\Phi = 0. \tag{171}$$

The derivative W_{Kj} may be expressed in terms of $\Gamma[\Phi, G]$ by using the generalization of relation (42) for the two arguments j, K. To simplify the formulas, let us define, for a moment,

$$\begin{aligned} \Phi_1 &= \Phi \\ \Phi_2 &\equiv \frac{1}{2}(G + \Phi^2) \end{aligned} \tag{172}$$

and the auxiliary effective action

$$\Gamma^{\text{aux}}[\Phi_1, \Phi_2] = \Gamma[\Phi, 2\Phi_2 - \Phi^2]. \tag{172'}$$

This has the simpler inverse Legendre transforms

$$\begin{aligned} \Gamma^{\text{aux}}_{\Phi_1} &= -j \equiv -j_1 \\ \Gamma^{\text{aux}}_{\Phi_2} &= -K \equiv -j_2 \end{aligned} \tag{173}$$

such that we may immediately generalize (42) to $W_{j_i j_j}\Gamma_{\Phi_j \Phi_k} = -\delta_{ik}$ or

$$W_{jj}\Gamma^{\text{aux}}_{\Phi\Phi} + W_{jK}\Gamma^{\text{aux}}_{\Phi_2\Phi} = -1 \tag{174}$$

$$W_{jj}\Gamma^{\text{aux}}_{\Phi\Phi_2} + W_{jK}\Gamma^{\text{aux}}_{\Phi_2\Phi_2} = 0. \tag{175}$$

Now we use

$$\begin{aligned} \Gamma^{\text{aux}}_{\Phi_2} &= 2\Gamma_G \\ \Gamma_\Phi{}^{\text{aux}} &= \Gamma_\Phi - 2\Phi\Gamma_G \\ \Gamma^{\text{aux}}_{\Phi_2\Phi_2} &= 4\Gamma_{GG} \\ \Gamma^{\text{aux}}_{\Phi\Phi_2} &= 2\Gamma_{\Phi G} - 4\Phi\Gamma_{GG} \\ \Gamma^{\text{aux}}_{\Phi\Phi} &= \Gamma_{\Phi\Phi} - 4\Phi\Gamma_{\Phi G} - 2\Gamma_G \end{aligned} \tag{176}$$

to solve (174), (175) for W_{jj} and W_{jK}:

$$W_{jj} = -[\Gamma^{\text{aux}}_{\Phi\Phi} - \Gamma^{\text{aux}}_{\Phi\Phi_2}\Gamma^{\text{aux}\,-1}_{\Phi_2\Phi_2}\Gamma^{\text{aux}}_{\Phi_2\Phi}]^{-1} \tag{177}$$

$$W_{jK} = -W_{jj}\Gamma^{\text{aux}}_{\Phi\Phi_2}\Gamma^{\text{aux}\,-1}_{\Phi_2\Phi_2} = -\frac{1}{2}\,W_{jj}(\Gamma_{\Phi G} - 2\Phi\Gamma_{GG})\,\Gamma^{-1}_{GG}. \tag{178}$$

The latter equation may be used together with (155) to turn (171) into an equation for $\Gamma[\Phi, G]$

$$iG_0^{-1}\Phi - \frac{1}{2}\,V_3(G + \Phi^2) - \frac{1}{3!}\,V_4[-G\Gamma_{\Phi G}\Gamma^{-1}_{GG} + 3G\Phi + \Phi^3] - \Gamma_\Phi = 0. \tag{179}$$

Since j and K are not independent sources but related by the constraint (148), there exists an equivalent restriction upon the combined functional dependence of $\Gamma[\Phi, G]$ on the two arguments. The constraint itself has already been used to derive the equation of motion (179) from (29). Thus we need another functional identity which is independent of equ. (179). For this we may use directly (177) after inserting $W_{jj} = iG$:

$$i = G[\Gamma_{\Phi\Phi} - 2\Gamma_G - \Gamma_{\Phi G}\Gamma^{-1}_{GG}\Gamma_{G\Phi}]. \tag{180}$$

For a free theory, the first equation reads

$$\Gamma_\Phi = iG_0^{-1}\Phi \tag{181}$$

and may be solved trivially by

$$\Gamma^0[\Phi, G] = \frac{1}{2}\,\Phi iG_0^{-1}\Phi + f[G] \tag{182}$$

where $f[G]$ is an arbitrary functional of G. This is determined by the constraint (180) which reads

$$f_G[G] = \frac{1}{2}\,iG_0^{-1} - \frac{1}{2}\,iG^{-1} \tag{183}$$

and is solved by

$$f[G] = \frac{1}{2}\,\text{tr}\,(iG_0^{-1}\,G) + \frac{i}{2}\,\text{tr}\log iG^{-1} + \text{const.} \tag{184}$$

In order to proceed with the solution it is convenient to split Γ into free and interacting part

$$\begin{aligned}\Gamma[\Phi, G] &= \Gamma^0[\Phi, G] + \Gamma^{\text{int}}[\Phi, G] \\ &= \frac{1}{2}\,\Phi iG_0^{-1}\Phi + \frac{i}{2}\,\text{tr}\,(G_0^{-1}G) + \frac{i}{2}\,\text{tr}\log iG^{-1} + \Gamma^{\text{int}}[\Phi, G].\end{aligned} \tag{185}$$

Then we have

$$\Gamma_G = -\frac{i}{2}\,G^{-1} + \Gamma_G{}^{\text{int}} \tag{186}$$

$$\Gamma_{GG} = \frac{i}{2}\,G^{-1}G^{-1} + \Gamma^{\text{int}}_{GG} \tag{187}$$

$$\Gamma_{G\Phi} = \Gamma^{\text{int}}_{G\Phi} \tag{188}$$

such that the interacting part satisfies the differential equation

$$\Phi\Gamma_\Phi{}^{\rm int} = -\frac{V_4}{3!}\,[2iG^3\Phi\Gamma^{\rm int}_{\Phi G}(1 - 2iGG\Gamma^{\rm int}_{GG})^{-1}] - \frac{1}{2}\,V_3\Phi^3 - \frac{1}{3!}\,V_4\Phi^4 - \frac{1}{2}\,V_3G\Phi - \frac{1}{2}\,V_4G\Phi^2 \tag{189}$$

where we have multiplied by Φ and contracted indices as we did before with (51). The right-hand side may be expanded in powers of $\Gamma^{\rm int}_{GG}$ such that the equation takes the form displayed graphically in Fig. 5. It can be solved iteratively if one respects at each order the equation of constraint (180) which for the interacting part becomes

$$G\Gamma_G{}^{\rm int}[\Phi, G] = \frac{1}{2}\,G\Gamma^{\rm int}_{\Phi\Phi} + iG\Gamma^{\rm int}_{\Phi G}GG(1 - 2iGG\Gamma^{\rm int}_{GG})^{-1}\,\Gamma^{\rm int}_{G\Phi}. \tag{190}$$

Again, the right-hand side is expanded in powers of $\Gamma^{\rm int}_{GG}$ and shown in Fig. 5. To first order we neglect $\Gamma^{\rm int}$ on the right-hand side of (190) and have

$$\Phi\Gamma_\Phi{}^1 = -\frac{1}{2}\,V_3\Phi^3 - \frac{1}{3!}\,V_4\Phi^4 - \frac{1}{2}\,V_3G\Phi - \frac{1}{2}\,V_4G\Phi^2. \tag{191}$$

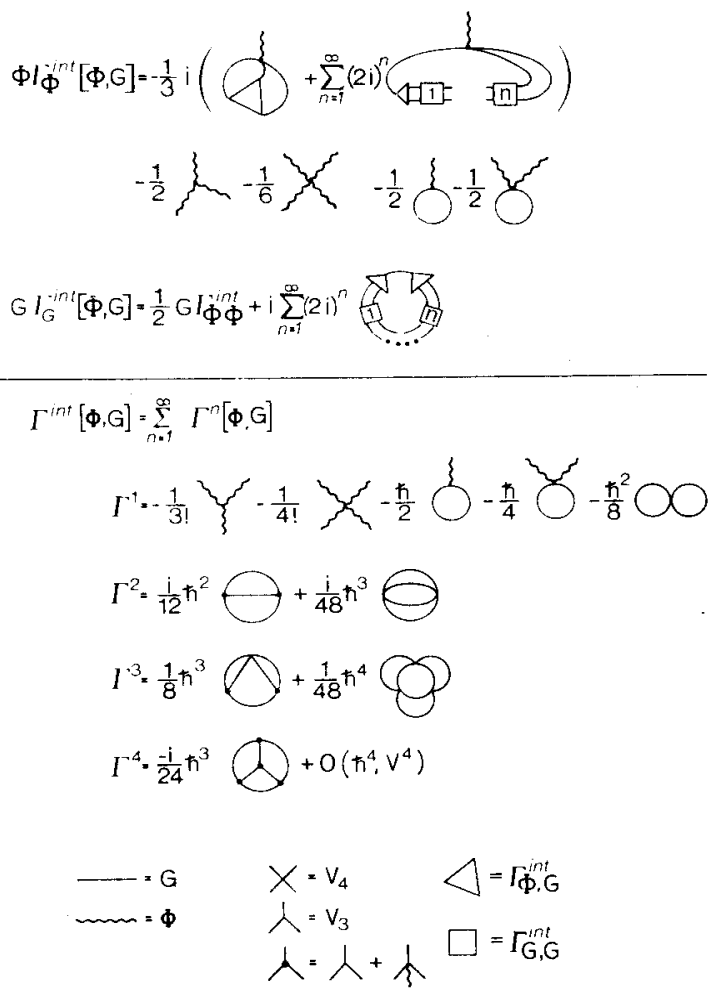

Fig. 5. The integral equation and constraint for the interacting part of the effective action of second type. The iterative solution gives the results Γ^a organized according to powers of V^n. For completeness we have also written down the explicit power $\hbar$ such that one gains control over the semiclassical expansion. The expansion is complete up to V^3 or $\hbar^3$

This can be integrated in Φ to get the first order result (see Fig. 5 for a graphical representation)

$$\Gamma^1 = -\frac{1}{3!} V_3\Phi^3 - \frac{1}{4!} V_4\Phi^4 - \frac{1}{2}\left(V_3 + \frac{1}{2} V_4\Phi\right) G\Phi + f^1[G] \tag{192}$$

where $f^1[G]$ an arbitrary functional of G only. Inserting this into (190) we find the constraint:

$$Gf_G{}^1 = -\frac{1}{4} V_4G^2 \tag{193}$$

such that

$$f^1[G] = -\frac{1}{8} V_4G^2 + \text{const.} \tag{194}$$

The result for Γ^1 is shown in Fig. 1.
To second order we insert the first order derivative

$$\Gamma^1_{\Phi G} = -\frac{1}{2}(V_3 + V_4\Phi) \tag{195}$$

into (189) and find

$$\Phi\Gamma_\Phi{}^2 = \frac{i}{3!} V_4(V_3 + V_4\Phi)\, G^3\Phi \tag{196}$$

which is integrated to

$$\Gamma^2 = \frac{i}{3!} V_4\left(V_3 + \frac{1}{2} V_4\Phi\right) G^3\Phi + f^2[G]. \tag{197}$$

Inserted into the constraint we find

$$G\Gamma_G{}^2 = \frac{i}{12} V_4{}^2G^4 + \frac{i}{4}(V_3 + V_4\Phi)^2\, G^3 \tag{198}$$

such that

$$\Gamma^2 = \frac{i}{48} V_4{}^2G^4 + \frac{i}{12}(V_3 + V_4\Phi)^2\, G^3. \tag{199}$$

To third order we have two contributions in (189)

$$\Phi\Gamma_\Phi{}^3 = -\frac{i}{3} G^3\Phi(\Gamma^2_{\Phi G} + 2i\Gamma^1_{\Phi G}G^2\Gamma^1_{GG}). \tag{200}$$

From (195) and (199) we derive

$$\Gamma^2_{\Phi G} = \frac{i}{2} V_4{}^2\Phi G^2 + \frac{i}{2} V_4V_3G^2$$

$$\Gamma^1_{\Phi G} = -\frac{1}{2}(V_3 + V_4\Phi) \tag{201}$$

$$\Gamma^1_{GG} = -\frac{1}{4} V_4$$

and obtain

$$\Phi\Gamma_{\Phi}{}^{3} = \frac{1}{4} V_4{}^2(V_3 + V_4\Phi)\, \Phi G^5. \tag{202}$$

This integrates to

$$\Gamma^3 = \frac{1}{8} V_4(V_3 + V_4\Phi)^2\, G^5 + f^3[F]. \tag{203}$$

Now the equation of constraint becomes

$$\begin{aligned} G\Gamma_G{}^3 &= \frac{1}{2} G\Gamma^3_{\Phi\Phi} + 2i[2\Gamma^2_{\Phi G} G^2 \Gamma^1_{G\Phi} G + 2i\Gamma^1_{\Phi G} G^2 \Gamma^1_{GG} G^2 \Gamma^1_{G\Phi} G) \\ &= \frac{1}{8} V_4{}^3 G^6 + \frac{5}{8} V_4(V_3 + V_4\Phi)^2\, G^5 \end{aligned} \tag{204}$$

and may be integrated to

$$\Gamma^3 = \frac{1}{48} V_4{}^3 G^6 + \frac{1}{8} V_4(V_3 + V_4\Phi)^2\, G^5. \tag{205}$$

If we continue this procedure we see the interacting part of $\Gamma[\Phi, G]$ consists of all vacuum graphs formed with the full propagators G and vertices V_4 and $V_3 + V_4\Phi$ except for an important topological restriction: They do not fall apart when cutting two lines. Such graphs are called two-particle irreducible (TPI).

Our calculation of Γ^n has proceeded iteratively in powers of the potential. It can easily be verified that the same result would have been obtained by starting out with the previously derived expansion for the first effective action $\Gamma[\Phi]$ but using the propagator G^K instead of G_0, and performing the second Legendre transform sequentially according to (160), (161). An alternative derivation would go via the loop expansion where we summed all OPI vaccum graphs consisting of propagators $G^{\Phi,K} = i\hbar[i\hbar G_{\Phi}{}^{-1} + K]$ and vertices $V_4/\hbar$, $(V_3 + V_4\Phi)/\hbar$ organizing according to the explicit power of $\hbar$. In this case the expansion of $\Gamma[\Phi, G]$ would come out to be the same as in Fig. 5 except for the organization of the graphs: Now the explicit powers of $\hbar$ count the number of independent loops of the fully interacting Green's function G. Thus the second order correction Γ^2 combines a two and a three loop diagram while Γ^3 contains three and a four loops. If we want to be complete up to the three loop level we have to take one more diagram from Γ^4 and obtain:

$$\begin{aligned} \Gamma[\Phi, G] = A[\Phi] &+ \hbar\frac{i}{2}\,\mathrm{tr}\,(G_0{}^{-1} G) + \hbar\frac{i}{2}\,\mathrm{tr}\log G^{-1} - \frac{\hbar}{2} V_3 G\Phi - \frac{\hbar}{4} V_4 G\Phi^2 \\ &- \frac{\hbar^2}{8} V_4 G^2 + \frac{i}{12}\hbar^2(V_3 + V_4\Phi)^2\, G^3 + \frac{i}{48}\hbar^3 V_4{}^2 G^4 + \frac{1}{8}\hbar^3 V_4(V_3 + V_4\Phi)^2\, G^5 \\ &+ \frac{1}{48}\hbar^4 V_4{}^3 G^6 - \frac{i}{24}\hbar^3(V_3 + V_4\Phi)^4\, G^6 + O(\hbar^4, V^4). \end{aligned} \tag{206}$$

Here we have displayed the explicit powers of $\hbar$ and kept the term of order $\hbar^4 V^3$ such that one may have the complete expansion also up to powers V^3 if desired. Our convention is such that $\hbar G_0$, $\hbar G$ are the propagators $\langle T\varphi\varphi\rangle$ in free and interacting cases. The iterative procedure can be somewhat rationalized by going to reduced variables as is shown in the Appendix.

5.4. Equations of motion

The new effective action has two explicit arguments Φ and G such that the extrema determine physical configurations of field expectation and Green's function. With Γ known up to the 3 loop level, the extremality condition (167) reads

$$(iG_0^{-1} + K^{\text{c.p.}})\,\Phi = -\Gamma_\Phi{}^{\text{int}}[\Phi, G] = \frac{1}{2} V_3\Phi^2 + \frac{1}{6} V_4\Phi^3 + \frac{\hbar}{2}(V_3 + V_4\Phi)\,G$$
$$- \frac{i}{6}\hbar^2 V_4(V_3 + V_4\Phi)\,G^3 - \frac{1}{4} V_4{}^2(V_3 + V_4\Phi)\,G^5$$
$$+ \frac{i}{6}\hbar^3 V_4(V_3 + V_4\Phi)^3\,G^6 + O(\hbar^4, V^4). \tag{207}$$

The other condition is most conveniently written by introducing the quantity

$$\Sigma \equiv -\frac{2}{\hbar}\Gamma_G{}^{\text{int}} = \left(V_3 - \frac{1}{2} V_4\Phi\right)\Phi + \frac{\hbar}{2} V_4 G - \frac{i}{2}\hbar(V_3 + V_4\Phi)^2\,G^2 - \frac{i}{6}\hbar^2 V_4{}^2 G^3$$
$$- \frac{5}{4}\hbar^2 V_4(V_3 + V_4\Phi)^2\,G^4 - \frac{1}{4}\hbar^3 V_4{}^3 G^5 + \frac{i}{2}\hbar^2(V_3 + V_4\Phi)^4\,G^5 + O(\hbar^3, V^4) \tag{208}$$

The equation (168) determines how the interacting Green's function differs from the free one

$$G = i[iG_0^{-1} + K^{\text{c.p.}} - \Sigma]^{-1}. \tag{209}$$

such that Σ coincides with the self energy introduced in (54). The equations are shown in Fig. 6.

The diagrams contributing to (208) are obtained by differentiating the interacting part of $\Gamma[\Phi, G]$ with respect to G. Graphically this amounts to cutting one line and truncating the two open ends. What remains is called a self energy diagram.

Since this operation is done on two-particle irreducible vacuum graphs the results must be one-particle irreducible. Thus we conclude that Γ^{int} collects precisely all OPI self-energy diagrams. These are often referred to as *proper*.

Equs. (209), (208) are known as Dyson's equation. To lowest order in $\hbar$,

$$(iG_0^{-1} + K^{\text{c.p.}})\,\Phi = \frac{1}{2} V_3\Phi^2 + \frac{1}{6} V_4\Phi^3 - \frac{\hbar}{2}(V_3 + V_4\Phi)\,G \tag{210}$$

$$G = i\left[iG_0^{-1} + K^{\text{c.p.}} - (V_3 + V_4\Phi)\,\Phi - \frac{\hbar}{2} VG\right]^{-1}. \tag{211}$$

We have seen before in Sect. 2.5 that the equ. of motion (210), even that the zero loop level, can accommodate phase transitions which are signalized by the appearance of a non-vanishing field expectation Φ. This corresponds to the presence of a condensate of φ particles in the ground state. Equ. (211) extends this picture and allows also for non-zero density $\langle T\psi\psi^+\rangle$ and pair correlation functions $\langle T\psi\psi\rangle$ in the ground state. It is this property which has led to the first successful description of the superconductive state by Bardeen, Cooper, and Schrieffer and of the Fermi superfluid ^{3}He (for a review see [7]).

At first sight, the presence of a two point function $\langle T\psi\psi\rangle$ seems to contradict the property of particle number conservation of the original action. As a matter of fact, the extrema of the truncated effective action do violate this fundamental law. The relative fluctuations in particle number are of the order of 1/(total number)$^{1/3}$ and can be made arbitrarily small only in the thermodynamic limit if infinitely many particles. Of course

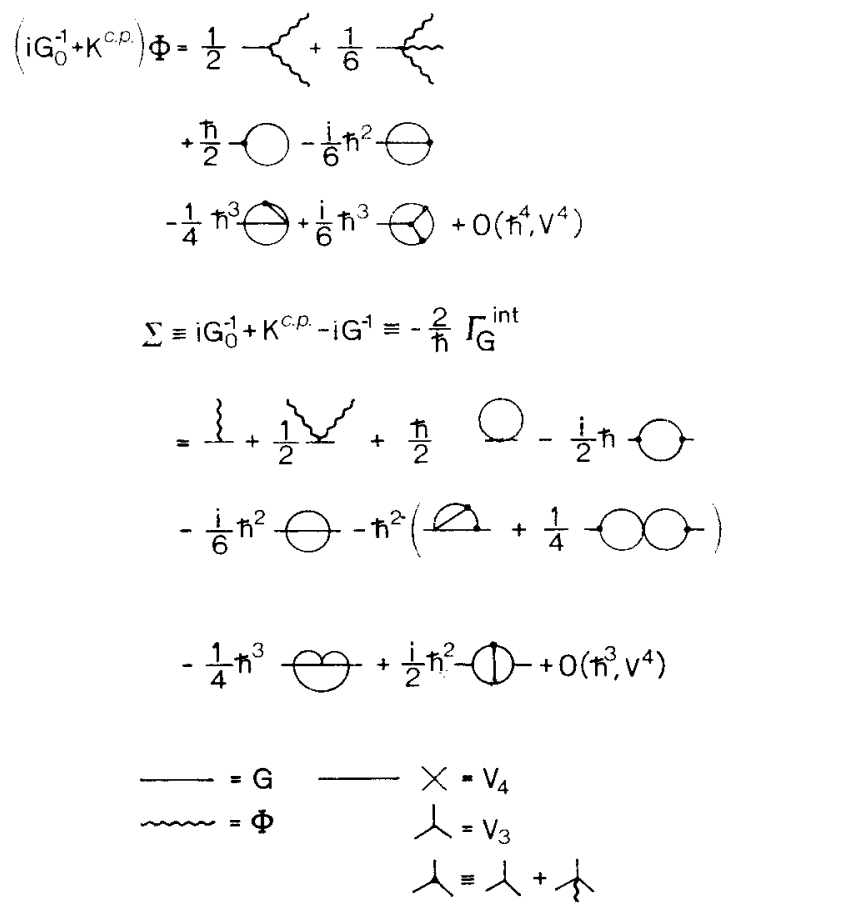

Fig. 6. The equations of motion for field expectation Φ and connected Green's function G as it follows from the effective action $\Gamma[\Phi, G]$ of the second type. The result may be used either up to V^3 or up to $\hbar^3$

this is a property of the approximation. If we would known $\Gamma[\Phi, G]$ exactly, particle number would certainly be conserved also for a finite system.

Notice that the solution of equs. (207), (208), (209) are completely non perturbative in nature since both Φ and G collect an infinite set of powers in V_3, V_4. This happens also if we choose the expansion of $\Gamma[\Phi, G]$ in powers of V as a starting point rather than the loop expansion. Which of the two leads to a better approximation cannot be decided without focussing attention upon a specific physical system. It obviously depends on the relative size of the couplings V_3 and V_4.

It should be pointed out that the whole formalism derived here holds also for fermions if we watch out that the contraction of indices are done in the proper sequence. Only the two traces in Γ^0 appear with a changed sign. This leads to same equations of motion since the source $K(x_1x_2)$ and the Green's function $G(x_1, x_2)$ are antisymmetric in the two indices such that

$$\frac{\delta}{\delta G(x_1x_2)} = \pm \frac{\delta}{\delta G}(x_1x_2) \quad \text{for} \quad \begin{matrix}\text{bosons}\\ \text{fermions}\end{matrix} \tag{212}$$

and

$$\frac{\delta}{\delta G(x_1x_2)}\left(\pm i \operatorname{tr}\log iG^{-1} \pm \frac{i}{2}\operatorname{tr} G_0^{-1}G\right) = (iG_0^{-1} - iG^{-1})(x_1x_2) \tag{213}$$

for both, fermions and bosons while for, example, the one loop correction $-V_4/8\,G^2$ is contracted as

$$-\frac{1}{8} V_4(x_1x_2x_3x_4)\, G(x_1x_2)\, G(x_2x_4)$$

and therefore adds $-(1/2)\ V(x_1x_2)\ G(x_3x_4)$ to (213) in either case. The only difference is that the fermion system cannot have a cubic coupling V_3 nor a non-vanishing field expectation in the ground state such that (207), (210) is absent and (209), (211), reduce to the well known Hartree Fock Bogoljubov equations and their generalization (see Ref. [4]).

VI. The Higher Effective Actions

6.1. Higher Legendre transforms

The formation of a condensate is always accompanied by a whole chain of non-perturbative effects which runs through all higher correlations function. We have already seen that the non-vanishing expectation value $\Phi \equiv \langle\varphi\rangle$ makes cubic and quartic couplings release contributions to linear and quadratic pieces in the action (59) by effectively setting parts of the field operators equal to Φ. At the next level, when minimizing the effective action of second type $\Gamma[\Phi, G]$, the propagator determines a non perturbative distribution of particle density and of pairs in ground state. This distribution has an important feature: It may be of a form which was not present in the original action. For example, if the action has only a particle number conserving quadratic piece $\psi^+\psi$, the ground state solution G may contain non-zero expectations for the composite pair field $\psi\psi$.

It is easy to convince oneself that once such anomalous expectations exist they may generate even more new couplings. Even if the original action contains only particle number conserving vertices $(1/2)\ v_4\psi^+\psi\psi^+\psi$, a non-vanishing anomalous Green's function allows, to second order, for processes in which four particles come together and annihilate, i.e. for a vertex function $\langle T\psi\psi\psi\psi\rangle$. The physical reason is, of course, the presence of a condensate of pairs, which may add and subtract two particles to any interaction. The argument can be continued to clusters of any higher number of particles.

Mechanisms of this type are very important in understanding the properties of nuclei in which four particle correlations may be very strong as a consequence of the high stability of alpha particles.

In order to describe such higher non-perturbative effects we may systematically extend the technique of Legendre transformations and include into the effective action the most important higher connected Green's functions of three and four particles. Then the extremization comprises also these observable quantities and one may expect a significant improvement at the level of approximate calculations. Actually, the higher Green's functions themselves are not the most economic variables to be included into the list of arguments. The reason is that some of the information they carry is trivial. We have seen before in (45) that the connected Green's function $G_c^{(3)}$ contains singularities in the external legs which are those already known from the connected two-point function $G_c^{(2)} = G$. Therefore it is preferable to factor these out and go to the smoother objects, the three point vertex functions $\Gamma_{\Phi\Phi\Phi}$. Since at the zero-loop level $\Gamma_{\Phi\Phi\Phi} = A_{\varphi\varphi\varphi}[\Phi]$ we find

$$\alpha_3 \equiv -\Gamma_{\Phi\Phi\Phi}$$

to be the best variable from which the connected three point function is obtained as (compare (45))

$$G_c^{(3)} = -i\alpha_3 G^3. \tag{214}$$

Similary we shall take the vertex function

$$\alpha_4 \equiv -\Gamma_{\Phi\Phi\Phi\Phi}$$

as the variable characterizing four-particle correlations. According to (84) the connected four point function is obtained from this as

$$G_c^{(4)} = -i\alpha_4 G^4 - (G^2\alpha_3 G\alpha_3 G^2 + 2 \text{ permutations}). \tag{215}$$

We are now ready to introduce the new effective action $\Gamma[\Phi, G, \alpha_3, \alpha_4]$. For this we add to the original action source terms also for three and four particle vertices.

$$A^{\text{source}} = -\frac{1}{3!} L_3\varphi^3 - \frac{1}{4!} L_4\varphi^4 \tag{216}$$

and define the generating functional

$$Z[j, K_1, L_3, L_4] = e^{iW[j,K,L_3,L_4]} = \langle 0|Te^{i(j\varphi+(1/2)\varphi K\varphi-(1/3!)L_3\varphi^3-(1/4!)L_4\varphi^4)}. \tag{217}$$

The higher effective action is then obtained as

$$\Gamma[\Phi, G, \alpha_3, \alpha_4] = W[j, K, L_3, L_4] - W_j j - W_K K - W_{L_3}L_3 - W_{L_4}L_4. \tag{218}$$

The new arguments α_3, α_4 are simply related to the new derivatives of W. Since $iZ^{-1}Z_{L_{3,4}}$ are obviously the three and four-point Green's functions, we have

$$W_{L_3} = -\frac{1}{3!} G^{(3)} \qquad W_{L_4} = -\frac{1}{4!} G^{(4)}. \tag{219}$$

Using relations (33), (34) the right-hand sides may be decomposed into their connected pieces as

$$W_{L_3} = -\frac{1}{3!}[G_c^{(2)} + (G_c^{(2)}G_c^{(1)} + 2 \text{ permutations}) + G_c^{(1)3}] \tag{220}$$

$$W_{L_4} = -\frac{1}{4!}[G_c^{(3)} + (G_c^{(3)}G_c^{(1)} + 3 \text{ permutations}) + (G_c^{(2)}G_c^{(2)} + 2 \text{ permutations}) + (G_c^{(2)}G_c^{(1)}G_c^{(1)} + 5 \text{ permutations}) + G_c^{(1)4}].$$

With (214) and (215) this leads to the following relations

$$W_{L_3} = -\frac{1}{3!}(-i\alpha_3 G^3 + 3G\Phi + \Phi^3) \tag{221}$$

$$W_{L_4} = -\frac{1}{4!}(-i\alpha_4 G^4 - 3\alpha_3{}^2 G^5 - 4i\alpha_3 G^3\Phi + 3G^2 + 6G\Phi^2 + \Phi^4). \tag{222}$$

By definition, the new effective action $\Gamma[\Phi, G, \alpha_3, \alpha_4]$ satisfies the following equations

$$\Gamma_{\alpha_4} = -\frac{1}{4!} L_3 G^4 \tag{223}$$

$$\Gamma_{\alpha_3} = -\frac{i}{3!} G^3(L_3 + L_4\Phi) - \frac{1}{4}\alpha_3 G^5 L_4 \tag{224}$$

$$\Gamma_G = -\frac{1}{2}K + \frac{1}{2}\Phi\left(L_3 + \frac{1}{2}L_4\Phi\right) + \frac{1}{4}GL_4 - \frac{i}{2}\alpha_3 G^2(L_3 + L_4\Phi) - \frac{i}{6}\alpha_4 G^3 L_4 - \frac{5}{8}\alpha_3{}^2 G^4 L_4 \tag{225}$$

$$\Gamma_\Phi = -j - K\Phi + \frac{1}{2}\Phi^2 L_3 + \frac{1}{6}\Phi^3 L_4 + \frac{1}{2}G(L_3 + L_4\Phi) - \frac{i}{6}\alpha_3 G^3 L_4. \tag{226}$$

Thus $\Gamma[\Phi, G, \alpha_3, \alpha_4]$ is extremal for vanishing external sources and represents the proper higher effective action.
Let us calculate the new Γ by using the previous results for $\Gamma[\Phi, G]$. For this we notice that the external sources L_3, L_4 may simply be added to the potentials V_3, V_4. Therefore at any *fixed* external L_3, L_4 we may directly go over to the effective action of the second type $\Gamma[\Phi, G]$ by replacing, in the final results, V_3, V_4 by

$$\overline{V}_3 = V_3 + L_3, \qquad \overline{V}_4 = V_4 + L_4. \tag{227}$$

Using the same argument as in (162), (164) we may perform the further Legendre transformations immediately on $\Gamma^{L_3 L_4}[\Phi, G]$ and define

$$\Gamma[\Phi, G, \alpha_3, \alpha_4] = \Gamma^{L_3 L_4}[\Phi, G] - \Gamma^{L_3 L_4}_{L_3} \cdot L_3 - \Gamma^{L_3 L_4}_{L_4} \cdot L_4 \tag{228}$$

where

$$\Gamma^{L_3 L_4}_{L_3} = W_{L_3} \qquad \Gamma^{L_3 L_4}_{L_4} = W_{L_4}.$$

The formalism becomes slightly more elegant if we do not use the additional sources $L_{3,4}$ but consider the potentials themselves as non-vanishing external sources. Then all formulas hold with L_i replaced by V_i. This has the superficial disadvantage that the equations (223—226) no longer determine the physical configurations via a proper extremum of Γ, since the right-hand sides have the true potentials V_3, V_4 in place of the vanishing sources L_3, L_4. This is completely analogous to the previous discussion of the chemical potential. It may be included either in iG_0^{-1}, in which case we may set $K = 0$ at the end, or in the external source, in which case $K = K^{\text{c.p.}}$ corresponds to the physical situation. Because of this analogy we shall reserve the symbol $\Gamma^{\text{g.c.}}$ for the true effective action (215) with the extremality properties (223)—(226). The symbol Γ without superscript will be used for the more economic quantity which is given by

$$\Gamma[\Phi, G, \alpha_3, \alpha_4] = \Gamma[\Phi, G] - \Gamma_{V_3}[\Phi, G]\, V_3 - \Gamma_{V_4}[\Phi, G]\, V_4 \tag{229}$$

where α_3, α_4 are related to the derivatives as

$$\Gamma_{V_3}[\Phi, G] = -\frac{1}{3!}\,(-i\alpha_3 G^3 + 3G\Phi + \Phi^3) \tag{230}$$

$$\Gamma_{V_4}[\Phi, G] = -\frac{1}{4!}\,(-i\alpha_4 G^4 - 3\alpha_3{}^2 G^5 - 4i\alpha_3 G^3\Phi + 3G^2 + 6G\Phi^2 + \Phi^4) \tag{231}$$

and the equations of motion are the same as (223)—(226) except with L_3, L_4 replaced by V_3, V_4.
The connection between $\Gamma[\Phi, G, \alpha_3, \alpha_4]$ and $\Gamma^{\text{g.c.}}[\Phi, G, \alpha_3, \alpha_4]$ is trivial as it was before in (169). Since the subtraction (229) is done with the full potential V rather than only the source L the relation (169) extends to the new effective action as

$$\Gamma^{\text{g.c.}}[\Phi, G, \alpha_3, \alpha_4] = \Gamma[\Phi, G, \alpha_3, \alpha_4] + \frac{1}{2}\,(\Phi^2 + G)\, K^{\text{c.p.}} - \frac{1}{3!}\,(-i\alpha_3 G^3 + 3G\Phi + \Phi^2)\, V_3$$
$$- \frac{1}{4!}\,(-i\alpha_4 G^4 - 3\alpha_3{}^2 G^5 - 4i\alpha_3 G^3\Phi + 3G^2 + 6G\Phi^2 + \Phi^4)\, V_4. \tag{232}$$

6.2. Calculation of $\Gamma[\Phi, G, \alpha_3, \alpha_4]$

From defining equation (229) we see that only the interacting part of Γ is affected by the new Legendre transformations such that we may write

$$\Gamma[\Phi, G, \alpha_3, \alpha_4] = \Gamma^0[\Phi, G] + \Gamma^{\text{int}}[\Phi, G, \alpha_3, \alpha_4] \tag{233}$$

with (compare (185))

$$\Gamma^0[\Phi, G] = \frac{1}{2}\Phi i G_0^{-1}\Phi + \frac{i}{2}\hbar \operatorname{tr}\log iG^{-1} + \frac{\hbar}{2}\operatorname{tr}(iG_0^{-1}G)$$

and calculate $\Gamma^{\text{int}}[\Phi, G, \alpha_3, \alpha_4]$ from (229). Using (206) we find

$$\begin{aligned}\Gamma^{\text{int}}_{V_3} = &-\frac{1}{3!}\Phi^3 - \frac{\hbar}{2}G\Phi + \frac{i}{6}\hbar^2(V_3 + V_4\Phi)\,G^3 \\ &+ \frac{1}{4}\hbar^2\,V_4(V_3 + V_4\Phi)\,G^5 - \frac{i}{6}\hbar^3(V_3 + V_4\Phi)^3\,G^6 + O(\hbar^4, V^3)\end{aligned} \tag{234}$$

$$\begin{aligned}\Gamma^{\text{int}}_{V_4} = &-\frac{1}{4!}\Phi^4 - \frac{\hbar}{4}G\Phi^2 - \frac{\hbar^2}{8}G^2 + \frac{i}{6}\hbar^2\Phi(V_3 + V_4\Phi)\,G^3 + \frac{i}{24}\hbar^3 V_4 G^4 \\ &+ \frac{1}{8}\hbar^3(V_3 + V_4\Phi)^2\,G^5 + \frac{1}{4}\hbar^3 V_4\Phi(V_3 + V_4\Phi)\,G^5 + \frac{1}{16}\hbar^4 V_4^2 G^6 \\ &- \frac{i}{6}\hbar^3\Phi(V_3 + V_4\Phi)^3\,G^6 + O(\hbar^4, V^3).\end{aligned} \tag{235}$$

Comparing this with (230), (231) we may identity the vertex functions as (carrying $\hbar$ along also there)

$$\alpha_3 = (V_3 + V_4\Phi) - \frac{3}{2}iV_4(V_3 + V_4\Phi)\,G^2 - \hbar(V_3 + V_4\Phi)^3\,G^3 + O(\hbar^2, V^3) \tag{236}$$

$$\begin{aligned}\alpha_4 - 3i\alpha_3^2 G + 4\alpha_3\Phi G^{-1}/\hbar = {} & V_4 - 3i(V_3 + V_4\Phi)^2\,G - \frac{3}{2}i\hbar V_4^2 G^2 + 4V_3\Phi G^{-1}/\hbar \\ & - GiV_4\Phi(V_3 + V_4\Phi)\,G - 4\Phi(V_3 + V_4\Phi)^3\,G^2 + O(\hbar, V^3)\end{aligned} \tag{237}$$

or, inserting (236),

$$\alpha_4 = V_4 - \frac{3}{2}i\hbar V_4 G^2 V_4 + O(\hbar, V^3).$$

These relations can be inverted as

$$V_3 + V_4\Phi = \alpha_3 + \frac{3}{2}i\hbar\alpha_3\alpha_4 G^2 + \hbar\alpha_3^3 G^3 + O(\hbar^2, \alpha^3) \tag{238}$$

$$V_4 = \alpha_4 + \frac{3}{2}i\hbar\alpha_4^2 G^2 + O(\hbar, \alpha^3). \tag{239}$$

When subtracting the two pieces on the right hand side of (229), the terms linear in the potentials cancel and those of n-th order in the potential multiply by $(1 - n)$. Therefore

we find

$$\Gamma^{\text{int}}[\Phi, G, \alpha_3, \alpha_4] = -\frac{i}{12}\hbar^2(V_3 + V_4\Phi)^2 G^3 - \frac{i}{48}\hbar^3 V_4{}^2 G^4 - \frac{1}{4}\hbar^3 V_4(V_3 + V_4\Phi)^2 G^5 - \frac{1}{24}\hbar^4 V_4{}^3 G^6 + \frac{i}{8}\hbar^3(V_3 + V_4\Phi)^4 G^6 + O(\hbar^4, V^4). \tag{240}$$

Inserting (236), (237) this takes the simple form

$$\Gamma^{\text{int}}[\Phi, G, \alpha_3, \alpha_4] = -\frac{i}{12}\hbar^2\alpha_3{}^2 G^3 - \frac{i}{48}\hbar^3\alpha_4{}^2 G^4 + \frac{1}{48}\hbar^4\alpha_4{}^3 G^6 - \frac{i}{24}\hbar^3\alpha_3{}^4 G^6 + O(\hbar^4, \alpha^4). \tag{241}$$

Notice that there are no more Φ fields in the effective interaction Moreover, since each term comes from a vacuum graph in which all G lines have to end in one of the vertices α_3 or α_4, the number of G's in each term must be equal to $(3n_3 + 4n_4)/2$ or, expressed differently, G, α_3, α_4 can occur only in the combination $\alpha_3 G^{3/2}$, $\alpha_4 G^2$, i.e.

$$\Gamma^{\text{int}}[\Phi, G, \alpha_3, \alpha_4] = f[\alpha_3 G^{1/2}G^{1/2}G^{1/2}, \alpha_4 G^{1/2}G^{1/2}G^{1/2}G^{1/2}].$$

This may be expressed in differential form as

$$G\Gamma_G{}^{\text{int}} = \frac{3}{2}\alpha_3\Gamma^{\text{int}}_{\alpha_3} + 2\alpha_4\Gamma^{\text{int}}_{\alpha_4} \tag{242}$$

which will be of use later.

6.3. The equations of motion

Let us now use the new effective action and calculate the equations of motion (223) to (226) at $j = 0$, $K = K^{\text{c.p.}}$ and with L replaced by V. Since there are no Φ fields in the interaction, equ. (226) becomes directly (including the powers of $\hbar$)

$$(iG_0{}^{-1} + K^{\text{c.p.}})\,\Phi = \frac{1}{2}V_3\Phi^2 + \frac{1}{6}V_4\Phi^3 + \frac{\hbar}{2}(V_3 + V_4\Phi)\,G - \frac{i}{6}\hbar^2 V_4 G^3\alpha_3. \tag{243}$$

Notice that due to the absence of Φ in $\Gamma^{\text{int}}[\Phi, G, \alpha_3, \alpha_4]$ this equation has only a finite number of terms. All radiative corrections are absorbed in the exact propagator G and vertex α_3. This is a great advantage over the previous equation (207) which followed from the effective action of the second type $\Gamma[\Phi, G]$. A similar advantage arises in the equation for the self-energy $\Sigma \equiv iG_0{}^{-1} + K^{\text{c.p.}} - iG^{-1}$. According to the equation of motion (225) this is given by

$$\Sigma = V_3\Phi + \frac{1}{2}V_4\Phi^2 + \frac{\hbar}{2}V_4 G - i\hbar\alpha_3 G^2(V_3 + V_4\Phi) - \frac{i}{3}\hbar^2\alpha_4 G^2 V_4 - \frac{5}{4}\hbar^2\alpha_3{}^2 G^4 V_4 - \frac{2}{\hbar}\Gamma_G{}^{\text{int}} \tag{244}$$

and involves, at first the whole infinite series arising from the derivative of (241). Due to the homogeneity equation (242), however, this simplifies since the derivatives $\Gamma^{\text{int}}_{\alpha_3}$, $\Gamma^{\text{int}}_{\alpha_4}$ are directly related to the potentials via the other two equations of motion (223), (224)

for the vertices, i.e.

$$G\Gamma_G^{\text{int}} = -\frac{i}{4}\hbar^2\alpha_3 G^3(V_3 + V_4\Phi) - \frac{3}{8}\hbar^3\alpha_3^2 G^5 V_4 - \frac{i}{12}\hbar^3\alpha_4 G^4 V_4. \tag{245}$$

Thanks to this circumstance, the self-energy takes the short form

$$\Sigma = V_3\Phi + \frac{1}{2}V_4\Phi^2 + \frac{\hbar}{2}V_4 G - \frac{i}{2}\hbar\alpha_3 G^2(V_3 + V_4\Phi) - \frac{i}{6}\hbar^2\alpha_4 G^4 V_4 - \frac{1}{2}\hbar^2\alpha_3^2 G^5 V_4. \tag{246}$$

Certainly, the full complexity of the expansion (241) for Γ^{int} is now carried by the equations for the vertices whose first terms are given by

$$V_4 = 4!\, i\hbar^{-3}G^{-4}\Gamma^{\text{int}}_{\alpha_4} = \alpha_4 + \frac{3}{2}i\hbar\alpha_4^2 G^2 + O(\hbar, \alpha^3) \tag{247}$$

$$V_3 + V_4\Phi - \frac{3}{2}i\hbar\alpha_3 G^2 V_4 = 3!\, i\hbar^{-2}G^{-3}\Gamma^{\text{int}}_{\alpha_3} = \alpha_3 + \hbar\alpha_3^3 G^3 + O(\hbar^2, \alpha^3) \tag{248}$$

and which coincide to this order in $\hbar$ or α with (238), (239), as they should. We have displayed the final result graphically in Fig. 7.

The important progress in these equations lies in the possibility of generating vertex functions non-perturbatively. In fact, there may be solutions of (247), (248) in channels

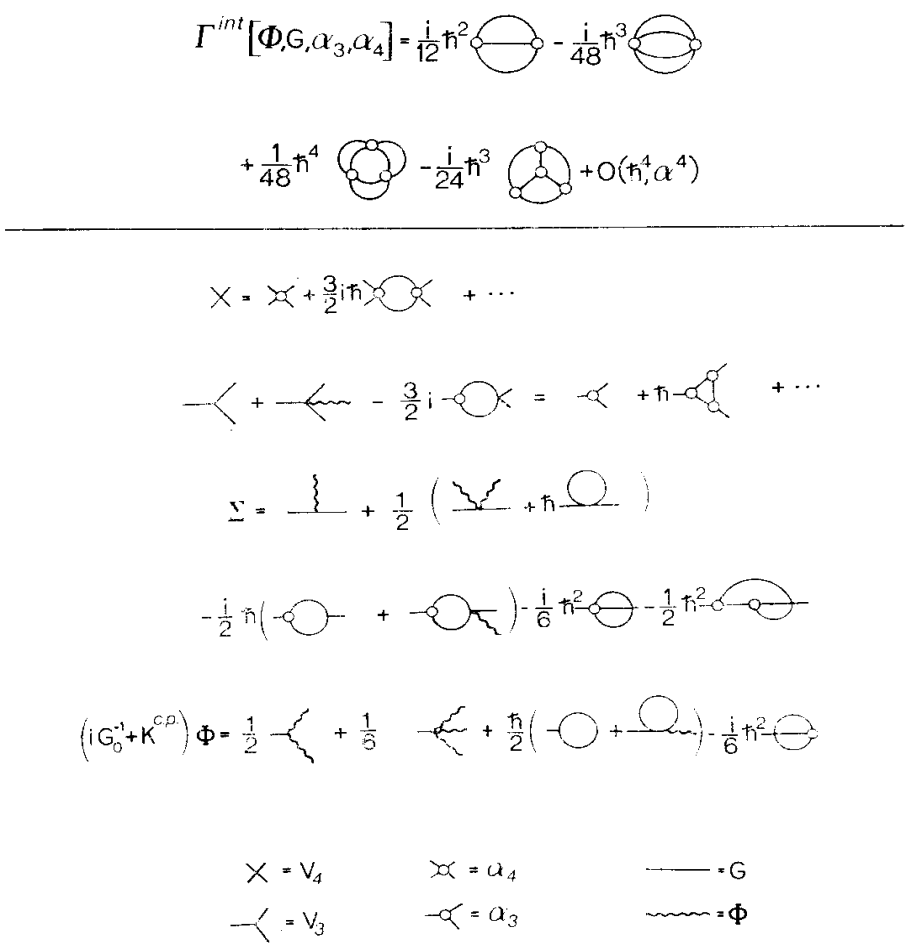

Fig. 7. The expansion of the effective interaction action of the fourth type and the resulting equations of motion (compare (243)–(248)). One may take either all terms up to $\hbar^3$ or up to α^3 depending on the relative size of α_3 and α_4

in which there is no coupling in the original action. We mentioned before that the presence of anomalous Green's functions gives automatically rise to non-vanishing expectations of four-particle correlation functions. Equ. (247) makes this statement quantitative. In the four-particle channel, $V_4 = 0$ such that the vertex function satisfies the homogeneous equation

$$\alpha_4 + \frac{3}{2} i\alpha_4{}^2 G + O(\hbar, \alpha^3)\Big|_{\psi^4\text{part}} = 0. \tag{249}$$

Similarly equ. (248) gives rise to three point vertices even if there is no potential V_3 at all. If there is a non-trivial solution for (247) or (248) with V_3, $V_4 = 0$ we may speak of a spontaneous generation of vertices via fluctuations.

This is completely analogous to the spontaneous generation of a pair correlation $\langle T\psi\psi\rangle$ in the self-consistent Hartree-Fock-Bogoljubov equation (211). The $\psi\psi$ part of (211) is referred to as gap equation. Correspondingly we might call (247), (248) "gap equations" for vertices.

We have said before that a non-vanishing gap $\langle T\psi\psi\rangle$ corresponds to formation of a condensate of Cooper pairs. In complete analogy, the new vertex functions α_3, α_4 signalize the presence of a condensate of three- and four-particle clusters in the many-body system. The latter are of particular importance in nuclear physics where the forces show strong attraction in four particle channels. (For a model discussion see the last two of Refs. [4]).

6.4. Singular potentials

In many realistic problems, the truncated expressions (247), (248) would lead to bad approximations. Standard intermolecular potentials increase rapidly for distances shorter than a few Angström and may be idealized by a hard core. Similarly, between charged particles there are Coolumb forces which are infinite at zero moment transfer k. In both cases, V is singular once in real and once in momentum space. The full interaction vertices, however, are rather smooth objects in these singular regions: For the hard core potential we know from the non-relativistic Schrödinger theory that the scattering amplitude is well behaved. For charged particles, the infinite range Coulomb interactions is screened after a finite distance (the Debeye length) and the scattering amplitude has no longer a $k = 0$ singularity.

This fact can only be accommodated by calculating α_3, α_4 to be inserted into the equations for Φ and Σ not from (247), (248) but finding a better approximation which sums up infinitely many diagrams contained in Γ^{int} in such a way that the vertices come out non-singular. Since the infinitely many diagrams in Γ^{int} become rapidly more and more involved, the most convenient procedure consists in deriving integral equations for the vertices in terms of the potentials which then may be approximated in an convenient fashion.

In order to do so we observe that the three-point vertex α_3 may be obtained from $W[j, K]$ as

$$2iW_{jK} = W_{jjj} + 2iW_{jj}W_j = -G_c{}^{(3)} - 2G\Phi \tag{250}$$

$$= i\alpha_3 G^3 - 2G\Phi. \tag{251}$$

Using equ. (178) we have

$$W_{Kj} = -\frac{i}{2} G(\Gamma_{\Phi G}\Gamma_{GG}^{-1} - 2\Phi) \tag{252}$$

such that α_3 can be obtained from $\Gamma[\Phi, G]$ as

$$i\alpha_3 G^3 = G\Gamma_{\Phi G}\Gamma_{GG}^{-1}. \tag{253}$$

Inserting here the separation between free and interacting parts (184) we see that

$$\alpha_3 = -2\Gamma^{\rm int}_{\Phi G}(1 - 2iGG\Gamma^{\rm int}_{GG})^{-1}. \tag{254}$$

Therefore the three-point vertex α_3 satisfies the integral equation

$$\alpha_3 = -2\Gamma^{\rm int}_{\Phi G} + 2iGG\Gamma^{\rm int}_{GG}\alpha_3. \tag{255}$$

Notice that this result can be employed to rewrite the equation of motion (189) as

$$\Phi\Gamma_{\Phi}{}^{\rm int} = i\frac{V_4}{3!}\Phi G^3\alpha_3 - \frac{1}{2}V_3\Phi^3 - \frac{1}{3!}V_4\Phi^4 - \frac{1}{2}V_3G\Phi - \frac{1}{2}V_4G\Phi^2. \tag{256}$$

This may be expressed in another way by separating $\Gamma^{\rm int}$ as

$$\Gamma^{\rm int}[\Phi, G] = -\frac{V_3}{3!}\Phi^3 - \frac{V_4}{4!}\Phi^4 - \frac{1}{2}V_3G\Phi - \frac{1}{4}V_4G\Phi^2 - \frac{1}{8}V_4G^2 + \bar{\Gamma}[\Phi, G]. \tag{257}$$

In other words, $\bar{\Gamma}$ is meant to collect all terms after the 6 th in equ. (206). This tail piece of the effective action is very simply related to the vertex α_3 by

$$\bar{\Gamma}_{\Phi}[\Phi, G] = \frac{i}{3!}V_4G^3\alpha_3. \tag{258}$$

Let us compare this equation with the definition of α_3 via (230). Inserting (257) we find

$$\bar{\Gamma}_{V_3}[\Phi, G] = \frac{i}{3!}G^3\alpha_3. \tag{259}$$

Thus we conclude that $\bar{\Gamma}[\Phi, G]$ can depend on Φ only via the combination $V_3 + V_4\Phi$, i.e. $\bar{\Gamma}[\Phi, G]$ must be an explicit functional only of the vertices V_4, $V_3 + V_4\Phi$ and G. Moreover, since the number of G lines has to be equal to twice the number of V_4 and 3/2 times the number of $V_3 + V_4\Phi$ vertices, the functional $\bar{\Gamma}$ must have the form

$$\bar{\Gamma}[\Phi, G] = F[V_4G^2, (V_3 + V_4\Phi)G^{3/2}] \tag{260}$$

which is indeed true (see (206) and the diagramatic rules for its construction).
Consider now the four-point vertex α_4. Let us use W_{KK} for its calculation. From (148) we derive

$$-4iW_{KK} = -2W_{Kjj} - 4iW_{jK}W_j. \tag{261}$$

From (149) we may rewrite

$$W_{Kjj} = \frac{1}{2i}W_{jjjj} + W_jW_{jjj} + W^2_{jj} = \frac{1}{2i}i^3G_c{}^{(4)} + G^2 + 2i\Phi(W_{Kj} - iG\Phi) \tag{262}$$

such that W_{KK} satisfies the identity

$$W_{KK} = \frac{i}{4}(G_c{}^{(4)} + 2G^2 - 4G\Phi^2) + 2W_{Kj}\Phi \tag{263}$$

which we shall record for a moment in the form

$$W_{KK} - W_{Kj}\Phi = \frac{i}{4}(G_c{}^{(4)} + 2G^2 - 4G\Phi^2) + W_{Kj}\Phi. \tag{264}$$

The left hand side may be evaluated in another way by considering one of the two unused components of the matrix equation

$$W_{j_i j_j} \Gamma^{\text{aux}}_{\Phi_j \Phi_k} = -\delta_{ik}$$

(compare (174), (175)), namely

$$W_{KK} \Gamma^{\text{aux}}_{\Phi_2 \Phi_2} + W_{Kj} \Gamma^{\text{aux}}_{\Phi \Phi_2} = -1 \tag{265}$$

which may be rewritten in terms of Γ as

$$W_{KK} - W_{Kj}\Phi = -\frac{1}{2} W_{Kj} \Gamma_{\Phi G} \Gamma^{-1}_{GG} - \frac{1}{4} \Gamma^{-1}_{GG}. \tag{266}$$

But we have just found out (253) such that

$$W_{KK} - W_{Kj}\Phi = -\frac{i}{2} W_{Kj} \alpha_3 G^2 - \frac{1}{4} \Gamma^{-1}_{GG}. \tag{267}$$

Equating this with (264) gives

$$\frac{i}{4} (G_c{}^{(4)} + 2G^2 - 4G\Phi^2) + W_{Kj}\Phi = -\frac{i}{2} W_{Kj} \alpha_3 G^2 - \frac{1}{4} \Gamma^{-1}_{GG}. \tag{268}$$

Now we insert (149) in the form recalling (27) and (214))

$$W_{Kj} = \frac{1}{2} \alpha_3 G^3 + iG\Phi \tag{269}$$

and use the relation

$$\Gamma^{-1}_{GG} = \left(\frac{i}{2} G^{-1} G^{-1} + \Gamma^{\text{int}}_{GG} \right)^{-1} \tag{270}$$

$$= \frac{2}{i} GG(1 - 2iGG\Gamma^{\text{int}}_{GG})^{-1} = \frac{2}{i} G^2 + 2iG^4 \Gamma^{\text{int}}_{GG} (1 - 2iGG\Gamma^{\text{int}}_{GG})^{-1}. \tag{271}$$

In order to abbreviate the notation it is useful to introduce the repeated Γ^{int}_{GG} interaction as an auxiliary vertex

$$t = -4\Gamma^{\text{int}}_{GG} (1 - 2iGG\Gamma^{\text{int}}_{GG})^{-1}. \tag{272}$$

In other words, t solves the integral equation

$$t = -4\Gamma^{\text{int}}_{GG} + 2i\Gamma^{\text{int}}_{GG} GGt. \tag{273}$$

Then

$$-\frac{1}{4} \Gamma^{-1}_{GG} = \frac{i}{2} GG + \frac{1}{4} G^4 t. \tag{274}$$

Inserting this into equ. (267) and using the decomposition (215) for $G_c{}^{(4)}$ we have

$$\frac{i}{4} (-i\alpha_4 G^4 - 3\alpha_3{}^2 G^5 - 4G\Phi^2) + \left(\frac{1}{2} \alpha_3 G^3 + iG\Phi \right) \Phi$$

$$= -\frac{i}{2} \left(\frac{1}{2} \alpha_3 G^3 + iG\Phi \right) \alpha_3 G^2 + \frac{1}{4} G^4 t \tag{275}$$

such that the four-point vertex α_4 may be obtained from t as

$$\alpha_4 = t + 2i\alpha_3 G\alpha_3 . \tag{276}$$

When calculating this expression we have to watch out for the proper index contractions. The term $2i\alpha_3 G^2\alpha_3$ has to be contracted with the propagator G in what we usually called t and u channels of the scattering amplitude. We can convince ourselves at low order in pertubation theory that the infinite sum (272)

$$t = -4\Gamma^{\rm int}_{GG}\left(1 + 2iG^2\Gamma^{\rm int}_{GG} - 4(G^2\Gamma^{\rm int}_{GG})^2 + \cdots\right) \tag{277}$$

consists of a one-particle irreducible piece which is symmetric in all four legs and equals α_4 and another one which falls apart by cutting a single line in the t or u channel. These channels are defined as follows: Remember that matrix multiplications in (277) is proceeds in such a way that $\Gamma^{\rm int}_{G_{12}G_{34}}$ is a matrix with a left index pair (12) and right one (34). Therefore $t_{12,34}$ is a matrix of such a type. If cutting one line separates the lines (12) from (34) we speak of an s-channel graph, while those separating (13) and (24) or (14) and (23) are called t and u channel graphs, respectively. The first terms of the sum are illustrated in Fig. 8. Notice that in theory with $\alpha_3 = 0$ (as is the case for the equilibrium solution of Fermi systems) the t matrix itself is one-particle irreducible and completely symmetric in all four lines even though the chain-like summation (277) gives apparently a preference to the particle index pairs (12), (34).
Given t we can calculate α_3 from

$$\alpha_3 = -2\Gamma^{\rm int}_{\Phi G}\left(1 - \frac{i}{2}\,GGt\right) \tag{278}$$

and α_4 from (276).

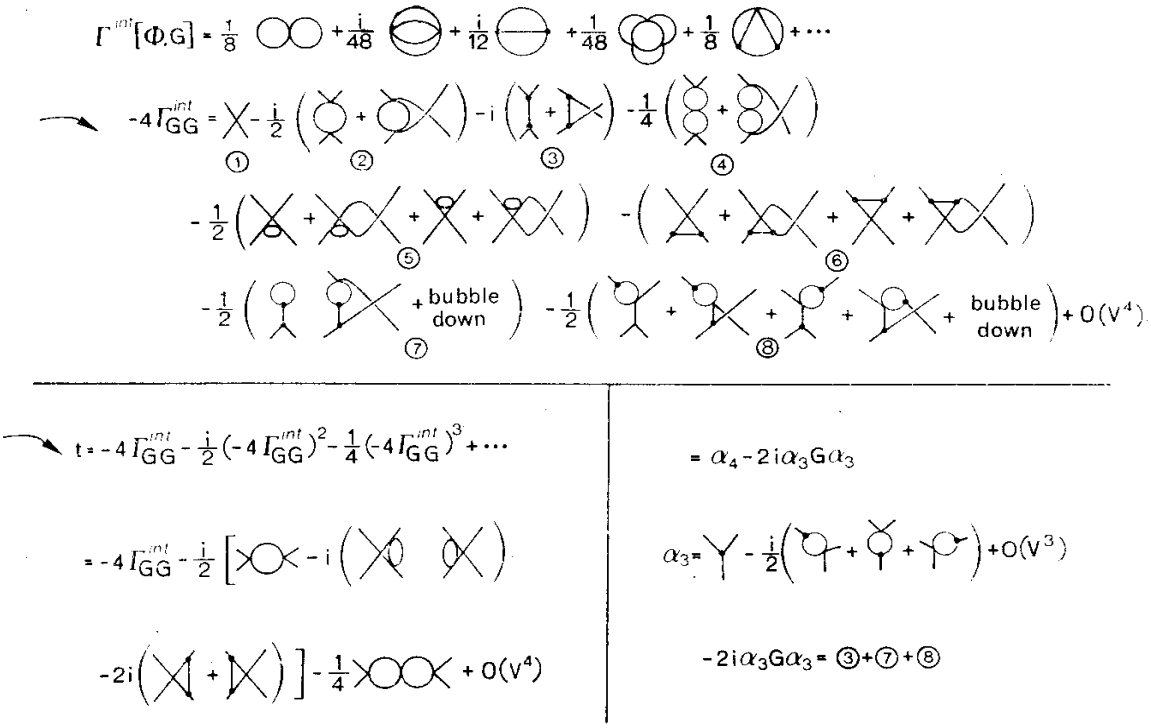

Fig. 8. The t and u channel one-particle reducible terms in the t matrix are removed by adding the term $2i\alpha_3 G\alpha_3$. Notice that up to cubic order, the quadratic piece $(\Gamma^{\rm int}_{GG})^2$ supplies those terms which make (2), (5), and (6)) completely symmetric in all four legs, while $(\Gamma^{\rm int}_{GG})^3$ does so for the graphs (4). A similar mechanism can be verified to any order.

The simplest approximation which can now be done is to take the lowest order form of $\Gamma^{\text{int}}[\Phi, G] \sim -1/8 V_4 G^2$ and insert it into (272). This gives the auxiliary vertex

$$t = V_4 \left(1 + \frac{i}{2} GGV_4\right)^{-1} \tag{279}$$

which sums up an infinite chain of bubble diagrams. Physically this accounts for the repeated scattering among two particles in the system before undergoing a process in which more particles interact simultaneously. We have remarked before that the summation gives preference to one channel. There are usually two possibilities and the selection has to be made on the basis of physical reasoning. For example, hard core and Coulomb singularities in V require the repetition of bubbles in the particle-particle and particle-hole channels, respectively. Up to now there exists no general criterium for the best choice which is still a matter of experience.

Given such a t matrix, we can find the three-particle vertex α_3 from (278). Taking also here the lowest order expression for $\Gamma^{\text{int}}_{\Phi G}$ we may use

$$\alpha_3 = V_3 \left(1 + \frac{i}{2} \hbar G^2 V_4\right)^{-1} = V_3 \left(1 - \frac{i}{2} \hbar G^2 V_4 - \cdots\right). \tag{280}$$

The vertices α_3 and $\alpha_4 = t + 2i\alpha_3 G\alpha_3$ may be inserted into (243) and (246) for a solution of the equations for field Φ and self-energy Σ.

It must be kept in mind that the vertices α_3, α_4 obtained from such an approximation are no longer symmetric in the external legs but exhibit a preferred channel.

VII. Conclusion

The structural framework presented in this paper can serve as a basis for an understanding of a hierarchy of physical phenomena. The equations for the ground state of a Bose system can systematically be extended to include multiparticle correlations. The gap-like equations for the vertices collect an infinite number of potential exchanges and can give rise to couplings which are not included in the original action. In particular, hard core potentials present no more difficulties and can be treated in a straightforward fashion.

The decomposition of the full Green's function $G^{(2)} = G_c^{(2)} + \Phi\Phi$ into connected and disconnected parts yields directly the two fluid description of the Bose systems with $G_c^{(l)}$ containing the normal and Φ^2 the superfluid densities. If there are condensates of pairs or higher bosonic multiparticle clusters, these will give additional contributions to the total current and the liquid requires a multi-fluid description for its proper understanding. The dynamical equations follow from the present approach.

For a description of small-amplitude collective excitations, the action $\Gamma[\Phi, G, \alpha_3, \alpha_4]$ can be expanded quadratically in all its arguments around the ground state solution and then extremized. Then not only normal and superfluid components oscillate but also the vertex functions. In Fermi systems this has already been observed and given rise to a significant extension [*8*] of Landau's theory of Fermi liquids [*9*]. It will be interesting to set up and study the analogous equations also for bosons.

Especially interesting will be the study of large amplitude collective equations which has already become fashionable in nuclear physics. Here the action provides the direct tool for quantizing any periodic time-dependent extremals olution [*10*]. Also for tunneling phenomena, $\Gamma[\Phi, G, \alpha_3, \alpha_4]$ is an important quantity since the action of solutions along the imaginary time axis measures directly the logarithm of the penetration amplitude [*11*]. All these aspects will be studied in future work.

Appendix

Calculation of $\Gamma[\Phi, G]$

If we split the effective action

$$\Gamma[\Phi, G] = \Gamma^0[\Phi, G] + \Gamma^{\text{int}}[\Phi, G] \tag{A.1}$$

according to (compare (257))

$$\Gamma[\Phi, G] = \Gamma^0[\Phi, G] - \frac{1}{3!} V_3\Phi^3 - \frac{1}{4!} V_4\Phi^4 - \frac{1}{2} V_3 G\Phi - \frac{1}{4} V_4 G\Phi^2 - \frac{1}{8} V_4 G^2 + \bar{\Gamma}[\Phi, G] \tag{A.2}$$

then we find from the equ. (189) (compare (256), (258)):

$$\bar{\Gamma}_\Phi = \frac{i}{3!} V_4 \alpha_3 G^3 \tag{A.3}$$

and from (190)

$$G\bar{\Gamma}_G = \frac{1}{2} G\bar{\Gamma}_{\Phi\Phi} - \frac{i}{2} G^3 \left[\bar{\Gamma}_{\Phi G} - \frac{1}{2}(V_3 + V_4\Phi)\right] \alpha_3 \tag{A.4}$$

where

$$\alpha_3 = -2\Gamma^{\text{int}}_{\Phi G}(1 - 2iG^2\Gamma^{\text{int}}_{GG})^{-1}$$

satisfies the integral equation

$$\alpha_3 = -2\Gamma^{\text{int}}_{\Phi G} + 2iG^2\Gamma^{\text{int}}_{GG}\alpha_3 = V_3 + V_4\Phi - 2\bar{\Gamma}_{\Phi G} - \frac{i}{2} V_4 G^2 \alpha_3 + 2iG^2\bar{\Gamma}_{GG}\alpha_3 . \tag{A.5}$$

In equs. (250)—(254) this is shown to be the three point vertex function. Its definition is given in (221) which becomes in terms of $\bar{\Gamma}$

$$\bar{\Gamma}_{V_3} = \frac{i}{3!} \alpha_3 G^3 . \tag{A.6}$$

Comparing this with equ. (A.3) we conclude that $\bar{\Gamma}[\Phi, G]$ contains the field Φ only in the functional combination

$$\bar{\Gamma}[\Phi, G] = F[G, V_3 + V_4\Phi, V_4] \tag{A.7}$$

in agreement with the expansion (206). We may now use the equations (A.3) to (A.6) to calculate the functional $\bar{\Gamma}$. The algebra simplifies by observing that the number of G lines in each vacuum graph is determined by the number of $V_3 + V_4\Phi$ and V_4 vertices n_3 and n_4 as

$$(4n_4 + 3n_3)/2$$

such that $\bar{\Gamma}$ is really only a functional of the reduced variables

$$x_3 = (V_3 + V_4\Phi)\, G^{3/2}, \qquad x_4 = V_4 G^2 \tag{A.8}$$

i.e.

$$\bar{\Gamma}[\Phi, G] = f(x_3, x_4), \qquad \alpha_3 = -3!\, iGf_{x_3} \tag{A.9}$$

such that (A.6) becomes

$$\alpha_3 = -3!\, iG^{-3/2} f_{x_3}. \tag{A.10}$$

Now, using

$$G\bar{\Gamma}_G = \frac{3}{2} x_3 f_{x_3} + 2x_4 f_{x_4} \tag{A.11}$$

$$G^2\bar{\Gamma}_{GG} = \frac{9}{4} x_3{}^2 f_{x_3x_3} + 6x_3x_4 f_{x_3x_4} + 4x_4{}^2 f_{x_4x_4} + \frac{3}{4} x_3 f_{x_3} + 2x_4 f_{x_4} \tag{A.12}$$

$$G\bar{\Gamma}_{\varphi\Phi} = GV_4{}^2\bar{\Gamma}_{V_3V_3} = x_4{}^2 f_{x_3x_3} \tag{A.13}$$

$$G^{3/2}\bar{\Gamma}_{\Phi G} = G^{3/2}V_4\bar{\Gamma}_{V_3G} = \frac{3}{2} x_3x_4 f_{x_3x_3} + 2x_4{}^2 f_{x_3x_4} + \frac{3}{2} x_4 f_{x_3} \tag{A.14}$$

equs. (A.5), (A.3) may be written as

$$f_{x_3} = \frac{i}{3!}\left\{1 + \frac{i}{2} x_4 - i\left(\frac{3}{2} x_3 f_{x_3} + x_4 f_{x_4} + \frac{9}{2} x_3{}^2 f_{x_3x_3} + 3x_3x_4 f_{x_3x_4} + 2x_4{}^2 f_{x_4x_4}\right)\right\}^{-1}$$
$$\times (x_3 - 3x_3x_4 f_{x_3x_4} - 4x_4{}^2 f_{x_3x_4} - 3x_4 f_{x_3}) \tag{A.15}$$

$$f_{x_4} = \frac{1}{4} x_4 f_{x_3x_3} - \frac{3}{2}\left\{\frac{3}{2} x_3 f_{x_3x_3} + 2x_4 f_{x_3x_4} + \frac{3}{2} f_{x_3}\right\} f_{x_3}. \tag{A.16}$$

The iteration may start with $h \equiv 0$ such that the lowest correction is obtained from (A.15) as

$$f = \frac{i}{12} x_3{}^2 + X(x_4) \tag{A.17}$$

where $X(x_4)$ is an arbitrary functional of x_4. Inserting this into (A.16) we find

$$f_{x_4} = \frac{i}{24} x_4 + O(x^2) \tag{A.18}$$

yielding the quadratic expression

$$f = \frac{i}{12} x_3{}^2 + \frac{i}{48} x_4{}^2 + O(x^3) \tag{A.19}$$

which may once more be iterated through (A.15), (A.16) to find the cubic terms.

$$\frac{1}{8} x_3{}^2 x_4 + \frac{1}{48} x_4{}^3 \tag{A.20}$$

and so on. The resulting series is in agreement with (206). The explicit power of $\hbar$ in a term $x_3{}^{n_3} x_4{}^{n_4}$ is

$$\hbar\left(\frac{1}{\hbar}\right)^{n_3} \hbar^{(3/2)n_3}\left(\frac{1}{\hbar}\right)^{n_4} \hbar^{2n_4} = \hbar^{(1/2)n_3 + n_4 + 1}. \tag{A.21}$$

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