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Quasiclassical Approach to Collective Nuclear Phenomena

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Abstract

We point out that large-amplitude collective excitations, tunneling phenomena, condensation of higher particle clusters, and excitation spectra beyond Landau's theory of Fermi liquid are most conveniently described in terms of successive effective actions. These are functionals depending explicitly on two-particle, four-particle, etc. correlations which have a simple quasi-classical expansion. Their extrema account for the above described phenomena. Contrary to the path integral approach to collective phenomena, the lowest approximation contains exchange and pairing effects and is therefore suited for systems in which the time-dependent Hartree-Fock-Bogoljubov equations are required for a proper understanding of the phenomena.

I. Introduction

The atomic nucleus may be seen as a result of an involved hierarchy of condensation processes: The self consistent interactions between particles and holes leads to the formation of a localized state of non-zero density. Paring forces generate a condensate of Cooper pairs. Moreover, four and more particles correlate strongly and may be considered as sub-clusters of particles or other small nuclei within the system. The final outcome is an object which at long distances and low energy looks rather classical and obeys the laws of hydrodynamics. When probed at shorter and shorter distances and higher energies, however, it reveals more and more quantum structure.

The goal of nuclear theory is to start out with the fundamental Hamiltonian and understand this hirarchy of phenomena using n-body quantum mechanics [*1*]. Due to the many degrees of freedom, the quantum mechanical problem can be well defined only perturbatively which amounts to a power series expansion in the strength of the potential V. The effects described above are, however, strongly non-perturbative in nature. It is gratifying to know that, still, each class in the above hierarchy can be understood by the leading role of a infinite subset of diagrams. For example, density and pair correlations are dominated by infinite sums of ring and ladder diagrams, respectively. Corresponding dominant subsets have been found also for higher clusters.

Once subsummations are performed and a single effect has been roughly explained, the question arises of how to go systematically beyond the lowest approximation in such a way that eventually the true answer may be recovered. The main problem arising is that of avoiding the double-counting of diagrams.

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There does exist a powerful method which permits systematizing such resummations. It is based on the path integral formulation of quantum field theory. The original fluctuating quantum fields are eliminated in favor of composite fields by changes of integration variables. In this way, the theory is reformulated completely in terms of new fluctuating collective quantum fields. This has led to successful quantitative descriptions of important many-body-systems such as superconductors and superfluid ^{3}He [2].
When it comes to studying nuclear phenomena, however, this method runs into severe difficulties. The reason lies in a too restrictive selection of the diagrams to be summed to lowest approximation. When introducing a collective field variable, a choice must be made once and for all whether one wants to resum, in two by two scattering amplitudes, a string of interactions in the direct or the two crossed channels. As long as one of them is dominant, this selection can be made without difficulty. For example, if the phenomena are principally caused by pairing forces, the method works perfectly and this is the reason for its success in the above mentioned areas of physics.
In nuclear physics, however, forces are usually of comparable importance in each of the three channel unless there is a high degree of degeneracy in the outer shells in which case exchange forces may be negligible as far as collective excitations of the valence nucleons are concerned. This is the reason why the author, when proposing the use of collective quantum fields for nuclear collective excitations [3], restricted himself to a degenerate shell model with pairing forces for illustration.
The question arises as to how one can find a method in which the crossed channel forces are properly included from the outset, while preserving all the attractive features of collective fields.

In order to answer this question, we must go back and remember what these were:

1) Collective quantum fields are Bose fields which can account for all dynamical properties of fermionic many-body problems. To lowest approximation, they obey classical equations of motion which amount to time dependent Hartree equations. There exist well-known self-consistency methods for their solution.
2) The path integral may be used to study small fluctuations around such solutions. The resulting spectra give the energies seen in small amplitude collective excitations. They correspond to the random phase approximation.
3) These excitations render radiative corrections to the Hartree result. Moreover, there exists an expansion of the path integral into powers of the fluctuation size which allows for higher radiative corrections. In particular, the forces in the crossed channels can be included in this way. The correction procedure can be formulated in terms of Feynman diagrams involving the propagator and vertices of the collective field only, with no further reference to the original fermions.
4) The path integral supplies an action along any periodic solution of the Hartree equation. These can therefore be quantized approximately [4] with methods which are by now standard in relativistic field theories [5]. Sometimes this is called semiclassical quantization which is slightly incorrect since the fluctuation expansion around the Hartree solution is not one in powers of $\hbar$. Rather: If the fermions had N degenerate energy levels, it would be one in powers of $1/N$.
5) The action yields tunneling amplitudes for metastable states by evaluating solutions of the Hartree equation which connect different minima along the imaginary time axis. These methods are well-known in statistical mechanics and have proven successful in problems such as the decay rate of a superconductive wire [6]. They have recently been applied in quantum field theories: for determining the ground state of quantum chromodynamics or the singularity structure in the complex coupling constant plane [7].

It is the purpose of this paper to point out that there is an alternative approach to collective nuclear phenomena which shares all attractive features of the collective quantum fields: It allows for a reformulation of the fermionic many-body problem in terms of bosonic observable quantities. There is an action whose extrema lead to equations of motion. This may be used to study and quantize large amplitude collective excitations and determine decay rates for tunneling phenomena. Small oscillations around extremal solutions give experimentally observable energy spectra. Thus there will be no problem in reproducing all results which have been obtained in previous discussions using collective quantum fields. In addition, the alternative approach has decisive advantages which make it far more powerful when it comes to problems of nuclear physics:

1) There is no preference of individual channels when summing strings of diagrams at the lowest approximation to scattering amplitudes. All direct exchange and pairing forces are included. Thus not the Hartree but the Hartree-Fock-Bogoljubov equations form the basis of all calculations and subsequent corrections.
2) While collective quantum fields can describe economically only one specific condensation process involving either particle-particle or particle-hole channels, the alternative approach to be desdribed allows for a systematic understanding of condensation processes involving also higher clusters of any number of particles. With the importance of four particle α-like clusters in nuclei this represents an important progress.

The organization of diagrams accounting for such condensation processes does not proceed, as for collective fields, according to powers of $1/N$ with N being the level degeneracy but in the same way as the experimental investigation: From the global hydrodynamic type of behaviour to the involved quantum phenomena. We shall see that the proper theoretical measure corresponding to this is the number of exact fermion loops in a graphical expansion in which lines and vertices represent the fully interacting propagators and scattering amplitudes, respectively. At the two-loop level, a droplet of nucleons can be formed which contains a condensate of Cooper pairs. Higher clusters arise at higher loop levels.

The alternative approach to be presented here is based on a theoretical quantity which provides the perfect tool for such an expansion: The effective action Γ. This is a functional of only observable quantities which are directly the two-point, four-point and higher correlation functions of the system. It shares an important property with standard classical actions: Physically admissable configurations move along the extrema of Γ. Therefore it is the most natural quantity for extracting hydrodynamic laws of motion. But, in addition, it contains all quantum aspects of the system and they can all be extracted from Γ in a simple and direct fashion. This is why it deserves the name effective action. Effective actions of the type to be used here have been introduced into many-body physics a long time ago and recurred in simplified versions in relativistic quantum field theory [8]. Since some of the original papers are quite high-brow and possibly difficult to read, it may be useful to present another derivation of the main results, not with the claim of doing any better, but with the hope that a different point of view may provide a better understanding for some readers. We shall first introduce the exact effective action and display its quantum dynamical content. Then we present the loop expansion and show how standard self-consistency formalisms arise at the lowest level of approximation.

The power of the higher effective action is demonstrated by giving:

1) Equations for large Amplitude Collective excitations and their corrections.
2) A prescription on how to quantize solutions in a semiclassical fashion and how to evaluate tunneling amplitudes for the decay of metastable states.

3) A model for the condensation of four-particle α-like clusters.
4) An extension of Landau's theory of Fermi liquids, in which not only densities but also vertex functions are subject to small dynamic oscillations.

Part of the content of this paper amounts to a detailed exposition and extension of the methods and ideas put forward in a series of previous letters by the author [*3, 9, 10, 11*] where also references to related work can be found.

II. The Generating Functional

The non-relativistic nuclear quantum theory is usually specified in terms of a Hamiltonian of the form

$$H \equiv H^0 + H^{\text{int}} = \sum_{\alpha,\beta} a_\alpha{}^+ \varepsilon_{\alpha\beta} a_\beta + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} v_{\alpha\beta\gamma\delta} a_\alpha{}^+ a_\beta a_\gamma{}^+ a_\delta \tag{1}$$

where $v_{\alpha\beta\gamma\delta}$ is the two-particle potential. The indices enumerate the orbitals in some complete (for example momentum-)basis and $\varepsilon_{\alpha\beta}$ are the corresponding kinetic energies. As relativistic effects grow important the Hamiltonian description becomes inconvenient and the action

$$\begin{aligned} \mathcal{A} &= \mathcal{A}^0 + \mathcal{A}^{\text{int}} \\ &= \sum_\alpha \int dt (a_\alpha{}^+ i\,\partial_t a_\alpha - a_\alpha{}^+ \varepsilon_{\alpha\beta} a_\beta) - \frac{1}{2} \sum_{\alpha\beta\gamma\delta} v_{\alpha\beta\gamma\delta} \int dt a_\alpha{}^+(t)\, a_\beta(t)\, a_\gamma{}^+(t)\, a_\delta(t) \end{aligned} \tag{2}$$

is the best starting point. In it, retardation effects can naturally be incorporated by allowing, in the potential $v_{\alpha\beta\gamma\delta}$, for four different times, i.e. by writing the interaction as

$$\mathcal{A}^{\text{int}} = -\frac{1}{2} \sum_{\alpha\beta\gamma\delta} \int dt_1\, dt_2\, dt_3\, dt_4 v_{\alpha\beta\gamma\delta}(t_1, t_2, t_3, t_4)\, a_\alpha{}^+(t_1)\, a_\beta(t_2)\, a_\gamma{}^+(t_3)\, a_\delta(t_4). \tag{3}$$

This general action forms the basis of a wide variety of quantum phenomena. The case (2) amounts to a time independent instantaneous potential

$$v_{\alpha\beta\gamma\delta}(t_1, t_2, t_3, t_4) = v_{\alpha\beta\gamma\delta} \delta(t_1 - t_2)\, \delta(t_1 - t_3)\, (t_1 - t_4) \tag{4}$$

and is particularly simple. It will be invoked if it helps solving equations. In general, since each index α is accompanied by its own time variable t, the notation may be simplified by absorbing t into α. Moreover, many integral- and summation-symbols can be saved by using matrix notation and writing

$$a^+ M a \equiv \sum_{\alpha,\beta} \int dt_1\, dt_2 a_\alpha{}^+(t_1)\, M_{\alpha\beta}(t_1, t_2)\, \alpha_\beta(t_2). \tag{5}$$

Thus the free part of the action is simply written as

$$\mathcal{A}^0 + a^+(i\partial_t - \varepsilon)\, a \tag{6}$$

where ε is a diagonal matrix in the time index

$$\varepsilon_{\alpha\beta}(t_1, t_2) = \varepsilon_{\alpha\beta} \delta(t_1 - t_2) \tag{7}$$

and $i\partial_t$ is the matrix

$$(i\partial_t)_{\alpha\beta}(t_1 t_2) \equiv \delta_{\alpha\beta} i\, \partial_t \delta(t_1 - t_2). \tag{8}$$

The interaction may be considered as a tensor in the space in which $a_\alpha^+(t_1)$ is a vector with indices (αt_1). Thus we shall write

$$\mathcal{A}^{\text{int}} \equiv -\frac{1}{2}\, a^+ava^+a \quad \text{or} \quad -\frac{1}{2}\, va^+aa^+a.$$

In what follows, another notation will turn out to be useful: Instead of complex creation and annihilation operators we shall employ quasi-real objects

$$\varphi_a(t) = \begin{pmatrix} \varphi_{\alpha\uparrow}(t) \\ \varphi_{\alpha\downarrow}(t) \end{pmatrix} = \begin{pmatrix} a_\alpha(t) \\ a_\alpha^+(t) \end{pmatrix} = C_{ab}\varphi_b^+(t) \tag{9}$$

in a doubled space where

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{10}$$

is the matrix by which φ and φ^+ are similar to each other C merely changes upper and lower components. With this convention, the free action may be written as

$$\mathcal{A}^0 \equiv \frac{1}{2}\, \varphi i G_0^{-1} \varphi \tag{11}$$

where iG_0^{-1} stands short for the off-diagonal matrix[2])

$$iG_0^{-1} \equiv \begin{pmatrix} 0 & i\,\partial_t + \varepsilon^T \\ i\,\partial_t - \varepsilon & 0 \end{pmatrix}. \tag{12}$$

The interaction will be abbreviated in the form

$$\mathcal{A}^{\text{int}} = -\frac{1}{4!}\, V\varphi^4 \tag{13}$$

where V has four doubled indices, each of which being contracted successively with one of the φ fields. In order to agree with (9), the potential V may be chosen as

$$V_{\alpha\downarrow\beta\uparrow\gamma\downarrow\delta\uparrow} = 2v_{\alpha\beta\gamma\delta}. \tag{14}$$

The remaining entries are determined from the antisymmetry in the doubled indices of V:

$$V_{abcd}(t_1, t_2, t_3, t_4) = -V_{bacd}(t_2, t_1, t_3, t_4) = V_{c\alpha ab}(t_3, t_4, t_1, t_2) \tag{15}$$

$$V_{abcd}(t_1, t_2, t_3, t_4) = 0; \qquad a + b + c + d \neq 0. \tag{16}$$

Here $a + b + c + d \neq 0$ means that the numbers of up and down components are not equal. Equ. (17) expresses particle number conservation of the interaction.
The complete quantum theory is given by the set of all Green's functions

$$G^{(n)}(x_1, x_2, \ldots, x_n) = \left\langle T\big(\varphi(x_1)\, \varphi(x_2) \cdot \cdots \cdot \varphi(x_n)\big)\right\rangle \tag{17}$$

where $\varphi(x_i)$ are the fully interacting Heisenberg fields: Here we have collected orbital, time, and up-down index in a single variable x_i which sometimes will be replaced even shorter by i itself. The advantage of this formulation is that many expressions will look

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[2]) The symbol T denotes transposition in all occurring indices.

very much like those of a simple φ^4 field theory except that φ is a Fermi field and the interaction $-1/4!\ V\varphi^4$ is strongly non-local in x. The symbol T denotes the time ordering operation with the effect that earlier times in x come before later times and a minus sign is implied for every permutation necessary to achieve this goal. The Green's function with n fields will be referred to as n-point function.
Because of the Fermi nature of the field φ, only Green's functions with an even number of fields can be non-zero.
Statements concerning the set of all these Green's functions are most easily collected in a single quantity called the generating functional [9]

$$Z[K] = \langle T e^{i(1/2)\varphi K \varphi} \rangle . \tag{18}$$

Here K is an antisymmetric matrix

$$K(x_1, x_2) = -K(x_2, x_1) = -K^T(x_1, x_2) \tag{19}$$

acting as an expansion parameter. It may be seen as an external source coupling to the bilocal quantity

$$\varphi(x_1)\,\varphi(x_2) = \begin{pmatrix} a_\alpha a_\beta & a_\alpha a_\beta^+ \\ a_\alpha^+ a_\beta & a_\alpha^+ a_\beta^+ \end{pmatrix} (x_1, x_2) . \tag{20}$$

Thus K may be written as

$$K(x_1, x_2) = \begin{pmatrix} \lambda_{\alpha\beta} & -\mu_{\alpha\beta}^T \\ \mu_{\alpha\beta} & \lambda_{\alpha\beta}^{T*} \end{pmatrix} (x_1, x_2) \tag{21}$$

where μ plays a similar role as a chemical potential in grand-canonical ensembles: It is similar in its coupling to the density, but different by its being bilocal. The source λ is the analogous quantity as far as pairs are concerned.
The important property of $Z[K]$ is that functional differentiations with respect to K generate all non-zero Green's functions

$$G^{(2n)}(x_1, \ldots, x_{2n}) = 2^n \frac{1}{Z[K]} \frac{\delta}{\delta i K(x_1, x_2)} \cdot \ldots \cdot \frac{\delta}{\delta i K(x_{2n-1}, x_{2n})} Z[K] \tag{22}$$

if this relation is evaluated at zero source $K = 0$. For the following discussion it will be useful to speak about the objects defined by (22) also for $K \neq 0$ and call them Green's functions in the presence of the external source K. This will have the advantage that we can look at the results alternatively in the canonical and grand-canonical ensemble. In the first case, K has to be set equal to zero at the end. In the second case we simply have to equate

$$K(x_1, x_2) = K^{\text{c.p.}}(x_1, x_2) \equiv \delta_{\alpha\beta}\delta(t_1 - t_2) \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} \tag{23}$$

where μ is the proper constant chemical potential which guarantees a certain particle number N. Notice that in the exact solution this number is always sharp though approximations may turn out to cause fluctuations of the order $1/\sqrt{N}$.
The set of all free Green's functions is well-known from Wick's theorem: It consists of sums of products of simple free Green's functions (the two-point functions)

$$G_0^{(2)}(x_1, x_2) \equiv \overbrace{\varphi_0(x_1)\,\varphi_0(x_2)} = G_0(x_1, x_2) \tag{24}$$

where G_0 is the inverse of the matrix in the free action (12)

$$G_0 = i \begin{pmatrix} 0 & i\partial_t + \varepsilon^T \\ i\partial_t - \varepsilon & 0 \end{pmatrix}^{-1} (x_1, x_2) \tag{25}$$

and this is, of course, the reason for chosing the notation (12). The hook symbol is called a contraction. Wick's rule reads

$$G_0^{(n)}(x_1, \ldots, x_n) = \sum_{\text{all contradictions}} \left\langle T\left(\overbrace{\varphi_0(x_1)}\, \overbrace{\varphi_0(x_2)} \cdot \cdots \cdot \overbrace{\varphi_0(x_{n-1})}\, \varphi_0(x_n)\right)\right\rangle \tag{26}$$

where the sum over all contractions is done successively: First $\varphi_0(x_1)$ is contracted with $\varphi_0(x_2)$ or $\varphi_0(x_3)$ or $\varphi_0(x_4)$ ..., then the next uncontracted φ_0 is contracted with all its free successors, and so on. There are $(n-1)!!$ terms, each of which is a product of single contractions (24).

The use of the generating functional $Z[K]$ has the advantage that this rule can be verified via differential manipilations. We shall see that the correct expression for $Z[K]$ is

$$Z^0[K] = \exp\left\{\frac{1}{2}\,\mathrm{tr}\log\left(iG_0^{-1} + K\right)\right\} \Big/ \exp\left\{\frac{1}{2}\,\mathrm{tr}\log\left(iG_0^{-1}\right)\right\}. \tag{27}$$

This has indeed the desired property of generating all free Green's functions via (22). (The superscript zero on Z records the absence of interactions). For the first few terms, the statement can be verified by simply expanding the logarithm in (27) in powers of K

$$Z^0[K] = \exp\left\{-\frac{1}{2}\,\mathrm{tr}\sum_{n=1}^{\infty}(iG_0K)^n/n\right\} \tag{28}$$

and seeing that in the series

$$Z^0[K] = 1 - \frac{1}{2}\,\mathrm{tr}\sum_{n=1}^{\infty}(iG_0K)^n/n + \frac{1}{2!}\left(-\frac{1}{2}\sum_{n=1}^{\infty}(iG_0K)^n/n\right)^2 + \ldots, \tag{29}$$

the coefficients of $(K^n/n!)\,(i/2)^n$ do give $G^{(2)}$, $G^{(4)}$, $G^{(6)}$, ... as the sum of all contractions of $G^{(2)}$. The general proof of (27) is most simply given in the path integral representation of the partition functions. Suppose for a moment φ were a Bose field. Then $Z^0[K]$ could be obtained as

$$\begin{aligned} Z^0[K] &= \int \mathscr{D}\varphi e^{(i/2)\varphi(iG_0^{-1}+K)\varphi} \Big/ \int \mathscr{D}\varphi e^{i\varphi iG_0^{-1}\varphi} = \left[\det\left(iG_0^{-1} + K\right)/\det\left(iG_0^{-1}\right)\right]^{-1/2} \\ &= e^{-(1/2)\mathrm{tr}\log(iG_0^{-1}+K)} \Big/ e^{-(1/2)\mathrm{tr}\log(iG_0^{-1})} \end{aligned} \tag{30}$$

where the result on the right-hand side is a straight-for-ward generalization of the Fresnel integral $\int dx/\sqrt{2\pi i}\; e^{(i/2)xax} = 1/\sqrt{a}$ to vectors $\boldsymbol{x}$ and matrices $\boldsymbol{a}$. Since we are dealing with Fermi fields, however, this formula is no longer true. But the modification can be achieved by hand: the expansion of the trace log in the exponent (30) may be represented graphically as shown in Fig. 1: It is the sum of one-loop diagrams which successively

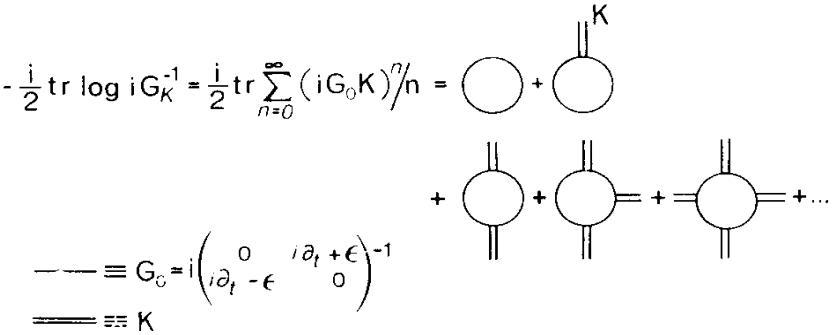

Fig. 1. The expansion of the generating functional $W^0[K]$ in powers of the external two-particle source K. Lines represent the free fermion Green's function G_0 and double lines the source K

emit 1, 2, 3, ... double lines representing K. Since fermion loops carry a minis sign with respect to boson loops, we many simply reverse the sign in the exponents of (30) and obtain the fermion version (27). Sometimes, anticommuting Grassmann variables are formally introduced to achieve this result but we shall not need this mathematical apparatus here.

For the reader not familiar with the functional integral representation we can derive the result in an alternative fashion which is somewhat less direct but has the advantage of preparing the grounds for later arguments. This is based on using the free operator equation of motion in the presence of the source K

$$(iG_0^{-1} + K)\,\varphi = 0 \tag{31}$$

together with the canonical anticommutation rule, which in our doubled notation reads

$$\{\varphi(x_1)\,\varphi(x_2)\}_{t_1=t_2} = C\delta_{\alpha\beta}\,. \tag{32}$$

Applying (30) to the free Green's function we find the standard result

$$\begin{aligned}(iG_0^{-1} + K)\,G^{(2)}(x_1, x_2) &= (iG_0^{-1} + K)\left\langle T\big(\varphi(x_1)\,\varphi(x_2)\big)\right\rangle = i\{C\varphi(x_1), \varphi(x_2)\}_{t_1=t_2} \\ &= i\delta(t_1 - t_2)\,\delta_{a,b} = i\delta_{x_1,x_2}\,.\end{aligned} \tag{33}$$

Representing G in the presence of K as

$$G^{(2)}(x_1, x_2) = \{Z^0[K]\}^{-1}\,2\,\frac{\delta}{\delta iK(x_1, x_2)}\,Z^0[K] \equiv \frac{2}{i}\,Z^0_{K(x_1,x_2)}/Z^0$$

we have the differential equation

$$(iG_0^{-1} + K)\,\frac{2}{i}\,Z_K{}^0 = iZ^0 \tag{34}$$

which holds also for Bose particles. This equation is immediately solved by

$$Z^0[K] = e^{(1/2)\mathrm{tr}\log(iG_0^{-1}+K)}/e^{(1/2)\mathrm{tr}\log(iG_0^{-1})} \tag{35}$$

where the multiplicative factor has been chosen to comply with the definition (18) according to which $Z[0] = 1$. Notice that the correct minus sign with respect to the Bose result arises from the fact that K is antisymmetric for fermions such that

$$\frac{\delta}{\delta K(x_1, x_2)} = -\frac{\delta}{\delta K(x_2, x_1)} = \frac{\delta}{\delta K^T(x_1, x_2)} = -\frac{\delta}{\delta K}\,(x_1, x_2) = \frac{\delta}{\delta K^T}\,(x_1, x_2) \tag{36}$$

and

$$\frac{\delta}{\delta K(x_1, x_2)}\,\mathrm{tr}\log\,(iG_0 + K) = -\frac{\delta}{\delta K}\,(x_1, x_2)\,\mathrm{tr}\log\,(iG_0 + K) = -(iG_0 + K)^{-1}\,.$$

The exponent of $Z^0[K]$ will be of so much importance later that it deserves an own symbol:

$$W^0[K] \equiv -i\log Z^0[K] = -\frac{i}{2}\,\mathrm{tr}\log\,(iG_0^{-1} + K)/(iG_0^{-1})\,. \tag{37}$$

Such a quantity is also introduced in the general interaction case

$$W[K] = -i\log Z[K]\,. \tag{38}$$

There are simple connections between derivatives of $W[K]$ and $Z[K]$. If subscripts denote functional differentiations, i.e. $\delta/\delta K(x_1, x_2)\, Z = Z_{K_{12}}$ we can derive

$$Z_K = iW_K Z, \qquad Z_{KK} = (iW_{KK} - W_K{}^2)\, Z$$
$$Z_{KKK} = (iW_{KKK} - 3W_{KK}W_K - iW_K{}^3)\, Z \tag{39a, b, c}$$
$$\vdots$$

such that

$$G^{(2)} = \frac{2}{i} Z_K/Z = 2W_K$$
$$G^{(4)} = \left(\frac{2}{i}\right)^2 Z_{KK}/Z = -4i(W_{KK} + iW_K{}^2)\, Z \tag{40a, b, c}$$
$$G^{(6)} = \left(\frac{2}{i}\right)^3 Z_{KKK}/Z = -8(W_{KKK} + 3iW_{KK}W_K - W_K{}^3)\, Z.$$
$$\vdots$$

All indices have been left out and the direct products of several matrices have shortly been written as a power, i.e.

$$W_{K_{12}}W_{K_{34}}W_{K_{56}} \to W_K{}^3. \tag{41}$$

Different even permutations of the same expression have been collected in a single term with a multiplicity factor. If the signs are to be correct, the indices have to be grouped in an order which is an even even permutation of those on the other side of the equation. The relations can be inverted to give

$$2W_K = G^{(2)} \tag{42a}$$
$$\left(\frac{2}{i}\right)^2 iW_{KK} = G^{(4)} - G^{(2)}G^{(2)} \tag{42b}$$
$$\left(\frac{2}{i}\right)^3 iW_{KKK} = G^{(6)} - 3G^{(4)}G^{(2)} + 2G^{(2)}G^{(2)}G^{(2)}. \tag{42c}$$
$$\vdots$$

The latter equations (42) can be verified by direct differentiation of (35). We leave it up to the reader to keep track of the indices and give only the examples of the right-hand side of (42b) where the indices are $G^{(4)}(x_1x_2x_3x_4) - G^{(2)}(x_1x_2)\, G^{(2)}(x_2x_4)$ while in (40b) they are $W_{K(x_1x_2)K(x_3x_4)} + iW_{K(x_1x_2)}W_{K(x_3x_4)}$.

It is simple to write down a formal expression for the generating functional also in the presence of interactions. The Green's functions are determined in the interaction picture of quantum field theory by the well-known formula

$$G^{(n)}(x_1 \cdot \cdots \cdot x_1) = \langle Te^{i\mathscr{A}^{\mathrm{int}}[\varphi_0]}\, \varphi_0(x_1) \cdot \cdots \cdot \varphi_0(x_n)\rangle / \langle Te^{i\mathscr{A}^{\mathrm{int}}[\varphi_0]}\rangle. \tag{43}$$

Expanding in powers of $\mathscr{A}^{\mathrm{int}}$ and performing again all Wick contractions yields the perturbation series for $G^{(n)}$. Formula (43) can be translated to the generating functional as

$$Z[K] = \langle Te^{i\mathscr{A}^{\mathrm{int}}[\varphi_0]+\varphi_0 K\varphi_0}\rangle / \langle Te^{i\mathscr{A}^{\mathrm{int}}[\varphi_0]}\rangle. \tag{44}$$

Since the essential property of Z in (22) is unaffected by the constant denominator we shall find it convenient to drop this trivial factor from the definition and use for the interacting case the functional

$$Z[K] = \langle Te^{i\mathscr{A}^{\mathrm{int}}[\varphi_0]+(i/2)\varphi_0 K\varphi_0}\rangle. \tag{45}$$

This expression can be evaluated further. Since $\mathcal{A}^{\text{int}}$ depends on four fields φ_0 and since these may simply be generated via two $\delta/\delta K$ differentiations, we can obviously write

$$Z[K] = e^{i(V/4!)4(\delta/\delta K)(\delta/\delta K)}\langle T e^{(i/2)\varphi_0 K \varphi_0}\rangle = e^{i(V/3!)(\delta/\delta K)(\delta/\delta K)} Z^0[K]. \tag{46}$$

Inserting the free expression (35) and expanding in powers of V generates the perturbation series.
There is a differential equation for $Z[K]$ analogous to (34) also in the interacting case. Applying $iG_0^{-1} + K$ to $\delta/\delta K\, Z$ and using (34) we find

$$(iG_0^{-1} + K)\frac{2}{i} Z_K[K] = iZ + \frac{2}{i}[K, e^{i(V/3!)(\delta/\delta K)(\delta/\delta K)}] Z_K = iZ - \frac{4}{3!} V Z_{KK} \tag{47}$$

which could also have been derived directly in the same way as (34) using the operator equation of motion

$$(iG_0^{-1} + K)\varphi - \frac{1}{3!} V\varphi\varphi\varphi = 0. \tag{48}$$

Equation (48) can also be translated to a differential equation for the exponent $W[K] = -i \log Z[K]$. Using (39) we find

$$(iG_0^{-1} + K) W_K + \frac{1}{3} V i(W_{KK} + iW_K{}^2) - \frac{i}{2} = 0. \tag{49}$$

It is convenient to separate out the known $V = 0$ part (37) of $W[K]$ and define

$$W[K] \equiv W_0[K] + W^{\text{int}}[K] \tag{50}$$

with the interacting part satisfying

$$(iG_0^{-1} + K)_{12} W^{\text{int}}_{K_{25}} + \frac{1}{3} V_{1234} i(W^{\text{int}}_{K_{23}K_{45}} + i W^{\text{int}}_{K_{23}} W^{\text{int}}_{K_{45}}) + \frac{1}{4} V_{1234} G^K_{34} G^K_{52} = 0. \tag{51}$$

Here we have abbreviated

$$G^K \equiv i(iG_0^{-1} + K)^{-1} \tag{52}$$

and used

$$\frac{\delta}{\delta K_{12}}\frac{\delta}{\delta K_{34}} iW^0[K] = \frac{1}{2}\frac{\delta}{\delta K_{12}} iG^K_{34} = -\frac{i}{2}\left(\frac{\delta}{\delta K}\right)_{12} G^K_{34} = \frac{1}{4}(G^K_{23}G^K_{41} - G^K_{31}G^K_{24}). \tag{53}$$

The latter result may alternatively be derived from formula (27) if $W[K]$ is expanded around $K \neq 0$ rather than $K = 0$, i.e. one inserts $K \to K + \delta K$ and writes

$$iW^0 = -\frac{1}{2}\,\mathrm{tr}\sum_{n=0}^{\infty}(iG^K\delta K)^n/n \tag{54}$$

with the quadratic term

$$\frac{1}{4} G^K_{23}\delta K_{34} G^K_{41}\delta K_{12} \tag{55}$$

giving directly (53).
Equation (51) can be solved iteratively. To lowest order we can neglect the second term and find

$$W^{(1)}[K] = -\frac{1}{8} V_{1234} G^K_{12} G^K_{34} \tag{56}$$

as can verified by differentiation (using (53))

$$\frac{\delta}{\delta K_{2'5}} W^{(1)}[K] = i \frac{1}{4} V_{1234} G^K_{2'1} G^K_{52} G^K_{34}. \tag{57}$$

Multiplying this with $i(G^K_{12'})^{-1}$ gives

$$\frac{1}{4} V_{1234} G^K_{34} G^K_{52} \tag{58}$$

in agreement with (51). We may now reinsert (56) into (51) and obtain $W^{(2)}$

$$W^{(2)}[K] = \frac{i}{48} V_{1234} G^K_{11'} G^K_{22'} G^K_{33'} G^K_{44'} V_{1'2'3'4'} \tag{59}$$

and further

$$W^{(3)}[K] = \frac{1}{48} (VG^K G^K)^3 \tag{60}$$

$$W^{(4)}[K] = -i \frac{5}{8 \cdot 12} (VG^K G^K)^4 \tag{61}$$

$$\vdots$$

etc. These terms can be represented graphically by the sum of all connected vacuum graphs shown in Fig. 2. They are one-particle irreducible due to the fermion nature of the

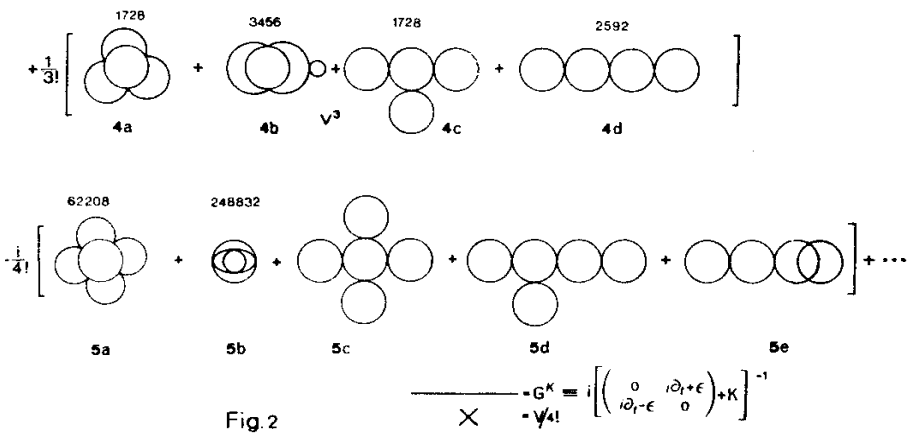

Fig. 2. The one-particle irreducible vacuum graphs for the interacting generating functional $W[K]$. The numbers on top of each term give the numbers of different contractions

fields which means that they do not fall apart by cutting one line. Of course, they amount to precisely all Wick contractions which would emerge by expanding (45) in a power series on $\mathcal{A}^{\text{int}}$ and going to the logarithm. The latter operation collects the subset of connected graphs. Notice that the multiplicities of all vacuum graphs to order V^n is $(4n-1)!!$ while that of the connected graphs satisfies the recursion relation

$$N_n{}^c + 3(n-1) N^c_{n-1} + 7 \cdot 5 \cdot 3 \binom{n-1}{2} N^c_{n-2} + \cdots$$

$$+ (4n-5)!! \binom{n-1}{n-1} N_1{}^c = (4n-1)!! \tag{62}$$

which follows directly from the exponential relation (see Fig. 3). Thus we have $N_1{}^c = 3$, $N_1{}^c = 96 = 24 + 72$, $N_3{}^c = 9504$, etc. Of course, the total coefficients before the term $\left(V(G^K)^2\right)^n$ are

$$a_n = \frac{(-i)^{n-1}}{n!} N_n \left(\frac{1}{4!}\right)^n \tag{63}$$

such that $a_1 = -1/8$, $a_2 = i/12$, $a_3 = 11/2 \cdot 3 \cdot 16$, etc. This expansion of the generating functional provides the basis for the discussion to follow.

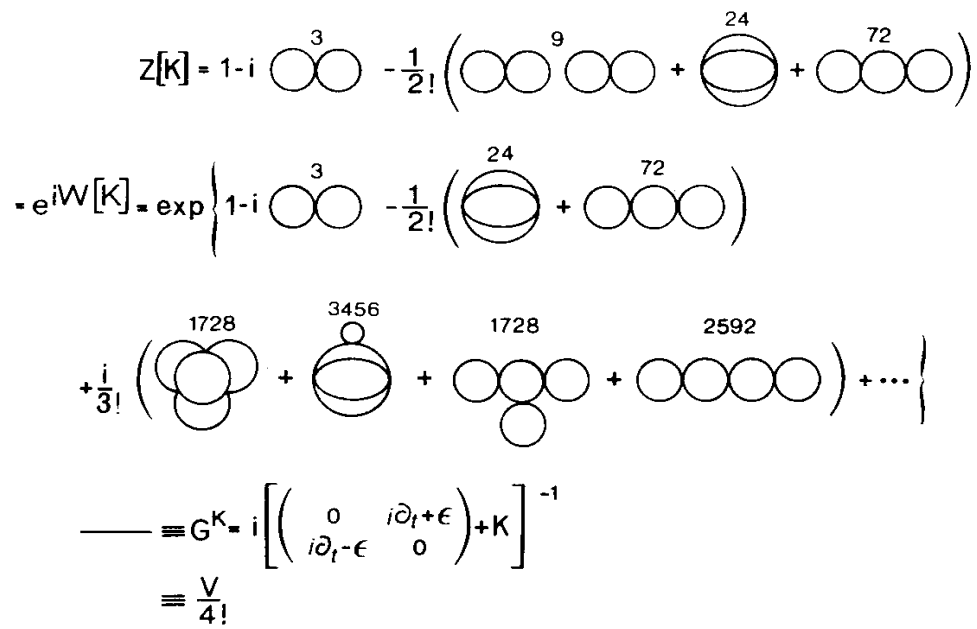

Fig. 3. The standard exponential connection between vacuum graphs and their connected subset

III. Effective Action for Two-Point Functions

Up to now we have merely rederived the usual perturbation series for $W[K]$. This does not yet permit any discussion of non-perturbative effects which are so important for the understanding of collective phenomena. We shall now demonstrate, however, that only a few rather simple manipulations are capable of turning the feeble perturbative object W into a new and very powerful functional Γ which will be called the effective action for two-point functions. It will permit the extraction of non-perturbative results even at the level of low order approximations.

Suppose for a moment, the full generating functional $W[K]$ were known. Then the exact two-point function $G \equiv G^{(2)}$ in the presence of the source K would be calculable from

$$G(x_1x_2) = \frac{1}{2} W_{K(x_1x_2)} = \langle T\varphi(x_1)\, \varphi(x_2)\rangle = G[K] \tag{64}$$

in the presence of the source K (not to be confused with the free Green's function G^K at $K = 0$). We now use G to define a Legendre transformed functional

$$\Gamma[G] = W[K] - \frac{1}{2} \operatorname{tr}(GK^T). \tag{65}$$

The explicit G dependence arises by inverting (64) to obtain $K = K[G]$ and substituting this for K on the right-hand side of (65). By construction, this object has an important

property: Its derivative with respect to the Green's function G satisfies

$$\frac{\delta \Gamma[G]}{\delta G(x_1 x_2)} = -\frac{1}{2} K(x_1 x_2). \tag{66}$$

Since the source K is just an auxiliary mathematical device which has to be set equal to zero at the end we conclude that the functional Γ is extremal for physical two-point functions G. Thus it has the same property with respect to this fully quantum theoretic quantity as any mechanical action does with respect to classical orbits. This is why it is called the effective action.

In a grand canonical ensemble the extremality conditions is changed to

$$\frac{\delta \Gamma[G]}{\delta G} = -\frac{1}{2} K^{\text{c.p.}} = -\frac{1}{2} \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} \tag{67}$$

where μ is the external chemical potential (see (23)). Thus Γ is not truely extremal. We can, however, introduce the slightly modified quantity

$$\Gamma^{\text{g.c.}}[G] \equiv \Gamma(G) + \frac{1}{2} \operatorname{tr} (G K^{\text{c.p.}\,T}) \tag{68}$$

which does satisfy

$$\frac{\delta \Gamma^{\text{g.c.}}[G]}{\delta G} = 0 \tag{69}$$

and deserves again being called effective action. Since the modification is so simple we shall keep working with Γ itself. Notice that $\Gamma^{\text{g.c.}}$ is related to Γ in the same way as the grand-canonical energy $H - \mu N$ is to the Hamiltonian H. Thus $\Gamma^{\text{g.c.}}$ should really be called the grand-canonical effective action which explains the superscript.

Let us now calculate $\Gamma[G]$ from the first known pieces (56)—(61) of $W[K]$. From the free part $W^0[K]$ we find

$$G = 2W_K{}^0 = G^K = i[iG_0{}^{-1} + K]^{-1} \tag{70}$$

and inverting this

$$K = iG^{-1} - iG_0{}^{-1} \tag{71}$$

such that

$$\Gamma^{(0)}[G] = -\frac{i}{2} \operatorname{tr} \log iG^{-1} + \frac{1}{2} \operatorname{tr} (iG_0{}^{-1} G^T) + \frac{i}{2} \operatorname{tr} 1. \tag{72}$$

The last term is an irrelevant constant and may be dropped. Consider now the first order correction $W^0[K] + W^{(1)}[K]$. Then

$$\begin{aligned} G &= 2W_K = G^K - \frac{i}{2} G^K (V G^K) G^K \\ K &= iG^{-1} - iG_0{}^{-1} + \frac{1}{2} VG \end{aligned} \tag{73}$$

such that we find

$$\Gamma^{(1)}[G] = -\frac{1}{8} VG^2. \tag{74}$$

This result may be pictured as a vacuum diagram which looks the same way as that of $W^{(1)}[K]$ except that a line stands for the fully interacting Green's function G rather than the free G^K.

If we go on to the third order, however, the situation changes. Now

$$G = 2W_K = G^K - \frac{i}{2} G^K (VG^K) G^K - \frac{2}{3} V^2 G^{K\,5} \tag{75}$$

and this may be solved for K as

$$K = iG^{-1} - iG_0^{-1} + \frac{1}{2} VG - \frac{i}{6} V^2 G^3. \tag{76}$$

Inserting this into $\Gamma[G] = W[K] - 1/2 \text{ tr } (GK^T)$ we find

$$\Gamma^{(2)} = \frac{i}{48} V^2 G^4. \tag{77}$$

We immediately see that the coefficient is smaller than that of $W^{(2)}[K]$. This indicates that in a diagrammatic representation not all graphs of $W[K]$ can also be present in $\Gamma[G]$. In order to find out which ones are absent we have to be more careful with the order of index contractions. In the present case these can be followed graphically but when proceeding to higher orders, work rapidly proliferates.
Let us therefore search for a more economic and direct determination of $\Gamma[G]$. This is possible if we use the equation of motion (49) as a starting point and transform it into an equation for Γ and G. First of all, $2W_K$ is simply G. Second we have

$$K = -2\Gamma_G \tag{78}$$

and finally

$$2W_{KK} = G_K = K_G{}^{-1} = -\frac{1}{2} \Gamma_{GG}^{-1} \tag{79}$$

by definition of Γ such that (49) becomes

$$(iG_0^{-1} - 2\Gamma_G) G - \frac{1}{3!} V(i\Gamma_{GG}^{-1} + G^2) - i = 0. \tag{80}$$

We now decompose

$$\Gamma[G] = \Gamma^{(0)}[G] + \Gamma^{\text{int}}[G] \tag{81}$$

and use (72) to calculate

$$\Gamma_G = -\frac{i}{2} G^{-1} + \frac{1}{2} iG_0^{-1} + \Gamma_G{}^{\text{int}} \tag{82}$$

$$\Gamma_{GG} = \frac{i}{2} G^{-1}G^{-1} + \Gamma_{GG}^{\text{int}}. \tag{83}$$

Therefore (80) may be rewritten as an equation for the interacting part only:

$$\Gamma_G{}^{\text{int}} = -\frac{1}{12} GV\{1 + 2[1 - 2iGG\Gamma_{GG}^{\text{int}}]^{-1}\}. \tag{84}$$

Multiplying this with G and contracting indices gives

$$\Gamma_G{}^{\text{int}} G = -\frac{1}{12} GGV\{1 + 2[1 - 2iGG\Gamma_{GG}^{\text{int}}]^{-1}\}. \tag{85}$$

The left-hand side is equal to $\Gamma^{\text{int}}[G]$ if in a power series expansion of $\Gamma^{\text{int}}[G]$, every term G^n is multiplied by a factor n^{-1}. It will be useful to rewrite (85) as

$$\Gamma_G{}^{\text{int}}G = -\frac{1}{4} VGG + \frac{i}{12} VG^4t \tag{86}$$

where

$$t = -4\Gamma^{\text{int}}_{GG}[1 - 2iGG\Gamma^{\text{int}}_{GG}]^{-1} = -4\Gamma^{\text{int}}_{GG} - 8i\Gamma^{\text{int}}_{GG}GG\Gamma^{\text{int}}_{GG} + 16\Gamma^{\text{int}}_{GG}(GG\Gamma^{\text{int}}_{GG})^2 + \cdots \tag{87}$$

satisfies the integral equation

$$t = -4\Gamma^{\text{int}}_{GG} + 2i\Gamma^{\text{int}}_{GG}GGt \tag{88}$$

and has a simple physical meaning: It is the scattering matrix of the theory. This quantity is conventionally defined as the amputated connected four-point function

$$G_c^{(4)} = -itGGGG. \tag{89}$$

We do know what the full four-point functions is from (40b):

$$G^{(4)} = -4iW_{KK} + G^2. \tag{90}$$

But this is a disconnected object. The two particles may run through the system independently in three combinations without interacting with each other, i.e.

$$G^{(4)} = G_c^{(4)} + 3G^2 \tag{91}$$

where we have used our short notation for the three permutations of GG. But then we have

$$G_c^{(4)} = -4iW_{KK} - 2G^2. \tag{92}$$

Expressing this in terms of Γ we see

$$G_c^{(4)} = -2iG_K - 2G^2 = -2iK_G{}^{-1} - 2G^2 = -i\Gamma^{-1}_{GG} - 2G^2. \tag{93}$$

Inserting our decomposition into free and interacting part (83), this becomes

$$G_c^{(4)} = 4iG^4\Gamma^{\text{int}}_{GG}[1 - 2iGG\Gamma^{\text{int}}_{GG}]^{-1}. \tag{94}$$

Thus with (87),

$$G_c^{(4)} = -itGGGG, \tag{95}$$

such that t is indeed the scattering matrix.

Notice that t is antisymmetric in all its four indices. At first sight, it may seem surprising that a string-like summation of matrices $G^2\Gamma^{\text{int}}_{GG}$ appearing in equ. (94) can lead to an antisymmetric result, but it is a direct consequence of $Z_{K_{12}K_{34}}$ being completely antisymmetric, which follows from the definition.[3])

The coupled integral equations (86), (87) are shown in Fig. 3. They may be solved recursively (similar to what we did for W^{int}). To lowest order, we neglect t on the right-hand side of (86) and obtain

$$\Gamma^{(1)}[G] = -\frac{1}{8} VGG \tag{96}$$

just as in (74). From this we find, using (87), the lowest approximation for the t matrix

$$t = V \tag{97}$$

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[3]) See also the remarks in Sect. IX and the illustration there on Fig. 11.

which is commonly known as Born approximation. Reinserting this into (86) yields the second order result

$$\Gamma^{(2)}[G] = \frac{i}{48} V^2 G^4 \tag{98}$$

in agreement with (77). In Fig. 4 we have displayed the iteration graphically. Only the first of the vacuum diagrams in $W^{(2)}[K]$ is contained in $\Gamma^{(2)}[G]$ (apart from the different meaning of a line). In the same figure we go on to the next orders. When calculating t we observe that the two derivatives of Γ with respect to G amount to cutting two lines and removing them. In this way, vacuum graphs are opened up and become vertex graphs. Inserting these into the boxes in Fig. 4 results in only a single final diagram of third order

$$\Gamma^{(3)}[G] = \frac{1}{48} (VG^2)^3 \tag{99}$$

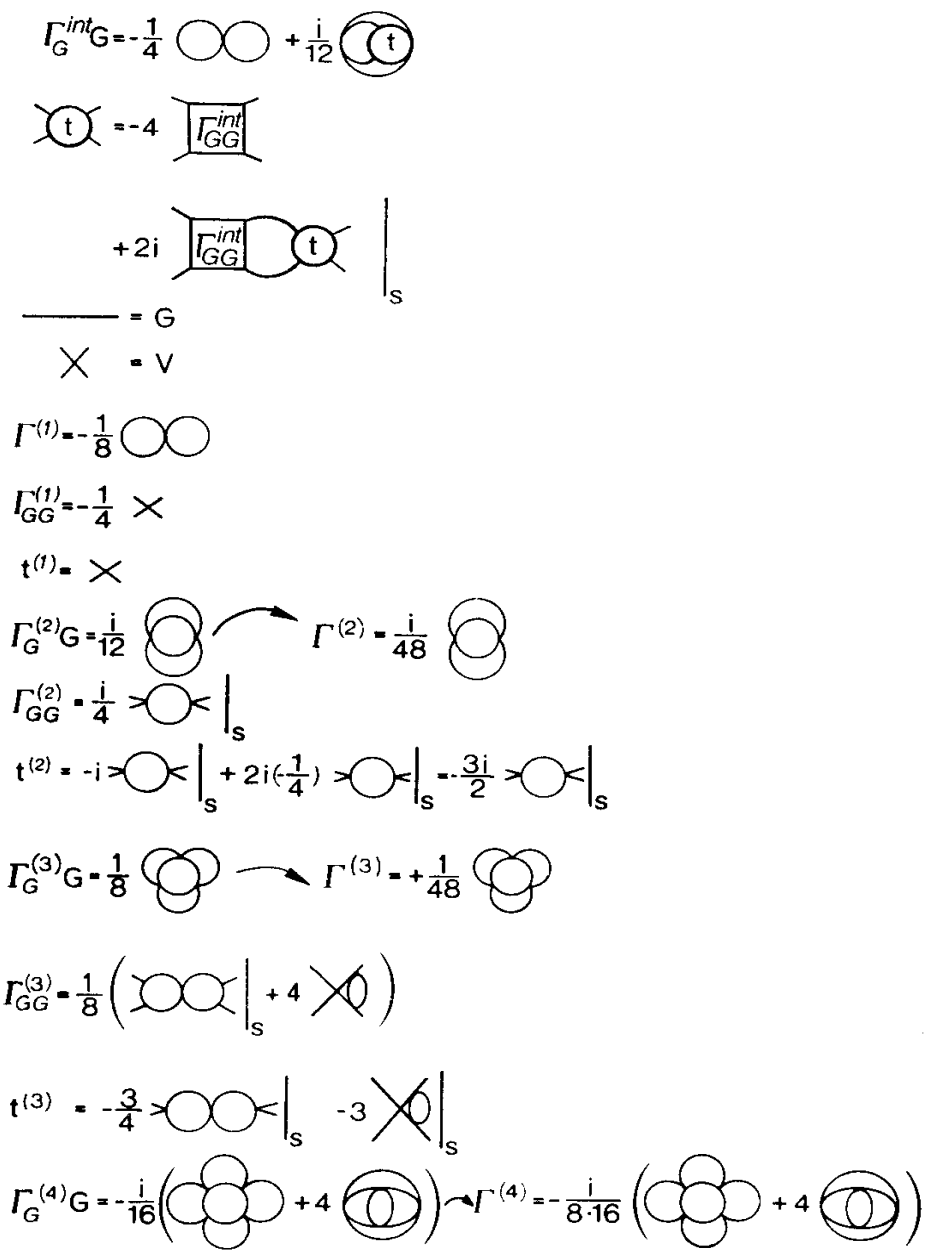


Fig. 4. The integral equation for the interacting part of the effective action and the four-particle vertex function, the t matrix and its iterative solution. The symbol $|_s$ means symmetrization in all four legs

which corresponds again to the first of the diagrams in $W^{(3)}[K]$. The derivative $\Gamma^{(3)}_{GG}$ has now two different pieces depending on whether one removes two lines of the same or of neighbouring bubbles. The expansion up to $\Gamma^{(3)}$ can again be iterated through the integral equation to obtain $\Gamma^{(4)}$ as shown in Fig. 4.:

$$\Gamma^{(4)}[G] = -i\,\frac{5}{8\cdot 16}\,(VG^2)^4 \tag{100}$$

with two different ways of contracting the indices as is obvious from the graphical derivation.

For higher orders, the number of different admissible diagrams increases rapidly but one property remains invariant: When forming Γ_{GG} i.e. when cutting two lines, the diagrams always remain in one piece and do not fall apart. Such diagrams are called two-particle irreducible (TPI). As a matter of fact, $\Gamma[G]$ collects precisely the TPI subset of all vacuum graphs which can be drawn with V and G.

The full coefficient of the $n\cdot$ th order term can easily be evaluated by multiplying (87) with the denominator in the equation, inserting the expansion

$$\Gamma^{\mathrm{int}}[G] = i\sum_{n=1}^{\infty} a_n(-iVG^2)^n \tag{101}$$

and finding the recursion relation

$$a_n = \frac{1}{2n}\left[-\frac{1}{6}\,a_{n-1}(2n-2)\,(2n-3) + 2\sum_{m+m'=n} 2ma_m\,2m'(2m'-1)\,a_{m'}\right]$$

$$a_1 = 1/8$$

which is solved by

$$a_2 = \frac{1}{48},\quad a_3 = \frac{1}{48},\quad a_4 = \frac{5}{8\cdot 16},\ \ldots \tag{102}$$

Let us now see the physical consequences of the effective action. The extremality condition (67) reads

$$\Gamma_G = -\frac{1}{2}\,K^{\mathrm{c.p.}} \tag{103}$$

in the presence of a non-vanishing chemical potential. Inserting our expansion for $\Gamma[G] = \sum_{n=0}^{4}\Gamma^{(n)}[G]$ gives the equation

$$-iG^{-1} + iG_0^{-1} + K^{\mathrm{c.p.}} - \frac{1}{2}\,VG + \frac{i}{6}\,V^2G^3 + \frac{1}{4}\,V^3G^5 - i\,\frac{5}{8}\,V^4G^7 = 0. \tag{104}$$

This may be written as

$$G = i\left[iG_0^{-1} + K^{\mathrm{c.p.}} - \Sigma[G]\right]^{-1} = i\left[\begin{pmatrix} 0 & i\,\partial_t + (\varepsilon^T - \mu) \\ i\,\partial_t - (\varepsilon - \mu) & 0 \end{pmatrix} - \Sigma[G]\right]^{-1} \tag{105}$$

with an auxiliary functional

$$\Sigma[G] \equiv -2\delta\Gamma^{\mathrm{int}}[G]/\delta G = \frac{1}{2}\,VG - \frac{i}{6}\,V^2G^3 - \frac{1}{4}\,V^3G^5 + i\,\frac{5}{8}\,V^4G^7 + \cdots. \tag{106}$$

We now realize that what we have derived analytically is nothing else but Dyson's equation: As we showed, $\Gamma^{\mathrm{int}}[G]$ collects all two-particle irreducible (TPI) vacuum

graphs. According to (106), the functional Σ arises by differentiation with respect to G which amounts to cutting and removing a single line from these TPI graphs. But these are just the diagrams required for the calculation of the self-energy in Dyson's equation: That they are of the self-energy type is obvious since they can be linked up with two legs at the previous end point of the line which was removed. Moreover they posses the second property of being "proper" which is an adjective reserved to those self-energy diagrams which are also OPI. But this is automatically true: A diagram which is TPI remains OPI after removing one leg.

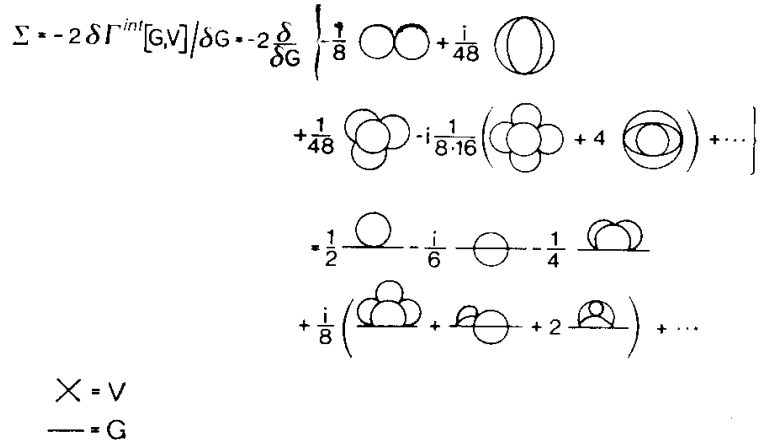


Fig. 5. The proper self-energy graphs as derivatives of the TPI vacuum graphs of the effective action

Notice the important feature of equation (105): It is completely non-perturbative. Even if Γ^{int} is determined only up to a finite order in V, the equation (105) for G, resums an infinity of self energy diagrams in which each line is again the fully interacting Green's function G itself.

From this discussion it is obvious why one particle-reducible diagrams disappeared when going from $W[K]$ to $\Gamma[G]$. These arise automatically when expanding the right-hand side of (105) in powers of Σ thereby chaining up an infinity of irreducible pieces and the chain can be disconnected by cutting a single line.

The effective action $\Gamma[G]$ is appropriate for a study of ground states and large-amplitude collective excitations and this will be shown in the next section.

After that, we shall discuss an even more powerful effective action which contains not only two-particle Green's functions but also four-particle vertex functions as explicit variables to be determined by extremization.

IV. Solution of Dyson's Equation

To lowest non-trivial approximation for Γ, Dyson's equation reads

$$G = i\left[\begin{pmatrix} 0 & i\,\partial_t + \varepsilon^T - \mu \\ i\,\partial_t - \varepsilon + \mu & 0 \end{pmatrix} - \frac{1}{2}\,VG\right]^{-1}. \tag{107}$$

This is recognized as the standard time-dependent Hartree-Fock-Bogoljubov self-consistency equation: It is an equation for the Green's function of particles moving in an external potential

$$\Sigma = \frac{1}{2}\,V_{1234}G_{12}G_{34} \tag{108}$$

which is again prepared by the Green's function itself. The important result of the last section was that using the expansion for Γ^{int}, the equation can be improved to increasing accuracy thereby including more and more quantum effects without ever double-counting any diagrams.

Let us summarize the method of solving the general equation (105) [9]. In order to cope with the most common situation we shall assume the potential to be instantaneous and time independent. Then the self-energy involves only the equal-time part of the Green's function

$$G_{ab}(t, t')|_{t=t'+\varepsilon} = \langle T\varphi_a(t)\, \varphi_b(t')\rangle|_{t=t'+\varepsilon} \tag{109}$$

such that the self consistency has to be achieved for only this part. It contains all information on what is usually referred to as the density matrix

$$\varrho_{\alpha\beta}(t) = \langle a_\alpha{}^+(t)\, a_\beta(t)\rangle \tag{110}$$

of the system. Using $\varrho_{\alpha\beta}$, the different matrix elements of the Green's function may be written as

$$G_{ab}(t, t')|_{t=t'+2} = \begin{pmatrix} \langle a_\alpha(t)\, a_\beta(t)\rangle & \langle a_\alpha(t)\, a_\beta{}^+(t)\rangle \\ \langle a_\alpha{}^+(t)\, a_\beta(t)\rangle & \langle a_\alpha{}^+(t)\, a_\beta{}^+(t)\rangle \end{pmatrix} = \begin{pmatrix} \Delta_{\alpha\beta}(t) & \delta_{\alpha\beta} - \varrho_{\beta\alpha}(t) \\ \varrho_{\alpha\beta}(t) & \Delta^+(t)_{\alpha\beta} \end{pmatrix} \tag{111}$$

where the diagonal elements are the pair correlation functions

$$\Delta_{\alpha\beta}(t) = \langle a_\alpha(t)\, a_\beta(t)\rangle, \quad \Delta^+(t)_{\alpha\beta} = \langle a_\alpha{}^+(t)\, a_\beta{}^+(t)\rangle = \Delta(t)_{\beta\alpha}{}^*. \tag{112}$$

From now on in this section, the symbol G will denote only this equal-time matrix.

A comment may be in order concerning the equal-time limit $t = t' + \varepsilon$. The reason for this choice rather than the opposite derives from our convention of contracting indices in (96): If we take the interaction $-1/4\, V_{abcd}\varphi_a\varphi_b\varphi_c\varphi_d$, the lowest loop correction to this expression is

$$-\frac{1}{2}\, V_{abcd}\varphi_a\varphi_b G_{cd}$$

where $\varphi_a\varphi_b$ appear in the same order as in (96) if we take the limit $t = t' + \varepsilon$.

If G is periodic in some time interval, say T, the inverse operator appearing on the right-hand side of equ. (105) can be calculated as follows: One multiplies the operator in the denominator by the matrix C which brings the $i\,\partial_t$ part to diagonal form and determines the eigenfunctions of the differential equation

$$C\left[\begin{pmatrix} 0 & i\,\partial_t + \varepsilon^T - \mu \\ i\,\partial_t - \varepsilon + \mu & 0 \end{pmatrix} - \Sigma[G]\right] \chi^l(t) = -\varkappa^l[G]\, \chi^l(t). \tag{113}$$

There exists a complete set of these with time independent eigenvalues $\varkappa^l$ which have the same period as G as a function of time. This is a consequence of translational invariance under the substitution $t \to t + T$. From this follows Bloch's theorem, according to which a homogeneous equation

$$(i\,\partial_t - M)\, \chi = 0 \tag{114}$$

with a periodic operator M can always be solved in the form

$$\chi(t) = e^{i\varkappa t} f(t) \tag{115}$$

where the frequency like quantity in the exponent is called Bloch-Floquet index and $f(t)$ has the same period as M.

Notice that if $\chi^l(t)$ is a solution with eigenvalue $\varkappa^l$ one immediately knows an infinity of solutions

$$\chi_n{}^l(t) = e^{-i\omega_n t}\chi^l(t) \tag{116}$$

with eigenvalue $\omega_n - \varkappa^l$ which differ from $\varkappa^l$ only by a frequency in the "reciprocal lattice" of the periodic functions, $\omega_n = 2\pi n/T$. In our case we are dealing with fermions. It is a well-known fact of statistical mechanics that fermion fields in periodic systems have to be antiperiodic. The additional minus sign drops out in all observables since these are even products of fermion operators. But there are consequences as far as the spectrum is concerned. The previously constructed eigenfunctions $\chi_n{}^l$ solve the antiperiodic problem if the frequencies are taken to be odd multiples of π/T:

$$\omega_n = \frac{2\pi}{T}\left(n + \frac{1}{2}\right). \tag{117}$$

These fermionic reciprocal lattice vectors are real time versions of what are known as Matsubara frequencies in statistical mechanics.
Using the complete set of wave functions, we may now explicity construct the Green's function on the right hand side of (105) as

$$i\left[C\begin{pmatrix} 0 & i\,\partial_t + \varepsilon^T - \mu \\ i\,\partial_t - \varepsilon + \mu & 0 \end{pmatrix} - C\Sigma[G]\right]^{-1} = \sum_{n,l} \frac{i}{\omega_n - \varkappa^l} e^{-i\omega_n(t-t')}\chi^l(t)\,\chi^l(t')^*. \tag{118}$$

That this is the correct way of forming the inverse operator (113) can be seen directly by applying (113) and using the completeness relation

$$\sum_l \chi_a{}^l(t)\,\chi_b{}^l(t')^* = \delta(t - t')\,\delta_{ab}. \tag{119}$$

With this result, Dyson's equation becomes

$$G_{ab}(t, t') = \sum_{n,l} \frac{i}{\omega_n - \varkappa^l} e^{-i\omega_n(t-t')}\,\chi_a{}^l(t)\,\chi_{b'}^l(t')^*\,C_{b'b}\,. \tag{120}$$

Before proceeding it is useful to realize that the differential equation (105) has the following symmetry property: With $\chi_n{}^l(t)$ also $\chi_n{}^{l*}(t)\,C$ is a solution with opposite eigenvalue. This is a consequence of the identity (9) for the doubled field φ. The proof follows by inspection for the free term and from antisymmetry of the potential V in the case of interactions. Thus all Bloch-Floquet indices come in pairs and we may write

$$G = \sum_{\substack{n \\ \varkappa^l < 0}} e^{-i\omega_n(t-t')}\left\{\frac{i}{\omega_n - \varkappa^l}\chi^l(t)\,\chi^l(t')^*\,C + \frac{i}{\omega_n + \varkappa^l}\,C\chi^l(t)^*\,\chi^l(t')\right\}. \tag{121}$$

Let us now perform the frequency sum by converting it into a contour integral

$$G = -\frac{1}{2\pi}\sum_{\varkappa^l<0}\int_C \frac{dz}{e^{izT}+1}\,e^{-iz(t-t')}\left\{\frac{i}{z - \varkappa^l}\chi^l(t)\,\chi^l(t')^*\,C + \frac{i}{z + \varkappa^l}\,C\chi^l(t)^*\,\chi^l(t')\right\} \tag{122}$$

where the factor $(e^{izT} + 1)^{-1}$ has poles precisely at the frequencies $z_n = \omega_n$ and the contour C encloses these poles but leaves out the eigenvalues $\varkappa^l$. In Fig. 6 we have achieved this by moving $\varkappa^l$ slightly into the complex plane. The prescription as to what direction has to be taken comes from standard arguments of statistical mechanics:

For free particles, all states with positive energy above the chemical potential are moved below the real axis, those with negative energies above. As in the interaction V is turned on the magnitudes of the energies change but the signs remain the same and so do the imaginary parts.

We now may open up the contour of integration. For $|t - t'| < T$, the upper and lower branch of the contour C can be deformed into the infinite semicircles shown in Fig. 6 where their contribution vanishes. When passing the poles at $\pm\varkappa^l$, however, we pick up the residues

$$G(t, t') = i[iG_0^{-1} + K^{\text{c.p.}} + \Sigma]^{-1}(t, t')$$
$$= \sum_{\varkappa^l<0} \{(e^{i\varkappa^l T} + 1)^{-1} e^{-i\varkappa^l(t-t')}\chi^l(t)\ \chi^l(t')^* C + (e^{-i\varkappa^l T} + 1)^{-1} e^{i\varkappa^l(t-t')} C\chi^l(t)^*\ \chi^l(t')\}. \tag{123}$$

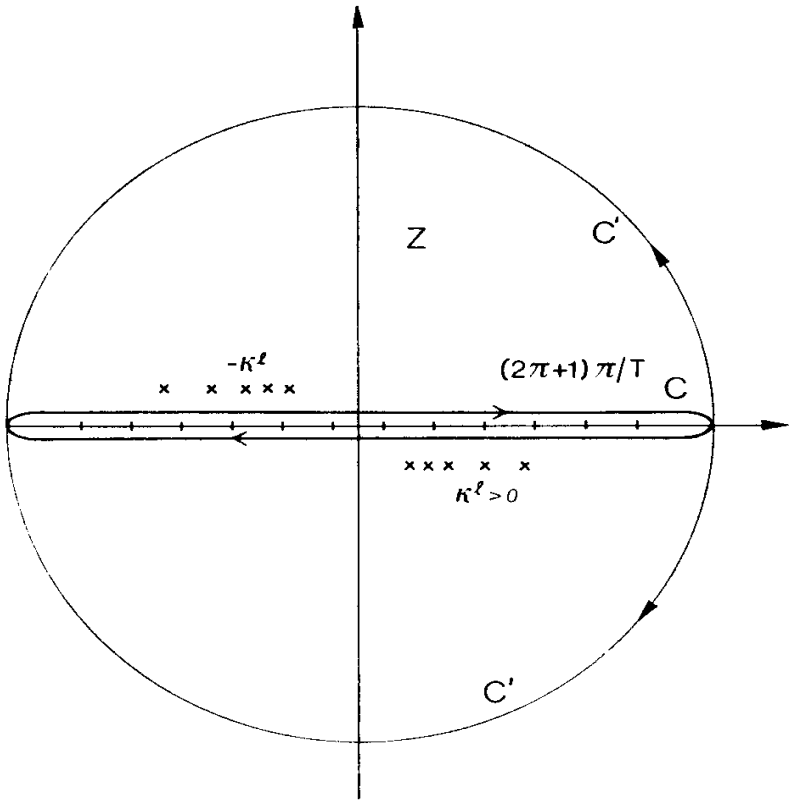


Fig. 6. The contour of integration for summing the discrete frequencies in the case of periodic orbits. When opening up C to C', one picks up the residues at the eigenvalues $\varkappa^l$ and the integral over the infinite semicircles vanishes

The weight functions

$$n(\varkappa^l) = (e^{i\varkappa^l T} + 1)^{-1}$$
$$1 - n(\varkappa^l) = (e^{-i\varkappa^l T} + 1)^{-1} \tag{124}$$

are recognized as the Fermi occupation numbers for an imaginary temperature

$$T_{\text{emp}} = -\varepsilon/T. \tag{125}$$

This is a reflection of the standard connection between periodic and thermal problems which usually arises the other way around: Thermal Green's functions are analytically continued time dependent Green's functions periodic along the imaginary time direction with period period $T = -i/T_{\text{emp}}$.

In the static limit, $T \to \infty$, the $i\varepsilon$ in the energies enforces $n(\varkappa^l) \to 1$ such that

$$i[iG_0^{-1} + K^{\text{c.p.}} - \Sigma[G]]^{-1} = \sum_{\varkappa^l<0} \chi^l(t)\, \chi^l(t')^*\, C \tag{126}$$

where the sum runs over all states below the Fermi surface which are all occupied states. The off diagonal part of (121), in the two by two scheme, becomes

$$\varrho_{\alpha\beta} = \sum_{\varkappa^l<0} \chi^l_{\alpha\uparrow}\chi^{l*}_{\beta\uparrow} \tag{127}$$

while the diagonal piece gives

$$\Delta_{\alpha\beta} = -\sum_{\varkappa^l<0} \chi^l_{\alpha\uparrow}\chi^{l*}_{\beta\downarrow}. \tag{128}$$

If this is non-zero, the system contains a condensate of bound states between two particles which are referred to as Cooper pairs.

Equations (113), (127), (128) are the well-known time-dependent Hartree-Fock-Bogoljubov equations: For some initial trial distributions ϱ, Δ, the wave functions χ are found by solving (113). Then improved distributions ϱ, Δ, are recovered from (127) and (128) and the process is iterated.

For systems with large amplitude collective motion, equation (126) remains valid approximately if the motion is very slow as compared to the initial orbital time. This is called the adiabatic limit. The quality of this approximation can be estimated as follows: We expand, for large T.

$$\begin{aligned} n(\varkappa^l) &= 1 - e^{i(\varkappa^l+i\varepsilon)T} + \cdots \\ 1 - n(\varkappa^l) &= e^{i(\varkappa^l+i\varepsilon)T} + \cdots \end{aligned} \tag{129}$$

such that the first correction to (123) becomes

$$-\sum_{\varkappa^l<0} e^{i(\varkappa^l+i\varepsilon)T}\, \{\chi^l(t)\, \chi^l(t')^*\, C - C\chi^l(t)^*\chi^l(t')\}.$$

This has to be much smaller than (123) in order that the adiabatic limit is acceptable. Since for large T, $e^{i(\varkappa^l+i\varepsilon)T}$ is an oscillating function in $\varkappa^l$, the sum has appreciable contributions only from a neighborhood of the Fermi surface. The clean cut particle occupation in the static solution is smeared out around the chemical potential with a width of the order of $1/T$. This situation is very similar to the thermal effects except that the additional piece is oscillatory.

If the motion is not sufficiently slow, the distributions (124) influence strongly the periodic motion and this is probably of prime importance for the understanding of giant dipole resonances.

V. Semiclassical Quantization and Tunneling

We now come to an important feature of the effective action: It permits a direct semiclassical quantization of the periodic orbits obtained from the TDHF equations. For this we remember the basic result which is obtained in path integral discussions of this problem: The trace of the time evolution operator

$$\text{tr}\, U(T) = \text{tr}\, (e^{-iHT/\hbar})$$

contains the information on the complete energy spectrum, bound as well as continuous, since its Fourier transform reads

$$G(E) = \int \frac{dT}{2\pi\hbar}\, e^{iET/\hbar}\, \text{tr}\,\big(U(T)\big)$$

has the spectral decomposition

$$G(E) = \sum_n \frac{i}{E - E_n}.$$

In semiclassical treatments the same object is obtained in the following form [4, 5]

$$G(E) = \frac{\partial W(E)}{\partial E} \sum_{n=1}^{\infty} e^{inW(E)/\pi} = \frac{\partial W(E)}{\partial E} \frac{e^{iW(E)/\hbar}}{1 - e^{iW(E)/\hbar}} \tag{130}$$

where $W(E)$ is the Legendre transform of the action of the orbits of fundamental period T,

$$\mathcal{A}(T) = \mathcal{A} \text{ [classical solutions of period } T\text{]},$$

which is related to $\mathcal{A}(T)$ by

$$W(E) = \mathcal{A}(T) - \mathcal{A}'(T)\, T$$

with the energy variable E being defined as

$$E = -\mathcal{A}'(T).$$

From this we see that the derivative $\partial W(E)/\partial E$ in (130) coincides with the period of the orbit of energy E

$$\frac{\partial W(E)}{\partial E} = T.$$

The factors n in the expansion of (130) have their origin in multiple repetitions of the same classical orbit of energy E.
The semiclassical quantization condition is obtained from (130) by locating the poles E_n in the energy plane [4, 5]

$$W(E_n) = 2\pi n\hbar \tag{131}$$

close to which $G(E)$ has indeed the form

$$G(E) \sim \frac{i}{E - E_n}.$$

But the effective action of a periodic orbit differs from $\mathcal{A}(T)$ only by terms of order $\hbar$. Therefore, we obtain the condition that the Legendre transform $\Gamma[G]$ for periodic orbits has to satisfy this condition. Explicitly, if

$$\Gamma(T) = \Gamma[G]\,|_{\text{orbit of period } T}$$

then

$$W(E) = \Gamma(T) - \Gamma'(T)\, T$$

with

$$E \equiv -\Gamma'(T)$$

has to satisfy approximately the condition (131). The corrections are of order $\hbar$ but these were supposed to be small, otherwise we could not have used the loop expansion of $\Gamma[G]$ for the calculation of the orbit in the first place. [9]
Actually, there are subtleties when it comes to realistic calculations since it has to be decided whether the orbits have or do not have turning points in which case there is a sign change in front of exp $(iW(E)/\hbar)$ in the denominator in (130) and the condition (131) must be solved with half-integer values of n.

Certainly, this method supplies only the principal quantum number for the periodic orbit. The resolution of the remaining degeneracies requires the study of fluctuations orthogonal to the periodic solution and their quantization. We shall come back to this problem later.

Related to the issue of quantization is the direct access given by the effective action to tunneling amplitudes [9]. It is well-known from semiclassical approximations to path integrals that solutions of the classical equations of motion along the imaginary time axis are capable of connecting different degenerate minima of the potential [6]. The exponential of the classical action of this path determines the level splitting caused by the tunneling effect. Similarly, if one of the minima lies lower than the other, and the system is caught in the higher one, there is a solution for imaginary time which is the analogue of the critical bubble in the boiling of an overheated liquid [6]: It starts out at the higher minimum, runs uphill of the potential barrier (due to the imaginary time) approaching the other minimum, but with its energy being insuffiecient to arrive there it returns to where it came from. The classical action of this solution differs from that of the static solution by an imaginary amount, say

$$i\mathcal{A}\,[\text{bubble solution}] - i\mathcal{A}\,[\text{static solution}] = -F_b .$$

Then the decay rate of the metastable state is to semiclassical approximation

$$\text{rate} = \sqrt{\frac{F_b}{2\pi i\hbar}}\,\frac{1}{2}\left[\frac{\det iG^{-1}\,(\text{static solution})}{|\det {}^{1}iG^{-1}\,(\text{bubble solution})|}\right]^{1/2} e^{-F_b/\hbar}. \tag{132}$$

The exponential $e^{-F_b/\hbar}$ is the probability of a critical bubble appearing. The rate factor in front is called the frequency of attempt and has its origin in the quadratic fluctuation determinant

$$\left(\frac{\det iG^{-1}\,(\text{static solution})}{\det iG^{-1}\,(\text{bubble solution})}\right)^{1/2}$$

when comparing the bubble solution with the static one in the metastable state. The determinant consists of the product of eigenvalues $\varkappa^l$. Since the bubble solution breaks translational invariance in time, its quadratic fluctuations must have a zero frequency mode which correspond to the translations of the "bubble" as a whole along the time axis. Its contribution to the determinant needs a careful treatment and one finds that its square vanishes inversely proportional to the total time T under consideration [6], namely

$$\frac{1}{\sqrt{\text{"zero frequency"}}} = T\sqrt{\frac{F_b}{2\pi i\hbar}}.$$

This is the reason why we have removed this mode from the product of all the others in $\det iG^{-1}$ (bubble solution), denoted this process by a prime, and written down the zero frequency contribution separately. Division by T gives the rate.

There is another subtlety in equ. (132) which is the factor 1/2 as well as the absolute bars around $\det {}^{1}iG^{-1}$ (bubble solution). They have the following explanation: The set of eigenvalues has one negative member which is the one responsible for the decay of the metastable state. It signalizes the fact that the bubble is a saddle point in the functional space with one direction along which the system can slide down into a more stable configuration. The opposite direction does not lead to a decay but to a dis-

[4]) More intuitively, define $\omega \equiv 2\pi/T$ and $L(\omega) = \mathcal{A}(T)/T$ as an effective Lagrangian of the orbit. Then $E \equiv (\partial L/\partial\omega)\,\omega - L$ is the Legendre transform in analogy with $H = \text{"}p\cdot q\text{"} - L$ and the quantization condition (131) is $\partial L/\partial\omega = \text{"}p\text{"} = n\hbar$ just as for cyclic coordinates.

appearing of the bubble in favor of a return to the initial metastable state. The factor 1/2 accounts for the selection among these two of the direction of decay. The point is now that the effective action supplies correctly up to $\hbar^2$ the rate formula

$$\text{rate} = \frac{1}{2}\frac{1}{T}\exp\left(i\{\Gamma\,[\text{bubble solution}] - \Gamma\,[\text{static solution}]\}\right) \tag{133}$$

if we only keep in mind that the trace log term in Γ has to be treated with care as far as the zero and the negative frequency modes are concerned.

VI. Effective Action for Two- and Four-Particle Correlations

After these preparations, we are now ready to embarque on a further important improvement of the non-perturbative result of the last section. We have seen that the Legendre transform of the generating functional W with respect to the two-particle source K immediately furnishes a resummation of infinitely many diagrams which lead to the self-consistent density and pair correlations of the Hartree-Fock-Bogoljubov equations and specify successive corrections. It is now crucial to realize that by including more and more terms in the self-energy $\Sigma[G]$, the improvements remain within the same class of resummed diagrams. There is an important set of phenomena which can never be explained even by taking a large number of corrections $\Gamma^{(n)}[G]$ into account, as long as this is finite: It is associated with the condensation, in the ground state, of clusters of more than two particles. Since the formation of Cooper pairs was incorporated by the Legendre transformation with respect to the two-particle source K it is suggestive to try and continue the procedure to sources of four and more particles and expect thereby the non-perturbative description of larger subclusters. In nuclear physics, alpha particles play a rather special role due to their great stability in a tight bound state. Therefore one expects a strong enhancement of four nucleon correlation functions

$$G^{(4)}(x_1 \cdots x_4) = \left\langle T\left(\varphi(x_1) \cdots \varphi(x_4)\right)\right\rangle$$

also inside larger nuclei.

We are thus led to search for an effective action in which this Green's function appears as a functional variable such that it can be considered explicitly in the extremization process [*10*].

Actually, the four-point function (132) itself is not the best variable to measure such correlations. First of all, it is disconnected such that it is advisable to study only the proper four-particle information which is contained in its connected part

$$G_c^{(4)} = G^{(4)} - (GG + 2 \text{ permutations}).$$

Here G is the full (connected) two-point function $G_c^{(2)}$ as before. Moreover, G has singularities in the energy variables of each external leg of the diagram exactly where $G = G^{(2)}$ does. Therefore a smoother function to consider is the amputated connected Green's function or four-point vertex function

$$-i\alpha \equiv G^{-1}G^{-1}G^{-1}G^{-1}G_c^{(4)} \tag{134}$$

where the singularities have been removed by contracting each leg with G^{-1}. But this quantity coincides exactly with the scattering matrix t (see (95)) which was introduced before as an intermediate object when solving the integral equation (84) for $\Gamma^{\text{int}}[G]$.

Actually, due to our doubled field notation, α contains several informations: One of them concerns the true four-particle correlation $\langle T\psi\psi\psi\psi\rangle$. Only this is relevant for alpha-

like particle clusters. Another part deals with the interacting two-particle two-hole correlation $\langle T\psi^+\psi\psi^+\psi\rangle$. Only this part is really what is commonly referred to as a t matrix. A third part is the density-pair correlation $\langle T\psi^+\psi\psi\psi\rangle$. In our doubled field notation, both appear at the same level and it will not be necessary to distinguish between them except later. Thus we shall use alternatively the symbol α or t for both of them.

The variable α can be incorporated into the scheme of Legendre transformations as follows: Let us add to the action a source for four-particle Green's functions

$$\mathscr{A}^{\text{source}} = -\frac{1}{4!} L\varphi\varphi\varphi\varphi \tag{135}$$

and consider the corresponding partition function

$$Z[K, L] = e^{iW[K,L]}. \tag{136}$$

Here L is a matrix depending on four doubled indices just as the potential. Derivatives with respect to L obviously generate the Green's function $G^{(4)}$

$$-\frac{1}{4!} G^{(4)} = Z^{-1}[K, L] \frac{1}{i} \frac{\delta}{\delta L} Z[K, L]. \tag{137}$$

Using the vertex function α this amounts to

$$W_L[K, L] = -\frac{1}{4!} (-i\alpha GGGG + 3GG) = \frac{i}{4!} \alpha G^4 - \frac{1}{8} G^2 \tag{138}$$

where in the second row we have gone to our previous short-hand notation. There is no new problem in calculating $W[K, L]$. All we have to do is replace in all formulas V by $V + L$. The same thing holds for the first Legendre transformed effective action at fixed L.

$$\Gamma[G, L] = W[K, L] - \text{tr}\,(W_K K^T). \tag{139}$$

Since L appears added to V everywhere it may simply be absorbed into it such that we may use only the single symbol V and deal directly with $W[K, V]$ and

$$\Gamma[G, V] = W[K, V] - \frac{1}{2} \text{tr}\,(GK^T). \tag{140}$$

With this convention the next Legendre transformation becomes simply

$$\Gamma[G, \alpha] \equiv \Gamma[G, V] - \Gamma_V[G, V] \tag{141}$$

where (139) implies that Γ_V is the same as W_V:

$$\Gamma_V[G, V] = \left(W_K - \frac{1}{2} G\right) K_V + W_V[K, V] = W_V[K, V] = \frac{i}{4!} \alpha G^4 - \frac{1}{8} G^2. \tag{142}$$

Therefore the Legendre transformation (141) may be written explicitly as as

$$\Gamma[G, \alpha] = \Gamma[G, V] - \frac{i}{4!} \alpha G^4 V + \frac{1}{8} VG^2. \tag{143}$$

It is obvious that only the interacting part is affected such that we may write

$$\Gamma[G, \alpha] = \Gamma^{(0)}[G] + \Gamma^{\text{int}}[G, \alpha] = -\frac{i}{2} \text{tr}\log G^{-1} + \frac{1}{2} \text{tr}\,(iG_0^{-1}G) + \Gamma^{\text{int}}[G, \alpha] \tag{144}$$

with

$$\Gamma^{\text{int}}[G, \alpha] = \Gamma^{\text{int}}[G] - \frac{i}{4!}\alpha G^4 V + \frac{1}{8} VG^2. \tag{145}$$

The new functional $\Gamma[G, \alpha]$ has again the typical property of an effective action. If differentiated with respect to G and α it yields

$$\Gamma_G[G, \alpha] = -\frac{1}{2} K - \frac{1}{6} i\alpha G^3 V + \frac{1}{4} VG \tag{146}$$

$$\Gamma_\alpha[G, \alpha] = -\frac{i}{4!} VG^4 \tag{147}$$

where the second equation involves only the interacting part of Γ:

$$\Gamma_\alpha^{\text{int}}[G, \alpha] = -\frac{i}{4!} VG^4 \tag{148}$$

while the first may be written in the form

$$\Gamma_G^{\text{int}}[G, \alpha] = \frac{i}{2} G^{-1} - \frac{1}{2}(iG_0^{-1} + K) - \frac{1}{6} i\alpha G^3 V + \frac{1}{4} VG. \tag{149}$$

The physical situation is given by equating K with the chemical potential $K = K^{\text{c.p.}}$ and setting the variable V equal to the physical potential V.
Notice that similar to the previous case in the presence of a chemical potential, $\Gamma[G, \alpha]$ is not truely extremal. But this can again be adjusted by a simple redefinition analogous to (69). Obviously

$$\Gamma^{\text{g.c.}}[G, \alpha] = \Gamma[G, \alpha] + \frac{1}{2} \text{tr}\,(GK^{\text{c.p.}T}) + \frac{i}{4!}\alpha G^4 V - \frac{1}{8} VG^2 \tag{150}$$

is extremal for physical G, α configurations. Again, because of simplicity, we shall keep working with (143) having (150) in mind when it comes to really knowing the action itself.
Let us now calculate the lowest contributions to $\Gamma^{\text{int}}[G, \alpha]$. Forming the derivative with respect to V we determine

$$\Gamma_V[G, V] = -\frac{1}{8} G^2 + \frac{i}{24} VG^4 + \frac{1}{16} V^2 G^6 - i\frac{5}{4 \cdot 16} V^3 G^8 \tag{151}$$

which is equal to

$$= \frac{i}{4!}\alpha G^4 - \frac{1}{8} G^2$$

by definition. From this we find

$$\alpha[G, V] = V - \frac{3}{2} iV^2 G^2 - \frac{15}{4} iV^3 G^4 + \cdots \tag{152}$$

which agrees with the previous solution for the t matrix (remember $\alpha \equiv t$) as shown in Fig. 6 where the proper index contractions arevisible. The relation can be inverted with the result [*10*]

$$V[G, \alpha] = \alpha + \frac{3}{2} i\alpha^2 G^2 - \frac{3}{4} \alpha^3 G^4 + \cdots. \tag{153}$$

The last term has two possible index contractions and we have to resort to the graphical method to find out the different contributions. We write the last term in (153) as

$$-\frac{3}{4}a\alpha^3G^4 - \frac{3}{4}b\alpha^3G^4$$

with $a + b = 1$ where a and b distinguish the different contractions and reinsert (152) into (153) to find out where cancellation takes place. This is shown in Fig. 7. The result is $a = 1$, $b = 0$.

We can now determine $\Gamma^{\text{int}}[G, \alpha]$. For this it is not necessary to use the defining equation (145). The reason is that we already know $V[G, \alpha]$ from inverting (152). But then we may simply insert (153) into (148) and integrate in α to find

$$\Gamma^{\text{int}}[G, \alpha] = -\frac{i}{48}\alpha^2G^4 + \frac{1}{48}\alpha^3G^6 + \frac{i}{8\cdot 16}\alpha^4G^8 + \cdots. \tag{154}$$

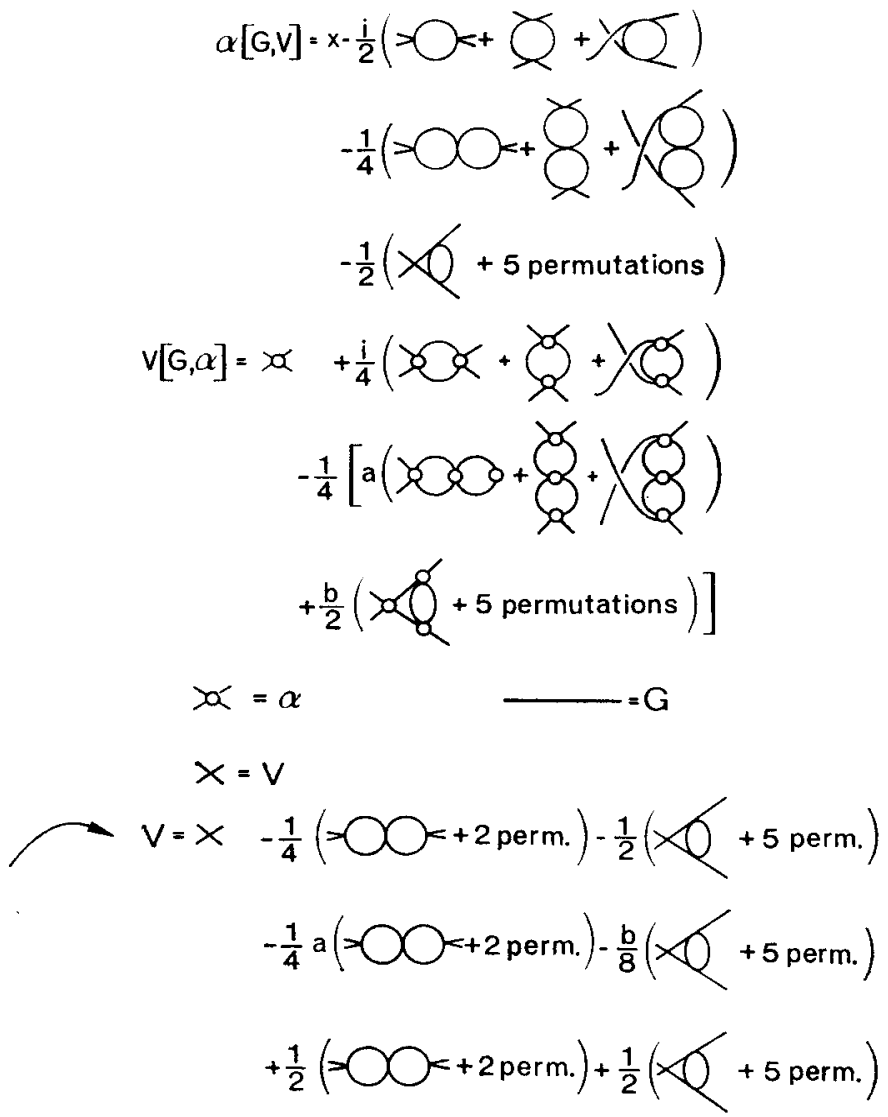
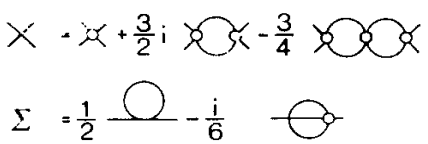


Fig. 7. The equation $\alpha[G, V]$ and its graphical inversion to find $V[G, \alpha]$. The last term in $V[G, \alpha]$ may have two different contractions which are distinguished by a and b. Inserting $\alpha[G, V]$ into $V[G, \alpha]$ we obtain an identity $V = \times$ for $a = 1, b = 0$

Graphically, we simply have to connect, in the expansion for $V[G, \alpha]$ in Fig. 8, the four free ends and divide each term by the order of the graph. We see that, contrary to $\Gamma^{\text{int}}[G, V]$, the last term has only one way of contracting the indices corresponding to the first graph for $\Gamma^{(4)}$ in Fig. 4.

If we want to know the effective action to all orders we have to resort again to an integral equation. For this we consider the derivative

$$\Gamma_G^{\text{int}}[G, \alpha] = \Gamma_G^{\text{int}}[G, V] + \left(-\frac{i}{6}\alpha G^3 + \frac{1}{4} G\right) V + \left[\Gamma_V^{\text{int}}[G, V] + \left(-\frac{i}{4!}\alpha G^4 + \frac{1}{8} \quad ^2\right)\right] V_G\Big|_\alpha \tag{155}$$



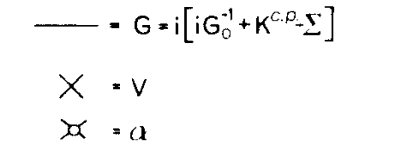


Fig. 8. The extremality condition following from the new effective action $\Gamma[G, \alpha]$. The second is an equation of the gap type which leads to non-trivial solutions for the self-energy, the first is an analogous self-consistency condition for the vertex function such that there may be spontaneous generation of coupling

and observe that the last term in brackets vanishes due to (142). Therefore

$$\Gamma_G^{\text{int}}[G, \alpha] = \Gamma_G^{\text{int}}[G, V] + \left(-\frac{i}{6}\alpha G^3 + \frac{1}{4} G\right) V. \tag{156}$$

Now we insert (86) and have the simple formula

$$\Gamma_G^{\text{int}}[G, \alpha] = -\frac{i}{12} V G^3 \alpha. \tag{157}$$

If we differentiate $\Gamma_G^{\text{int}}[G, \alpha]$ once more we obtain

$$\Gamma_{GG}^{\text{int}}[G, \alpha] = \Gamma_{GG}^{\text{int}}[G, V] + \left(-\frac{i}{2}\alpha G^2 + \frac{1}{4}\right) V + \frac{i}{4!} G^4 \alpha_G\Big|_V V_G|_\alpha. \tag{158}$$

The right hand side can be expressed in terms of α and G: Inverting (88) we find $\Gamma_{GG}^{\text{int}}[G, V]$ as a functional of $\alpha \equiv t$

$$\Gamma_{GG}^{\text{int}}[G, V] \equiv -\frac{1}{4}\alpha\left(1 - \frac{i}{2} GG\alpha\right)^{-1}. \tag{159}$$

Further we use (147) to derive

$$\alpha_{G|V} = -\frac{1}{\Gamma_{\alpha x}^{\text{int}}}\left(\Gamma_{\alpha G}^{\text{int}} - 4G^{-1}\Gamma_\alpha^{\text{int}}\right) \tag{160}$$

and

$$V_G|_\alpha = -4!\, iG^{-4}(\Gamma^{\rm int}_{\alpha G} - 4G^{-1}\Gamma_\alpha{}^{\rm int}) \tag{161}$$

such that

$$\frac{i}{4!}\, G^4\alpha_G\Big|_V V_G|_\alpha = \Gamma^{{\rm int}-1}_{\alpha\alpha}(\Gamma^{\rm int}_{\alpha G} - 4G^{-1}\Gamma_\alpha{}^{\rm int}). \tag{162}$$

With these relations we solve (157) for V as

$$V = 4(1 - 2iG^2\alpha)^{-1}\left\{\Gamma^{\rm int}_{GG}[G\alpha] + \frac{1}{4}\,\alpha\left(1 - \frac{i}{2}\,G^2\alpha\right)^{-1} - \Gamma^{{\rm int}-1}_{\alpha\alpha}(\Gamma^{\rm int}_{\alpha G} - 4G^{-1}\Gamma_\alpha{}^{\rm int})^2\right\}. \tag{163}$$

This is used in (157) to obtain the recursion relation

$$\Gamma_G{}^{\rm int}[G, \alpha]\, G = -\frac{i}{3}\,\alpha G^4(1 - 2iG^2\alpha)^{-1} \times \left\{\Gamma^{\rm int}_{GG}[G, \alpha] + \frac{1}{4}\,\alpha\left(1 - \frac{i}{2}\,G^2\alpha\right)^{-1} - \Gamma^{{\rm int}-1}_{\alpha\alpha}(\Gamma^{\rm int}_{\alpha G} - 4G^{-1}\Gamma_\alpha{}^{\rm int})^2\right\}. \tag{164}$$

We may bring this to a somewhat more managable form by observing that (157) and (147) together imply

$$2\Gamma_\alpha{}^{\rm int}[G, \alpha]\,\alpha = \Gamma_G{}^{\rm int}[G, \alpha]\, G \tag{165}$$

which actually amounts to the trivial statement that $\Gamma^{\rm int}[G, \alpha]$ is a pure function of αG^2. But then we have

$$\Gamma^{\rm int}_{GG}[G, \alpha]\, G^2 = 4\Gamma^{\rm int}_{\alpha\alpha}\alpha\alpha + 2\Gamma_\alpha{}^{\rm int}\alpha \tag{166}$$

$$\Gamma^{\rm int}_{G\alpha} - 4G^{-1}\Gamma_\alpha{}^{\rm int} = 2(\Gamma^{\rm int}_{\alpha\alpha}\alpha - \Gamma_\alpha{}^{\rm int}) \tag{167}$$

and (164) becomes

$$\Gamma_\alpha{}^{\rm int}[G, \alpha]\,\alpha = -\frac{i}{24}\,\alpha^2G^4\left(1 - \frac{i}{3}\,G^2\alpha\right)^{-1}\left(1 - \frac{i}{2}\,G^2\alpha\right)^{-1} + \frac{2}{3}\,iG^2\alpha\left(1 - \frac{i}{3}\,\alpha G^2\right)^{-1}\Gamma^{{\rm int}-1}_{\alpha\alpha}\Gamma_\alpha{}^{\rm int2}. \tag{168}$$

This integral equation may be used to verify the first three terms (154) and to go beyond, up to any further approximation.
The general $\Gamma^{\rm int}[G, \alpha]$ has the form

$$\Gamma^{\rm int}[G, \alpha] = i\sum_{n=1}^{\infty}(i\alpha G^2)^n\, a_n \tag{169}$$

and we may multiply (168) by $\Gamma^{\rm int}_{\alpha\alpha}$ to derive the recursion relation for the coefficients a_n

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}(m+2)\,a_{m+2}(n-m+2)\,(n-m+1)\,a_{n-m+2}\\
&-\frac{1}{3}\sum_{m=0}^{n-1}\binom{n-1}{m}(m+2)\,a_{m+2}(n-m+1)\,(n-m)\,a_{n-m+1}\\
&-\frac{2}{3}\sum_{m=0}^{n-1}\binom{n-1}{m}(m+2)a_{m+2}\,(n-m+2)\,a_{n-m+2}\\
&-\frac{1}{3}\sum_{m=0}^{n}\binom{n}{m}2^{-m-3}(n-m+2)\,(n-m+1)\,a_{n-m+2}.
\end{aligned} \tag{170}$$

Inserting $a_2 = 1/48$ we find again $a_3 = 1/48$, $a_4 = 1/(8\cdot 16)$, etc.

Notice that the algebra can be somewhat simplified by using the property of $\Gamma^{\text{int}}[G]$ and $\Gamma^{\text{int}}[G, \alpha]$ being functions, say $F[x]$, $F^{\text{LT}}[\xi]$, of the reduced arguments $x \equiv VG^2$, $\xi = \alpha G^2$ only. Then the Legendre transform becomes shortly

$$F^{\text{LT}}[\xi] = F[x] - F_x[x]\, x \tag{171}$$

with

$$F_x = \frac{i}{24}\,\xi - \frac{1}{8}. \tag{172}$$

Now, since

$$G\Gamma_G{}^{\text{int}}[G] = 2xF_x \tag{173}$$

$$GG\Gamma_{GG}^{\text{int}}[G] = 4x^2F_{xx} + 2xF_x \tag{174}$$

we may rewrite (159) as

$$4x^2F_{xx} + 2xF_x = -\frac{1}{4}\,\xi\left(1 - \frac{i}{2}\,\xi\right)^{-1}. \tag{175}$$

Inserting (172) brings this to the form

$$x[\xi] = \left[\xi\left(1 - \frac{i}{2}\,\xi\right)^{-1} + \frac{2}{3}\,ix^2x_\xi{}^{-1}\right]\left[1 - \frac{i}{3}\,\xi\right]^{-1} \tag{176}$$

in which we recognize directly (168) upon using the equation of motion (147):

$$F_\xi{}^{\text{L.T.}}[\xi] = -\frac{i}{24}\,x[\xi].$$

Let us end this section by writing down the grand canonical effective action

$$\Gamma^{\text{g.c.}}[G, \alpha] = -\frac{i}{2}\,\text{tr}\log G^{-1} + \frac{1}{2}\,\text{tr}\,[(iG_0{}^{-1} + K^{\text{c.p.}})\,G]$$
$$-\frac{1}{8}\,VG^2 + \frac{i}{48}\,(2\alpha V - \alpha^2)\,G^4 + \frac{1}{48}\,\alpha^3G^6 + \frac{i}{8\cdot 16}\,\alpha^4G^8 + \cdots \tag{177}$$

which has the property that its equations of motion follow from true extremization rather than equations (146), (147). The expansion of the interacting part of $\Gamma^{\text{g.c.}}[G, \alpha]$ is shown in Fig. 9.

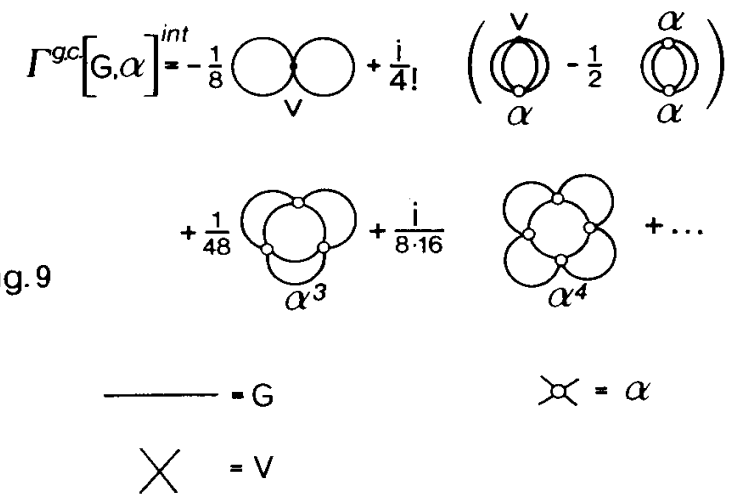


Fig. 9

Fig. 9. The graphical expansion for the interacting part of the grand-canonical effective action $\Gamma^{\text{g.c.}}[G, \alpha]$ from which physical propagators and vertex function follow by extremization

VII. Condensation of Alpha-like Clusters

What have we gained from this effective action $\Gamma[G, \alpha]$? Consider the extremality conditions (146), (147). If we use the expansion for Γ only up to $\Gamma^{(2)}$

$$\Gamma[G, \alpha] = -\frac{i}{2} \operatorname{tr} \log G^{-1} + \frac{1}{2} \operatorname{tr} (iG_0^{-1}G) - \frac{i}{48} \alpha^2 G^2 + \frac{1}{48} \alpha^3 G^6 + \frac{i}{8 \cdot 16} \alpha^4 G^8 + \cdots \tag{178}$$

the equations of motion for G and α become

$$\Gamma_G[G, \alpha] = -\frac{i}{2} G^{-1} + \frac{i}{2} G_0^{-1} - \frac{i}{12} \alpha^2 G^3 = -\frac{1}{2} K + \left(-\frac{i}{3!} \alpha G^3 + \frac{1}{4} G \right) V \tag{179}$$

$$\Gamma_\alpha[G, \alpha] = -\frac{i}{24} \alpha G^4 = -\frac{i}{24} V G^4. \tag{180}$$

In analogy with (105), (106), the first equation, (179), can be cast in the form

$$G = i[iG_0^{-1} + K^{\text{c.p.}} - \Sigma]^{-1} \tag{181}$$

where the self energy is

$$\Sigma[G, \alpha] = \frac{1}{2} VG - \frac{1}{3} VG^3 \alpha - 2\Gamma_G^{\text{int}}[G, \alpha]. \tag{182}$$

Fortunately, this expression simplifies considerably by making use of (165) together with the other equation of motion (180) which gives

$$-2\Gamma_G^{\text{int}}[G, \alpha] = -4\Gamma_\alpha G^{-1} \alpha = \frac{i}{6} VG^3 \alpha \tag{183}$$

such that

$$\Sigma[G, \alpha] = \frac{1}{2} VG - \frac{i}{6} VG^3 \alpha \tag{184}$$

and the exact self-consistent time dependent Hartree-Fock-Bogoljubov equation reads

$$G = i \left[iG_0^{-1} + K^{\text{p.p.}} - \frac{1}{2} VG + \frac{i}{6} VG^3 \alpha \right]^{-1} \tag{185}$$

which has to be solved together with [*10*]

$$V = 24i\Gamma_\alpha^{\text{int}}[G, \alpha] \tag{186}$$

$$= \alpha + \frac{3}{2} i \alpha G^2 \alpha - \frac{3}{4} \alpha^3 G^4 + \cdots. \tag{187}$$

It can be checked that if we invert (186) to find a power series for α (which is of course (152)) and insert this into (185) this equation coincides with (105), (106).
The new and important feature of equations (186), (187) lies in the possibility of having solutions for (186) which are not related to V in a perturbative way. The potential V has matrix elements only in the two-particle two-hole part. But α may be non-zero also in the four-particle and four-hole componentes. In fact, this necessarily happens if there

is pair formaton. For that case there is a second order radiative correction $[v/2\ \psi^+\psi\psi^+\psi]^2$ which is of the type $\psi^4 + \psi^{+4}$. Such effects are all accounted for by equ. (186) which may be considered as an equation of the gap type for the vertex function α and there are several analogies:
Just as the correlation function $\langle\psi\psi\rangle \neq 0$ can be interpreted as a signal for the condensation of Cooper pairs, non-zero four-particle matrix elements in α indicate the presence of alpha-like clusters in the condensate.
Just as equ. (185) can give rise to a spontaneous generation of mass from radiative corrections, equ. (186) can do so for coupling.
Certainly, non-perturbative effects are also present in the two-particle two-hole channels but their consequences are less dramatic.
This mechanism can best be illustrated via an exactly soluble model [*11*]. Consider N degenerate relativistic fermion fields in $D = 2 + \varepsilon$ dimensions with a Lagrangian

$$\mathscr{L}(x) = \bar{\psi}_\alpha(x)\, i\, \partial\!\!\!/\, \psi_\alpha(x) + \frac{g_0}{2N}\, \bar{\psi}_\alpha(x)\, C\bar{\psi}_\alpha{}^T(x)\, \psi_\alpha{}^T(x)\, C\psi_\alpha(x) \tag{188}$$

where C is the matrix of charge conjugation which satisfies

$$C\gamma^\mu C^{-1} = -\gamma^{\mu T}. \tag{189}$$

In two dimensions we make take

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \gamma^1 = (C^T)^{-1} = -C^T. \tag{190}$$

In $D = 2 + \varepsilon$ dimensions we employ anticommutation rules

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 2^{D/2} g^{\mu\nu}. \tag{191}$$

The label $\alpha = 1, \ldots, N$ denotes the degeneracy of each energy level. In the limit $N \to \infty$, the model becomes exactly soluble. It may be considered as an idealization of a nucleus with a highly degenerate outer and a stable inner shell. A little preparation is in order to bring (188) into a form in which we can directly apply our results. We introduce the doubled field

$$\varphi_a = \varphi_{\alpha',\alpha} = \begin{pmatrix} \psi_\alpha \\ C\bar{\psi}_\alpha{}^T \end{pmatrix} \tag{192}$$

where the first index α' distinguishes the two, upper and lower, entries. When there are different index pairs (α_1', α_1), (α_2', α_2) we shall sometimes write $(1, \alpha_1)$, $(2, \alpha_2)$. With this notation, the free action can be written in the doubled form

$$\mathscr{A} = \frac{1}{2}\, \varphi_\alpha{}^T i G^{-1}_{0ab} \varphi_b \tag{193}$$

with

$$iG^{-1}_{0ab} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}_{\alpha',\beta'} \begin{pmatrix} 0 & i\partial\!\!\!/ \\ i\partial\!\!\!/ & 0 \end{pmatrix} \delta_{\alpha\beta}. \tag{194}$$

Its inverse is the free propagator of the φ field

$$\overbrace{\varphi_a(x)\, \varphi_b(x)} = G_0(x, y)_{ab} = \int \frac{d^D p}{(2\pi)^D}\, e^{-ip(x-y)}\, \frac{i}{p^2} \left[\begin{pmatrix} 0 & p\!\!\!/ \\ p\!\!\!/ & 0 \end{pmatrix} \begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} \right]_{\alpha'\beta'} \delta_{\alpha\beta}. \tag{195}$$

It is useful to introduce the matrices

$$\mathscr{C}^u = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathscr{C}^d = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}, \qquad \mathscr{C} = \mathscr{C}^u + \mathscr{C}^d \tag{196}$$

which are antisymmetric

$$\mathscr{C}^T = -\mathscr{C} \tag{197}$$

and whose square satisfies

$$\begin{aligned} \mathscr{C}^{u2} &= -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \mathscr{C}^{d2} &= -\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathscr{C}^2 &= -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{198}$$

Then we may write

$$iG^{-1}_{0ab} = \left[\mathscr{C}\begin{pmatrix} 0 & i\partial\!\!\!/ \\ i\partial\!\!\!/ & 0 \end{pmatrix}\right]_{\alpha'\beta'} \delta_{\alpha\beta} \tag{199}$$

and the Green's function in momentum space reads

$$G_0(p) = \frac{i}{p^2}\begin{pmatrix} 0 & p\!\!\!/ \\ p\!\!\!/ & 0 \end{pmatrix}\mathscr{C}^{-1}. \tag{200}$$

Consider now the interaction. Using the doubled fields (192) it may be written in the form

$$\mathscr{L}^{\text{int}} = \frac{g_0/N}{12}\{(\mathscr{C}^u_{12}\mathscr{C}^d_{34} + \mathscr{C}^d_{12}\mathscr{C}^u_{34})\,\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} + 2 \text{ permutations}\}\,\varphi_{a_1}\varphi_{a_2}\varphi_{a_3}\varphi_{a_4}. \tag{201}$$

Taking advantage of the antisymmetry we may write this in analogy with (13) as

$$\mathscr{L}^{\text{int}} = -\frac{1}{4!}\,VA_{1234}S_{\alpha_1\alpha_2\alpha_3\alpha_4}\varphi_{a_1a_2a_3a_4} \tag{202}$$

where we have split the interaction into a number

$$V = -36g_0 \tag{203}$$

and collected all its matrix properties in the symmetric and antisymmetric tensors

$$\begin{aligned} S_{\alpha_1\alpha_2\alpha_3\alpha_4} &= \frac{1}{3N}\,(\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} + \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3} + \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}) \\ A_{1234} &= \frac{1}{6}\,\{(\mathscr{C}^u_{12}\mathscr{C}^d_{34} + \mathscr{C}^d_{12}\mathscr{C}^u_{34}) + (14)\,(23) - (13)\,(24)\}. \end{aligned} \tag{204}$$

The symmetric tensor satisfies the following obvious identities

$$S_{\alpha\beta\gamma\gamma} = \frac{1}{3N}\,(N+2)\,\delta_{ab} \tag{205}$$

$$S_{\alpha\alpha\beta\beta} = \frac{1}{3}\,(N+2) \tag{206}$$

$$S_{\alpha\beta\gamma\delta}S_{\alpha'\beta'\gamma\delta}|_{\text{symmetrized}} = S_{\alpha\beta\alpha'\beta'}\frac{N+8}{9N} \tag{207}$$

$$S_{\alpha\beta\gamma\delta}S_{\alpha'\beta\gamma\delta} = \frac{N+2}{3N}\delta_{\alpha\alpha'}. \tag{208}$$

The external source K may now be introduced as follows

$$\frac{1}{2}\varphi^T K\varphi = \frac{1}{2}(\psi^T, \bar{\psi}C^T)\begin{pmatrix} C\lambda & \mu^T C^T \\ -C\mu & C\bar{\lambda}\end{pmatrix}\begin{pmatrix}\psi \\ C\bar{\psi}^T\end{pmatrix}. \tag{209}$$

Thus the four different blocks of K are related to each other by

$$C^{-1}K_{11} = C^{-1}\gamma_0 K_{11}^{+}\gamma_0 \tag{210}$$

$$K_{21} = -K_{12}^T.$$

If we now form the generating functional $W(K)$ it is obvious that this general property will be shared also by the full Green's function

$$G_{ab} = -2iW_{K_{ab}} \tag{211}$$

i.e., the Green's function will have the form

$$G = \left\langle T\begin{pmatrix}\psi \\ C\bar{\psi}^T\end{pmatrix} \times \overbrace{\psi^T\bar{\psi}C^T}\right\rangle \equiv \begin{pmatrix} C\Delta & \varrho C^T \\ -C\varrho^T & C\bar{\Delta}\end{pmatrix} \tag{212}$$

where $\varrho(x, y)$ denotes the Green's function $\langle T\psi(x)\,\bar{\psi}(y)\rangle$ and $\Delta(x, y)$ the anomalous part $\langle T\psi(x)\,\psi(y)\rangle$. In order to see that the $N \to \infty$ limit leads to a soluble problem it is useful to realize that due to the factored nature of free propagator and interaction, each Feynman diagram involving G_0 and V decomposes into a product in which one factor involves the 4×4 doubled spinor space $\varphi(x)$ while the other concerns only the trivial N dimensional space of degeneracy labels. For this reason, as $W(K)$ is expanded in powers of K_{ab}, the exact Green's function $G_{ab} = -2iW_{K_{ab}}$ appears with factor $\delta_{\alpha\beta}$, i.e.

$$G_{ab} \to G_{\alpha'\beta'}\delta_{\alpha\beta}. \tag{213}$$

But then we can conclude that the diagrams which make up the effective action $\Gamma[G]$ factorize just in the same way as those in $W[K]$: For the free effective action (72) this is trivial and N appears as an overall factor such that per field component, there is the finite limit:

$$\frac{1}{N}\Gamma^{(0)}[G] = -\frac{1}{2}\operatorname{tr}\log iG^{-1} - \frac{1}{2}\operatorname{tr}(iG_0^{-1}G) + \text{const.} \tag{214}$$

where G and iG_0^{-1} are the 4×4 parts of the propagators. Consider now the interacting part per field component $1/N\,\Gamma^{\text{int}}[G]$. The lowest order correction factorizes as

$$-\frac{1}{N}\frac{1}{8}VGG \to -\frac{1}{8}VAGGS_{\alpha\alpha\beta\beta}/N \tag{215}$$

and we see from (206) that the second factor involving the degeneracy labels has a finite limit for $N \to \infty$

$$S_{\alpha\alpha\beta\beta}/N \to \frac{1}{3}. \tag{216}$$

The important point to realize now is that all higher terms on $1/N\,\Gamma^{\text{int}}[G]$ are supressed by at least a factor $1/N$ with respect to this. Looking, for example, at the next correction

(98), $1/N\ i/48\ V^2G^2$, we see that it is accompanied by a tensor contraction

$$S_{\alpha\beta\gamma\delta}S_{\alpha\beta\gamma\rho}/N \to \frac{1}{3N}. \tag{217}$$

It is easy to convince ourselves that all other two particle irreducible diagrams are suppressed at least such a factor. Notice that in contradistinction to this, the expansion for $W[K]$ contains infinitely many more terms to leading order in N: All two-particle reducible diagrams (see Fig. 2) which consist of trees of bubbles appear at the same level. The Legendre transformation assembles all these in one expression $-1/8VG^2$ since the lines involve the full propagator G which includes all insertions of bubbles into bubble. Thus we remain with the limiting action per field component

$$\frac{1}{N}\Gamma[G]_{N\to\infty} = -\frac{i}{2}\,\mathrm{tr}\log G^{-1} - \frac{1}{2}\,\mathrm{tr}\,(iG_0^{-1}G) + \frac{3g_0}{4}\int dx A G(x,x)\,G(x,x). \tag{218}$$

The equilibrium configuration of G follows from the extremum

$$G = i\left[iG_0^{-1}(x,y) + \frac{g_0}{6}\,\delta(x-y)\,A^1 G(x,x)\right]^{-1}. \tag{219}$$

Only the first of the three terms in the antisymmetric matrix A contributes, because of the antisymmetry of $G_{\alpha'\beta'}(x,x)$ and we have indicated this by writing a superscript, A^1. Equ. (219) may be put into the form of a gap equation for the self-energy

$$\Sigma(x,y) = -\delta(x-y)\,\frac{g_0}{6}\,A^1 G_\Sigma(x,x). \tag{220}$$

This equation allows for two kinds of constant solutions for $\Sigma = \Sigma^0$. Since Σ must have the same symmetry as all 2×2 matrices in this doubled notation, we may write according to (209), (210), (212)

$$\Sigma^0 = \begin{pmatrix} C\Delta^{0+} & \varrho^T C^T \\ -C\varrho & C\Delta^0 \end{pmatrix}. \tag{221}$$

But looking at (220) we see that the antisymmetric matrix A permits only the Δ^0 entries to be present. Therefore we must solve

$$\begin{pmatrix} C\Delta^{0+} & 0 \\ 0 & C\Delta^0 \end{pmatrix} = -3g_0A^1\int\frac{d^Dp}{(2\pi)^D}\,\frac{1}{p^2+M^2}\begin{pmatrix} \Delta^0 & \not{p} \\ \not{p} & \Delta^{0+} \end{pmatrix}\begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} \tag{222}$$

where we have set $M = \sqrt{|\Delta^0|^2}$.
The integral can be performed with the result

$$\int\frac{d^Dp}{(2\pi)^D}\,\frac{1}{p^2+M^2} = 2^{-D/2}\,\frac{D}{2}\,b_\varepsilon M^\varepsilon \tag{223}$$

where

$$b_\varepsilon \equiv \frac{2}{D}\,\frac{1}{(2\pi)^{D/2}}\,\Gamma(1-D/2) \tag{224}$$

is a number which diverges as the dimension $D = 2+\varepsilon$ approaches 2

$$b_a \sim -\frac{1}{\pi\varepsilon}. \tag{225}$$

The off diagonal entries in (222) vanish upon integration and we have

$$\begin{pmatrix} \Delta^{0+} & 0 \\ 0 & \Delta^0 \end{pmatrix} = g_0 \frac{D}{2} b_\varepsilon M^\varepsilon \begin{pmatrix} \Delta^{0+} & 0 \\ 0 & \Delta^0 \end{pmatrix}. \tag{226}$$

This has either the trivial solution $\Delta^0 = 0$ (normal phase) or $\Delta^0 \neq 0$ which may be chosen to be real (condensed phase). In the latter case, $M = \sqrt{|\Delta^0|^2}$ satisfies

$$1 = g_0 b_\varepsilon \frac{D}{2} M^\varepsilon. \tag{227}$$

We may calculate the effective action in the neighborhood of these two solutions. Inserting into (218) the Green's function

$$G_\Delta(x, y) = \int \frac{d^D p}{(2\pi)^D} e^{-ip(x-y)} \begin{pmatrix} -\Delta^+ & \not{p} \\ \not{p} & -\Delta \end{pmatrix}^{-1} \begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} \tag{228}$$

in which Δ is an arbitrary constant we find the effective potential

$$\frac{1}{N} V(\Delta) = -\frac{1}{N} \Gamma[N]/(\text{volume} \cdot \text{time}) = \frac{i}{2} \int \frac{d^D p}{(2\pi)^D} \operatorname{og} \begin{pmatrix} -\Delta^+ & \not{p} \\ \not{p} & -\Delta \end{pmatrix} + \frac{1}{2g_0} |\Delta|^2. \tag{229}$$

The matrix in the logarithm has the eigenvalues $\pm\sqrt{p^2 + |\Delta|^2}$, each one appearing twice, in D dimensions $2^{D/2}$ times, such that

$$\frac{1}{N} V(\Delta) = \frac{1}{2g_0} |\Delta|^2 - \frac{1}{2} 2^{D/2} \int \frac{d^D p}{(2\pi)^D} \log (p^2 + |\Delta|^2) = \frac{1}{2} \left(\frac{|\Delta|^2}{g_0} - b_\varepsilon |\Delta|^{2+\varepsilon} \right). \tag{230}$$

For $\varepsilon \to 0$, b_ε diverges and the expression may be made finite by introducing a renormalized coupling $g(\mu)$, which depends on some arbitrary mass scale μ, as

$$\frac{1}{g_0 \mu^2} = \frac{1}{g(\mu)} + b_\varepsilon. \tag{231}$$

Then the potential reads

$$\frac{1}{N} V(\Delta) = \frac{\mu^\varepsilon}{2} \left[\frac{|\Delta|^2}{g} + b_\varepsilon \left(1 - \left(\frac{|\Delta|}{\mu} \right)^\varepsilon \right) \mu^2 \right] \tag{232}$$

and becomes in the limit $\varepsilon \to 0$

$$\frac{1}{N} V(\Delta) = \frac{1}{2} \left(\frac{|\Delta|^2}{g} + \frac{1}{\pi} |\Delta|^2 \log |\Delta|/\mu \right). \tag{233}$$

The extremum of V gives again $\Delta = \Delta^0 = 0$ or $\Delta = \Delta^0 = M \neq 0$ of (227).
But now we can see that the extremum at $\Delta^0 = 0$ is unstable for

$$g_0 < 0 \quad \text{or} \quad g > -b_\varepsilon^{-1} \equiv g^* \tag{234}$$

while $\Delta^0 \neq 0$ is unstable for

$$g_0 > 0 \quad \text{or} \quad g < -b_\varepsilon^{-1} \equiv g^*. \tag{235}$$

Thus we may continue the discussion for $g_0 > 0$ in the normal and for $g_0 < 0$ in the condensed phase. The condensed phase has $\Delta^0 = \langle \psi\psi \rangle \neq 0$. Such a non-vanishing ex-

pectation of a composite two-particle operator is interpreted as a signal for the presence of a condensate of pairs of particles in the ground state. The model provides a simple illustration for the fact that the value of $\Delta^0 \neq 0$ arises in a completely nonperturbative fashion. Inverting (227) we see

$$|\Delta^0| = M = (g_0 b_\varepsilon D/2)^{-1/\varepsilon} \tag{236}$$

which cannot be expanded in a power series around $g_0 = 0$. In terms of the renormalized coupling, the gap equation becomes

$$\frac{1}{g} = b_\varepsilon \left[\frac{D}{2}\left(\frac{M}{\mu}\right)^\varepsilon - 1\right] \tag{237}$$

which has the finite limit for $\varepsilon \to 0$

$$\frac{1}{g} = \frac{1}{\pi}\left(\log \frac{M}{\mu} - \frac{1}{2}\right) \tag{238}$$

and the solution

$$M = \mu e^{-(1/2+\pi/g)}. \tag{239}$$

It must be said that this soluble model is actually an example for the power of the collective quantum field approach and the discussion in terms of effective actions is somewhat more clumbsy than the direct path integral approach. Let us see, for comparison, how simple the same result would have been obtained in that approach. The partition function would read

$$Z[K] = \int \mathcal{D}\varphi e^{(i/2)(\mathcal{A}+\varphi^T K\varphi)} \tag{240}$$

with the fluctuating action

$$\mathcal{A} = \int dx \left(\frac{1}{2}\,\varphi^T i G_0^{-1}\varphi + \frac{g_0}{2N}\,\bar{\psi} C \bar{\psi}^T \psi^T C \psi\right). \tag{241}$$

The quadratic piece can be eliminated via a fluctuating pair field with an auxiliary trivial partition function

$$Z_{\text{aux}} = \int \mathcal{D}\Delta \mathcal{D}\Delta^+ e^{-i(N/2g_0)|\Delta - \varphi^T C\psi|^2} = \text{const.} \tag{242}$$

which can be multiplied into (240) without changing the functional behaviour of $Z[K]$. Therefore, we can write

$$Z[K] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\Delta \mathcal{D}\Delta^+ \exp\left\{\frac{i}{2}\int dx \varphi^T i G^{-1}_{\Delta,K}\varphi - \frac{N}{2g_0}\int dx\, |\Delta|^2\right\}. \tag{243}$$

where

$$iG^{-1}_{\Delta,K}(p) = iG_0^{-1}(p) - \mathscr{C}\begin{pmatrix}\Delta^+ & 0\\ 0 & \Delta\end{pmatrix} + K = \begin{pmatrix}-C\Delta^+ & C\not{p}\\ C\not{p} & -C\Delta\end{pmatrix} + K \tag{244}$$

is the Green's function of the fermions in the presence of the sources Δ and K. The fermions can be integrated out, leading to the new representation of $Z[K]$:

$$Z[K] = \int \mathcal{D}\Delta \mathcal{D}\Delta^+ e^{i\mathcal{A}[\Delta,K]} \tag{245}$$

in which all quantum fluctuations are accounted for by the collective field Δ with the action

$$\mathcal{A}[\Delta, K] = N\left[-\frac{i}{2}\,\text{tr}\log iG^{-1}_{\Delta,K} - \frac{1}{2g_0}\int dx\, |\Delta|^2\right]. \tag{246}$$

We now see why great simplification occurs in the limit $N \to \infty$: The exponential oscillates infinitely rapidly for the slightest fluctuations. As a consequence, these are frozen, only the extremum of $\mathcal{A}[\Delta]$ can contribute, and we have the limiting result

$$Z[K] \xrightarrow[N\to\infty]{} \exp\{i\mathcal{A}[\Delta, K]|_{\Delta=\Delta_{ex}}\} \tag{247}$$

where Δ_{ex} is the functional of K which satisfies i.e.

$$\Delta_{ex}(x) = \frac{g_0}{2} \operatorname{tr}\left(G_{\Delta_{ex},K}(x, x)\right). \tag{248}$$

In the absence of an external source $K = 0$ this is precisely the gap equation (222). The expression (247) also determines the generating functional $W[K]$

$$W[K] = \mathcal{A}[\Delta_{ex}[K], K] \tag{249}$$

from which we can evaluate the full Green's functions

$$G = -2iW_K = G_{\Delta_{ex},K} \tag{250}$$

which can be inverted to determine

$$K = iG^{-1} - iG_0^{-1} + \mathscr{C}\begin{pmatrix} \Delta_{ex}^{+} & 0 \\ 0 & \Delta_{ex} \end{pmatrix}. \tag{251}$$

We may therefore calculate the Legendre transform directly from (249):

$$\begin{aligned} \frac{1}{N}\Gamma[G] &= \frac{1}{N}\left(W[K] - \frac{1}{2}\operatorname{tr}(GK)\right) \\ &= -\frac{i}{2}\operatorname{tr}\log iG^{-1} - \frac{1}{2g_0}|\Delta_{ex}|^2 - \frac{1}{2}\operatorname{tr}\left(G\mathscr{C}\begin{pmatrix}\Delta^{+} & 0\\ 0 & \Delta\end{pmatrix}\right) - \frac{1}{2}\operatorname{tr}(iG_0^{-1}G) \end{aligned} \tag{252}$$

which reduces to (2.18) by inserting (248), (250).
We shall now use this model to illustrate the presence of four-particle clusters in the condensate. For this, we take equ. (249) and calculate the vertex function. Writing K in the 2×2 form specified in (209) and taking it to be diagonal in space, time, and degeneracy labels

$$K_{ab} = \delta(x - y)\, K_{\alpha',\beta'}(x)\, \delta_{\alpha\beta} \tag{253}$$

we may differentiate $W[K]$ with respect to $\lambda, \bar{\lambda}$ and obtain, by definition, the pair expectations

$$\langle T\psi_\alpha{}^T(x)\, C\psi_\alpha(x)\rangle = 2W_\lambda \tag{254}$$

$$\langle T\bar{\psi}_\alpha(x)\, C\bar{\psi}_\alpha{}^T(x)\rangle = 2W_{\bar{\lambda}}. \tag{255}$$

Differentiating once more gives the four-point functions (compare (40b))

$$\langle T\psi_\alpha{}^T(x)\, C\psi_\alpha(x)\, \psi_\beta{}^T(y)\, C\psi_\beta(y)\rangle = -4i(W_{\lambda\lambda} + iW_\lambda{}^2), \tag{256}$$

$$\langle T\psi_\alpha{}^T(x)\, C\psi_\alpha(x)\, \bar{\psi}_\beta(y)\, C\bar{\psi}_\beta{}^T(y)\rangle = -4i(W_{\lambda\lambda} + iW_\lambda W_{\bar{\lambda}}). \tag{257}$$

In order to evaluate the derivatives on the right hand sides, we use (249), thereby watching out that there are two sources for the dependence of $\mathcal{A}[\Delta_{ex}[K], K]$ on λ, one from the explicit source and one from the fact that Δ depends on K via (248): Thus

$$G = 2(\mathcal{A}_\Delta \Delta_K + \mathcal{A}_{\bar{\Delta}}\bar{\Delta}_K + \mathcal{A}_K)|_{\Delta=\Delta_{ex}}. \tag{258}$$

In particular for the source λ:

$$W_\lambda = [\mathcal{A}_\Delta \Delta_\lambda + \mathcal{A}_{\bar\Delta} \bar\Delta_\lambda + \mathcal{A}_\lambda)|_{\Delta = \Delta_{\rm ex}}. \tag{259}$$

Therefore, we have

$$\begin{aligned} W_{\lambda\lambda} = (&\mathcal{A}_\Delta \Delta_{\lambda\lambda} + \mathcal{A}_{\bar\Delta} \bar\Delta_{\lambda\lambda} \\ &+ \mathcal{A}_{\Delta\Delta} \Delta_\lambda \Delta_\lambda + \mathcal{A}_{\Delta\bar\Delta} \Delta_\lambda \bar\Delta_\lambda + \mathcal{A}_{\bar\Delta\Delta} \bar\Delta_\lambda \Delta_\lambda + \mathcal{A}_{\bar\Delta\bar\Delta} \bar\Delta_\lambda \bar\Delta_\lambda \\ &+ \mathcal{A}_{\lambda\Delta} \Delta_\lambda + \mathcal{A}_{\lambda\bar\Delta} \bar\Delta_\lambda + \mathcal{A}_{\lambda\lambda})|_{\Delta_{\rm ex}} \end{aligned} \tag{260}$$

with the first terms vanishing due to the extremality property of $\Delta_{\rm ex}$:

$$\mathcal{A}_\Delta[\Delta_{\rm ex}[K], K]|_{\Delta_{\rm ex}} = 0. \tag{261}$$

This can be used to simplify (260) further since it implies

$$(\mathcal{A}_{\Delta\Delta} \Delta_K + \mathcal{A}_{\Delta K})|_{\Delta_{\rm ex}} = 0 \tag{262}$$

where, for brevity, we have considered $\Delta \to (\Delta, \bar\Delta)$ as a two componente vector. In this notation we may write (260) as

$$W_{\lambda\lambda} = \mathcal{A}_{\Delta\Delta} \Delta_\lambda \Delta_\lambda + 2\mathcal{A}_{\lambda\Delta} \Delta_\lambda + \mathcal{A}_{\lambda\lambda} \tag{263}$$

and evaluate it, using (262), as

$$W_{\lambda\lambda} = -\mathcal{A}_{\lambda\Delta} \mathcal{A}^{-1}_{\Delta\Delta} \mathcal{A}_{\Delta\lambda} + \mathcal{A}_{\lambda\lambda}. \tag{264}$$

The last term is a disconnected contribution just as the terms $W_\lambda{}^2$, $W_\lambda W_{\bar\lambda}$ in (256), (257). Therefore, we obtain the result that the connected parts of the four-point functions are

$$\langle T\psi^T_{(x)} C\psi(x)\ \psi^T(y)\ C\psi(y)\rangle = -\mathcal{A}_{\lambda\Delta} \mathcal{A}^{-1}_{\Delta\Delta} \mathcal{A}_{\Delta\lambda} \tag{265}$$

$$\langle T\psi^T_{(x)} C\psi(x)\ \bar\psi(y)\ C\bar\psi^T(y)\rangle = -\mathcal{A}_{\lambda\Delta} \mathcal{A}^{-1}_{\Delta\Delta} \mathcal{A}_{\Delta\bar\lambda}. \tag{266}$$

Now we observe the derivatives

$$\mathcal{A}_\lambda = -\frac{1}{2} \operatorname{tr} (\mathscr{C}^u G) \tag{267}$$

$$\mathcal{A}_{\lambda\Delta} = i\,\frac{1}{2} \operatorname{tr} (\mathscr{C}^u G \mathscr{C}^u G)$$

$$\mathcal{A}_{\lambda\bar\Delta} = i\,\frac{1}{2} \operatorname{tr} (\mathscr{C}^u G \mathscr{C}^a G) \tag{268}$$

to be simply the Green's functions corresponding to the external legs such that we remain with vertex functions of the ψ^4, $\bar\psi^4$, $\psi^2\bar\psi^2$ type

$$-i \begin{pmatrix} \alpha_{\psi^4} & \alpha_{\psi^2\bar\psi^2} \\ \alpha_{\bar\psi^2\psi^2} & \alpha_{\bar\psi^4} \end{pmatrix} = (\mathcal{A}_{\Delta\Delta})^{-1}. \tag{269}$$

Let us calculate this matrix at $\Delta = \Delta^0$. Looking back at (246) we see that

$$\frac{1}{N} \mathcal{A}_{\substack{\bar\Delta\Delta \\ \Delta\Delta}} = -\left\{ \begin{pmatrix} 1/g_0 \\ 0 \end{pmatrix} + i \int \frac{d^D k}{(2\pi)^D} \operatorname{tr} \left\{ \left[\mathscr{C}^u \mathscr{C}^{d-1} \begin{pmatrix} -\Delta^0 & \not k \\ \not k & -\Delta^{0+} \end{pmatrix} \mathscr{C}^d \mathscr{C}^{-1} \begin{pmatrix} -\Delta^0 & \not k - \not q \\ \not k - \not q & -\Delta^{0+} \end{pmatrix} \right] \right\} \right. \tag{270}$$

$$\left. \times \frac{i}{k^2 - |\Delta_0|^2} \frac{i}{(k-q)^2 - |\Delta_0|^2} \right. . \tag{271}$$

This can be written as

$$\frac{1}{N}\mathcal{A}_{\bar{\Delta}\Delta} = -\left(\frac{1}{g_0} + A\right) \tag{272}$$

$$\frac{1}{N}\mathcal{A}_{\Delta\Delta} = -B \tag{273}$$

with the following integrals

$$A = i \operatorname{tr} \int \frac{d^D k}{(2\pi)^D} \not{k}(\not{k} - \not{q}) \frac{i}{(k^2 - M^2)} \frac{i}{(k-q)^2 - M^2}$$

$$= i \cdot 2^{D/2} \int \frac{d^D k}{(2\pi)^D} \left[\left(k(k-q) + M^2\right) - M^2\right] \frac{i}{(k^2 - M^2)} \frac{i}{(k-q)^2 - M^2} \tag{274}$$

and

$$B = iM^2 2^{D/2} \int \frac{d^D k}{(2\pi)^D} \frac{i}{(k^2 - M^2)} \frac{i}{(k-q)^2 - M^2}. \tag{275}$$

Both can be Wick rotated to Euclidean space and evaluated with Feynman's methods. The second integral gives directly

$$B = -\frac{D}{2} b_2 M^\varepsilon \frac{\varepsilon}{2} J_2(z) \tag{276}$$

where we have abbreviated $z \equiv q_E^2/M^2$ and used the subscript E to denote Euclidean scalar products $q_E^2 \equiv \boldsymbol{q}^2 - q_0^2 > 0$. The symbol J_2 stands for

$$J_2(z) = \int_0^1 [z \cdot x(1-x) + 1]^{(D/2)-2}. \tag{277}$$

The first integration of A needs a little more work. Taking the spinor trace, it becomes in Euclidean space

$$-2^{D/2} \int \frac{d^D k}{(2\pi)^D} \frac{k(k-q)_E + M^2}{(k_E^2 + M^2)\left((k-q)_E^2 + M^2\right)} \tag{278}$$

which leads to

$$A = -\frac{D}{2}(D-1) b_\varepsilon M^\varepsilon \int dx \left[\frac{q_E^2}{M^2} x(1-x) + 1\right]^{(D/2)-1} - B$$

$$= -\frac{D}{2}(D-1) b_\varepsilon M^\varepsilon \left[(D-1) J_1(z) - \left(\frac{D}{2} - 1\right) J_2(z)\right]. \tag{279}$$

In analogy with (277), we have introduced

$$J_1(z) = \int_0^1 [z \cdot x(1-x) + 1]^{(D/2)-1}. \tag{280}$$

Using the gap equation (227), we may also write

$$\frac{1}{g_0} + A = \frac{D}{2} b_\varepsilon M^\varepsilon \left[1 - (1+\varepsilon) J_1(z) + \frac{\varepsilon}{2} J_2(z)\right]. \tag{281}$$

We now invert the matrix

$$\begin{pmatrix} \mathcal{A}_{\bar{\Delta}\Delta} & \mathcal{A}_{\Delta\Delta} \\ \mathcal{A}_{\bar{\Delta}\bar{\Delta}} & \mathcal{A}_{\Delta\bar{\Delta}} \end{pmatrix} \tag{282}$$

and find for the vertex functions

$$\alpha_{\psi^2\bar{\psi}^2} = \frac{\frac{1}{g_0} + A}{\left(\frac{1}{g_0} + A\right)^2 - B^2} \tag{283}$$

$$\alpha_{\psi^4} = -\frac{B}{\left(\frac{1}{g_0} + A\right)^2 - B^2}. \tag{284}$$

Inserting (276), (281) this becomes more explicitly

$$\alpha_{\psi^2\bar{\psi}^2} = \frac{1}{\frac{D}{2} b_\varepsilon M^\varepsilon} \frac{1 - (1+\varepsilon) J_1(z) + \frac{\varepsilon}{2} J_2(z)}{\left(1 - (1+\varepsilon) J_1(z)\right)\left(1 - (1+\varepsilon) J_1(z) + \varepsilon J_2(z)\right)} \tag{285}$$

$$\alpha_{\psi^4} = \frac{1}{\frac{D}{2} b_\varepsilon M^\varepsilon} \frac{\frac{\varepsilon}{2} J_2(z)}{\left(1 - (1+\varepsilon) J_1(z)\right)\left(1 - (1+\varepsilon) J_1(z) + \varepsilon J_2(z)\right)}. \tag{286}$$

The denominators display a pole at $z = 0$ since there $J_1(0) = J_2(0)$. This is, of course, a consequence of the Nambu-Goldstone theorem, since the ground state expectation $\Delta = \Delta^0$ breaks the symmetry of the Lagrangian (188) under constant gauge transformations

$$\psi \to e^{i\alpha}\psi. \tag{287}$$

The formulas become particularly simple in the limit $\varepsilon \to 0$, $D = 2$ where we can expand

$$J_1(z) = 1 + \frac{\varepsilon}{2} J(z) + O(\varepsilon^2) \tag{288}$$

where $J(z)$ is the integral

$$J(z) = \int_0^1 dk \log\left(z \cdot x(1-x) + 1\right). \tag{289}$$

It can be expressed in terms of an angle θ

$$\theta = \operatorname{Arth} \sqrt{\frac{z}{z+4}} \tag{290}$$

as

$$J(z) = -2 + 2\theta \operatorname{cth} \theta. \tag{291}$$

It turns out that J_2 is simply related to the same expression

$$J_2(z)|_{D=z} = \frac{1}{z} \frac{2\sqrt{z}}{z+4} \operatorname{Arth} \sqrt{\frac{z}{z+4}} = \frac{2}{z+4}\left(J(z) + 2\right). \tag{292}$$

Therefore we find

$$\alpha_{\psi^2\bar{\psi}^2} = 2\pi \frac{z+2}{z} \frac{1}{J(z)+2} \tag{293}$$

$$\alpha_{\varphi^4} = -4\pi \frac{1}{z} \frac{1}{J(z)+2}. \tag{294}$$

Again, there is the Nambu-Goldstone pole in the denominators. It should be noted that in two dimensions this is an artifact of the $N \to \infty$ limit which suppresses fluctuations of the Δ field. These would wipe out the condensate [*12*] (just as there is no second order pion condensate in three dimensions due to fluctuations [*13*]).
Equs. (283), (284) can be inverted and written in the form

$$g_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \alpha \left[1 - \begin{pmatrix} B & A \\ A & B \end{pmatrix} \alpha \right]^{-1} \tag{295}$$

where α is the matrix

$$\alpha = \begin{pmatrix} \alpha_{\psi^4} & \alpha_{\psi^2\bar{\psi}^2} \\ \alpha_{\bar{\psi}^2\psi^2} & \alpha_{\bar{\psi}^4} \end{pmatrix}. \tag{296}$$

Equ. (295) is this model's version of the gap-type of equation (187). We observe that the four-particle elements on the diagonal of $g_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (i.e. of V) vanish. They would do so to any finite order in perturbation theory. But here, the spontaneous symmetry breakdown does generate four-particle clusters. It is easy to see why this result escapes perturbative calculations. Inserting for small g

$$z = \frac{{q_E}^2}{\mu^2} e^{(1+(2\pi/g))} \tag{297}$$

we find the limiting behavior of the vertices at fixed ${q_E}^2$, μ^2

$$\alpha_{\psi^4} \sim -\frac{g}{{q_E}^2/\mu^2} e^{-2\pi/q} + \cdots, \quad \alpha_{\psi^2\bar{\psi}^2} \sim g + O(g^2, e^{-2\pi/g}) \tag{298}$$

which cannot be expanded in a powers series around $g = 0$.
Notice that the four-particle clusters in this model are a direct consequence of the formation of Cooper pairs.
It should be mentioned that a number of years ago, the idea was very popular that all strong interactions were self-generated and that no input interaction V was needed at all to find non-trivial soutions for the scattering matrix of strong interactions (bootstrap) [*14*]. In nuclear physics, however, the non-perturbative origin of α particle formation is probably more of the pair induced type described by the model at hand.

VIII. Extension of Landau's Theory of Fermi Liquids

In order to illustrate the power of the higher effective action we would like to show that it leads, in a natural way, to an important generalization [*15*] of Landau's theory of Fermi liquids [*16*]. Suppose we have obtained a ground state solution G, α to equs. (185) (187). Let $\varkappa_i^l$ be the single particle energies in the self consistent potential (see (113)). We may then study the dynamics of small oscillations around this solution by replacing

$$\begin{aligned} G &\to G + \delta G \\ \alpha &\to \alpha + \delta\alpha \end{aligned} \tag{299}$$

and expanding $\Gamma[G + \delta G, \alpha + \delta\alpha]$ to second order in $\delta G, \delta\alpha$. Using the functional $V[G, \alpha]$ (see equ. (153)), this expansion can be written in the following compact form

$$\delta^2\Gamma[G, \alpha] = \frac{i}{4}\,\delta G G^{-1} \times G^{-1}\delta G - \frac{1}{8}\,V\delta G\delta G + \frac{i}{8}\left(\alpha G^2\delta G\delta G V - \frac{1}{3}\,\alpha G^3\delta G V_G \delta G\right)$$
$$- \frac{i}{48}\,\delta\alpha G^4 V_\alpha \delta\alpha + \frac{i}{12}\,(\delta\alpha G^3\delta G V - \alpha G^3\delta G V_\alpha \delta\alpha). \tag{300}$$

This result is exact. For application one may use some truncated form of $V[G, \alpha]$ and obtain an approximate equation. The index contractions are best recorded graphically as illustrated in Fig. 10.

If we extremize $\delta^2\Gamma$ with respect to $\delta G, \delta\alpha$ we obtain coupled equations of motion for the oscillations of particle densitypair correlations, and vertex functions.

We now realize that these may be considered as the natural extension of Landau's theory of Fermi liquids: Landau considers a translationally invariant ground state in

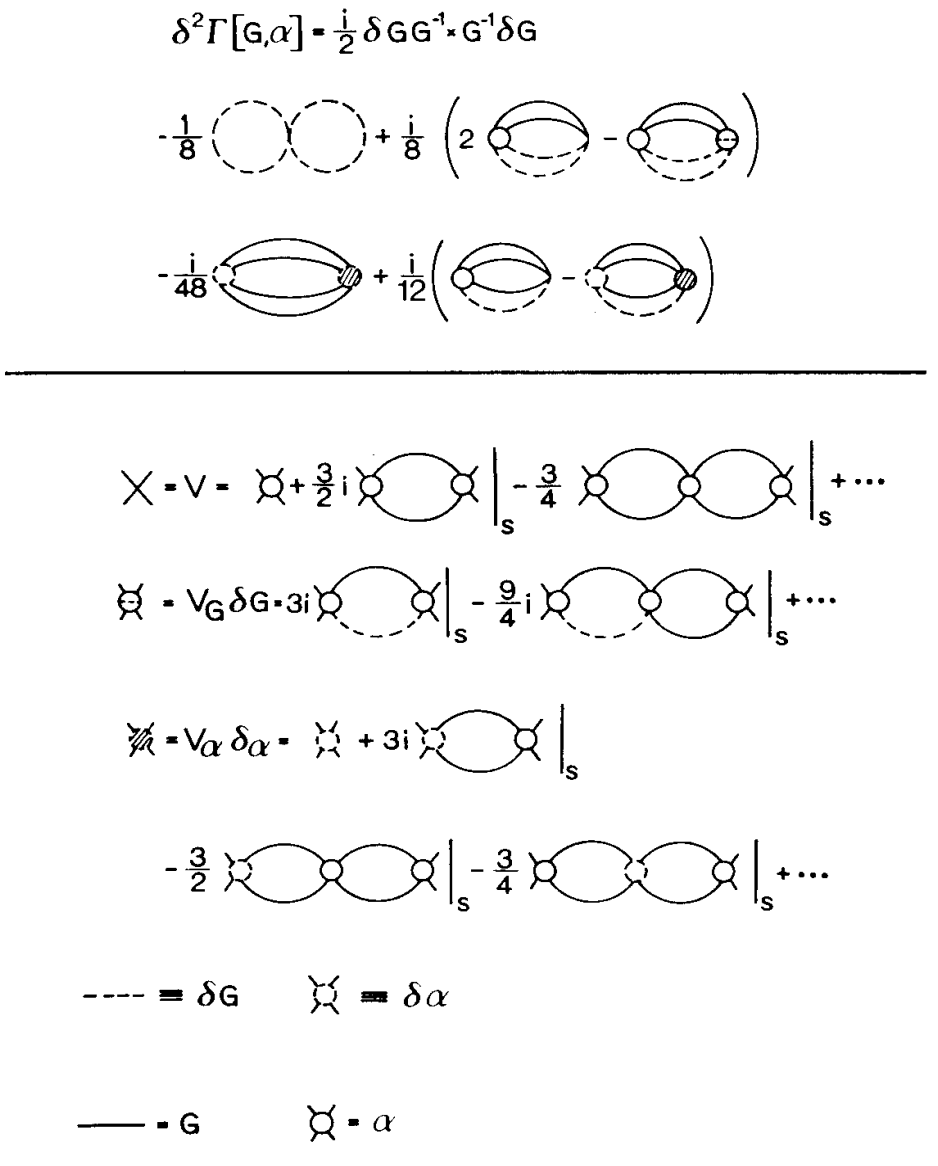


Fig. 10. The expansion of the higher effective action $\Gamma^{\text{s.c.}}[G, \alpha]$ around the ground state solution up to quadratic order in density, pair, and vertex oscillations. The first two terms correspond to Landau's theory of Fermi liquids. The others render corrections in the form of collision integrals and determine the dynamical properties of vertices

which the levels $\varkappa^l$ may be labelled by the single particle momenta $\boldsymbol{p}$. At very low temperature, the levels are filled up to some Fermi momentum p_F. At that momentum the group velocity of the particles

$$\boldsymbol{v} \equiv \partial\varkappa(\boldsymbol{p})/\partial\boldsymbol{p} \tag{301}$$

is defined as Fermi-velocity

$$\boldsymbol{v}_F = |\boldsymbol{v}|_{|\boldsymbol{p}|=p_F}. \tag{302}$$

Landau then assumes the following expansion of the energy of the normal Fermi liquid according to density oscillations

$$\delta E = \sum_{\boldsymbol{p}} \varkappa(\boldsymbol{p})\, \delta n_{\boldsymbol{p}}(\boldsymbol{x}) + \frac{1}{2} \sum_{\boldsymbol{p},\boldsymbol{p}'} V_{\boldsymbol{p},\boldsymbol{p}'} \delta n_{\boldsymbol{p}}(\boldsymbol{x}) \delta n_{\boldsymbol{p}'}(\boldsymbol{x}). \tag{303}$$

Where $\delta n_{\boldsymbol{p}}(x)$ is the change in the density of particles of momentum $\boldsymbol{p}$ at the place $\boldsymbol{x}$. By rewriting this expression as

$$\delta E = \sum_{\boldsymbol{p}} \tilde{\varkappa}(\boldsymbol{p})\, \delta n_{\boldsymbol{p}}(\boldsymbol{x}) \tag{304}$$

the quantities

$$\tilde{\varkappa}(\boldsymbol{p}) = \varkappa(\boldsymbol{p}) + \sum_{\boldsymbol{p}'} V_{\boldsymbol{p},\boldsymbol{p}'} \delta n_{\boldsymbol{p}'}(\boldsymbol{x}) \tag{305}$$

may be considered as local energies of the particles in the oscillating density field and can be used to derive the classical equation of motion for the total density of particles in phase space

$$\frac{\partial}{\partial t}\, n_{\boldsymbol{p}}(\boldsymbol{x}, t) + \{n_{\boldsymbol{p}}, \tilde{\varkappa}_{\boldsymbol{p}}\} = 0 \tag{306}$$

where

$$\{n_{\boldsymbol{p}}, \tilde{\varkappa}_{\boldsymbol{p}}\} = \nabla_{\boldsymbol{x}} n_{\boldsymbol{p}} \cdot \nabla_{\boldsymbol{p}} \tilde{\varkappa}_{\boldsymbol{p}} - \nabla_{\boldsymbol{p}} n_{\boldsymbol{p}} \cdot \nabla_{\boldsymbol{x}} \tilde{\varkappa}_{\boldsymbol{p}} \tag{307}$$

is the Poisson bracket. Inserting here the decomposition of $n_{\boldsymbol{p}}(\boldsymbol{x}, t)$ into space time independent Fermi distributions

$$n_{\boldsymbol{p}}{}^0 = (e^{\varkappa(\boldsymbol{p})/kT} - 1)^{-1} \tag{308}$$

of the ground state and small oscillations

$$n_{\boldsymbol{p}}(\boldsymbol{x}, t) = n_{\boldsymbol{p}}{}^0 + \delta n_{\boldsymbol{p}}(\boldsymbol{x}, t) \tag{309}$$

and keeping only linear terms in $\delta n_{\boldsymbol{p}}(\boldsymbol{x}, t)$, Landau finds the transport equation

$$\frac{\partial}{\partial t}\, \delta n_{\boldsymbol{p}}(\boldsymbol{x}, t) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \delta n_{\boldsymbol{p}}(\boldsymbol{x}, t) = \frac{\partial n_{\boldsymbol{p}}{}^0}{\partial \varkappa} \sum_{\boldsymbol{p}'} V_{\boldsymbol{p},\boldsymbol{p}'} \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \delta n_{\boldsymbol{p}'}(\boldsymbol{x}, t). \tag{310}$$

Let us now demonstrate that the same equation is obtained from (300) if we keep only the first two terms

$$\delta^2 \Gamma[G, \varkappa] \approx \frac{i}{4}\, \delta G G^{-1} \times G^{-1} \delta G - \frac{1}{8}\, V \delta G \delta G. \tag{311}$$

The first term can be evaluated by inserting the solution for G for a translationally invariant system. If we consider a normal liquid such that pair correlations are absent, we may write the self-consistent solution (118) for a translationally invariant system

as ($p^0 \equiv \varepsilon$)

$$G_{12}(p) = \frac{i}{\varepsilon - \varkappa(\boldsymbol{p})} \tag{312}$$

where 12 is the $\langle T\psi\psi^+\rangle$ matrix element of G. We may then form the loop integral (at zero temperature)

$$\begin{aligned} &-i\int \frac{dp_0}{2\pi}\, G\left(p + \frac{q}{2}\right) G\left(p - \frac{q}{2}\right) \\ &= i\int \frac{d\varepsilon}{2\pi} \frac{1}{\varepsilon + \frac{q_0}{2} - \varkappa\left(\boldsymbol{p} + \frac{\boldsymbol{q}}{2}\right)} \frac{1}{\varepsilon - \frac{q_0}{2} - \varkappa\left(\boldsymbol{p} - \frac{\boldsymbol{q}}{2}\right)} \end{aligned} \tag{313}$$

where $\boldsymbol{q}_0$ is the total and $\boldsymbol{p}$ the relative momentum of the particle-hole pair in the loop. This may be rewritten as

$$\begin{aligned} &= \left[q_0 - \varkappa\left(\boldsymbol{p} + \frac{\boldsymbol{q}}{2}\right) + \varkappa\left(\boldsymbol{p} - \frac{\boldsymbol{q}}{2}\right)\right]^{-1} \\ &\times \int \frac{d\varepsilon}{2\pi} \left[-\frac{i}{\varepsilon + \frac{q_0}{2} - \varkappa\left(\boldsymbol{p} + \frac{\boldsymbol{q}}{2}\right)} + \frac{i}{\varepsilon - \frac{q_0}{2} - \varkappa\left(\boldsymbol{p} + \frac{\boldsymbol{q}}{2}\right)} \right]. \end{aligned} \tag{314}$$

The integrals may be performed with the correct ε prescription (levels above the Fermi energy $\varepsilon_F \approx \mu$ are slightly below the axis and vice versa) and yield the Fermi distributions

$$n\left(\boldsymbol{p} + \frac{\boldsymbol{q}}{2}\right) - n\left(\boldsymbol{p} - \frac{\boldsymbol{q}}{2}\right). \tag{315}$$

In the long wavelength limit $\boldsymbol{q} \to 0$, this reduces to

$$\frac{\partial n}{\partial \varkappa} \frac{\partial \varkappa}{\partial \boldsymbol{p}} \boldsymbol{q} = \frac{\partial n}{\partial \varkappa} \boldsymbol{v} \cdot \boldsymbol{q} \approx -\boldsymbol{v} \cdot \boldsymbol{q} \delta(\varkappa - \mu). \tag{316}$$

Therefore we find

$$-iG \times G = -\frac{\boldsymbol{v} \cdot \boldsymbol{q}}{q_0 - \boldsymbol{v} \cdot \boldsymbol{q}} \frac{\partial n}{\partial \varkappa}. \tag{317}$$

Taking the inverse of this we may write (311) as

$$\delta^2 \Gamma[G, \alpha] \approx \frac{1}{4} \delta G \left(\frac{q_0}{\boldsymbol{v} \cdot \boldsymbol{q}} - 1\right) \left(\frac{\partial n}{\partial \varkappa}\right)^{-1} \delta G - \frac{1}{8} V \delta G \delta G \tag{318}$$

whose extremum corresponds to an equation of motion

$$(q_0 - \boldsymbol{v} \cdot \boldsymbol{q})\, \delta G = \boldsymbol{v} \cdot \boldsymbol{q} \frac{\partial n}{\partial \varkappa} \frac{1}{2} V \delta G \tag{319}$$

which is precisely the energy-momentum form of (310) with $\delta G_{\boldsymbol{p}}(q_0, \boldsymbol{q})$ being the Fourier transform of $\delta n_{\boldsymbol{p}}(\boldsymbol{x}, t)$

$$\delta G_{\boldsymbol{p}}(q_0, \boldsymbol{q}) = \int dx e^{i(q_0 t - \boldsymbol{q}\boldsymbol{x})} \delta n_{\boldsymbol{p}}(\boldsymbol{x}, t). \tag{320}$$

By extending the same consideration to include also the pair correlation parts of G we may derive the extension of Landau's theory to superfluid systems. The important feature of the new expansion (300) is that not only does it provide for more contributions to $\delta G \delta G$, which are commonly referred to as collision integrals and whose inclusion is standard in this theory; there are also dynamical oscillations of the vertex strengths, i.e. of the Landau parameters themselves, which are coupled with the original density and pair modes.
The study of this coupled set of equations promises to render important new insights into the fundamental excitations of Fermi liquids.

IX. Treatment of Singular Potentials

There are many physical systems where the potential cannot be considered as weak but still, the interaction does not really have dramatic effects for sufficiently low densities. An example is the hard core repulsion between atoms in liquid ^{3}He where the potential is practically infinite for distances smaller than 0.25 nm. In this case the equation (187) is useless as it stands since with V also α becomes infinite as long as the expansion is truncated after a finite number of terms. The standard way to circumvent this problem is by unitarizing the interaction via rescattering corrections, which may take place for pairs of particles between further many-body interactions, i.e. one sums up an infinite number of bubble diagrams. The roughest and simplest approach might start with the first order expression (74) for $\Gamma^{\text{int}}[G]$

$$\Gamma^{\text{int}}[G] \approx -\frac{1}{8} VG^2 \tag{321}$$

inserting this into (88) we obtain the approximate α matrix

$$\alpha^{\text{B.S.}} = V \frac{1}{1 + \frac{i}{2} G^2 V} \tag{322}$$

(BS stands for bubble sum) which may be used in the Hartree-Fock-Bogoljubov equation (185). Actually, the self-energy can here be simplified a little by writing

$$-\frac{1}{2} VG + \frac{i}{6} VG^3 \alpha = -\frac{1}{6} VG - \frac{1}{3} \alpha^{\text{B.S.}} G. \tag{323}$$

Of course, the summation (322) corresponds to solving the Schrödinger equation for the hard core potential.
It must be mentioned that formula (322) violates a general important property of α namely that of being symmetric in all four external lines. Since the iteration of V proceeds in a string like fashion, one may wonder how the defining formula (87) manages to ensure this symmetry even though it also amounts to a string like iteration of $\Gamma^{\text{int}}_{GG}[G]$ with $\Gamma^{\text{int}}_{G_{12}G_{34}}$ being read as a matrix with left indices (12) and right indices (34) whose topology does not look symmetric. The important difference, however, is that Γ^{int} contains infinitely many higher order terms in V which combine with iterated lower order terms in such a way that the α matrix is indeed symmetric. This is illustrated to lower orders in Fig. 11.
A more symmetric but analytically harder to handle approach would consist in summing an infinite set of graphs in $\Gamma^{\text{int}}[G, \alpha]$. It is easy to convince oneself that the sunflower

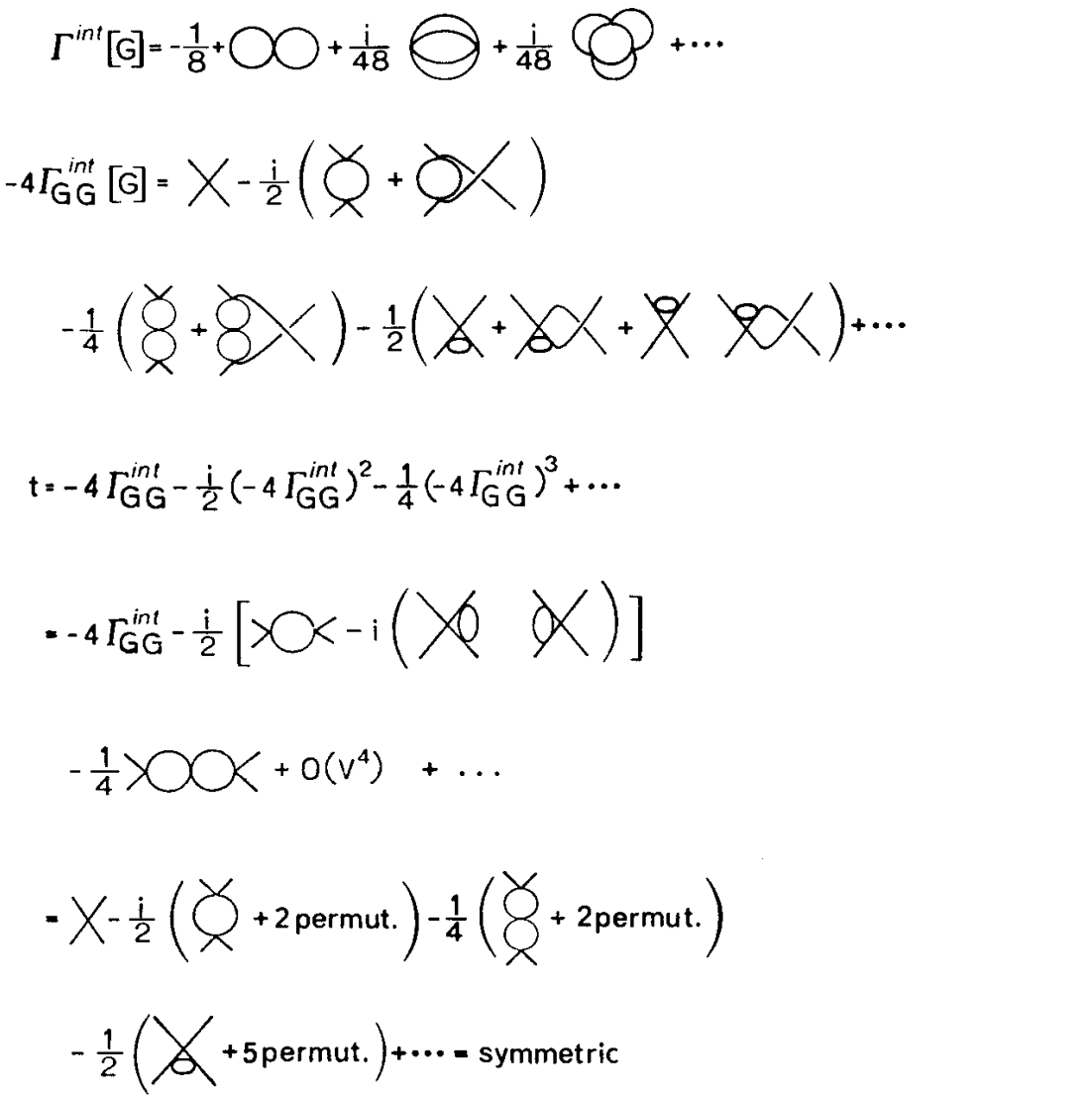


Fig. 11. Illustration of the symmetry of the scattering amplitude up to third order in V. The symmetry is achieved only due to higher orders contained in Γ^{int}_{GG} combining with iterated lower orders

like diagrams, whose leading examples are the last two graphs shown in Fig. 9, may be summed up in the form

$$\frac{i}{2}\sum_{n\geq 3}\left(\frac{i}{2}\alpha G^2\right)^n\Big/n \tag{324}$$

which can be written as

$$-\frac{i}{2}\,\mathrm{tr}\,\log\left(1-\frac{i}{2}G^2\alpha\right)+\frac{1}{4}\alpha G^2+\frac{i}{16}\alpha^2 G^4. \tag{325}$$

Now the equ. of motion reads

$$V=\alpha+\frac{3}{2}i\alpha G^2\left(1-\frac{i}{2}G^2\alpha\right)^{-1}\alpha \tag{326}$$

which is symmetric in all four legs since the infinite string of bubbles occurs in all three channels. The factor three accounts for the three permutations of the external

legs. If one channel is selected as especially relevant, (326) again reduces to (322)

$$V \to \lambda + \frac{i}{2}\,\lambda G^2 \left(1 - \frac{i}{2} G^2 \alpha\right)^{-1} \lambda = \lambda \left(1 - \frac{i}{2} G^2 \lambda\right)^{-1}. \tag{327}$$

Another example is the plasma in which the bubble sum (322) provides for a screening of the Coulomb potential of large distances and removes the infrared problem at zero momentum.

X. Outlook

We hope to have demonstrated that for the discussions of large amplitude phenomena the higher effective actions are far better suited than the collective quantum fields which have recently become fashionable in nuclear physics. These actions possess all the attractive features of that popular approach, while lacking its disabilities, the most serious being the failure to cope naturally with exchange forces. Moreover, they permit a straight-forward description of multiparticle correlations at the same level as pairing. Therefore they can extend our understanding of nuclear oscillations by a hierarchy of higher cluster phenomena. The generalization of Landau's theory of Fermi liquids was just an example of the many possibilities of applications ahead [*17*].

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