Restrictions on Pion Condensate.

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We prove the following theorem: In a system of charged nonrelativistic bosons with a finite particle number per unit volume and an energy spectrum $\omega \sim (|\boldsymbol{q}|-q_c)^2 + 1 - \varrho/\varrho_c$, no condensate of momentum $q = q_c \neq 0$ can form in a second-order phase transition. The theorem is intended as a further step towards understanding the properties of pions in dense nuclear matter. It implies that if a pion condensate is to be discovered experimentally, it must arise in a first-order phase transition or the spectrum must acquire an anomalous behaviour $(|\boldsymbol{q}|-q_c)^{2-\eta}$ at the transition point due to fluctuations.

Recently [1], we have argued that neither thermal nor quantum fluctuations would tolerate the formation of a pion condensate of momentum $q = q_c \approx 2\mu$ ($\mu \equiv$ pion mass) in a second-order phase transition. Its existence was proposed on the basis of mean-field studies of nuclear matter, if densities exceed a critical value of the order $0.5\mu^3$. Our argument was proved only for the somewhat artificial limit that the pion fields π_a ($a = 1, \ldots, N$) obey a theory with isotopic O_N symmetry with $N \to \infty$ rather than O_3. The violent nature of fluctuations suggested, however, that the result should not depend on this limit and that there must be a theorem of a rigor similar to those prohibiting condensates in one and two dimensions [3].

It is the purpose of this note to lend further support to this conjecture by presenting a no-go theorem for the thermal condensate with $N = 2$, *i.e.* when there are only charged pions, say π^+. Thus we are quite close to the true physical situation and demonstrate the inessential nature of the limit $N \to \infty$. The present proof has, however, another unphysical feature: It assumes the pions to be nonrelativistic with a finite-particle number per unit volume which could, in principle, be untrue in the would-be pion condensate. This assumption should be removable by a more detailed study of the interaction.

The physical system to which our theorem does apply is superfluid ^{4}He in which case it excludes the condensation of rotons in a second-order transition.

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(1) H. KLEINERT: *Phys. Lett. B*, **102**, 1 (1981). There is a similar problem in solid-state physics where a $q \neq q_c = 0$ condensate cannot form in magnetic superconductors (the so-called cryptoferromagnetism), H. KLEINERT: *Phys. Lett. A*, **83**, 294 (1981) and in press.

(2) A. B. MIGDAL: *Rev. Mod. Phys.*, **50**, 107 (1978).

(3) N. D. MERMIN and H. WAGNER: *Phys. Rev. Lett.*, **17**, 1133 (1966); P. C. HOHENBERG: *Phys. Rev.*, **158**, 383 (1967).

Let ψ be the charged interacting nonrelativistic Bose field quantized according to the canonical rule.

(1) $$[\psi_{\boldsymbol{q}}(\tau), \psi^{+}_{\boldsymbol{q}'}(\tau)] = \delta_{\boldsymbol{q},\boldsymbol{q}'} .$$

Its dependence on imaginary time is given by

(2) $$\psi_{\boldsymbol{q}}(\tau) = \exp[H\tau]\psi_{\boldsymbol{q}}\exp[-H\tau] ,$$

where H is the Hamiltonian.

Consider the thermal Green's function

(3) $$G(\omega_n, \boldsymbol{q}) = \int\limits_0^{1/T} \mathrm{d}\tau \exp[i\omega_n\tau]\langle\psi_{\boldsymbol{q}}(\tau)\psi^{+}_{\boldsymbol{q}}(0)\rangle ,$$

where $\langle\theta\rangle \equiv \mathrm{tr}\,(\exp[-H/T]\theta)/\mathrm{tr}\,(\exp[-H/T])$ denotes the thermal average.

There is a spectral representation

(4) $$G(\omega_n, \boldsymbol{q}) = \int \frac{\mathrm{d}\omega}{2\pi}\,\varrho(\omega)\,\frac{-1}{\omega_n - \omega}$$

with

(5) $$\varrho(\omega) = \left\{\sum_{n,m}\exp[-(E_n + E_m)/2T]|\langle n|\psi_{\boldsymbol{q}}|m\rangle|^2\cdot \cdot 2\pi\delta(\omega - E_m + E_n)\,2\sinh\frac{\omega}{2T}\right\}\Big/\sum_n \exp[-E_n/T] ,$$

which ensures that $g \equiv G(0, \boldsymbol{q})$ can never be negative. The same spectral function determines also the commutator

(6) $$c \equiv \langle[\psi_{\boldsymbol{q}}, \psi^{+}_{\boldsymbol{q}}]\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\,\varrho(\omega) = 1$$

and anticommutator

(7) $$a \equiv \langle\{\psi_{\boldsymbol{q}}, \psi^{+}_{\boldsymbol{q}}\}\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\,\varrho(\omega)\,\mathrm{ctgh}\,\frac{\omega}{2T} .$$

Our no-go theorem will be a consequence of a simple but powerful inequality [4] which holds among the three quantities a, c, g as a consequence of convexity of the function $f(\omega) \equiv \omega\,\mathrm{ctgh}(\omega/2T)$. Recall that any convex function $f(\omega)$ satisfies

(8) $$f(\mu_1\omega_1 + \mu_2\omega_2) \leqslant \mu_1 f(\omega_1) + \mu_2 f(\omega_2)$$

for arbitrary positive μ_1, μ_2 with $\mu_1 + \mu_2 = 1$. By going to infinitely many μ_i's with $\sum_{i=1}^{\infty}\mu_i = 1$ we arrive at Jensen's inequality

(9) $$f\left(\int \frac{\mathrm{d}\omega}{2\pi}\,\mu(\omega)\,\omega\right) \leqslant \left(\int \frac{\mathrm{d}\omega}{2\pi}\,\mu(\omega)\,f(\omega)\right)$$

([4]) G. Roepstorff: *Commun. Math. Phys.*, **53**, 143 (1977); *J. Stat. Phys.*, **18**, 191 (1978).

in which $\mu(\omega)$ can be an arbitrary positive function of unit area, *i.e.* $\int (d\omega/2\pi)\mu(\omega) = 1$. But $\mu(\omega) \equiv g^{-1}\varrho(\omega)/\omega$ has this property, due to (4), (6), such that we may conclude

$$\frac{c}{g}\,\text{ctgh}\,\frac{c}{2Tg} \leqslant \frac{a}{g}\,. \tag{10}$$

Moreover, because of the canonical commutation rule, $a = 1 + 2\langle \psi_{\boldsymbol{q}}^+ \psi_{\boldsymbol{q}} \rangle$ such that (10) implies

$$\langle \psi_{\boldsymbol{q}}^+ \psi_{\boldsymbol{q}} \rangle \geqslant (\exp[g^{-1}/T] - 1)^{-1}\,. \tag{11}$$

Thus the number of interacting particles at each momentum state is bounded from below by the Bose distribution with $g^{-1} = G(0, \boldsymbol{q})^{-1}$ as an effective energy. Notice that the equal sign is achieved for the free case where

$$g^{-1} = G(0, \boldsymbol{q})^{-1} = \frac{q^2}{2m} - \mu\,.$$

We note in passing that the inequality (11) immediately eliminates conventional Bose condensates in two dimensions, since at the critical point $G(0, \boldsymbol{q})^{-1}$ would have to vanish like $\boldsymbol{q}^2$ resulting in a divergence of the number of particles per unit volume ([4]).

Consider now the case at hand. Suppose there exists a condensate of the rotonlike nature suggested by mean-field studies in nuclear physics ([2]) (or magnetic superconductors ([1])). By rotonlike we mean that the interacting inverse pion propagator would behave, near some critical density ϱ_c, as ([2])

$$G^{-1}(0, \boldsymbol{q}) \approx (|\boldsymbol{q}| - q_c)^2 + \tau\,, \qquad \tau \equiv 1 - \frac{\varrho}{\varrho_c}\,, \tag{12}$$

if the momenta lie in the neighborhood of a spherical shell of radius $q_c \neq 0$. Inserting this into (11) we see that under the condition of a finite number of particles for unit volume, ϱ can never reach ϱ_c.

Our theorem should not be abused to discourage experimental search for a condensate. There are two possibilities by which nature can escape the conclusion: one possibility is that the transition must take place to first order and this would, in many ways, facilitate its detection even though it would not announce itself by marked pretransitional fluctuations (since these would remain small even at the transition point itself). Rather, one should look for typical first-order characteristics (« supercompression », « superexpansion », transition enthalpy) ([5]).

As a matter of fact, a first-order transition was deduced in a completely different study of the condensation process a long time ago ([6]). Our theorem, therefore, favours those author's results over the more standard ones quoted in ref. ([1]).

The other possibility is that our assumption of a rotonlike spectrum (12) was too restrictive and that the energy vanishes in some wedgelike fashion, say $G^{-1}(\boldsymbol{q}) \sim (|\boldsymbol{q}| - q_0)^{2-\eta} + \tau$. Then there can be a continuous phase transition with very interesting properties which would take place in the same region in $\varrho \sim \varrho_c$, where

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([5]) H. KLEINERT and F. PALUMBO: to be published.

([6]) F. CALOGERO: in *The Nuclear Many-Body Problem*, edited by F. CALOGERO and C. CIOFI DEGLI ATTI, Vol. **2** (Roma, 1972), p. 535; F. CALOGERO and F. PALUMBO: *Lett. Nuovo Cimento*, **6**, 663 (1973).

pion condensation was expected previously. The high-density phase would not be truly ordered but nevertheless be characterized by an increased « stiffness » of nuclear matter against local isotopic spin rotations. This is in complete analogy to a thin surface of superfluid ^{4}He which has bending energies (7) for local phase changes, even though there exists no long-range order, in agreement with the no-go theorem of ref. (3). It is well known that in this system there is a phase transition characterized by a breakdown of local stiffness (7,8) which is due to thermal generation of freely moving vortices. Since such objects exist also in the would-be pion condensate, they could give rise to an interesting set of new and unexpected phenomena (9).

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(7) J. M. Kosterlitz and D. J. Thouless: *J. Phys. C*, **6**, 1181 (1973); J. M. Kosterlitz: *J. Phys. C*, **7**, 1046 (1974).

(8) See A. F. Hebard and A. T. Fiory: *Phys. Rev. Lett.*, **44**, 291 (1980).

(9) This order-disorder transition is discussed in H. Kleinert: *Lett. Nuovo Cimento*, **34**, 103 (1982).