Linelike Defects in Pion Condensates.

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We want to point out the existence of linelike defects in pion condensates and develop their disorder field theory as a basis for studying fluctuation phenomena and phase transitions.

There is no doubt that fluctuations cause important qualitative changes to mean-field results on pion condensates as a consequence of the Landau-Peierls theorem (1,2). Rough estimates were first undertaken, in analogy with similar work on cholesteric liquid crystals (3), by DYUGAEV (4) and revived recently (5). Their convincingness was, however, limited due to the use of selective sums of Feynman diagrams in a regime where each term diverges. These difficulties were absent in an independent investigation by the author (6) who used the well-known trick of idealizing nature to contain infinitely many « isospin » symmetric pions in which case there is an exact solution to the fluctuation problem. In this slightly unphysical scenario it was possible to prove that for a mean-field behaviour of the energy $\omega \sim (|\boldsymbol{q}| - q_0)^2 + 1 - \varrho/\varrho_c$ in the neighbourhood of $q \sim q_0$, $\varrho \sim \varrho_c$ no condensate could form in a second-order phase transition. Moreover, it was noticed that the $N \to \infty$ limit could not be essential to the argument, since for normal charged pions there is an exact inequality leading to the same conclusion (7), albeit in another slightly unphysical idealization that pions are nonrelativistic. Convinced by these arguments from two directions it was concluded that either the transition must be of first order (as imagined in an initial work along quite different lines (8)), or there must be a continuous transition of the Berezinskii-Kosterlitz-Thouless type (9)

(1) R. E. PEIERLS: *Helv. Phys. Acta Suppl.*, **7**, 81 (1934).
(2) L. D. LANDAU and E. M. LIFSHITZ: *Statistical Physics* (Reading, Mass., 1969), p. 402.
(3) S. A. BRAZOVSKII: *Sov. Phys. JETP*, **41**, 85 (1975).
(4) A. M. DYUGAEV: *Sov. Phy . JETP Lett.*, **22**, 83 (1975).
(5) G. BAYM: *Proceedings of the Hakoné Symposium on High-Energy Nuclear Interactions and the Properties of Nuclear Matter*, edited by K. NAKAI and A. S. GOLDHABER, part III (Kyoto, and Tokyo), p. 173. A more detailed account is available as an Urbana preprint 1981 by K. KOLEMAINEN and G. BAYM.
(6) H. KLEINERT: *Phy . Lett. B*, **102**, 1 (1982).
(7) H. KLEINERT: *Lett. Nuovo Cimento* (in press), Lecture and Berlin preprint presented at the *1981 Conference of the European Physical Society, Lisboa, July 1981.*
(8) F. CALOGERO: in *The Nuclear Many-Body Problem*, edited by F. CALOGERO and C. CIOFI DEGLI ATTI, Vol. 2 (Roma, 1972), p. 535; F. CALOGERO and F. PALUMBO: *Lett. Nuovo Cimento*, **6**, 663 (1973).
(9) L. BEREZINSKII: *Žurn. Èksp. Teor. Fiz.*, **59, 32**, 423 (1971 ; J. M. KOSTERLITZ and D. J. THOULESS: *J. Phys. C*, **6**, 1181 (1973); J. M. KOSTERLITZ: *J. Phys. C*, **7**, 1046 (1974).

in which the condensate has only pseudo-long-range order characterized by « stiffness » to field bending a phenomenon well known in films of superfluid ^{4}He and smectic liquid crystals [10]. This suggestion was taken up and studied in detail by BAYM *et al.* [11] who calculated the typical algebraic long-range behaviour of correlation functions. In this note we shall use the analogy further by demonstrating the existence and physical relevance of linelike defects in the pion condensate.

For simplicity we restrict ourselves to thermal fluctuations and consider only π^+-fields. Suppose the mean field condenses at $q = q_0$. Then the relevant $q \sim q_0$ fluctuations are described by a partition function $Z = \int \mathcal{D}\pi\, \mathcal{D}\pi^+ \exp[-(1/T)F]$ with a free energy functional

$$F[\pi] = \tfrac{1}{2}\int d^3x\{|\partial^2\pi|^2 - 2q_0^2|\partial\pi|^2 + q_0^4|\pi|^2 - |\pi|^2\} + F_{\text{int}}[\pi]\,. \tag{1}$$

Freezing out the massive size fluctuations in $|\pi|$ we can reparametrize

$$\pi = \sqrt{B}\exp\left[i(q_0\tau + \varphi(\boldsymbol{x}))\right] \tag{2}$$

and find the bending energy [12,13] (setting $\lambda^{-2} \equiv 4q_0^2$)

$$F = \frac{B}{2}\int d^3x\{(\partial_z\varphi)^2 + \lambda^2(\partial_\perp^2\varphi)^2\}\,. \tag{3}$$

From (2) we can see that φ and $\varphi + 2\pi n$ are indistinguishable such that in any given plane $\varphi(\boldsymbol{x})$ may have point singularities, around which

$$\oint d\varphi = \oint dl_i\,\partial_i\varphi = 2\pi n, \qquad n = \pm 1, \pm 2, \ldots, \tag{4}$$

just as the magnetic potential φ with $H_i = \partial_i\varphi$ and $\oint dl_i\,\partial_i\varphi = I$ in the presence of current loops. In three dimensions, these points trace out closed lines. These lines have long-range interactions due to the exchange of soft fluctuations of the phase $\varphi(\boldsymbol{x})$. Of course, the singularity is actually smoothed out as soon as variations of $|\pi(\boldsymbol{x})|$ are permitted. This will be taken into account by attributing to each line element a finite core energy ε. The $|\pi(\boldsymbol{x})|$ effects will also cause short-range repulsion, since the energy of a line grows quadratically in n.

We shall now use the method of duality transformations [14] to reformulate the partition function of the pion condensate as a grand-canonical ensemble of defect lines in which the long-range (« elastic ») and the short-range (« steric ») interactions are properly taken into account.

For this we first redefine the energy (3) on a lattice. Let $\boldsymbol{x}$ be the points on a simple cubic lattice and i be the oriented links in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions. Then

$$\nabla_i\varphi(\boldsymbol{x}) \equiv \varphi(\boldsymbol{x}+i) - \varphi(\boldsymbol{x})\,, \qquad \bar{\nabla}_i = \varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}-i)$$

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[10] See A. F. HEBARD and A. T. FIORI: *Phys. Rev. Lett.*, **44**, 291 (1980); for ^{4}He and J. ALS NIELSEN, R. J. BIRGENEAU, M. KAPLAN, J. D. LITSTER, C R. SAFINYA, A. LINDEGAARD-ANDERSEN and S. MATHIESEN: *Phys. Rev. B*, **22**, 312 (1980).

[11] G. BAYM, B. L. FRIMAN and G. GRINSTEIN: Urbana preprint (January 1982).

[12] P G. DE GENNES: *Solid State Commun.*, **10**, 753 (1972 ; *Mol. Cryst. Liq. Cryst.*, **21**, 49 (1973).

[13] A. CAILLÉ: *C. R. Acad. Sci. Ser. B*, **274**, 891 (1971); G. GRINSTEIN and R. A. PELCOVITS: *Phys. Rev. Lett.*, **47**, 856 (1981).

[14] For a review of this technique see R. SAVIT: *Rev. Mod. Phys.*, **52**, 453 (1980).

are the lattice derivatives and (3) is the $a \to 0$ limit of the energy

$$F = \frac{aB}{2} \sum_{x} \{(\nabla_3 \varphi)^2 + \lambda^2 (\boldsymbol{\nabla}_\perp(\nabla_1 \varphi))^2 + \lambda^2 (\boldsymbol{\nabla}_\perp(\nabla_2\varphi))^2\} . \tag{5}$$

In this form it is trivial to include the defects. They arise if the difference between neighbouring φ's have jump by $2\pi n$ with arbitrary integer $n = \pm 1, \pm 2, \ldots$. This brings us to the alternative partition function [15] ($\beta^{-1} \equiv T =$ temperature)

$$Z = \prod_x \int_{-\pi}^{\pi} \frac{d\varphi(\boldsymbol{x})}{2\pi} \sum_{\{n_i(\boldsymbol{x})\}} \exp\left[-\frac{\beta a B}{2} \sum_x \left[(\nabla_3 \varphi - 2\pi n_3)^2 + \right.\right.$$
$$\left.\left. + \lambda^2 (\boldsymbol{\nabla}_\perp(\nabla_1 \varphi - 2\pi n_1))^2 \lambda^2 (\boldsymbol{\nabla}_\perp(\nabla_2 \varphi - 2\pi n_2))^2\right]\right], \tag{6}$$

which may be used instead of the original one.

Introducing an auxiliary field b_i this may be rewritten as

$$Z = \prod_x \int_{-\pi}^{\pi} \frac{d\varphi(\boldsymbol{x})}{2\pi} \int \frac{db_i(\boldsymbol{x})}{\sqrt{2\pi\beta a B}} \sum_{\{n_i(\boldsymbol{x})\}} \exp\left[\sum_x -\frac{1}{2\beta a B}(b_3^2 - \lambda^{-2} b_1 (\overset{*}{\boldsymbol{\nabla}}_\perp \cdot \boldsymbol{\nabla}_\perp)^{-1} b_1 - \right.$$
$$\left. - \lambda^{-2} b_2 (\overset{*}{\boldsymbol{\nabla}}_\perp \cdot \boldsymbol{\nabla}_\perp)^{-1} b_2) + i b_i (\nabla_i \varphi - 2\pi n_i)\right]\Bigg] . \tag{7}$$

Performing the sums over $n_i(\boldsymbol{x})$ the b_i-fields are squeezed to integer values (recall $\sum_m \exp[2\pi i v m] = \sum_n \delta(v - n)$) and integrating out the $\varphi(\boldsymbol{x})$'s gives the constraint $\nabla_i b_i = 0$. Thus b_i is like an integer magnetic field and we can introduce a vector potential a_i via $\boldsymbol{b} \equiv (\boldsymbol{\nabla} \times \boldsymbol{a})$, imposing the gauge condition $\boldsymbol{\nabla}_\perp \boldsymbol{a}_\perp = 0$, for convenience.

This gives

$$Z = \prod_x \int \frac{da_i(\boldsymbol{x})}{\sqrt{2\pi\beta a B}} \exp\left[-\frac{1}{2\beta a B} \sum_{\boldsymbol{k}} \left[\lambda^{-2} |a_3(\boldsymbol{k})|^2 + \left(K_\perp^2 + \frac{K_3^2}{\lambda^2 K_\perp^2}\right) |\boldsymbol{a}_\perp(\boldsymbol{k})|^2\right]\right] \cdot$$
$$\cdot\, \delta(\boldsymbol{\nabla}_\perp \boldsymbol{a}_\perp) \sum_{\{l_i(\boldsymbol{x})\}} \delta_{\nabla_i l_i(\boldsymbol{x}), 0} \exp\left[2\pi i \sum_x l_i(\boldsymbol{x}) a_i(\boldsymbol{x})\right], \tag{8}$$

where the sum over $l_i(\boldsymbol{x})$ makes sure that only integer values of $a_i(\boldsymbol{x})$ are really contributing and $\delta_{\nabla_i l_i, 0}$ is a consequence of the gauge invariance of (7). The symbols K_i^2 denote the lattice versions of k_i^2, namely $K_i^2 \equiv 2(1 - \cos a k_i)/a^2$, $K_\perp^2 \equiv K_1^2 + K_2^2$.

The sum over l_i can be interpreted physically as the grand canonical partition function of an arbitrary number of closed defect loops coupled minimally to a vector potential with a somewhat unusual energy.

There is a simple way of summing these loops based on the fact that the vacuum diagrams of a complex fluctuating field $\psi(\boldsymbol{x})$ represents a complete sum of random loops of precisely those configurations which are pictured in these diagrams [16]. This

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[15] This trick has first been used in the *xy* model by J. VILLAIN: *J. Phys. (Paris)*, **36**, 581 (1977).

[16] For a derivation on the lattice see H. KLEINERT: *Duality transformations for defect melting*, Berlin preprint, Dez. 1981, *Phys. Lett. A* (in press). See also *Gauge theory of dislocated solides*, in press to *Phys. Lett. A*.

leads to the new field-theoretic form of Z ([17])

$$Z = \int \mathscr{D}A(\boldsymbol{x})\,\mathscr{D}\psi(\boldsymbol{x})\,\mathscr{D}\psi^+(\boldsymbol{x}) \exp\left[-\frac{1}{T}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\lambda^{-2}\,|A_3(\boldsymbol{k})|^2 + \left(k_\perp^2 + \frac{k_3^2}{\lambda^2 k_\perp^2}\right)|A_\perp(\boldsymbol{k})|^2\right]\right]\delta(\partial_\perp \boldsymbol{A}_\perp)\exp\left[-\int \mathrm{d}^3x\left[\left|\left(\partial_i - i\,\frac{2\pi\sqrt{K}}{T}\,A_i\right)\psi\right|^2 + m^2|\psi|^2 + \frac{g}{2}\,|\psi|^4\right]\right], \tag{9}$$

where we have returned to continuum notation, rescaled $a_i = (\sqrt{a^2B}/T)A_i$ and inserted a quartic term to account for the steric repulsion discussed before. The mass term consists of a core energy minus an entropy per link $m^2 = (1/T)(\varepsilon - 2D\log 2DT)$, where D is the space dimension. The field ψ describes disorder and this manifests itself in the mass term changing sign *above* a critical temperature $T_c = \varepsilon/2D\log^2 D$. There defect lines proliferate destroying the pseudo-long-range order in the condensate.

It should be remarked that defect lines can form networks thus giving rise to three-dimensional periodic structures. These have been observed in solids ([18]) and cholesteric liquid crystals ([19]) and are, of course, the dual analogues of the periodic mean-field solutions with three-dimensional latticelike condensates ([20]).

([17]) S. AMELINCKX: *Acta Metall.*, **6**, 34 (1958).

([18]) D. L. JOHNSON, J. H. FLACK and P. P. CROOKER: *Phys. Rev. Lett.*, **45**, 641 (1981).

([19]) A. B. MIGDAL: *Rev. Mod. Phys.*, **158**, 383 (1967). For the analogous discussion in cholesterics see the references in H. KLEINERT and K. MAKI: *Fortschr. Phys.*, **29**, 295 (1981).

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