GAUGE FIELD THEORY OF VORTEX LINES IN ^{4}He AND THE SUPERFLUID PHASE TRANSITION

H. KLEINERT
Freie Universität Berlin, Arnimallee 14, 1000 Berlin 33, Germany

Received 2 July 1982

We use the XY model to derive a disorder gauge field theory of the Ginzburg–Landau type II and calculate the transition temperature. As a side result we find the complete phase transition line of a lattice superconductor $t - 3 + 0.253\, e^2 = 0$, where t is the temperature and e the electric charge, in good agreement with recent Monte Carlo numbers.

In many physical systems, order–disorder phase transitions are caused by line-like defects. Recently it was pointed out that their grand-canonical statistical mechanics can conveniently be described by a gauge field theory of the Ginzburg–Landau type [1] in which the complex field accounts for the defect loops and the gauge field for their long-range interactions. For $T > T_c$ the disorder field acquires a non-zero expectation, signalizing the proliferation of defects which cause the gauge fields to lose their long-range propagation, in analogy with the Meissner–Higgs effect of superconductivity.

The simplest example is superfluid ^{4}He where the disorder is caused by vortex loops. These have long-range "elastic" interactions, due to their coupling to sound waves, with the forces being the same as the magnetic Biot–Savart forces between current loops (apart from a minus sign) [2]. The proliferation of vortices leads to the loss of superfluidity. In solids, the defects are dislocation lines and these cause the phase transitions of melting [1] [‡1].

The construction of an approximate disorder field theory is quite straightforward [1]. There are three unknown parameters, the core energy, the short-range steric repulsion between defect lines, which in principle can be determined experimentally, and the entropy per link which for a pure random line is log $2D$ where D is the spatial dimension [1]. In the previous qualitative development, the first two parameters were left open for experimental determination and the entropy was approximated by ln $2D$, for simplicity.

It would be useful to find a way of determining the accurate values of these parameters a priori. This is what we shall do in this note for the vortex lines of superfluid ^{4}He close to the critical temperature.

It is well known that the critical properties of superfluid ^{4}He can well be described by the so-called XY model on a simple, cubic lattice whose partition function reads

$$Z_{XY}(T) = \prod_{\boldsymbol{x}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta(\boldsymbol{x})}{2\pi} \exp\left(\frac{1}{T} \sum_{\boldsymbol{x},i} [\cos \nabla_i \theta(\boldsymbol{x}) - 1]\right), \tag{1}$$

where $\boldsymbol{x}$ are the lattice points, $\boldsymbol{i}$ the basic lattice vectors, and $\nabla_i \theta(\boldsymbol{x}) = \theta(\boldsymbol{x}+\boldsymbol{i}) - \theta(\boldsymbol{x})$ the lattice derivatives. In the Villain approximation, good for $T \lesssim 1$, this can be written as

$$Z_{XYV}(T) = \mathrm{e}^{-N/T} \prod_{\boldsymbol{x}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta(\boldsymbol{x})}{2\pi} \sum_{\{n_i(\boldsymbol{x})\}} \exp\left(-\frac{1}{2T} \sum_{\boldsymbol{x},i} [\nabla_i \theta(\boldsymbol{x}) - 2\pi n_i(\boldsymbol{x})]^2\right), \tag{2}$$

‡1 For the crucial role if disclinations in making melting a first order transition, see H. Kleinert, Lett. Nuovo Cimento (in press).

where N is the total number of lattice sites. Introducing an auxiliary field via a quadratic completion, this becomes [4]

$$Z_{XYV}(T) = e^{-N/T} \prod_{x} \int_{-\pi}^{\pi} \frac{d\theta(x)}{2\pi} \int_{-\infty}^{\infty} \frac{db_i(x)}{\sqrt{2\pi/T}} \sum_{\{n_i(x)\}} \exp\left(-\frac{T}{2}\sum_{x,i} b_i^2(x) + i\sum_{x,i} b_i[\nabla_i\theta(x) - 2\pi n_i(x)]\right). \quad (3)$$

Summing over all $n_i(x)$ constrains the integrals over b_i to integers whereupon integration over $d\theta(x)$ enforces $\Sigma_i \nabla_i^* b_i(x) = 0$, where $\nabla_i^* b_i(x) = b_i(x) - b_i(x - i)$ is the dual lattice derivative. Thus we arrive at the dual representation of the XY partition function

$$Z_{XYV}(T) = (e^{2T}2\pi/T)^{-N/2} \sum_{\{b_i(x)\}} \delta_{\Sigma_i\nabla_i^* b_i(x),0} \exp\left(-\frac{T}{2}\sum_{x,i} b_i(x)^2\right). \quad (4)$$

Due to the vanishing lattice divergence, b_i may be considered as an integer valued magnetic field. Correspondingly, we introduce an integer vector potential a_i and write

$$b_i \equiv (\nabla \times a)_i \equiv \epsilon_{ijk}\nabla_j^* a_k(x - k), \quad (5)$$

an expression which is invariant under gauge transformations $a_i \to a_i + \nabla_i \Lambda$ with arbitrary integer functions $\Lambda(x)$. With this, (4) takes the alternative form [3,4]

$$Z_{XYV}(T) = e^{-N/T} \prod_{x,i} \int_{-\infty}^{\infty} \frac{dA_i(x)}{\sqrt{2\pi/T}} \exp\left(-\frac{T}{2}\sum_{x,i} (\nabla\times A)_i^2\right) Z^A_{\text{loops}}, \quad (6)$$

with

$$Z^A_{\text{loops}} \equiv \sum_{\{l_i(x)\}} \delta_{\Sigma_i\nabla_i^* l_i(x),0} \exp\left(-2\pi i \sum_{x,i} l_i(x)A_i(x)\right), \quad (7)$$

where the sum over integers $l_i(x)$ ensures that only integer values of $A_i(x)$ contribute to the integrals $\int dA_i(x)$. The condition $\Sigma_i \nabla_i^* l_i(x) = 0$ is required by gauge invariance. Certainly, the integrations over the $A_i(\gamma)$ fields have to be done after some gauge fixing. Since the XY model is identified with superfluid ^{4}He, the sum over $l_i(x)$ configuration with $\Sigma_i \nabla_i^* l_i(x) = 0$ accounts for the random set of non-backtracking vortex loops of unit vorticity [1]. The A_i field generates the long-range Biot–Savart forces between these [2].

It was observed by Peskin [4] that for $A_i = 0$ this sum has the same form as that over integer magnetic fields in (4) which in turn may be considered as the dual version of an auxiliary XY model in the limit of zero auxiliary temperature, which we call t to distinguish it from the proper temperature T, i.e.

$$Z^0_{\text{loops}} = \sum_{\{l_i(x)\}} \delta_{\Sigma_i\nabla_i^* l_i(x),0} = \lim_{t\to 0} (2\pi/t)^{N/2} Z_{XYV}(t). \quad (8)$$

Going back to the form (3) but with b_i, T, θ replaced by l_i, t, γ, to stress the auxiliary nature of this XY model, the sum becomes

$$Z^0_{\text{loops}} = \lim_{t\to 0}\left(\frac{2\pi}{t}\right)^{N/2} \prod_{x} \int_{-\pi}^{\pi} \frac{d\gamma(x)}{2\pi} \int_{-\infty}^{\infty} \frac{dl_i(x)}{\sqrt{2\pi/t}} \sum_{\{n_i(x)\}} \exp\left(-\frac{t}{2}\sum_{x,i} l_i^2 + i\sum_{x,i} l_i(\nabla_i\gamma - 2\pi n_i)\right).$$

In this form it is trivial to add the minimal coupling to the vector field A_i which gives

$$Z^A_{\text{loops}} = \lim_{t\to 0}\left(\frac{2\pi}{t}\right)^{N/2} \prod_{x} \int_{-\pi}^{\pi} \frac{d\gamma(x)}{2\pi} \int_{-\infty}^{\infty} \frac{dl_i(x)}{\sqrt{2\pi/t}} \sum_{\{n_i(x)\}} \exp\left(-\frac{t}{2}\sum_{x,i} l_i^2 + i\sum_{x,i} l_i(\nabla_i\gamma - 2\pi A_i - 2\pi n_i)\right), \quad (9)$$

or, after integrating out the l_i fields,

$$Z^A_{\text{loops}} = \lim_{t\to 0} \left(\frac{2\pi}{t}\right)^{N/2} \prod_x \int_{-\pi}^{\pi} \frac{d\gamma(x)}{2\pi} \sum_{\{n_i(x)\}} \exp\left(-\frac{1}{2t}\sum_{x,i} (\nabla_i\gamma - 2\pi A_i - 2\pi n_i)^2\right) . \tag{10}$$

This may be looked at as the Villain approximation to an XY model in an external vector potential (which for $t \to 0$ is actually exact).

$$Z^A_{\text{loops}} = \lim_{t\to 0} (e^{2t}\, 2\pi/t)^{N/2} \prod_x \int_{-\pi}^{\pi} \frac{d\gamma(x)}{2\pi} \exp\left(\frac{1}{t}\sum_{x,i} [\cos(\nabla_i\gamma - 2\pi A_i) - 1]\right) . \tag{11}$$

Thus

$$Z_{XYV}(T) = \lim_{t\to 0} \left(\frac{e^{2t}}{t}\frac{T}{e^{2T}}\right)^{N/2} \prod_{x,i} \int \frac{dA_i(x)}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\sum_{x,i} (\nabla\times A)_i^2\right)$$

$$\times \prod_x \int_{-\pi}^{\pi} \frac{d\gamma(x)}{2\pi} \exp\left(\frac{1}{t}\sum_{x,i} [\cos(\nabla_i\gamma - eA_i) - 1]\right) , \tag{12}$$

with $e \equiv 2\pi/\sqrt{T}$, where we have absorbed $\sqrt{T}$ into the A field. This is Peskin's result: The XY model corresponds to a lattice superconductor with $t \to 0$ and charge $e = 2\pi/\sqrt{T}$.

Let us now use this formula to derive the disorder field theory of vortex loops. For this we neglect, for a moment, the gauge field and rewrite

$$\prod_x \int \frac{d\gamma(x)}{2\pi} \exp\left(\frac{1}{t}\sum_{x,i} \cos\nabla_i\gamma\right) = \prod_x \int \frac{d\gamma(x)}{2\pi} \exp\left[\frac{D}{t}\sum_x S^\dagger \left(1 + \frac{1}{2D}\sum_i (\nabla_i - \nabla_i^*)\right) S\right] , \tag{13}$$

where $S(x) = e^{i\gamma(x)}$. The vector potential can be included by simply replacing ∇_i, ∇_i^* by the covariant derivatives

$$D_i S(x) = S(x+i) \exp[-ieA_i(x)] - S(x) , \qquad D_i^* S(x) = S(x) - \exp[ieA(x-i)] S(x-i) , \tag{14}$$

apart from the A field energy. Using $D_i - D_i^* \equiv D_i^* D_i$, the exponent takes the form

$$\frac{D}{t}\sum_{x,a} S_a \left(1 + \frac{1}{2D} D_i^* D_i\right) S_a , \tag{15}$$

where S_a is the real two-vector

$$S_a(x) = (\cos\gamma(x), \sin\gamma(x)) , \qquad a = 1, 2 . \tag{16}$$

The complex disorder field ψ_a may now be introduced by means of a quadratic completion in (13)

$$Z^A_{\text{loops}} = \lim_{t\to 0} (e^{2t}\, 2\pi/t)^{N/2} \prod_x \int_{-\pi}^{\pi} \frac{d\gamma(x)}{2\pi} \iint_{-\infty}^{\infty} \frac{d\psi_1\, d\psi_2(x)}{4\pi D/t} \exp\left(-\frac{t}{4D}\sum_x \psi_a^2 + \sum_x \varphi_a S_a\right) , \tag{17}$$

with $\varphi_a \equiv (1 + \Sigma_i D_i^* D_i)^{1/2}\, \psi_a$. Performing the $\gamma(x)$ integrals and inserting the result in (6) leads to

$$Z_{XYV}(T) = \lim_{t\to 0} \left(\frac{e^{2t}}{t}\frac{T}{e^{2T}}\right)^{N/2} \int_{-\infty}^{\infty} \frac{dA_i(x)}{\sqrt{2\pi}} \prod_x \iint_{-\infty}^{\infty} \frac{d\psi_1\, d\psi_2(x)}{4\pi D/t} \exp\left(-\frac{1}{2}\sum_{x,i} (\nabla\times A)_i^2\right)$$

$$\times \exp\left(-\frac{t}{4D}\sum_x \psi_a^2 + \sum_x \log I_0(\sqrt{\varphi_a^2})\right) . \tag{18}$$

This is the desired gauge theory of disorder. For $t \to 0$, the field ψ describes the vortex lines of the XY model, i.e. those of superfluid ^{4}He in the critical regime [1].

If the phase transition is of second order we can approximate the energy close to the phase transition by a Landau expansion

$$F[\psi, \psi^\dagger, A]/T \approx \sum_{x} \{\tfrac{1}{4}(t/D - 1)|\psi|^2 + (1/8D)|D_i\psi|^2 + \tfrac{1}{64}|\psi|^4 + \tfrac{1}{2}(\nabla \times A)_i^2\} . \tag{19}$$

In the absence of magnetism, this has the mean field phase transition at

$$t = D . \tag{20}$$

The magnetic field energy changes this as follows: Assuming $\psi \approx$ const and taking the gauge ψ = real, the A field can be integrated out and gives an additional "classical black-body" energy:

$$\Delta F = \sum_{k} \log[1 + (e^2/4DK^2)|\psi|^2] , \tag{21}$$

where $K^2 \equiv \Sigma_i\, 2(1 - \cos k_i)$. This results in an additional $|\psi|^2$ term

$$(e^2/4D)\Big(\sum_{k} 1/K^2\Big)|\psi|^2 . \tag{22}$$

Thus we conclude that the lattice superconductor has a straight line of phase transition which satisfies

$$t = D + e^2 \sum_{k} 1/K^2 . \tag{23}$$

For $D = 3$, $\Sigma_k\, 1/K^2 = 0.253$ and this line fits well the Monte Carlo data points of ref. [5]. The original XY model corresponds to $t = 0$ and we obtain a phase transition at $e_c^2 = 11.86$ or $T_c = 3.3$. This is only 10% larger than the direct mean field result for the XY model $T_c = D = 3$. Considering the fact that we have employed the Villain approximation, the agreement is excellent. For $D = 3$, the energy (21) also has a cubic term

$$\Delta F = -(e^3/6\pi)(\tfrac{1}{12}|\psi|^2)^{3/2} , \tag{24}$$

which suggests a first order phase transition at [6]

$$\tfrac{1}{3}t - 1 - \tfrac{1}{3}e^2 \sum_{k} 1/K^2 \sim [e^6/(6\pi)^2]/[\tfrac{1}{64}(12)^3] . \tag{25}$$

The right-hand side is $\leqslant 0.157$ for $e^2 \leqslant e_c^2$. The Ginzburg criterium [7], on the other hand, says that fluctuations in the size of the $|\psi|$ field are large for all

$$|\tfrac{1}{3}t - 1 - \tfrac{1}{3}e^2 \sum_{k} 1/K^2| \leqslant 1 . \tag{26}$$

Thus the approximation $|\psi| \approx$ const needed to calculate (24) is not trustworthy [‡2] and the transition can remain second order, which is confirmed by the Monte Carlo data. In fact, the Ginzburg theory has a K parameter [8]

$$K \equiv \text{mass}\,|\psi|/\sqrt{2}\ \text{mass}\,A = D/e , \tag{27}$$

which determines the range of the magnetic versus that of the $|\psi|$ fluctuations in the $\psi \neq 0$ phase. For $e^2 < e_c^2$ this is $K > 0.87 > 1/\sqrt{2}$ such that the Ginzburg–Landau expression (19) is of type II and is therefore expected to maintain a second order phase transition in spite of the A field fluctuations.

Let us conclude this discussion by noting that it is possible to incorporate a non-zero core energy per link, ϵ_{core}, of vortex lines and thus go beyond the XY model for superfluid ^{4}He. All one has to do is identify $\exp(-\tfrac{1}{2}t\, \Sigma_{x,i}\, l_i^2)$ with a Boltzmann factor $\exp[-(\epsilon_{\text{core}}/\pi)\, \Sigma_{x,i}\, l_i^2]$ which generates a positive t in (19) and moves

‡2 Only the $|\psi|^2$ term in (22) is, since it is merely the seagull diagram $\langle A_i^2(x)\rangle\, |\psi(x)|^2$, valid for arbitrary $|\psi(x)|^2$.

the phase transition in (23) to lower e^2, i.e. higher T [9] [‡3].

The author thanks T. Banks for an illuminating discussion.

‡3 This model is investigated further in ref. [9]. It leads to Ginzburg–Landau theories with different K and specifies the value $K_{1,2}$ where the superconductive transition becomes first order.

References

[1] In solids: H. Kleinert, Phys. Lett. 89A (1982) 246; 91A (1982) 295; Lett. Nuovo Cimento 34 (1982) 464, 471;
In pion condensates: H. Kleinert, Lett. Nuovo Cimento 34 (1982) 209;
In superfluid ^{4}He: H. Kleinert, Lett. Nuovo Cimento 34 (1982) 209;
In magnetic superconductors: H. Kleinert, Phys. Lett. 90A (1982) 259;
In smectic liquid crystals: H. Kleinert, J. de Phys. (Paris), to be published (February 1983).
[2] For a review see A. Fetter, in: The physics of liquid and solid helium, Part II, eds. K.H. Bennemann and J.B. Ketterson (Wiley, New York, 1976) p. 207.
[3] J. Villain, J. de Phys. (Paris) 36 (1975) 581;
See also T. Banks, R. Myerson and J. Kogut, Nucl. Phys. B129 (1977) 493.
[4] M.E. Peskin, Ann. Phys. (NY) 113 (1978) 122.
[5] C. Dasgupta and B.I. Halperin, Phys. Rev. Lett. 47 (1981) 1556.
[6] This point has been stressed by B.H. Halperin, T.C. Lubensky and S.K. Ma, Phys. Rev. Lett. 32 (1974) 292.
[7] V.L. Ginzburg, Fiz. Tverd. Tela 2 (1960) 2031 [Sov. Phys. Solid State 2 (1961) 1824].
[8] M. Tinkham, Introduction to superconductivity (McGraw-Hill, Kogakusha, Tokyo, 1975) p. 113;
S. Saint-James, G. Sarma and E.J. Thomas, Type II superconductivity (Pergamon, New York, 1969) p. 29.
[9] H. Kleinert, Lett. Nuovo Cimento, to be published (November 1982).