GAUGE FIELDS IN CONDENSED MATTER

Hagen Kleinert

Vol. I SUPERFLOW AND VORTEX LINES

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INTRODUCTION

Most physical systems follow complicated nonlinear equations. For a small number of degrees of freedom, numerical methods may lead to a satisfactory theoretical understanding. In macroscopic many-body systems, however, this number is immensely large and such an approach is hopeless. The best we can achieve is an approximate understanding of the statistical behavior, averaged either with respect to thermal or to quantum mechanical fluctuations.

The chance of gaining such an approximate understanding depends crucially on a successful separation of linear and nonlinear excitations. A good example to illustrate this is crystalline material: For zero absolute temperature, the equilibrium configuration consists of a perfect regular array of atoms.[†] This array is called the *ground state* of the system. If the crystal is perturbed weakly, the atoms are displaced slightly from their equilibrium position. If released, they begin to vibrate. For small enough displacements these vibrations are governed by the expansion of the energy up to quadratic terms in the displacements. The resulting equations of motion are all linear and can be diagonalized by a Fourier analysis. The eigenmodes correspond to a decoupled set of independent harmonic oscillators. They are observable in the form of elastic sound

[†]Neglecting quantum effects.

waves, or *phonons*, two being transverse, of the shear type, and one longitudinal, of the pressure type. Phonons have the characteristic that they exist for arbitrarily low excitation energy. Such excitation modes are called *massless*.

If the perturbation increases in strength, the vibrations acquire larger amplitudes. Nonlinear terms in the expansion of the energy grow increasingly important. They may be described as an interaction between phonons, measured by some dimensionless coupling strength g.

As long as this interaction strength is small, one may attempt to calculate the corrections to the linear approximation in the form of a power series in g. This is called a *perturbation expansion*. Mathematically, such an expansion has severe difficulties. It does not converge in any small interval around $g = 0$, i.e., it has a zero radius of convergence. In general there are contributions which cannot be expanded into a power series since they are of the type $e^{-1/g}$ (whose power series is $0 + 0 + 0 \ldots$) for $g \geq 0$.

As it happens quite often in theoretical physics, the mathematical difficulties have quite a physical origin. The example of the crystal can teach us what physical effects are necessarily missed by perturbation theory. Suppose a single atom is taken away from its equilibrium position while leaving all the others in their places. It so happens that in most crystals there exist additional local potential minima, outside the regular array, where an atom can find a new metastable position. If the atom arrives at such a position it can remain there for a long time. The crystal is no longer perfect. What has been created is a so-called *interstitial atom*. The empty lattice place left behind is called a *vacancy*. Interstitial atoms can move rather easily through the crystal, by two possible mechanisms to be discussed in more detail later. On their way they may either run into a vacancy and recombine, or they may arrive at the surface, at which there are many empty crystal sites, where they can settle down. After all, the space outside the crystal may be considered as a "crystal" consisting only of vacancies (see Fig. 2.1 in Part III).

It is obvious that for each point defect there is a whole neighborhood where the crystalline order is disturbed. Therefore, it becomes easier to create a second point defect close to it. Continuing this procedure, it is possible to imagine that an entire section of an atomic layer can disappear by interstitial migration. It is easy to see that such a missing layer is energetically favored over an equal number of disjoint defects. The reason is that the crystal is capable of repairing the defect over most of the layer (see Fig. 2.2 of Part III). Only at the circumference is such a

repair impossible. This is where most of the energy of the defect layer is stored. In fact, the final crystal has no memory where the layer was. Only the circumference is a physical observable. What has been obtained is a *line-like defect* called a *dislocation line*.

The energy of a straight dislocation line is proportional to the length of the line while the sum of energies of defects from which it was created would be proportional to the number of removed atoms, i.e., to the surface of the layer. As a consequence, larger numbers of point defects always prefer to combine and form line-like defects.

It is intuitively obvious that a perturbative treatment of the anharmonic parts of the crystalline forces cannot possibly be capable of describing such strongly nonlinear phenomena. Only if the expansion in powers of g could be carried out to *all* orders in g would one hope to gain some information. Since this is usually very hard it is necessary to develop a more economical way of directly including this type of macroscopic excitations.

It is the purpose of this work to develop a general theoretical framework for doing so. Since defects are discrete, an integer-valued field theory will turn out to be the most convenient tool for describing the defects. The fact that the crystal carries no memory of the layer of atoms removed to generate it, but only of the circumference, has important structural implications. We shall see that the shift of the layer from one position to another can be formulated as gauge transformations on the integer field variables defining the layer. Correspondingly, these field variables will be called *defect gauge fields*. The irrelevance of the layers amounts to a defect gauge invariance of the field theory of defects. This will be one of the important gauge structures to be studied in this text.

The other gauge structure will come about from the need of calculating the elastic distortions caused by the presence of the line-like defects. The associated fields will again be gauge fields, but now continuous fields. They will be called *stress gauge fields*.

The result will be a *double gauge theory of stresses and defects*.

Since integer-valued gauge fields are not quite easy to deal with analytically, progress will be made by observing that there is a way of transforming the defect fields into a complex field theory. Its Feynman loop diagrams will turn out to be direct pictures of the line-like defects. In this way the integer-valued defect gauge structure can be eliminated and what remains is a field theory of a continuous complex *disorder field* coupled to the gauge field of stresses. The resulting combined field theory of line-like defects and gauge fields of stresses will turn out to have a simple structure

which is familiar from other theories of completely different phenomena: A complex scalar field theory coupled to an Abelian gauge field theory is known as *scalar quantum electrodynamics*. Theories of this type have been introduced first in the context of superconductivity by Ginzburg and Landau and have led to a simple and correct description of all phenomena observable in superconductivity close to the phase transition.

The features of nonlinear forces illustrated above in the case of a crystal turn out to be rather universal. Similar phenomena are present in a great number of quite distinct physical systems, for example, superfluids, superconductors, liquid crystals, and pion condensates. In all these and many more systems it is possible to develop a theory of the same basic structure.

It is worth pointing out that this very structure is shared also be the presently popular theories of elementary particles which attempt to explain the phenomena of nature at the level of quarks, gluons, and leptons. Thus the theoretical framework to be developed has quite a wide range of applicability which greatly increases the chance of its being true.

The last part of the book looks at the gauge fields of defects in the continuum limit, where it can be viewed as a linearized version of a special type of gravitational theory in a metric space with curvature and torsion. In order to work out this connection clearly, the theory of metric spaces and gravitation will be developed once more from the defect point of view, as far as is necessary for our purpose.

PART I

FLUCTUATING FIELDS AND RANDOM CHAINS

Contemplator enim, cum solis lumina cumque
inserti fundunt radii per opaca domorum.
Multa minuta modis multis per inane videbis
corpora misceri radiorum lumine in ipso.

(*Observe what happens when sunbeams*
Are admitted into a building.
And shed light on its shadowy places.
You will see a multitude of particles
Mingling in a multitude of ways
In the empty space in the light of the beam.)

C. Lucretius, *De Rerum Natura*, Rome, 57 B.C.

CHAPTER ONE

STATISTICAL PHYSICS

The phenomena that will be studied take place in real space. The most economical way of describing such phenomena is based on attributing to each space point $\mathbf{x}$ one or more variables, collectively denoted by $\phi_\alpha(\mathbf{x})$. The set of these variables is called a *field*. The field $\phi_\alpha(\mathbf{x})$ undergoes thermal as well as quantum fluctuations and the development of this part of our text will be devoted to reviewing the general description of the statistical properties of such a fluctuating field. Since several good textbooks are already available on this subject we can afford to be somewhat sketchy. Wherever statements require elaborate proofs these will be omitted and we shall be content with simple illustrations and examples. Hopefully, our short treatment will be sufficient to serve as an introduction for those unfamiliar with the relevant basic field theoretical techniques. At the very least we hope to clarify the notation, conventions, and language to be used in the further development of the theory.

1.1. CLASSICAL STATISTICS

Consider a physical system whose configurations can be parametrized by a set of real continuous statistically independent variables ϕ_ℓ which can take values in the interval $\phi_\ell \in (-\infty, \infty)$. The index ℓ may run over a finite

or infinite set of labels. Given the energy as a function of these variables ϕ_ℓ,

$$E = E(\phi_\ell) \equiv E[\phi], \tag{1.1}$$

all global thermodynamic properties of the system can be derived from the knowledge of the *partition function*

$$Z = \prod_\ell \left[\int_{-\infty}^{\infty} \frac{d\phi_\ell}{\sqrt{2\pi T}} \right] \exp\{-E[\phi]/T\}, \tag{1.2}$$

where T is the temperature measured in energy units.[a] The partition function defines a free energy F via

$$Z \equiv e^{-F/T}. \tag{1.3}$$

Given Z, all global thermodynamic observables follow from further differentiation with respect to the parameters characterizing the system. The *entropy*, for example, is given by the derivative

$$S \equiv -\frac{\partial F}{\partial T}, \tag{1.4}$$

while the *internal energy* U is the Legendre transform

$$U = F - T\frac{\partial F}{\partial T} = F + TS. \tag{1.5}$$

The *specific heat* C at constant volume is defined by

$$C = T\frac{\partial}{\partial T}S = \frac{\partial}{\partial T}U. \tag{1.6}$$

The local properties of a thermodynamic system are given by the so-called *correlation functions* of the field variables, also called *n-point*

[a] T is the usual temperature times the Boltzmann constant $k_B = 1.380622(59) \times 10^{-16}$erg/K $= 0.861708(70) \times 10^{-4}$eV/K.

functions for short. These are defined by the expectation values

$$G^{(n)}(\ell_1, \ldots, \ell_n) = \langle \phi_{\ell_1} \ldots \phi_{\ell_n} \rangle$$
$$\equiv Z^{-1} \prod_{\ell} \left[\int_{-\infty}^{\infty} \frac{d\phi_\ell}{\sqrt{2\pi T}} \right] \phi_{\ell_1} \ldots \phi_{\ell_n} e^{-E[\phi]/T}. \tag{1.7}$$

It is convenient to include these also into the general thermodynamic rules of differentiation. For this we add to the energy in the exponent of (1.2) a so-called *external source term*

$$E_{\text{source}} = -\sum_{\ell} j_\ell \phi_\ell. \tag{1.1'}$$

We shall include this source term in the partition function by writing Z as a function $Z(j_\ell) \equiv Z[j]$. It is then easy to see that the correlation functions are simply the derivatives of $Z[j]$ with respect to the sources, i.e.,

$$G^{(n)}(\ell_1, \ldots, \ell_n) = Z^{-1}[j] T^n \frac{\partial}{\partial j_{\ell_1}} \cdots \frac{\partial}{\partial j_{\ell_n}} Z[j] \Big|_{j_\ell = 0}. \tag{1.8}$$

Because of this property, $Z[j]$ is called the *generating functional* of correlation functions.

Notice that the normalization of Z is completely arbitrary since every statistical average is divided by Z at the end. For this reason we shall often use the proportionality sign rather than the equal sign in relations involving Z.

It is generally assumed that the energy can be expanded in a power series, i.e.,

$$E[\phi] = E^{(0)} + \sum_{\ell} E_\ell^{(1)} \phi_\ell + \frac{1}{2!} \sum_{\ell_1 \ell_2} \phi_{\ell_1} D_{\ell_1 \ell_2} \phi_{\ell_2} + \frac{1}{3!} \sum_{\ell_1 \ell_2} V_{3,\ell_1 \ell_2 \ell_3} \phi_{\ell_1} \phi_{\ell_2} \phi_{\ell_3}$$
$$+ \frac{1}{4!} \sum_{\ell_1 \ell_2 \ell_3} V_{4,\ell_1 \ell_2 \ell_3 \ell_4} \phi_{\ell_1} \phi_{\ell_2} \phi_{\ell_3} \phi_{\ell_4} + \ldots \tag{1.9}$$

The set of variables ϕ_ℓ for all ℓ may be considered as components of a vector. Then $E_\ell^{(1)}$ is a vector, $D_{\ell_1 \ell_2}$ a matrix, and $V_{n,\ell_1 \ell_2 \cdots \ell_n}$ a tensor of rank n. The energy E itself is a scalar formed by contraction of all indices.

For some of the later discussions it will be convenient to define also the so-called *correlation functions in the presence of the external sources* by a

relation like (1.8) except that the sources are left *arbitrary* rather than set equal to zero at the end. Thus

$$G^{(n)}(\ell_1 \ \ldots \ \ell_n|j) = Z^{-1}[j]\, T^n \frac{\partial}{\partial j_{\ell_1}} \cdots \frac{\partial}{\partial j_{\ell_n}} Z[j]. \tag{1.10}$$

For brevity, we shall often omit the argument j as long as this can cause no confussion.

1.2. CLASSICAL FIELDS

As mentioned above, a many-body system will be described by a set of real field variables, $\phi_\alpha(\mathbf{x})$, defined at each space point. In the following we shall omit the label α. A field is really a straightforward generalization of the variables ϕ_ℓ, where the label ℓ has become continuous, $\ell \to \mathbf{x}$. In analogy with (1.9), the energy takes the form

$$\begin{aligned} E[\phi] = E^{(0)} &+ \int d^3x E^{(1)}(\mathbf{x})\phi(\mathbf{x}) + \frac{1}{2!}\int d^3x_1 d^3x_2 \phi(\mathbf{x}_1) D(\mathbf{x}_1, \mathbf{x}_2)\phi(\mathbf{x}_2) \\ &+ \frac{1}{3!}\int d^3x_1 d^3x_2 d^3x_3 V_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3) \\ &+ \frac{1}{4!}\int d^3x_1 d^3x_2 d^3x_3 d^3x_4 V_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4) \\ &+ \ldots \end{aligned} \tag{1.11}$$

In general, functions of a variable which carry a continuous label are called *functionals*. The energy functional (1.11) arises from (1.9) by simply replacing the sums by integrals.

In analogy with the discrete case, $\phi(\mathbf{x})$ may be viewed as the components of a functional vector labeled by $\mathbf{x}$. Correspondingly, the expressions $E^{(1)}(\mathbf{x})$, $D(\mathbf{x}_1, \mathbf{x}_3)$, ..., $V_n(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, ... are *functional vector, matrix, tensors* of rank n. Integrals over $\mathbf{x}$ correspond to index contractions and thus to scalar products in functional space.

Most field systems have a particular property called *translational invariance*, i.e., the energy is the same whether one calculates it for the field $\phi(\mathbf{x})$ or for the translated field $\phi(\mathbf{x}+\mathbf{a})$. This implies that the coefficient function $E^{(1)}(\mathbf{x})$ can only be a constant while all higher func-

tional tensors depend at most on the differences between the arguments, i.e.,

$$\begin{aligned} D(\mathbf{x}_1, \mathbf{x}_2) &= \bar{D}(\mathbf{x}_1 - \mathbf{x}_2), \\ &\vdots \qquad\qquad \vdots \\ V_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) &= \bar{V}_n(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3, \ldots, \mathbf{x}_{n-1} - \mathbf{x}_n). \end{aligned} \tag{1.12}$$

In order to calculate correlation functions we again add to $E[\phi]$ a source term analogous to (1.1′). The corresponding functional is obviously

$$E_{\text{source}} = -\int d^3x\, j(\mathbf{x})\, \phi(\mathbf{x}). \tag{1.1″}$$

The treatment of field systems $\phi(\mathbf{x})$ is reduced to that of discrete systems ϕ_ℓ by a simple limiting process. Space is decomposed into a narrow lattice

$$\mathbf{x}_\ell = (\ell_1, \ell_2, \ell_3)\, a = \boldsymbol{\ell} a,$$

where a is a lattice constant. If a is sufficiently small, the functional scalar products can be approximated by discrete scalar products

$$\int d^3k\, j(\mathbf{x})\, \phi(\mathbf{x}) \sim a^3 \sum_\ell j(\mathbf{x}_\ell)\, \phi(\mathbf{x}_\ell).$$

Correspondingly, the energy functional is approximately equal to the sum

$$\begin{aligned} E[\phi] = E^{(0)} &+ \sum_\ell \sqrt{a^3} E^{(1)}(\mathbf{x}_\ell) \sqrt{a^3} \phi(\mathbf{x}_\ell) \\ &+ \frac{1}{2!} \sum_{\ell_1 \ell_2} \sqrt{a^3} \phi(\mathbf{x}_{\ell_1}) \sqrt{a^3}^2 D(\mathbf{x}_{\ell_1}, \mathbf{x}_{\ell_2}) \sqrt{a^3} \phi(\mathbf{x}_{\ell_2}) \\ &+ \frac{1}{3!} \sum_{\ell_1 \ell_2 \ell_3} \sqrt{a^3}^3 V_3(\mathbf{x}_{\ell_1}, \mathbf{x}_{\ell_2}, \mathbf{x}_{\ell_3}) \sqrt{a^3} \phi(\mathbf{x}_{\ell_1}) \sqrt{a^3} \phi(\mathbf{x}_{\ell_2}) \sqrt{a^3} \phi(\mathbf{x}_{\ell_3}) \\ &+ \frac{1}{4!} \sum_{\ell_1 \ell_2 \ell_3 \ell_4} \sqrt{a^3}^4 V_4(\mathbf{x}_{\ell_1}, \mathbf{x}_{\ell_2}, \mathbf{x}_{\ell_3}, \mathbf{x}_{\ell_4}) \sqrt{a^3} \phi(\mathbf{x}_{\ell_1}) \sqrt{a^3} \phi(\mathbf{x}_{\ell_2}) \\ &\times \sqrt{a^3} \phi(\mathbf{x}_{\ell_3}) \sqrt{a^3} \phi(\mathbf{x}_{\ell_4}) + \ldots \end{aligned} \tag{1.13}$$

The connection with the previous form (1.9) is established by the identification

$$\begin{aligned} \phi_\ell &\equiv \sqrt{a^3}\,\phi(\mathbf{x}_\ell), \\ D_{\ell_1\ell_2} &\equiv a^3 D(\mathbf{x}_{\ell_1}, \mathbf{x}_{\ell_2}), \\ &\vdots \quad \vdots \quad \vdots \\ V_{n,\ell_1\ldots\ell_n} &\equiv \sqrt{a^3}^n V_n(\mathbf{x}_{\ell_1}, \ldots, \mathbf{x}_{\ell_n}). \end{aligned} \tag{1.14}$$

The partition function of this grated field system is given by (1.2) which now reads

$$Z[j] = \prod_\ell \left[\int_{-\infty}^{\infty} \frac{d\phi(x_\ell)}{\sqrt{2\pi T/a^3}} \right] \exp\left\{ -\frac{1}{T}(E[\phi] + E_{\text{source}}) \right\}. \tag{1.15}$$

The partition function of the continuous field system may now be defined as the limit of this grated expression for $a \to 0$.

In the limit one writes

$$\lim_{a\to 0} \prod_\ell \left[\int_{-\infty}^{\infty} \frac{d\phi(x_\ell)}{\sqrt{2\pi T/a^3}} \right] \equiv \int \mathscr{D}\,\phi(\mathbf{x}) \tag{1.16}$$

and calls (1.15) a *functional integral* or *path integral*. In the following we shall assume $E^{(0)} = 0$ since it merely produces an irrelevant overall factor to the partition function Z.

1.3. FREE FIELDS

As an example, take the simplest case in which the energy contains only fields to quadratic order, such that it has the form (including the source term)

$$E_0 + E_{\text{source}} = \frac{1}{2!} \int d^3x_1 d^3x_2\, \phi(\mathbf{x}_1)\, D(\mathbf{x}_1, \mathbf{x}_2)\, \phi(\mathbf{x}_2) - \int dx\, j(\mathbf{x})\, \phi(\mathbf{x}), \tag{1.17}$$

where $D(\mathbf{x}_1, \mathbf{x}_2)$ is symmetric under the exchange of variables, $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$. A theory with a quadratic energy form is called a *free field theory* and the subscript 0 indicates this property. In the grated form, (1.17) reads

$$E_0 + E_{\text{source}} = \frac{1}{2!}\sum_{\ell_1\ell_2} \phi_{\ell_1} D_{\ell_1\ell_2}\phi_{\ell_2} - \sum_{\ell} j_\ell \phi_\ell \tag{1.18}$$

and the partition function is

$$Z_0 = \prod_{\ell}\left[\int \frac{d\phi_\ell}{\sqrt{2\pi T}}\right] \exp\left\{-\frac{1}{T}\left(\frac{1}{2}\sum_{\ell_1\ell_2} \phi_{\ell_1} D_{\ell_1\ell_2}\phi_{\ell_2} - \sum_{\ell} j_\ell \phi_\ell\right)\right\}.$$

Since energy is a quadratic form, it can be written as a complete square, i.e.,

$$\begin{aligned} E_0 + E_{\text{source}} = \frac{1}{2}\sum_{\ell_1\ell_2} &\left(\phi_{\ell_1} - \sum_{\ell_1'} D^{-1}_{\ell_1\ell_1'} j_{\ell_1'}\right) D_{\ell_1\ell_2} \\ &\times \left(\phi_{\ell_2} - \sum_{\ell_2'} D^{-1}_{\ell_2\ell_2'} j_{\ell_2'}\right) - \frac{1}{2}\sum_{\ell_1\ell_2} j_{\ell_1} D^{-1}_{\ell_1\ell_2} j_{\ell_2}. \end{aligned} \tag{1.19}$$

Here $D^{-1}_{\ell\ell'}$ is the inverse matrix of $D_{\ell\ell'}$, i.e.,

$$\sum_{\ell'} D^{-1}_{\ell\ell'} D_{\ell'\ell''} = \delta_{\ell\ell''}. \tag{1.20}$$

Since the ϕ_ℓ integrations all run from $-\infty$ to ∞, the shifted variables $\phi_\ell - \Sigma_{\ell'} D^{-1}_{\ell\ell'} j_{\ell'}$ can be replaced by ϕ'_ℓ and Z_0 becomes

$$Z_0 = \prod_{\ell}\left[\int_{-\infty}^{\infty} \frac{d\phi'_\ell}{\sqrt{2\pi T}}\right] \exp\left\{-\frac{1}{2T}\sum_{\ell,\ell'} \phi'_\ell D_{\ell\ell'}\phi'_{\ell'}\right\} \exp\left\{\frac{1}{2T}\sum_{\ell\ell'} j_\ell D^{-1}_{\ell\ell'} j_{\ell'}\right\}. \tag{1.21}$$

In order to evaluate these integrals we now make use of the fact that $D_{\ell\ell'}$ is a symmetric matrix and hence can be diagonalized by an orthogonal transformation

$$D_{\ell\ell'} = \sum_{\ell''} \mathcal{O}_{\ell\ell''} D_{\ell''} \mathcal{O}^{-1}_{\ell''\ell'}. \tag{1.22}$$

With the transformed fields

$$\phi''_\ell = \sum_{\ell'} \mathcal{O}^{-1}_{\ell\ell'} \phi'_{\ell'} \tag{1.23}$$

the exponent factorizes as

$$\prod_{\ell} \exp\left\{-\frac{1}{2T}\phi''_{\ell} D_{\ell} \phi''_{\ell}\right\}.$$

The measure of integration remains invariant

$$\prod_{\ell} d\phi'_{\ell} = \prod_{\ell} d\phi''_{\ell}, \tag{1.24}$$

since the Jacobian

$$\left|\frac{\partial \phi'_{\ell'}}{\partial \phi''_{\ell''}}\right| = \det \mathcal{O} \tag{1.25}$$

is unity, as follows directly from $\mathcal{O}^{-1} = \mathcal{O}^{\mathrm{T}}$. But then the calculation of the partition function reduces to an infinite product of Gaussian integrals,

$$Z_0 = \prod_{\ell}\left[\int \frac{d\phi''_{\ell}}{\sqrt{2\pi T}} \exp\left\{-\frac{1}{2T}\sum_{\ell} \phi''_{\ell} D_{\ell} \phi''_{\ell}\right\}\right] \exp\left\{\frac{1}{2T}\sum_{\ell\ell'} j_{\ell} D^{-1}_{\ell\ell'} j_{\ell'}\right\}. \tag{1.26}$$

Each factor can be integrated to give $1/\sqrt{D_{\ell}}$ so that

$$Z_0 = \left[\prod_{\ell} D_{\ell}\right]^{-1/2} \exp\left\{\frac{1}{2T}\sum_{\ell\ell'} j_{\ell} D^{-1}_{\ell\ell'} j_{\ell'}\right\}. \tag{1.27}$$

The product of eigenvalues D_{ℓ} can be written in a form which is independent of its diagonal basis, namely,

$$\prod_{\ell} D_{\ell} = \det D. \tag{1.28}$$

In this way we arrive at the important formula

$$Z_0[j] = (\det D)^{-1/2} \exp\left\{\frac{1}{2T}\sum_{\ell\ell'} j_{\ell} D^{-1}_{\ell\ell'} j_{\ell'}\right\}. \tag{1.29}$$

As it stands, this formula is derived for the grated lattice version of the field theory. We can, however, convince ourselves that it has a proper meaning in the limit of zero lattice spacing, $a \to 0$. For the last factor,

this is trivial since using the replacement (1.14) we see that the exponent goes directly over into

$$\frac{1}{2T}\int d^3x d^3x'\, j(\mathbf{x})\, D^{-1}(\mathbf{x}, \mathbf{x}')\, j(\mathbf{x}'). \tag{1.30}$$

Here $D^{-1}(\mathbf{x}, \mathbf{x}')$ is defined via the limiting form of the inverse matrix (1.20); according to (1.14) we can identify

$$\frac{1}{a^3} D_{\ell\ell'} \equiv D(x_\ell, x_{\ell'}). \tag{1.31}$$

The grated version satisfies

$$a^3 \sum_{\ell'} D^{-1}(\mathbf{x}_\ell, \mathbf{x}_{\ell'})\, D(\mathbf{x}_{\ell'}, \mathbf{x}_{\ell''}) = \frac{1}{a^3}\delta_{\ell\ell''}. \tag{1.32}$$

In the limit $a \to 0$, the left-hand side can be rewritten as an integral

$$\int d^3x'\, D^{-1}(\mathbf{x}, \mathbf{x}')\, D(\mathbf{x}', \mathbf{x}''). \tag{1.33}$$

The right-hand side has the property of vanishing for $\ell \neq \ell''$ or $\mathbf{x}_\ell \neq \mathbf{x}_{\ell''}$, while becoming singular for $\mathbf{x}_\ell = \mathbf{x}_{\ell''}$. The singularity is such that

$$a^3 \sum_{\ell''} \frac{1}{a^3}\delta_{\ell\ell''} = 1. \tag{1.34}$$

In the limit $a \to 0$, $a^3\Sigma_\ell$ turns into an integral $\int d^3x$. The corresponding limit of $(1/a^3)\,\delta_{\ell\ell''}$ is denoted by $\delta(\mathbf{x}_\ell - \mathbf{x}_{\ell''})$; it may be characterized by

$$\delta(\mathbf{x} - \mathbf{x}'') = \begin{cases} 0 & \mathbf{x} \neq \mathbf{x}'', \\ \dfrac{1}{a^3} \to \infty & \mathbf{x} = \mathbf{x}'', \end{cases} \tag{1.35}$$

with the infinity normalized in such a way that

$$\int d^3x''\, \delta(\mathbf{x} - \mathbf{x}'') = 1. \tag{1.36}$$

This improper function is known as *Dirac's δ-function*. It is the functional version of the unit matrix $\delta_{\ell\ell''}$ and shares all its properties if sums are replaced by integrals (≡ functional sums). In terms of the δ-function, the inverse functional matrix $D^{-1}(\mathbf{x}, \mathbf{x}')$ satisfies

$$\int d^3x'\, D^{-1}(\mathbf{x}, \mathbf{x}')\, D(\mathbf{x}', \mathbf{x}'') = \delta(\mathbf{x} - \mathbf{x}''). \tag{1.37}$$

Let us now see whether we can make sense out of the determinant in Eq. (1.28) for a general functional matrix $D(\mathbf{x}, \mathbf{x}')$. At first sight this looks rather difficult since a determinant involves the product of matrix elements whose number tends to infinity in the limit $a \to 0$. However, a generalization is possible using the fact that a determinant can be expanded in a power series. Our starting point is the well-known formula for finite matrices

$$(\det D)^{-1/2} = \exp\left[-\frac{1}{2}\operatorname{tr}\log D\right]. \tag{1.38}$$

This formula holds trivially for diagonal matrices if the logarithm is defined as

$$\log D = \begin{pmatrix} \log D_1 & \cdot & \cdot & \cdot \\ \cdot & \log D_2 & \cdot & \cdot \\ \cdot & \cdot & \log D_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{1.39}$$

An extension of the logarithm to nondiagonal matrices D can be given by going first, in the diagonal equation (1.39), to a power series and expanding each logarithm around some common value, say 1. Thus

$$\begin{aligned} \log D_\ell &= \log(1 + D_\ell - 1) \\ &= D_\ell - 1 - \frac{1}{2}(D_\ell - 1)^2 + \frac{1}{3}(D_\ell - 1)^3 - \dots \\ &= -\sum_{n=1}^{\infty} (-)^n (D_\ell - 1)^n / n. \end{aligned} \tag{1.40}$$

Thus (1.39) can be rewritten as a power series of the diagonal matrices

$$\log D = -\sum_{n=1}^{\infty} (-)^n (D - 1)^n / n. \tag{1.41}$$

Now observe that the right-hand side of this equation has meaning for *any nondiagonal* matrix D; hence this expansion can be used as a *definition* of the logarithm of a general matrix and we arrive at

$$\operatorname{tr}\log D = -\sum_{n=1}^{\infty}(-)^n \operatorname{tr}(D-1)^n/n \tag{1.42}$$

as a basis independent representation of $\Sigma_\ell \log D_\ell$. This allows us to calculate $\det D = e^{\operatorname{tr}\log D}$.

The important property of the expansion (1.42) is that it can now be generalized from matrices to functional matrices in the limit $a \to 0$. Writing out the first few terms of the series with all indices explicitly shown, we have

$$\begin{aligned}\operatorname{tr}\log D = &\sum_{\ell}(D_{\ell\ell}-\delta_{\ell\ell}) - \frac{1}{2}\sum_{\ell_1\ell_2}(D_{\ell_1\ell_2}-\delta_{\ell_1\ell_2})(D_{\ell_2\ell_1}-\delta_{\ell_2\ell_1})\\ &+\frac{1}{3}\sum_{\ell_1\ell_2\ell_3}(D_{\ell_1\ell_2}-\delta_{\ell_1\ell_2})(D_{\ell_2\ell_3}-\delta_{\ell_2\ell_3})(D_{\ell_3\ell_1}-\delta_{\ell_3\ell_1}) - \ldots\end{aligned} \tag{1.43}$$

Using the correspondence (1.14) this becomes

$$\begin{aligned}\operatorname{tr}\log D = &\sum_{\ell} a^3\left(D(\mathbf{x}_\ell,\mathbf{x}_\ell) - \frac{1}{a^3}\delta_{\ell\ell}\right)\\ &-\frac{1}{2}\sum_{\ell_1\ell_2} a^3\left(D(\mathbf{x}_{\ell_1},\mathbf{x}_{\ell_2}) - \frac{1}{a^3}\delta_{\ell_1\ell_2}\right) a^3\left(D(\mathbf{x}_{\ell_2},\mathbf{x}_{\ell_1}) - \frac{1}{a^3}\delta_{\ell_2\ell_1}\right) + \ldots\end{aligned} \tag{1.44}$$

In the limit $a \to 0$, the matrix products $(a^3)^n D^n$ all tend to the corresponding products of functional matrices, i.e.,

$$\begin{aligned}\sum_{\ell_1}(a^3)^n D^n_{\ell_1\ell_1} \xrightarrow[a\to 0]{} &\int d^3x_1 d^3x_2 \ldots d^3x_{n-1}d^3x_n\\ &\times D(\mathbf{x}_1,\mathbf{x}_2)\,D(\mathbf{x}_2,\mathbf{x}_3)\ldots D(\mathbf{x}_{n-1},\mathbf{x}_n)\,D(\mathbf{x}_{n-1},\mathbf{x}_1).\end{aligned} \tag{1.45}$$

The expression $(1/a^3)\delta_{\ell\ell'}$, on the other hand, turns into Dirac's δ-function [see (1.35)] so that

$$\begin{aligned}\sum_{\ell_1}(a^3)^n\left(D-\frac{1}{a^3}\delta\right)^n_{\ell_1\ell_1} \xrightarrow[a\to 0]{} &\int d^3x_1 d^3x_2 \ldots d^3x_{n-1}d^3x_n\\ &\times (D(\mathbf{x}_1,\mathbf{x}_2)-\delta(\mathbf{x}_1-\mathbf{x}_2))\ldots(D(\mathbf{x}_n,\mathbf{x}_1)-\delta(\mathbf{x}_n-\mathbf{x}_1)).\end{aligned} \tag{1.46}$$

Summarizing the result we see that the partition function (1.15) for a free field energy (1.17), which is a quadratic functional in $\phi(\mathbf{x})$, is

$$Z_0[j] = \prod_{\ell}\left[\int_{-\infty}^{\infty}\frac{d\phi(\mathbf{x}_\ell)}{\sqrt{2\pi T/a^3}}\right]\exp\left\{-\frac{1}{T}(E[\phi]+E_{\text{source}})\right\}$$
$$\xrightarrow[a\to 0]{}\exp\left\{-\frac{1}{2}\operatorname{tr}\log D\right\}\exp\left\{\frac{1}{2T}\int d^3x\, d^3x'\, j(x)\, D^{-1}(\mathbf{x},\mathbf{x}')\, j(\mathbf{x}')\right\}. \tag{1.47}$$

The trace of the logarithm is defined by a power series expansion

$$\operatorname{tr}\log D = -\int d^3x \sum_{n=1}^{\infty}\frac{(-)^n}{n}(D-1)^n(\mathbf{x},\mathbf{x}), \tag{1.48}$$

where the matrix products are of the functional type and 1 stands for the unit matrix in the functional space, the δ-function.

1.4. FIELD CORRELATION FUNCTIONS

In the grated theory, there was a simple way of finding all correlation functions by merely forming derivatives of Z with respect to the external source vector j_ℓ [recall (1.8) and (1.10)]. Now there is a functional version of this. Applying formula (1.10) to the grated version of the energy functional (1.13) we see that the correlation function of the fields $\phi(\mathbf{x})$ in the presence of the sources is obtained by forming the derivatives

$$G^{(n)}(\mathbf{x}_{\ell_1},\ldots,\mathbf{x}_{\ell_n}|j) = Z^{-1}[j]\,T^n\frac{1}{(a^3)^n}\frac{\partial}{\partial j(\mathbf{x}_{\ell_1})}\cdots\frac{\partial}{\partial j(\mathbf{x}_{\ell_n})}E[j]. \tag{1.49}$$

These partial derivatives have a proper continuum limit. This can be seen formally by noticing that partial derivatives of discrete variables j_ℓ follow from two fundamental rules:

(i) independence of all variables, i.e.,

$$\frac{\partial j_\ell}{\partial j_{\ell'}} = \delta_{\ell,\ell'}. \tag{1.50}$$

(ii) chain rule of differentiation. For example,

$$\frac{\partial}{\partial j_\ell}\frac{1}{2}\sum_{\ell_1\ell_2} j_{\ell_1} D^{-1}_{\ell_1\ell_2} j_{\ell_2} = \frac{1}{2}\sum_{\ell_1\ell_2} \delta_{\ell\ell_1} D^{-1}_{\ell_1\ell_2} j_{\ell_2} + \frac{1}{2}\sum_{\ell_1\ell_2} j_{\ell_1} D^{-1}_{\ell_1\ell_2}\delta_{\ell\ell_2}$$

$$= \sum_{\ell'} D^{-1}_{\ell\ell'} j_{\ell}. \tag{1.51}$$

The same rules hold for the grated version of field theory where the fields $\phi(\mathbf{x}_\ell)$ are generated by $(1/a^3)(\partial/\partial j(\mathbf{x}_\ell))$. By (1.50), these derivatives can all be obtained from

$$\frac{1}{a^3}\frac{\partial j(\mathbf{x}_\ell)}{\partial j(\mathbf{x}_{\ell'})} = \frac{1}{a^3}\delta_{\ell\ell'}. \tag{1.52}$$

But in the continuum limit $a \to 0$, the right-hand side tends towards the δ-function so that

$$\frac{1}{a^3}\frac{\partial j(\mathbf{x}_\ell)}{\partial j(\mathbf{x}_{\ell'})} \xrightarrow[a\to 0]{} \delta(\mathbf{x}_\ell - \mathbf{x}_{\ell'}). \tag{1.53}$$

The limit of the partial derivative on the left-hand side, including the factor $(1/a^3)$, is defined as the *functional derivative* and denoted by $(\delta j(\mathbf{x})/\delta j(\mathbf{x}'))$. Thus, we arrive at the rule for forming general functional derivatives:

(i) independence of all variables, i.e.,

$$\frac{\delta j(\mathbf{x})}{\delta j(\mathbf{x}')} = \delta(\mathbf{x} - \mathbf{x}'); \tag{1.54}$$

(ii) chain rule of differentiation. For example,

$$\frac{\delta}{\delta j(\mathbf{x})}\frac{1}{2}\int d^3x_1 d^3x_2 j(\mathbf{x}_1) D^{-1}(\mathbf{x}_1, \mathbf{x}_2) j(\mathbf{x}_2)$$

$$= \frac{1}{2}\int d^3x_1 d^3x_2\, \delta(\mathbf{x} - \mathbf{x}_1) D^{-1}(\mathbf{x}_1, \mathbf{x}_2) j(\mathbf{x}_2)$$

$$+ \frac{1}{2}\int d^3x_1 d^3x_2 j(\mathbf{x}_1) D^{-1}(\mathbf{x}_1, \mathbf{x}_2)\, \delta(\mathbf{x} - \mathbf{x}_2)$$

$$= \int dx'\, D^{-1}(\mathbf{x}, \mathbf{x}') j(\mathbf{x}'). \tag{1.55}$$

Consequently, the general correlation function of a field theory in the presence of sources can be obtained, in complete analogy with (1.10), from the functional derivatives

$$G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n|j) = Z^{-1}[j]\, T^n \frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_n)} Z[j], \tag{1.56}$$

where j has to be set equal to zero if there are no external currents in the physical problem.

Let us apply this formula to the free field case. For $n = 1$, where $G^{(1)}(\mathbf{x})$ is simply the expectation value for the field, we have from (1.8) and (1.29)

$$\langle \phi_\ell \rangle = G^{(1)}(\ell) = Z_0^{-1}[j]\, T \frac{\partial}{\partial j_\ell} Z_0[j] = \sum_{\ell'} D^{-1}_{\ell\ell'} j_{\ell'}. \tag{1.57}$$

For $j_\ell = 0$ this vanishes but in the presence of the sources it is non-zero. In the continuum case we obtain from (1.56) and (1.47)

$$\begin{aligned} \langle \phi(\mathbf{x}) \rangle &= G^{(1)}(\mathbf{x}) = Z_0^{-1}[j]\, T \frac{\delta}{\delta j(\mathbf{x})} Z_0[j] \\ &= \int d^3x'\, D^{-1}(\mathbf{x}, \mathbf{x}')\, j(\mathbf{x}'). \end{aligned} \tag{1.58}$$

For comparison, let us go through a direct evaluation of this expectation value which is given in the discrete form by

$$\langle \phi_\ell \rangle = Z_0^{-1} \prod_{\ell'} \left[\int_{-\infty}^{\infty} \frac{d\phi_{\ell'}}{\sqrt{2\pi T}} \right] \phi_\ell \exp\left\{ -\frac{1}{T}(E[\phi] + E_{\text{source}}) \right\}. \tag{1.59}$$

By following the same steps as those for Eqs. (1.18) and (1.19) we arrive at the equivalent of Eq. (1.21) for the field expectation

$$\begin{aligned} \langle \phi_\ell \rangle = Z_0^{-1}[j] \\ \times \left\{ \prod_{\ell'} \int_{-\infty}^{+\infty} \frac{d\phi_{\ell'}}{\sqrt{2\pi T}} (\phi'_\ell + \Sigma_{\ell'} D^{-1}_{\ell\ell'} j_{\ell'}) e^{-(1/2T)\Sigma_{\ell',\ell''} \phi'_{\ell'} D_{\ell'\ell''} \phi'_{\ell''}} \right\} e^{(1/2T)\Sigma_{\ell',\ell''} j_{\ell'} D^{-1}_{\ell'\ell''} j_{\ell''}}. \end{aligned} \tag{1.60}$$

Of the integrals, the first piece,

$$\prod_{\ell'}\left[\int_{-\infty}^{+\infty}\frac{d\phi'_{\ell'}}{\sqrt{2\pi T}}\right]\phi'_{\ell}\,e^{-(1/2T)\Sigma_{\ell',\ell''}\phi'_{\ell'}D_{\ell'\ell''}\phi'_{\ell''}} \tag{1.61}$$

vanishes since the integrand is odd in the ϕ'_{ℓ}'s. The remaining piece gives

$$\sum_{\ell'} D^{-1}_{\ell\ell'} j_{\ell'} \det(D)^{-1/2} e^{(1/2T)\Sigma_{\ell',\ell''} j_{\ell'} D^{-1}_{\ell'\ell''} j_{\ell''}} = Z_0[j] \sum_{\ell'} D^{-1}_{\ell\ell'} j_{\ell'} \tag{1.62}$$

so that we obtain,

$$\langle \phi_{\ell} \rangle = \sum_{\ell'} D^{-1}_{\ell\ell'} j_{\ell'} \tag{1.63}$$

as in (1.57).

Taking each step to the continuum limit gives (1.58). It is obvious that after exchanging everywhere the indices ℓ by $\mathbf{x}$ and sums over ℓ by integrals over $\mathbf{x}$, all formulas look exactly alike in the discrete and continuum versions. For this reason we shall, from now on, consider only fields.

Let us now proceed to study the correlation function of two fields, i.e., the *two-point function* $G^{(2)}(\mathbf{x}_1, \mathbf{x}_2)$. Using (1.56) and (1.47), we can directly calculate it:

$$\begin{aligned}
G^{(2)}(\mathbf{x}_1, \mathbf{x}_2) &= \langle \phi(\mathbf{x}_1)\, \phi(\mathbf{x}_2) \rangle \\
&= Z_0^{-1}[j]\, T^2 \frac{\delta}{\delta j(\mathbf{x}_1)} \frac{\delta}{\delta j(\mathbf{x}_2)} Z_0[j] \\
&= Z_0^{-1}[j] \det(D)^{-1/2} T \frac{\delta}{\delta j(\mathbf{x}_1)} \left(\int d^3x'_2\, D^{-1}(\mathbf{x}_2, \mathbf{x}'_2)\, j(\mathbf{x}'_2) \right. \\
&\quad \left. \times \exp\left\{ \frac{1}{2T} \int d^3x''_1 d^3x''_2\, j(\mathbf{x}''_1)\, D^{-1}(\mathbf{x}''_1, \mathbf{x}''_2)\, j(\mathbf{x}''_2) \right\} \right) \\
&= TD^{-1}(\mathbf{x}_1, \mathbf{x}_2) + \int d^3x'_1\, D^{-1}(\mathbf{x}_1, \mathbf{x}'_1)\, j(\mathbf{x}'_1) \\
&\quad \times \int d^3x'_2\, D^{-1}(\mathbf{x}_2, \mathbf{x}'_2)\, j(\mathbf{x}'_2) \\
&= TD^{-1}(\mathbf{x}_1, \mathbf{x}_2) + \langle \phi(\mathbf{x}_1) \rangle \langle \phi(\mathbf{x}_2) \rangle .
\end{aligned} \tag{1.64}$$

The second term is the product of field expectation values

$\langle\phi(\mathbf{x}_1)\rangle\langle\phi(\mathbf{x}_2)\rangle$, which is non-zero in the presence of the external current $j(\mathbf{x})$. Since this is a quantity known from our previous calculation it is useful to separate it out and define the so-called *connected two-point function*

$$G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = G^{(2)}(\mathbf{x}_1, \mathbf{x}_2) - G^{(1)}(\mathbf{x}_1)\, G^{(1)}(\mathbf{x}_2). \tag{1.65}$$

The physical relevance of this subtraction is seen by rewriting it as

$$G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = \langle(\phi(\mathbf{x}_1) - \langle\phi(\mathbf{x}_1)\rangle)(\phi(\mathbf{x}_2) - \langle\phi(\mathbf{x}_2)\rangle)\rangle, \tag{1.66}$$

which shows that the connected two-point function describes the correlations of the *deviations* of the fields from their mean values. This is what is observed most directly in experiments. It is also what is most easily calculated. Equation (1.64) implies that the connected correlation function of a free field is simply the inverse of the quadratic coefficient in the energy functional, times T:

$$G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = TD^{-1}(\mathbf{x}_1, \mathbf{x}_2). \tag{1.67}$$

It is the Green function for $\int d^3x'\, D(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}') = j(\mathbf{x})$.

Because of the frequent occurrence of this free connected two-point function we shall use a special short notation for it and simply call it $G_0(\mathbf{x}_1, \mathbf{x}_2)$. In terms of this quantity, we can rewrite the partition function (1.47) in the alternative form

$$Z_0[j] = \exp\left\{-\frac{1}{2}\operatorname{tr}\log TG_0^{-1}\right\}\exp\left\{\frac{1}{2T^2}\int d^3x\, d^3x'\, j(\mathbf{x})\, G_0(\mathbf{x}, \mathbf{x}')\, j(\mathbf{x}')\right\}. \tag{1.68}$$

This form has some important structural information. If Z_0 is expanded in powers of j, then the coefficients of $T^{-n} j(\mathbf{x}_1) \ldots j(\mathbf{x}_n)$ are, by the general rule (1.56), just the n-point correlation functions of this theory when there is no external source. But given the explicit form (1.68) we see two things: first, all $G^{(n)}(x_1, \ldots, x_n)$ vanish for $j = 0$ unless n is even. Second, these even n-point correlation functions can consist only of products of $n/2$ connected two-point correlation functions $G_0 \ldots G_0$. In other words, the free field n-point function for zero external current must have a product representation of the type

$$G_0^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_q G_0(\mathbf{x}_{q(1)}, \mathbf{x}_{q(2)}) G_0(\mathbf{x}_{q(3)}, \mathbf{x}_{q(4)}) \ldots G_0(\mathbf{x}_{q(n-1)}, \mathbf{x}_{q(n)}), \tag{1.69}$$

where the subscript 0 of $G_0^{(n)}$ is a reminder of the absence of interactions and the sum over q denotes certain configurations of index pairs out of the set of indices $\mathbf{x}_1, \ldots, \mathbf{x}_n$, where n is even. Precisely which pairs occur can be deduced from executing the partial derivatives and the result is the content of an important theorem, due to G. Wick, which will be discussed in detail in the next section.

1.5. WICK'S THEOREM

The theorem may be stated as follows: For a free field theory with no external source there are only even-point correlation functions and these are the sums of $(n-1)!!$ products of Green functions of the n variables $\mathbf{x}_1, \ldots, \mathbf{x}_n$, one for each *pair contraction*.

These are counted in the following manner: Given the arguments $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, then $\mathbf{x}_1$ is to be paired once with each of the other arguments, $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$. Such a pair is denoted by a hook and one records the first set of $n-1$ pairs by writing $(\overbrace{\mathbf{x}_1, \mathbf{x}_2}, \mathbf{x}_3, \ldots, \mathbf{x}_n) + (\overbrace{\mathbf{x}_1, \mathbf{x}_2}, \mathbf{x}_3, \ldots, \mathbf{x}_n) + \ldots$ The hook is what is called a contraction and the remaining variables are said to be uncontracted. Among these uncontracted variables one now takes again the first one and contracts it successively with all the rest in the same way. Altogether, there are $(n-1)(n-3) \ldots 1 = (n-1)!!$ pair contractions. For example,

$$\begin{aligned}
(\mathbf{x}_1, \mathbf{x}_2) &\to (\overbrace{\mathbf{x}_1, \mathbf{x}_2}) \\
(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &\to (\overbrace{\mathbf{x}_1, \mathbf{x}_2}, \mathbf{x}_3, \mathbf{x}_4) + (\overbrace{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3}, \mathbf{x}_4) + (\overbrace{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4}) \\
&\to (\overbrace{\mathbf{x}_1, \mathbf{x}_2}, \overbrace{\mathbf{x}_3, \mathbf{x}_4}) + (\overbrace{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4}) + (\overbrace{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4}).
\end{aligned} \tag{1.70}$$

In this way, the free field n-point functions for zero current become indeed a sum over products of two-point functions. The two simplest results are

$$G_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = G_0(\mathbf{x}_1, \mathbf{x}_2), \tag{1.71}$$

$$G_0^{(4)}(\mathbf{x}_1, \mathbf{x}_2) = G_0(\mathbf{x}_1, \mathbf{x}_2)\, G_0(\mathbf{x}_3, \mathbf{x}_4) + G_0(\mathbf{x}_1, \mathbf{x}_3)\, G_0(\mathbf{x}_2, \mathbf{x}_4) + G_0(\mathbf{x}_1, \mathbf{x}_4)\, G_0(\mathbf{x}_2, \mathbf{x}_3). \tag{1.72}$$

Let us illustrate the proof of Wick's theorem for these simple examples. Expanding $Z_0[j]$ up to the fourth order in j, we have

$$Z_0[j] \propto 1 + \frac{1}{2T^2} \int d^3x_1 d^3x_2\, j(\mathbf{x}_1)\, G_0(\mathbf{x}_1, \mathbf{x}_2)\, j(\mathbf{x}_2) + \frac{1}{2!} \frac{1}{(2T^2)^2} \int d^3x_1 d^3x_2 j(\mathbf{x}_1)\, G_0(\mathbf{x}_1, \mathbf{x}_2)\, j(\mathbf{x}_2) \times \int d^3x_1' d^3x_2' j(\mathbf{x}_1')\, G_0(\mathbf{x}_1', \mathbf{x}_2')\, j(\mathbf{x}_2') + \ldots, \tag{1.73}$$

from which (1.71) follows trivially from the differentiation rule (1.56):

$$G_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = Z_0^{-1}[j]\, T^2 \frac{\delta}{\delta j(\mathbf{x}_1)} \frac{\delta}{\delta j(\mathbf{x}_2)} Z_0[j] \Big|_{j=0} = G_0(\mathbf{x}_1, \mathbf{x}_2). \tag{1.74}$$

Similarly, the four-point function is obtained from

$$G_0^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = Z_0^{-1}[j]\, T^n \frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_4)} Z_0[j] \Big|_{j=0} = \frac{1}{2!} \frac{1}{2^2} \frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_4)} \int d^3x_1' d^3x_2' d^3x_3' d^3x_4' \times G_0(\mathbf{x}_1', \mathbf{x}_2')\, G_0(\mathbf{x}_3', \mathbf{x}_4')\, j(\mathbf{x}_1')\, j(\mathbf{x}_2')\, j(\mathbf{x}_3')\, j(\mathbf{x}_4'). \tag{1.75}$$

The four functional derivatives on the four currents $j(\mathbf{x}_1')$, $j(\mathbf{x}_2')$, $j(\mathbf{x}_3')$ and $j(\mathbf{x}_4')$ give 4! terms with δ-functions $\delta(\mathbf{x}_1 - \mathbf{x}_{p(1)}')\, \delta(\mathbf{x}_2 - \mathbf{x}_{p(2)}')\, \delta(\mathbf{x}_3 - \mathbf{x}_{p(3)}')\, \delta(\mathbf{x}_4 - \mathbf{x}_{p(4)}')$ with $p(i)$ covering all permutations of the indices 1, 2, 3, 4. Performing the integrals, we arrive at 24 terms of the form $(1/2!)(1/2^2) \sum_{p(i)} G_0(\mathbf{x}_{p(1)}, \mathbf{x}_{p(2)})\, G_0(\mathbf{x}_{p(3)}, \mathbf{x}_{p(4)})$. Explicitly, the variables $\mathbf{x}_1, \ldots, \mathbf{x}_4$ appear with the index sequences

$$\begin{array}{llll} (12)(34), & (12)(43), & (21)(34), & (21)(43), \\ (13)(24), & (13)(42), & (31)(24), & (31)(42), \\ (14)(23), & (14)(32), & (41)(23), & (41)(32), \end{array} \tag{1.76}$$

and once more the same terms with the first and second pair interchanged. But the free correlation functions $G_0(\mathbf{x}, \mathbf{x}')$ are symmetric in their arguments so that the terms in each row are identical. Moreover, the two functions $G_0 G_0$ are indistinguishable so that the combinations shown in (1.76) and the interchanged ones are the same. Hence only the three independent pair contractions of (1.72) remain. The factors come out correctly: there are altogether $(1/2!)(1/2^2)\,4! = 3$ terms.

After this exercise it is easy to see that Wick's theorem also holds true for the general n-point function. Only even n can occur, receiving contributions from the $(n/2)$-th expansion term of $Z[j]$. This has an expansion factor $(1/(n/2)!)(1/2^{n/2})$. Of the $n!$ derivatives, $(n/2)!\ 2^{n/2}$ are identical pair contractions so that the denominator is precisely cancelled and what remains are $n!/((n/2)!\ 2^{n/2}) = (n-1)!!$ different pair contractions, as stated in the theorem.

1.6. QUANTUM STATISTICS

For some systems, the statistical behavior depends significantly upon quantum fluctuations. Technically, their inclusion does not require any *fundamentally* new treatment. There is a simple way of *formally* reducing the *quantum statistics* of a system to the *classical statistics of a system with one more space dimension of finite size*. In order to see this, let us recall that the study of quantum mechanical questions requires the knowledge of the energy not only as a function of the variables ϕ_ℓ but also of the canonically conjugate generalized momentum variables, say π_ℓ, i.e., one must know the full Hamiltonian

$$H = H(\pi_\ell, \phi_\ell). \tag{1.77}$$

Given $H(\pi_\ell, \phi_\ell)$, the states of the system can in principle be found from the Schrödinger equation

$$H\left(\frac{\hbar}{i}\frac{\partial}{\partial\phi_\ell}, \phi_\ell\right)\psi_m(\phi_\ell) = E_m\psi_m(\phi_\ell), \tag{1.78}$$

where $\psi_m(\phi_\ell)$ are *wave functions* which are orthonormal with respect to the scalar product

$$\left[\prod_\ell \int d\phi_\ell\right]\psi_m^\dagger(\phi_\ell)\,\psi_{m'}(\phi_\ell) = \delta_{m,m'} \tag{1.79}$$

and E_m are the associated energy eigenvalues. For field variables $\phi(\mathbf{x})$, $\pi(\mathbf{x})$, this differential equation generalizes to a functional Schrödinger equation

$$H\left[\frac{\hbar}{i}\frac{\delta}{\delta\phi},\ \phi\right]\psi_m[\phi] = E_m\,\psi_m[\phi], \tag{1.80}$$

where the wave functionals satisfy the orthonormality relation

$$\int \mathscr{D}\phi(\mathbf{x})\,\psi_m^{\dagger}[\phi]\,\psi_{m'}[\phi] = \delta_{m,\,m'}\,. \tag{1.81}$$

According to the results of statistical mechanics we can then form the *quantum statistical partition function*

$$Z = \sum_m \exp\left\{-\frac{1}{T}E_m\right\} \tag{1.82}$$

and all other thermodynamic quantities follow from this, just as before.

Let us now see how, from the formal viewpoint, the quantum statistical partition function (1.82) can be calculated via a simple extension of the path integral methods developed above for the classical partition function (1.15). For this we consider ϕ_ℓ, π_ℓ or, in the continuum version, $\phi(\mathbf{x})$, $\pi(\mathbf{x})$, as functions of an additional auxiliary, continuous variable τ, to be called *imaginary time*, and require $\phi_\ell(\tau)$, $\pi_\ell(\tau)$, or $\phi(\mathbf{x}, \tau)$, $\pi(\mathbf{x}, \tau)$ to be *periodic* in τ with period $\hbar/T$. Then we form the following functional

$$A_E = \int_0^{\hbar/T} d\tau \left\{\sum_\ell \left[-\pi_\ell(\tau)\, i\frac{\partial}{\partial\tau}\phi_\ell(\tau) + H(\pi_\ell(\tau),\ \phi_\ell(\tau))\right]\right\} \tag{1.83}$$

or, in the continuum version,

$$A_E = \int_0^{\hbar/T} d\tau \left\{\int d^3x \left(-\pi(\mathbf{x},\ \tau)\, i\frac{\partial}{\partial\tau}\phi(\mathbf{x},\ \tau) + H[\pi(\tau),\ \phi(\tau)]\right)\right\}. \tag{1.84}$$

This functional is called the *Euclidean action*. It can be thought of as being derived from the classical action of the system

$$A = \int_{t_a}^{t_b} dt \left\{ \sum_{\ell} \pi_\ell(t)\, \dot{\phi}_\ell(t) - H(\pi_\ell(t),\, \phi_\ell(t)) \right\}$$
$$= \int_{t_a}^{t_b} dt \left\{ \int d^3x \left(\pi(\mathbf{x},\, t) \frac{\partial}{\partial t} \phi(\mathbf{x},\, t) - H[\pi(t),\, \phi(t)] \right) \right\}, \quad (1.85)$$

by a formal substitution of the time t by $-i\tau$ and $t_b - t_a \to -i(\hbar/T)$. This is the reason why τ is called the imaginary time. Under this substitution A goes over into iA_E.

The point is now that quantum statistics can be thought of as "classical statistics" with respect to the pseudoenergy A_E at a pseudotemperature $\hbar$ as long as $\pi_\ell(\tau)$, $\phi_\ell(\tau)$ or $\pi(\mathbf{x},\, \tau)$, $\phi(\mathbf{x},\, \tau)$ are treated as *independent* statistical variables. In the case of a discrete discrete number of such variables, the quantum statistical partition function may be written symbolically as

$$Z = \prod_{\ell,\tau} \left[\int_{\phi_\ell(0) = \phi_\ell(\hbar/T)} d\phi_\ell(\tau) \int_{\pi_\ell(0) = \pi_\ell(\hbar/T)} \frac{d\pi_\ell(\tau)}{2\pi\hbar} \right] \exp\left\{ -\frac{1}{\hbar} A_E \right\}. \quad (1.86)$$

The continuous variable τ plays a similar role as the space variable $\mathbf{x}$ in the discussion of classical fields. The product of integrals at each τ is therefore defined by dividing the τ interval $(0,\, \hbar/T)$ into a large number, say N, small pieces, each of width $\hbar/NT$ sliced at

$$\tau_n = \frac{n}{N} \frac{\hbar}{T} \quad n = 0,\, \ldots,\, N, \quad (1.87)$$

and taking the limit $N \to \infty$ at the end. Correspondingly, the integral (1.86) must be properly written in a grated form in τ space and the partition function is given by

$$Z = \prod_{\ell} \prod_{n=1}^{N} \left[\int_{\phi_\ell(0) = \phi_\ell(\hbar/T)} d\phi_\ell(\tau_n) \int \frac{d\pi_\ell(\tau_n)}{2\pi\hbar} \right]$$
$$\times \exp\left\{ \frac{1}{\hbar} \sum_{n=1}^{N} \pi_\ell(\tau_n)\, i(\phi_\ell(\tau_n) - \phi_\ell(\tau_{n-1})) \right.$$
$$\left. - \frac{1}{T} \frac{1}{N} \sum_{n=1}^{N} H(\pi_\ell(\tau_n),\, \phi_\ell(\tau_n)) \right\}. \quad (1.88)$$

It is straightforward to verify that this expression converges to the original quantum statistical one (1.82). First we take the Hamiltonian outside the path integral and write

$$Z = \exp\left\{-\frac{1}{T}\frac{1}{N}\sum_{n=1}^{N} H\left(\frac{\hbar}{i}\frac{\partial}{\partial\phi_\ell(\tau_n)},\ \phi_\ell(\tau_n)\right)\right\}\prod_\ell\prod_{n=1}^{N}\left[\int d\phi_\ell(\tau_n)\frac{d\pi_\ell(\tau_n)}{2\pi\hbar}\right]$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{n=1}^{N}\pi_\ell(\tau_n)(\phi_\ell(\tau_n)-\phi_\ell(\tau_{n-1}))\right\}. \tag{1.89}$$

Now the variables $\pi_\ell(\tau_n)$ appear only in the trivial exponential. Thus they can be integrated, giving a product of δ-functions:

$$\prod_\ell\prod_{n=1}^{N}\left[\int\frac{d\pi_\ell(\tau_n)}{2\pi\hbar}\right]e^{i/\hbar\sum_{n=1}^{N}\pi_\ell(\tau_n)(\phi_\ell(\tau_n)-\phi_\ell(\tau_{n-1}))} = \prod_\ell\prod_{n=1}^{N}\delta(\phi_\ell(\tau_n)-\phi_\ell(\tau_{n-1})). \tag{1.90}$$

This brings Z to the form

$$Z = \prod_{n=1}^{N}\left\{\prod_\ell\left[\int_{-\infty}^{\infty}d\phi_\ell(\tau_n)\right]\exp\left\{-\frac{1}{T}\frac{1}{N}H\left(\frac{\hbar}{i}\frac{\partial}{\partial\phi_\ell(\tau_n)},\ \phi_\ell(\tau_n)\right)\right\}\right.$$
$$\left.\times\prod_\ell\delta(\phi_\ell(\tau_n)-\phi_\ell(\tau_{n-1}))\right\}. \tag{1.91}$$

Now we make use of the fact that the Schrödinger equation (1.78) has a *complete* set of orthonormal eigenfunctions $\psi_m(\phi_\ell)$. They satisfy the *completeness relation*

$$\sum_m\psi_m(\phi_\ell)\,\psi_m^*(\phi_\ell') = \prod_\ell\delta(\phi_\ell-\phi_\ell'). \tag{1.92}$$

Using the latter we can replace the δ-functions in (1.91) by a sum over pairs of eigenstates and calculate

$$\exp\left\{-\frac{1}{T}\frac{1}{N}H\left(\frac{\hbar}{i}\frac{\partial}{\partial\phi_\ell(\tau_n)},\ \phi_\ell(\tau_n)\right)\right\}\delta(\phi_\ell(\tau_n)-\phi_\ell(\tau_{n-1}))$$
$$=\exp\left\{-\frac{1}{T}\frac{1}{N}H\left(\frac{\hbar}{i}\frac{\partial}{\partial\phi_\ell(\tau_n)},\ \phi_\ell(\tau_n)\right)\right\}\sum_m\psi_m(\phi_\ell(\tau_n))\,\psi_m^*(\phi_\ell(\tau_{n-1}))$$

$$= \sum_m \exp\left\{-\frac{1}{T}\frac{1}{N}E_m\right\} \psi_m(\phi_\ell(\tau_n))\, \psi_m^*(\phi_\ell(\tau_{n-1})). \tag{1.93}$$

Performing the same operation for each of the N factors in (1.91) we arrive at

$$Z = \prod_{n=1}^{N} \left\{ \prod_\ell \left[\int d\phi_\ell(\tau_n) \right] \right.$$
$$\left. \times \sum_{m_n} \exp\left\{-\frac{1}{T}\frac{1}{N}E_{m_n}\right\} \psi_{m_n}(\phi_\ell(\tau_n))\, \psi_{m_n}^*(\phi_\ell(\tau_{n-1})) \right\}. \tag{1.94}$$

Now we can use the orthonormality condition (1.79) by which all $d\phi_\ell(\tau_n)$ integrations become trivial and force all E_{m_n} to become the same. This indeed leads to

$$Z = \sum_m \exp\left\{-\frac{1}{T}E_m\right\}, \tag{1.95}$$

in agreement with (1.82). Thus the path integral (1.86) is a proper representation of the quantum statistical partition function.

The same result holds, of course, for the continuous quantum field theory by going first through the grated form in $\mathbf{x}$ space, which leads to (1.95), and then taking the limit $a \to 0$. In the limit $a \to 0$, the quantum statistical partition function (1.86) can be written as a functional integral

$$Z = \int_{\phi(\mathbf{x},0)=\phi(\mathbf{x},\hbar/T)} \mathscr{D}\phi(\mathbf{x}, \tau) \int_{\pi(\mathbf{x},0)=\pi(\mathbf{x},\hbar/T)} \frac{\mathscr{D}\pi(\mathbf{x}, \tau)}{2\pi\hbar}$$
$$\times \exp\left\{-\frac{1}{\hbar}A_E[\pi, \phi]\right\}, \tag{1.96}$$

where $A_E[\pi, \phi]$ is given in (1.84) and the functional integration implies grating the *full* $\mathbf{x}$ *and* τ space and taking the limit of an infinitesimally small grating width at the end.

This expression can now be processed in complete analogy with classical functional integral (1.15), (1.16), the only difference being the extension of space by one more dimension, the imaginary time τ, and the use of the Euclidean action A_E instead of the energy E.

Because of this equivalence of quantum statistical systems with classical systems of one more imaginary time variable, we may restrict the forth-

coming general structural discussions to classical systems only. The extension to quantum systems will be straightforward, at least, in principle.

NOTES AND REFERENCES

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D.J. Amit, *Field Theory, The Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 1983),
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Wick's theorem is derived in all these texts.

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The general problem of convergence of a perturbation expansion was the subject of many investigations during the last 10 years. For a general review with a rather complete list of references, see
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CHAPTER TWO

PERTURBATION THEORY

2.1. GENERAL FORMALISM

If the field energy $E[\phi]$ contains higher terms beyond the quadratic order in the fields $\phi(\mathbf{x})$, the path integrals are no longer Gaussian and the partition function cannot be calculated exactly. As long as these higher terms can be considered small in some sense, we may attempt a series expansion of the partition function in powers of V_3, V_4,... We have mentioned in the introduction that this expansion really has a zero radius of convergence, due to non-expandable pieces of the type e^{-1/V_4}, e^{-1/V_3}. Still, as long as V_4, V_3 are small enough, the error in reproducing the exact expression can often be sufficiently small, such that the power series expansion can be useful for obtaining physically relevant information.

For simplicity, let us assume that there is only one *local* interaction of the V_4 type, by which we mean an energy of the form

$$E[\phi] = \frac{1}{2}\int d^3x_1 d^3x_2 \phi(\mathbf{x}_1) D(\mathbf{x}_1, \mathbf{x}_2) \phi(\mathbf{x}_2) + \frac{g}{4!}\int d^3x \, \phi(\mathbf{x})^4. \tag{2.1}$$

In other words, we shall assume a partition function of the form

$$Z[j] = \int \mathscr{D}\phi(\mathbf{x}) \exp\left\{-\frac{1}{T}E[\phi]\right\} \exp\left\{\frac{1}{T}\int d^3x j(\mathbf{x}) \phi(\mathbf{x})\right\}. \tag{2.2}$$

In the following, many formulas will be shortened by absorbing the temperature T into ϕ, j, and g with the rescaling

$$\phi \to \sqrt{T}\,\phi, \quad j \to \sqrt{T}\,j, \quad g \to \frac{1}{T} g. \tag{2.3}$$

We can then simply work with

$$Z[j] = \int \mathscr{D}\phi(\mathbf{x}) \exp\left\{ -\frac{1}{2}\int d^3x_1 d^3x_2\, \phi(\mathbf{x}_1)\, D(\mathbf{x}_1, \mathbf{x}_2)\, \phi(\mathbf{x}_2) - \frac{g}{4!}\int d^3x\, \phi^4(\mathbf{x}) + \int d^3x\, j(\mathbf{x})\, \phi(\mathbf{x}) \right\}, \tag{2.4}$$

where the measure of functional integration is now defined by $\lim_{a\to 0} \prod_{\ell} \int_{-\infty}^{\infty} \frac{d\phi(\mathbf{x}_\ell)}{\sqrt{2\pi/a^3}}$ [compare (1.16)]. The correlation functions are given by

$$G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = Z^{-1}[j] \frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_n)} Z[j] \Big|_{j=0}. \tag{2.5}$$

Since the formalism to be described is really independent of the number of space dimensions we shall, from now on, drop the superscripts "3" in the integration measures and write the spatial variables as x instead of $\mathbf{x}$.

The perturbation expansion is based on a formal expansion of $Z[j]$ in powers of g, up to a certain power g^N, say,

$$Z[j] \equiv \sum_{p=0}^{N} Z_p[j] = \sum_{p=0}^{N} \frac{1}{p!}\left(-\frac{g}{4!}\right)^p Z^{(p)}[j]. \tag{2.6}$$

The value at $j = 0$ supplies the free energy via $Z = \exp(-(1/T)F)$. It is called the *vacuum contribution* for reasons to become clear in the sequel. Similarly, the n-th functional derivative at $j = 0$ is a power series

$$\frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} Z[j]\Big|_{j=0} \equiv \sum_{p=0}^{N} Z_p(x_1, \ldots, x_n)$$

$$= \sum_{p=0}^{N} \frac{1}{p!}\left(-\frac{g}{4!}\right)^p \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} Z^{(p)}[j]\Big|_{j=0}. \tag{2.7}$$

According to the rule (2.5), the correlation functions up to order g^N are obtained by dividing this series by $Z[0]$ and retaining only powers up to g^N.

It is useful to absorb $Z_0[0]$ in the definition of the path integral, i.e., we define

$$\int \mathscr{D}^{\text{new}}\phi(x) = \frac{1}{Z_0[0]} \int \mathscr{D}\phi(x) \tag{2.8}$$

and drop the superscript "new". With this new measure, free field expectation values are directly found from the path integrals

$$\begin{aligned}\langle \phi(x_1) \ldots \phi(x_n) \rangle_0 &= G_0^{(n)}(x_1, \ldots, x_n) \\ &= \int \mathscr{D}\phi(x) \Big(\phi(x_1) \ldots \phi(x_n) \Big) \\ &\quad \times \exp\left(-\frac{1}{2} \int dx_1 dx_2 \phi(x_1) D(x_1, x_2) \phi(x_2) \right). \end{aligned} \tag{2.9}$$

Comparing (2.7) and (2.6) with (2.4) we see that the coefficient of $(1/p!)(-g^n/4!)^p$ in the perturbation series of the n-th functional derivative of $Z[j]$ (2.6) are given by the free field correlation functions as follows:

$$\begin{aligned}&\frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} Z^{(p)}[j] \Big|_{j=0} \\ &= \int dz_1 \ldots dz_p \, \langle \phi(x_1) \ldots \phi(x_n) \, \phi^4(z_1) \, \phi^4(z_2) \ldots \phi^4(z_p) \rangle_0 \\ &= \int dz_1 \ldots dz_p \, G_0^{(n+4p)}(x_1, \ldots, x_n, z_1, z_1, z_1, z_1, \ldots, z_p, z_p, z_p, z_p). \end{aligned} \tag{2.10a}$$

The vacuum contribution is a special case of this:

$$\begin{aligned}Z^{(p)}[0] &= \int dz_1 \ldots dz_p \, \langle \phi^4(z_1) \ldots \phi^4(z_p) \rangle_0 \\ &= \int dz_1 \ldots dz_n G_0^{(4p)}(z_1, z_1, z_1, z_1, \ldots, z_p, z_p, z_p, z_p). \end{aligned} \tag{2.10b}$$

The expressions (2.8)–(2.10) can now be evaluated further by using Wick's theorem, which tells us that the free n-point functions are sums of all pair contractions [recall Eq. (1.69)]

$$G_0^{(n)} = \sum \underbrace{G_0 \ldots G_0}_{n/2 \text{ factors}}. \tag{2.11}$$

In this way $Z[0]$ and $(\delta/\delta j(x_1)) \ldots (\delta/\delta j(x_n))\, Z[j]|_{j=0}$ can be reduced to a sum over products of free two-point functions G_0 which are simple and well-known objects.

Notice that the desired perturbation expansion of $Z[0]$ and $(\delta/\delta j(x_1)) \ldots (\delta/\delta j(x_n))\, Z[j]|_{j=0}$ could also have been extracted formally from (2.4) by removing the interaction from the path integral and writing

$$\begin{aligned} Z[j] &= \exp\left\{-\frac{g}{4!}\int dx \left(\frac{\delta}{\delta j(x)}\right)^4\right\} Z_0[j] \\ &= \exp\left\{-\frac{g}{4!}\int dx \left(\frac{\delta}{\delta j(x)}\right)^4\right\} \exp\left\{\frac{1}{2}\int dx_1\, dx_2\, j(x_1) D^{-1}(x_1, x_2) j(x_2)\right\}. \end{aligned} \tag{2.12}$$

Expanding this expression in powers of g and j we find

$$\begin{aligned} Z[j] = \left(1 - \frac{g}{4!}\int dz \left(\frac{\delta}{\delta j(z)}\right)^4 + \frac{1}{2!}\left(\frac{g}{4!}\right)^2 \int dz_1 dz_2 \left(\frac{\delta}{\delta j(z_1)}\right)^4 \left(\frac{\delta}{\delta j(z_2)}\right)^4 + \ldots\right) \\ \times \left(1 + \frac{1}{2}\int d^3y_1 d^3y_2\, G_0(y_1, y_2) j(y_1) j(y_2) + \frac{1}{2!}\frac{1}{2^2}\int d^3y_1 d^3y_2 d^3y_3 d^3y_4 \right. \\ \left. \times G_0(y_1, y_2)\, G_0(y_3, y_4) j(y_1) j(y_2) j(y_3) j(y_4) + \ldots\right). \end{aligned} \tag{2.13}$$

Performing appropriate functional differentiations with respect to the current $j(x)$ we may now obtain $Z[0]$ and all the Green functions.

2.2. VACUUM CONTRIBUTION $Z[0]$

Let us first evaluate $Z[0]$. To lowest order in p we find trivially

$$Z_0[0] = 1,$$

due to the new normalization (2.8) of the functional integral. To next order, $p = 1$, we have, from (2.10b),

$$Z_1[0] = -\frac{g}{4!}\int dz_1 \langle \varphi^4(z_1)\rangle = -\frac{g}{4!}\int dz_1 G_0^{(4)}(z_1, z_1, z_1, z_1). \tag{2.14}$$

Writing this as

$$Z_1[0] = -\frac{g}{4!}\int dz \int dx_1 dx_2 dx_3 dx_4$$
$$\times \delta(x_1 - z)\delta(x_2 - z)\delta(x_3 - z)\delta(x_4 - z) G_0^{(4)}(x_1, x_2, x_3, x_4),$$

we can insert the Wick expansion (1.72) and arrive at

$$Z_1[0] = -\frac{g}{4!}\int dz \int dz_1 dz_2 dz_3 dz_4 \delta(z_1 - z)\delta(z_2 - z)\delta(z_3 - z)\delta(z_4 - z)$$
$$\times (G_0(z_1, z_2)G_0(z_3, z_4) + 2 \text{ perm}), \tag{2.15}$$

where by "2 perm" we have abbreviated the other two pair contractions as given in (1.72). Each of the three terms leads to the same analytic expression so that the first order vacuum contribution to the partition function is

$$Z_1[0] = -\frac{g}{4!}3\int dz G_0(z, z)G_0(z, z). \tag{2.16}$$

Most field theories are translationally invariant and the coefficients $D(x_1, x_2)$ depend only on $x_1 - x_2$ [recall (1.12)]. As a consequence, the two-point correlation function satisfies

$$G_0(x_1, x_2) = \bar{G}_0(x_1 - x_2).$$

Thus the integration in (2.16) is trivial and gives

$$-\frac{g}{4!}\bar{G}_0^2(0)\int dz = -\frac{g}{4!}\bar{G}_0^2(0)\cdot V, \tag{2.17}$$

where V is the total volume of the system.

Consider now $Z[0]$ to second order in g. From (2.10b), this is given by

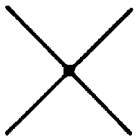
$$Z_2[0] = \frac{1}{2!}\left(-\frac{g}{4!}\right)^2 Z^{(2)}[0]\frac{1}{2!}\left(-\frac{g}{4!}\right)^2 \int dz_1 dz_2 \langle \phi^4(z_1)\phi^4(z_2)\rangle$$

$$= \frac{1}{2}\left(-\frac{g}{4}\right)^2 \int dz_1 dz_2 G^{(8)}(z_1, z_1, z_1, z_1, z_2, z_2, z_2, z_2).$$

Expanding the free 8-point function *à la* Wick gives

$$Z_2[0] = \frac{1}{2!}\left(-\frac{g}{4!}\right)^2 \int dz_1 dz_2 \int dz_{11} \ldots dz_{24}$$
$$\times \delta(z_{11} - z_1) \ldots \delta(z_{14} - z_1)\delta(z_{21} - z_2) \ldots \delta(z_{24} - z_2)$$
$$\times [G_0(z_{11}, z_{12})G_0(z_{13}, z_{14})G_0(z_{21}, z_{22})G_0(z_{23}, z_{24})$$
$$+ (7 \cdot 5 \cdot 3 \cdot 1 - 1) \text{ perm}]. \tag{2.18}$$

At this point it becomes useful to organize the calculation by means of graphical techniques and introduce the so-called *Feynman diagrams* or *graphs*. For each interaction we shall draw a graph



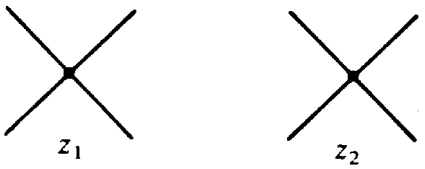
$$= -\frac{g}{4!}, \tag{2.19}$$

called a *vertex* with four legs. For each free two-point function $G_0(x_1, x_2)$, we draw a line between point x_1 and point x_2

$$= G_0(x_1, x_2), \tag{2.20}$$

called a *propagator*.

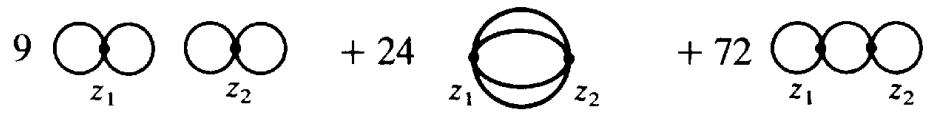
The evaluation of (2.18) amounts to the problem of connecting two vertices



(2.21)

by four propagators in all $7 \cdot 5 \cdot 3 \cdot 1 = 105$ different ways. Due to the

identity of their endpoints there are three classes of different topology which can be drawn as follows:



(2.22)

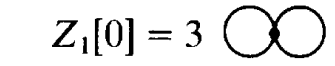
The factors in front denote the multiplicities of the Wick contractions with the same topology. They arise as follows: We split the 8 spatial arguments into $p = 2$ groups $(z_{11}z_{12}z_{13}z_{14})(z_{21}z_{22}z_{23}z_{24})$, which will afterwards be identified with the two points z_1 and z_2, respectively. Obviously, there are $3!! \cdot 3!! = 9$ contractions *within each pair of parentheses*, i.e., among z_{1i}'s and z_{2i}'s separately. These are represented by the first graph in (2.22). Then there are four contractions from one pair of parentheses to the other, i.e., from z_{1i} to the z_{2i}'s. For each of these, there are three from z_{12} to the remaining z_{2i}'s, etc. (two from z_{13}, and one from z_{14}). Altogether, there are 24 contractions associated with the second graph. The 72 contractions of the third graph arise from six single pair contractions within each vertex and two possible ways of interconnecting the two.

The analytic expression for the second order contribution to $Z[0]$ is

$$Z_2[0] = \frac{1}{2!}\left(-\frac{g}{4!}\right)^2\left[9\left(\int dz_1 G_0^2(z_1, z_1)\right)^2 + 24\int dz_1 dz_2 G_0^4(z_1, z_2) + 72\int dz_1 dz_2 G_0(z_1, z_1)G_0^2(z_1, z_2)G_0(z_2, z_2)\right], \tag{2.23}$$

where in a translationally invariant theory $G_0(z, z) = \tilde{G}_0(0)$ is a constant and $\int dz = V$.

Having introduced the graphical representation for representing the contributions to $Z[0]$ order-by-order we see that with the same rules, the first-order contribution (2.16) to $Z[0]$ has the graphical form



(2.24)

We can now proceed and find a graphical representation for the higher order contributions. The graphs making up the vacuum contribution to the partition function $Z[0]$ are called *vacuum graphs*. They have the

characteristic property of consisting of vertices whose legs are all connected with each other by lines.

Obviously, the calculation of higher orders of $Z[0]$ becomes increasingly tedious. There is, however, an important simplification which saves a great deal of labor during the calculation. Looking at the three topologies in the second order graphs (2.22) we see that the first of these is merely a combination of two first order graphs (2.24). Such a graph is said to be *disconnected*. In general, we can convince ourselves that in each p-th order vacuum graph one can find disconnected pieces containing all allowed combinations of lower order graphs. In fact, one can prove the following relation: The sum of all vacuum graphs in $Z[0]$ can be obtained from the exponential $e^{W[0]}$, where $W[0]$ collects only the connected vacuum graphs.

Up to second order we can verify this by inspection. Writing $Z[0]$ as

$$Z[0] = 1 + 3 \bigcirc\!\!\bigcirc + \frac{1}{2!}\left(9 \bigcirc\!\!\bigcirc \; \bigcirc\!\!\bigcirc + 24 \; \text{⦶} + 72 \bigcirc\!\!\bigcirc\!\!\bigcirc\right) + O(g^3)$$

$$= \exp\left[3 \bigcirc\!\!\bigcirc + \frac{1}{2!}\left(24 \; \text{⦶} + 72 \bigcirc\!\!\bigcirc\!\!\bigcirc\right) + O(g^3)\right] \qquad (2.25)$$

where $O(g^3)$ denotes the omitted terms of order higher than the second, we can therefore identify

$$W[0] = 3 \bigcirc\!\!\bigcirc + \frac{1}{2!}\left(24 \; \text{⦶} + 72 \bigcirc\!\!\bigcirc\!\!\bigcirc\right) + O(g^3). \qquad (2.26)$$

At this juncture it is useful to remember that the new measure of functional integration had the effect of dropping, in all formulas, an overall factor

$$Z_0[0] = \exp(-\tfrac{1}{2}\operatorname{tr}\log G_0^{-1}).$$

If we go back to the original measure, this amounts to an additional zeroth order term

$$W_0[0] = -\tfrac{1}{2}\operatorname{tr}\log G_0^{-1}.$$

It is customary to represent this by a single loop diagram

$$W_0[0] = \frac{1}{2}\bigcirc, \tag{2.27}$$

which has to be added to (2.26).

Notice that to this low order, only one term has been saved when calculating $Z[0]$ via the exponentiation of connected graphs in $\exp(W[0])$. The full advantage in using $W[0]$ will manifest itself only in higher order calculations.

Let us now turn to the perturbation expansion of correlation functions.

2.3. TWO-POINT FUNCTION AND SELF-ENERGY GRAPHS

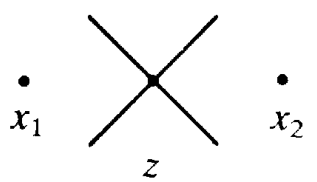
For this we have to calculate all terms in (2.10a) with two extra fields $\phi(x_1)\,\phi(x_2)$. Afterwards, we have to divide out the normalization factor $Z[0]$. The lowest contribution to $G(x_1, x_2)$ is trivial [recall (1.67)]:

$$G(x_1, x_2) = G_0(x_1, x_2) = D^{-1}(x_1, x_2). \tag{2.28}$$

The first order correction comes from the correlation function of six fields. In order to calculate this it is best to rewrite it in the form

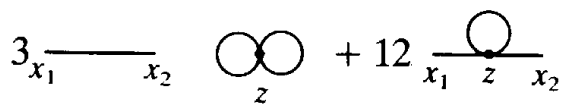
$$\begin{aligned} Z_1(x_1, x_2) &= -\frac{g}{4!}\int dz\, \langle \phi(x_1)\phi(x_2)\phi^4(z)\rangle_0 \\ &= -\frac{g}{4!}\int dz\, G^{(6)}(x_1, x_2, z, z, z, z) \\ &= -\frac{g}{4!}\int dz \int dz_1 dz_2 dz_3 dz_4\, \delta(z - z_1)\delta(z - z_2) \\ &\quad \times \delta(z - z_3)\delta(z - z_4) G_0^{(6)}(x_1, x_2, x_1, x_2, x_3, x_4). \end{aligned} \tag{2.29}$$

Then, by Wick-expanding the free six-point functions, one obtains $5 \cdot 3 \cdot 1$ different pair contributions. Four of the spatial arguments are forced to coincide with each other via the δ-functions. The procedure can be represented graphically by drawing a vertex at some place z and two points at places x_1 and x_3:



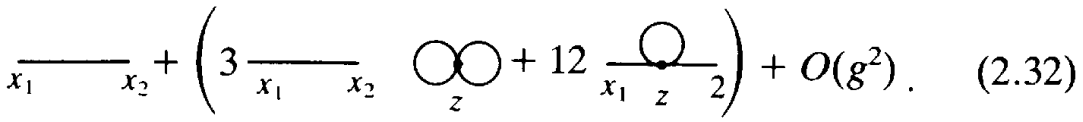
(2.30)

The Wick expansion amounts to all possible ways of connecting these objects by three G_0 lines. The result is



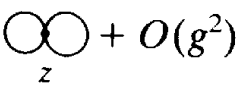
(2.31)

Going through the same type of counting as for the terms (2.22) we see that of the $5 \cdot 3 \cdot 1 = 15$ possible pair contractions, there are 3 of the first type and 12 of the second. Thus, up to first order, we have the expansion



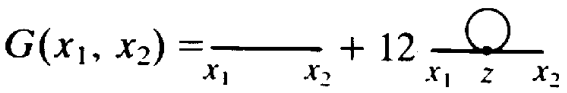
$+ O(g^2)$. (2.32)

In order to obtain $G(x_1, x_2)$, we have to divide this expression by $Z[0]$ to first order in g. Since



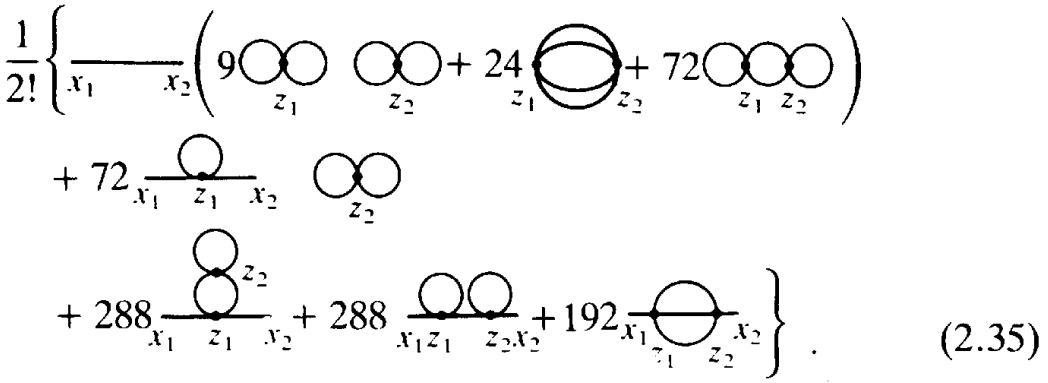
(2.33)

this amounts to removing the second graph so that we arrive at



(2.34)

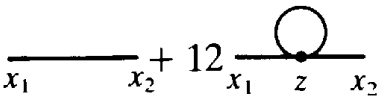
Also the next higher orders are easy to calculate. A careful counting of all possible graphs to second order gives

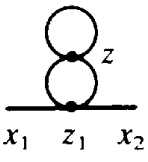
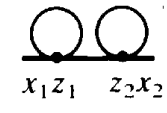
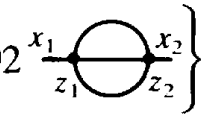


(2.35)

In view of the fact that the normalization factor $Z[0]$ must be divided out we have explicitly factored out the disconnected vacuum graphs. Thus, the first line contains the zeroth order graph for the correlation function multiplied with all second order vacuum graphs. The second line contains the product of two first-order graphs, one for the correlation function and

one from the vacuum. The third row, finally, consists only of connected second order graphs for the correlation function. In this way of writing, we can immediately normalize the expansion and remove a factor $Z[0]$, thus arriving at the two-particle correlation function

$G(x_1, x_2) =$  $+ 12$

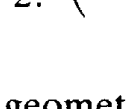
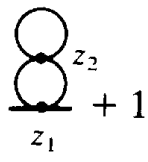
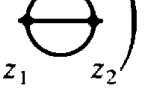
$+ \frac{1}{2!}\Big\{ 288$  $+ 288$  $+ 192$  $\Big\}$

$+ O(g^3)$. (2.36)

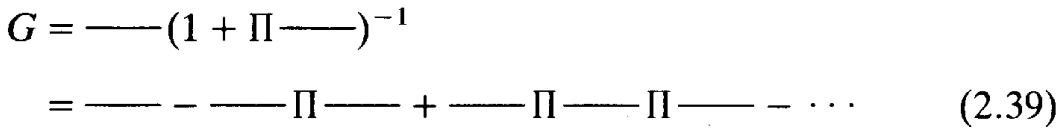
When calculating still higher orders a great deal of labor can be saved by observing that many graphs are simply linear repetitions of graphs which have already been calculated at lower orders. An example is the fourth term in (2.36), which is a repetition of two first order terms. It can be shown that such repeated graphs can simply be obtained as follows: One collects all graphs which do not have such repetitions. They can be characterized topologically by the fact that they do not fall apart when cutting one line. They are called *one-particle irreducible* (OPI). In each of these graphs one may cut off the two external legs. This process is called *amputation* or *truncation*. The amputated OPI graphs are called *proper self-energy graphs*. If their sum is denoted by $-\Pi$, the full two-particle correlation function is given by the geometric series

$$\begin{aligned} G &= (G_0^{-1} + \Pi)^{-1} = G_0(1 + \Pi G_0)^{-1} \\ &= G_0 - G_0 \Pi G_0 + G_0 \Pi G_0 \Pi G_0 + \ldots , \end{aligned} \tag{2.37}$$

where $G_0(-\Pi)G_0(-\Pi)G_0$, $G_0(-\Pi)G_0(-\Pi)G_0(-\Pi)G_0$, etc. are the proper repetitions of two, three, etc. proper self-energy graphs connected by a single free line G_0. As an example we can identify in (2.36) the proper self-energy graphs up to second order,

$-\Pi = 12$  $+ \frac{1}{2!}\Big(288$  $+ 192$  $\Big)$ (2.38)

and see that, indeed, the geometric series

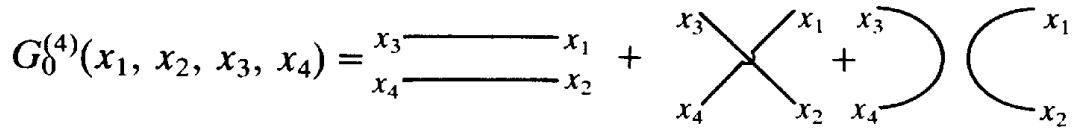


(2.39)

correctly reproduces (2.36) up to second order.

2.4. FOUR-POINT AND VERTEX FUNCTIONS

Here we have to calculate the expressions (2.10a) with four additional fields $\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_4)$, and divide again by $Z[0]$ in the end. The lowest contribution is the free four-point function which, by Wick's theorem, consists of three different pair contractions. These can be represented graphically by

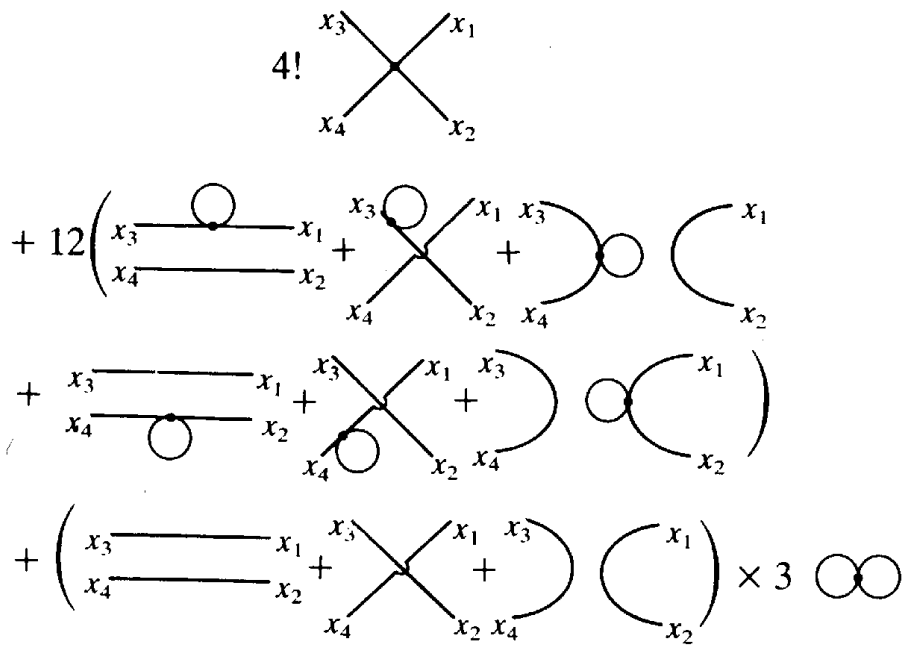


(2.40)

To first order, we need

$$Z_1(x_1, x_2, x_3, x_4) = -\frac{g}{4!}\int dz \,\langle \phi(x_1) \ldots \phi(x_4)\phi^4(z)\rangle .$$

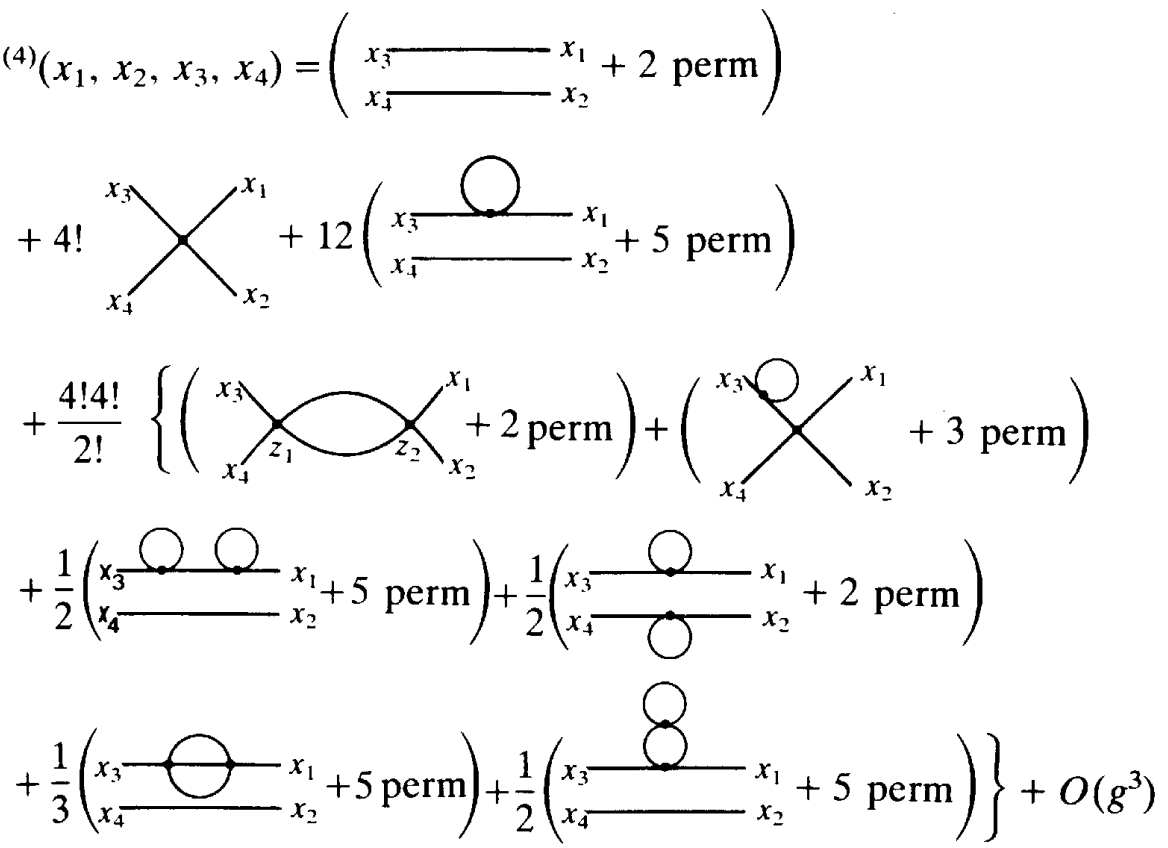
The graphs have three possible topologies:



(2.41)

Division by $Z[0]$ amounts to removing the last row in which there is a disconnected vacuum graph. The second and third row, have the following characteristic: they are disconnected, with pieces consisting of such

FIG. 2.1 The graphical expansion of the four-point function up to second order in the coupling g. Lines stand for $G_0(x, x')$, vertices for $-g/4!$.



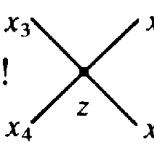
graphs which were already calculated in connection with the two-point function. In fact, they are nothing but first order corrections to the two-point functions appearing in the zeroth order graphs (2.40). In general, it can be shown that in a ϕ^4 theory,[a] all graphs contributing to the four-point function can be split into a connected piece, plus three disconnected pieces which are products of two-point functions, with three possible pair permutations,

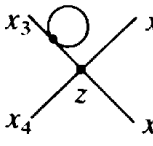
$$G^{(4)}(x_1, x_2, x_3, x_1) = G_c^{(4)}(x_1, x_2, x_3, x_4) + (G(x_1, x_2)G(x_3, x_4) + G(x_1, x_3)G(x_2, x_4) + G(x_1, x_4)G(x_2, x_3)). \tag{2.42}$$

Up to second order, the sum of all graphs to be calculated are shown in Fig. 2.1. We have already removed the disconnected graphs containing vacuum contributions which cancel out when dividing by $Z[0]$. In the graphs of Fig. 2.1 we can easily identify the disconnected pieces $G \cdot G$ using the graphical expansion in (2.36). In this way, the calculation of the

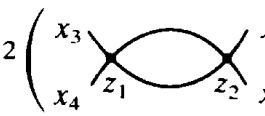
[a]This holds only for the ϕ^4 theory *without* spontaneous symmetry breakdown.

four-point function reduces to the following three types of connected graphs:

$$G_c^{(4)}(x_1, x_2, x_3, x_4) = 4!$$ 

$$+ \frac{1}{2!}4!^2 \Bigg($$ 

$$+ 3 \text{ perm} \Bigg)$$

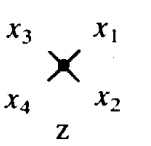
$$+ \frac{1}{2!}4!^2 \Bigg($$ 

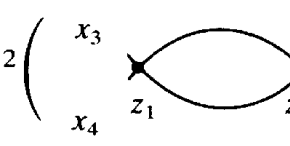
$$+ 2 \text{ perm} \Bigg) \qquad (2.43)$$

Inspecting these we realize one more important construction principle: the second graph is equal to the first graph with a first order correction applied to the two-point correlation at each leg [compare (2.34)]. This feature is a very general one. It will be shown in the next chapter that the connected diagrams for the full four-point function of the ϕ^4 theory can always be written in the form

$$G_c^{(4)}(x_1, x_2, x_3, x_4) = \int dz_1 dz_2 dz_3 dz_4 \, G(x_1, z_1) G(x_2, z_2) G(x_3, z_3)$$
$$\times G(x_4, z_4) \Gamma_c^{(4)}(z_1, z_2, z_3, z_4), \qquad (2.44)$$

where $G(x, z)$ are the full two-point function. The remainder is the four-point analogue of the proper self-energy introduced before. It contains only graphs without external legs, i.e., only amputated graphs. The legs are furnished by the G factors. Moreover, since the amputation involves the *full* two-point functions, these amputated graphs have the property of containing no further parts which are corrections to an external leg. They are one-particle irreducible (OPI) just as the proper self-energy graphs discussed above are.[b] In other words $\Gamma_c^{(4)}$ is given by the sum of all OPI amputated parts of the connected four-point functions. Such diagrams are referred to as *proper vertex graphs*. The functions $\Gamma_c^{(4)}$ are called *four-point vertex functions*. In the present case, the first and third diagrams in (2.43) are all proper vertex graphs

$$\Gamma_c^{(4)}(x_1 x_2 x_3 x_4) = 4!$$ 

$$+ \frac{1}{2!} \, 4!^2 \Bigg($$ 

$$+ 2 \text{ perm} \Bigg) \qquad (2.45)$$

[b]If there are ϕ^3 interactions, this is no longer true as we shall see in the next chapter.

2.5. SPECIAL FEATURES OF LOCAL THEORIES

The perturbation expansion derived up to now is applicable to a real field theory with $(g/4!)\,\phi^4$ interaction. The free part of the energy functional, $D(x_1, x_2)$, and thus the two-point correlation function $G_0(x_1, x_2) = D^{-1}(x_1, x_2)$ was left completely arbitrary. Only the comment after the vacuum graph (2.16) referred to the more special case of translationally invariant systems. The systems we shall be interested in this text will always have this property. Let us, therefore, study this situation further and see in which way translational invariance simplifies the perturbation expansion.

If a system is translationally invariant $D(x_1, x_2)$ depends only on the difference $x_1 - x_2$ and the free part of the energy reads

$$E_0 = \frac{1}{2}\int dx_1 dx_2 \phi(x_1)\bar{D}(x_1 - x_2)\phi(x_2). \tag{2.46}$$

This form has the advantage that it is diagonal in the Fourier components of the fields. If we expand $\phi(x)$ in momentum space,

$$\phi(x) = \int \frac{d^D k}{(2\pi)^D} e^{ik\cdot x}\phi(k) \tag{2.47a}$$

and, similarly,

$$D(x_1, x_2) = \int \frac{d^D q}{(2\pi)^D} e^{iq\cdot(x_1 - x_1)}\bar{D}(q), \tag{2.47b}$$

we may use the well-known identity

$$\int d^D x\, e^{ik\cdot x} = (2\pi)^D \delta^{(D)}(k) \tag{2.48}$$

and calculate

$$E_0 = \frac{1}{2}\int dx_1 dx_2 \int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\frac{d^D q}{(2\pi)^D} e^{i[k_1\cdot x_1 + k_2\cdot x_2 + q\cdot(x_1 - x_2)]}\phi(k_1)\bar{D}(q)\phi(k_2)$$

$$= \frac{1}{2}\int \frac{d^D q}{(2\pi)^D}\phi(-q)\bar{D}(q)\phi(q).$$

Since $\phi(x)$ is real, $\phi(q)$ satisfies

$$\phi(-q) = \phi(q)^* \tag{2.49}$$

and we can also write

$$E_0 = \frac{1}{2}\int \frac{d^D q}{(2\pi)^D} \phi(q)^* \bar{D}(q)\phi(q). \tag{2.50}$$

In all cases to be considered in this text, $\bar{D}(q)$ permits an expansion in a power series of the field momenta q. In mirror symmetric systems, only even powers can occur; then the expansion reads

$$\bar{D}(q) = m^2 + a_2 q^2 + a_4 q^4 + \ldots \tag{2.51}$$

The first term m^2 is called the *mass term*. It governs the energy of constant fields since these have $q = 0$. In this chapter it will be assumed to be positive such that among the $\phi = \text{const.}$ field configurations, $\phi = 0$ has minimal energy. What happens for negative m^2 will be studied in the next chapter. The higher terms measure the increase in energy if the field has $q \neq 0$ Fourier components, i.e., if the field varies in space. For this reason $a_1, a_2, \ldots$ are called *bending energies*. Bending which is very smooth in space requires only small momenta. We shall assume that such a smooth bending always costs energy. This implies $a_2 > 0$.[c] By an appropriate renormalization of the field and the interaction terms it is then always possible to choose $a_2 = 1$ and this is what will be done from now on.

The higher terms are required to describe the energy of fields with more violent spatial variations. If we restrict ourselves to smooth fields only, then the form

$$\bar{D}(q) = m^2 + q^2 \tag{2.52}$$

may serve as a good lowest approximation. Going back to x-space, this amounts to the energy expression

$$E_0 = \frac{1}{2}\int dx \phi(x)(-\partial^2 + m^2)\phi(x). \tag{2.53}$$

[c] In nature, this is not necessarily true. There are systems, such as antiferromagnets or laminar phases of liquid crystals, for which the lowest field configuration has a periodic structure. In this case, a_2 may be negative, preferring a periodic solution for its ground state.

Thus (2.52) corresponds in x-space to the functional matrix

$$D(x_1, x_2) = (-\partial^2 + m^2)\delta(x_1 - x_2) \tag{2.54}$$

a form which manifestly shows translational invariance. If the fields vanish at infinity, the $-\partial^2$ term in (2.53) can be partially integrated to give

$$E_0 = \frac{1}{2}\int dx((\partial\phi(x))^2 + m^2\phi^2(x)). \tag{2.55}$$

Thus in x-space, the bending energy manifests itself as a *gradient energy*.

Including now the interactions, the energy reads

$$E[\phi] = \int dx\left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\right]. \tag{2.56}$$

This ϕ^4 theory forms a very important example for an interacting field theory. Notice that the functional $E[\phi]$ consists of a simple spatial integral $E[\phi] = \int dx e(x)$ over an energy density $e(x)$ which is a function of $\phi(x)$, $\partial\phi(x)$ at the same place x:

$$e(x) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4. \tag{2.57}$$

Field theories with this property are referred to as *local* [compare with the definition of a local interaction in (2.1)].

2.6. MOMENTUM SPACE DIAGRAMS

In local theories, not only the free part of the energy but also all perturbation terms are simplified by working with Fourier transformed fields. Let us introduce the momentum space versions of the n-point correlation functions as follows:

$$G^{(n)}(q_1, \ldots, q_n) = \int d^D x_1 \ldots d^D x_n e^{i(q_1\cdot x_1 + \ldots + q_n\cdot x_n)} G^{(n)}(x_1, \ldots, x_n). \tag{2.58}$$

Since local theories are translationally invariant, the $G(x_1, \ldots, x_n)$'s can depend only on the differences of coordinates [compare (1.12)]

$$G^{(n)}(x_1, \ldots, x_n) = \bar{G}^{(n)}(x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n). \quad (2.59)$$

Inserting this into (2.58) we obtain

$$\begin{aligned} G^{(n)}(q_1, \ldots, q_n) &= \int d^D x_1 d^D x_2 \ldots d^D x_n \, e^{i(q_1 \cdot x_1 + \ldots + q_n \cdot x_n)} \\ &\quad \times \bar{G}^{(n)}(x_1 - x_n, x - x_n, \ldots, x_1 - x_n) \\ &= \int d^D x_n \, e^{i(q_1 + q_2 + \ldots + q_n) \cdot x_n} \int d^D x_1 \ldots d^D x_{n-1} \\ &\quad \times e^{i[q_1 \cdot (x_1 - x_n) + \ldots + q_{n-1} \cdot (x_{n-1} - x_n)]} \\ &\quad \times \bar{G}^{(n)}(x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n) \\ &= (2\pi)^D \delta^D(q_1 + \ldots + q_n) \bar{G}^{(n)}(q_1, q_2, \ldots, q_{n-1}). \quad (2.60) \end{aligned}$$

Thus translational invariance manifests itself in an overall momentum conservation δ-function $(2\pi)^D \delta^D(q_1 + \ldots + q_n)$ accompanying the Fourier transformed correlation functions.

In (2.59) we have singled out the variable x_n to form the differences shown. Certainly, we could have chosen any other variable among $x_1, \ldots, x_n$. We therefore find it useful to retain this symmetry in the notation and write $\bar{G}^{(n)}(q_1, \ldots, q_n)$ for $n > 2$ rather than $\bar{G}^{(n)}(q_1, \ldots, q_{n-1})$ with the tacit understanding that $\sum_{i=1}^{n} q_i = 0$.

Finally when there is no possibility of confusion, we shall drop all wiggles.

As an example, consider the free correlation function $G_0(x_1, x_2) = G_0(x_1 - x_2)$. Fourier transformation gives

$$\begin{aligned} G_0(q_1, q_2) &= \int dx_1 dx_2 \, e^{i(q_1 \cdot x_1 + q_2 \cdot x_2)} G_0(x_1 - x_2) \\ &= \int dx_2 \, e^{i(q_1 + q_2) \cdot x_2} \int dx_1 \, e^{iq \cdot (x_1 - x_2)} G_0(x_1 - x_2) \\ &= (2\pi)^D \delta^{(D)}(q_1 + q_2) G_0(q_1), \end{aligned}$$

where

$$G_0(q) = \int dx\, e^{iq\cdot x} G_0(x). \tag{2.61}$$

It is trivial to calculate the free correlation function in momentum space from $D(x_1, x_2) = D(x_1 - x_2)$ of (2.54). According to Eq. (2.28), $G_0(x_1, x_2) = G_0(x_1 - x_2)$ is the functional inverse, i.e., it satisfies

$$\int dx_2 D(x_1 - x_2) G_0(x_2 - x_3) = (-\partial_1^2 + m^2) G_0(x_1 - x_3) = \delta(x_1 - x_3). \tag{2.62}$$

In momentum space, the three functional matrices diagonalize and the relation becomes

$$G_0(q) D(q) = 1. \tag{2.63}$$

Thus, using (2.52), we have

$$G_0(q) = \frac{1}{D(q)} = \frac{1}{q^2 + m^2}. \tag{2.64}$$

It is now straightforward to Fourier transform all perturbative expansion terms.

In order to do so it is useful to introduce the concept of *external* and *internal* lines in a connected Feynman graph. External lines are those connecting a variable x with a vertex. Internal lines connect vertices with each other. There is a simple relation between the number of external lines, E, internal lines, I, and the number of vertices, V. Since each vertex has four lines emerging and since these can be either external, or two of them can combine as one internal line, one finds immediately

$$E + 2I = 4V. \tag{2.65}$$

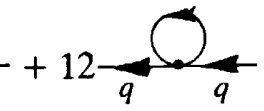
Consider now the Fourier transform of a Feynman graph. First of all, it factorizes into its disconnected pieces. Every connected piece has E x-variables and is a product of $E + I$ free correlation functions G_0. Each can be expressed in terms of its Fourier transform $G_0(q)$; it is accompanied by a momentum space integral $d^D q/(2\pi)^D$ and a phase factor $e^{iq\cdot(x_1 - x_2)}$ referring to the points which the lines connect. The Fourier transform of the connected graph is calculated by multiplying each

external line with a corresponding phase factor $e^{i\Sigma q_i \cdot x_i}$ and integrating over x_i. This gives a function which eliminates the momentum integral in the corresponding external $G_0(q)$'s. What remains are the integrals over the internal momenta. But all these meet at vertices. At each vertex there is a phase factor, say $e^{i(k_1+k_2+k_3+k_4)\cdot z}$, carrying the four associated momenta. This has to be integrated over z. The result is a δ-function $(2\pi)^D \delta^D(k_1 + k_2 + k_3 + k_4)$ which ensures the conservation of total momentum at each vertex. Of the δ-functions obtained in this way, one combination can be taken out as an overall momentum conservation factor. This is to be dropped when calculating $\bar{G}^{(n)}(q_1, \ldots, q_n)$ [remember (2.60)]. Then the remaining δ-functions eliminate $V-1$ internal momentum integrations such that only

$$L = I - V + 1 \tag{2.66}$$

integrations survive. These are called *loop integrations* and the corresponding momenta are referred to as *loop momenta.*

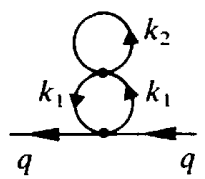
As an example, the Fourier transformed two-point correlation function $G(q)$ has the following expression, up to first order,

$G(q) =$ 

$$= \frac{1}{q^2+m^2} + \frac{1}{q^2+m^2}\left[12\left(-\frac{g}{4!}\right)\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+m^2}\right]\frac{1}{q^2+m^2}, \tag{2.67}$$

where the only loop momentum variable is k.

As another example, consider the second order graph which carries the momenta indicated below:

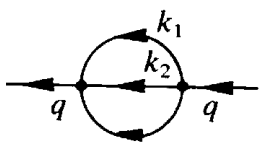
$\frac{1}{2!}288$ 

(2.68)

It has two-loop momenta and corresponds to the integral

$$\frac{1}{q^2+m^2}\left[\frac{288}{2!}\left(-\frac{g}{4!}\right)^2 \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{1}{(k_1^2+m^2)^2}\frac{1}{k_2^2+m^2}\right]\frac{1}{q^2+m^2}. \tag{2.69}$$

A third example is the second order graph

$$\frac{1}{2!}192$$ 

(2.70)

It also has two loop momenta and represents the integral

$$\frac{1}{q^2+m^2}\left[\left(\frac{192}{2!}\right)\left(-\frac{g}{4!}\right)^2\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{1}{k_1^2+m^2}\frac{1}{k_2^2+m^2}\right.$$
$$\left.\times\frac{1}{(q-k_1-k_2)^2+m^2}\right]\frac{1}{q^2+m^2}. \tag{2.71}$$

We have observed before that the two-point function can be obtained from a geometric series involving only proper self-energy graphs $-\Pi$. In formula (2.37), each term was a functional matrix and the evaluation of this series required functional multiplications $G_0\Pi G_0 \ldots \Pi G_0$. Now, in momentum space, the functional matrices are all diagonal and the series is simply an algebraic one for the diagonal elements. Thus, defining

$$(2\pi)^D\delta^D(q_1+q_2)\Pi(q) = \int dx_1dx_2\,e^{i(q_1\cdot x_1+q_2\cdot x_2)}\Pi(x_1-x_2)$$
$$= (2\pi)^D\delta^D(q_1+q_2)\int dx\,e^{iq\cdot x}\Pi(x), \tag{2.72}$$

the series (2.37) becomes

$$G(q) = \frac{1}{G_0(q)^{-1}+\Pi(q)}$$
$$= \frac{1}{q^2+m^2+\Pi(q)}$$
$$= \frac{1}{q^2+m^2}+\frac{1}{q^2+m^2}(-\Pi(q))\frac{1}{q^2+m^2}$$
$$+\frac{1}{q^2+m^2}(-\Pi(q))\frac{1}{q^2+m^2}(-\Pi(q))\frac{1}{q^2+m^2}+\ldots \tag{2.73}$$

Each higher term contains two factors of $1/(q^2 + m^2)$, one in front and one at the end, representing the two free lines entering each perturbative diagram. Comparison with (2.67)–(2.71) shows that the proper self-energy graphs drawn in Eq. (2.38) have a momentum representation which simply consists of the bracket expressions in Eqs. (2.67), (2.69), (2.71).

Finally, let us now look at the four-point function in momentum space. Since the functional multiplication is diagonal, the decomposition (2.44) becomes simply

$$G_c^{(4)}(q_1, q_2, q_3, q_4) = G(q_1)G(q_2)G(q_3)G(q_4)\Gamma^{(4)}(q_1, q_2, q_3, q_4), \quad (2.74)$$

where $\Gamma^{(4)}(q_1, q_2, q_3, q_4)$ is defined in terms of $\Gamma^{(4)}(x_1, x_2, x_3, x_4)$ by the same Fourier transform as $G_c^{(4)}(q_1, q_2, q_3, q_4)$ is in terms of $G_c^{(4)}(x_1, x_2, x_3, x_4)$.

Up to second order in g, the proper vertex diagrams in momentum space are found from (2.45) to be

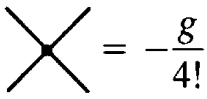
$$\Gamma_c^{(4)}(q_1, q_2, q_3, q_4) = -g + \frac{g}{2!}\left(\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} \frac{1}{(k - q_1 - q_2)^2 + m^2} + 2 \text{ perm}\right). \quad (2.75)$$

We thus arrive at the following general rules in momentum space: The connected contributions to a correlation function are calculated by:

(i) associating with every line a factor

$$\text{———} = \frac{1}{q^2 + m^2}; \quad (2.76)$$

(ii) associating with every vertex a factor


$$= -\frac{g}{4!}; \quad (2.77)$$

(iii) integrating over all loop momenta k (which are those not fixed by momentum conservation), i.e., inserting for each loop momentum k

$$\int \frac{d^D k}{(2\pi)^D}; \quad (2.78)$$

(iv) counting the proper number of independent pair contractions, according to the rules of Wick's theorem [cf. (1.70)].

2.7. RELATION BETWEEN VACUUM DIAGRAMS AND CORRELATION FUNCTIONS

There exists a simple relation between the connected diagrams contained in $W[0]$ and those making up the correlation functions $G^{(2)}(x_1, x_2)$ and $G^{(4)}(x_1, x_2, x_3, x_4)$. In order to see this we observe that the partition function (2.4) with the local energy (2.56) can be differentiated with respect to $-m^2/2$ thereby giving the expectation of the spatial integral over the composite field

$$\left\langle \int dz \phi^2(z) \right\rangle = -\frac{d}{d(m^2/2)} W[0] = -\frac{1}{Z[0]} \frac{d}{d(m^2/2)} Z[0]. \tag{2.79}$$

Thus the derivative of $W[0]$ coincides with the two-point function $\int dz G^{(2)}(z, z)$ integrated at equal arguments. In the graphical perturbation series for $W[0]$, the parameter m^2 appears only in the free field correlation functions $G_0(x_1, x_2)$, i.e., in the lines. Its derivative is equal to

$$-\frac{d}{dm^2} G_0(x_1, x_2) = \int dz G_0(x_1, z) G_0(z, x_2). \tag{2.80}$$

This amounts to the graphical rule,

$$-\frac{\partial}{\partial(m^2/2)} \text{———} = 2 \int dz \text{ ——⊗——}, \tag{2.81}$$

to be applied successively to each leg of a Feynman diagram. Taking $W[0]$ from (2.26), (2.27), we obtain

$$\left\langle \int dz \phi^2(z) \right\rangle = \left\{ \text{○⊗} + \underset{4 \cdot 3}{\text{⊗}} + \underset{2 \cdot 72}{\text{⊗}} + \underset{2 \cdot 72}{\text{⊗}} + \underset{4 \cdot 24}{\text{⊗}} \right\} \tag{2.82}$$

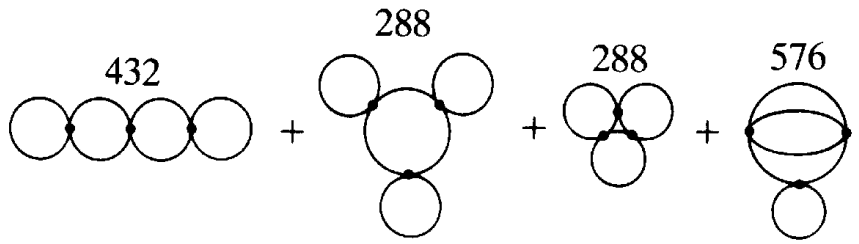
Comparison with the expansion (2.36) shows that this is, indeed, equal to the correlation function $G^{(2)}(x_1, x_2)$ with x_1 and x_2 brought to the same point and integrated over all space.

A similar relation can be found between the derivative of the vacuum diagrams with respect to the coupling constant and the disconnected four-point function $G_c^{(4)}$. Differentiating the partition function $Z[0]$ with respect to $-g/4!$ gives $\int dz G^{(4)}(z, z, z, z)$. Hence

$$-\frac{\partial}{\partial(g/4!)} W[0] = \int dz G^{(4)}(z, z, z, z). \tag{2.83}$$

Graphically, the differentiation corresponds to removing a vertex from each of the vacuum diagrams, where, due to the product rule of differentiation, each vertex has to be treated successively. Leaving the free ends open, the graphs have to be the same as those of $G^{(4)}(x_1, x_2, x_3, x_4)$. Indeed, performing this operation on $W[0]$ of (2.26), (2.27) we find $G^{(4)}$ up to first order (see Fig. 2.1). Notice that in this process the perturbative order of each graph is lowered by one unit. Actually, since we have given the graphical expansion of the four-point function up to second order in Fig. 2.1, the relation just described can be used in the opposite direction, namely, to calculate the third order contribution to $W[0]$, by simply joining the free ends of the four lines and making sure that differentiation gives the correct multiplicities. In this way we find, to third order in g,

$W_3[0] =$ 432 + 288 + 288 + 576 (2.84)



CHAPTER THREE

GAUGE INVARIANT FIELD THEORIES

3.1. FREE MAGNETIC FIELDS

The formalism developed in the last sections is quite general and applies to a wide variety of fluctuating fields. There exists, however, an important class of fields which requires a different treatment. Its most prominent representative is the *magnetic field* B_i.

Consider a set of stationary electric currents described by the current density $j_i(\mathbf{x})$. Since charge is conserved, stationary currents are conserved and therefore divergenceless, i.e.,

$$\partial_i j_i(\mathbf{x}) = 0. \tag{3.1}$$

The currents are the sources of a magnetic field $B_i(\mathbf{x})$ determined via Ampere's law, which reads

$$\nabla \times \mathbf{B} = \frac{\mu}{c}\mathbf{j}, \tag{3.2}$$

if we employ rationalized cgs units. The field energy E, in a medium of permeability μ, is given by

$$E = \frac{1}{2\mu} \int d^3x \, B_i^2(\mathbf{x}). \tag{3.3}$$

The magnetic field is divergenceless since there exist no magnetic sources:

$$\partial_i B_i(\mathbf{x}) = 0. \tag{3.4}$$

Now, it is well known that every vector field can be written as a sum of a gradient of a scalar, whose curl vanishes, plus a curl of a vector, whose gradient vanishes. Therefore it is customary to introduce a *vector potential*, A_i, such that B_i is the curl of A_i,

$$B_i = (\text{curl } \mathbf{A})_i = \varepsilon_{ijk} \partial_j A_k. \tag{3.5}$$

This decomposition is not unique. If we add to A_k the gradient of an arbitrary function $\Lambda(x)$,

$$A_i(\mathbf{x}) \to A_i(\mathbf{x}) + \partial_i \Lambda(\mathbf{x}), \tag{3.6}$$

then Λ does not appear in B_i if it satisfies the condition

$$\varepsilon_{ijk} \partial_j \partial_k \Lambda(\mathbf{x}) = 0. \tag{3.7}$$

In this case, (3.6) is called a *local gauge transformation*. The condition (3.7) can also be stated as follows:

$$(\partial_i \partial_j - \partial_j \partial_i) \Lambda(\mathbf{x}) = 0, \tag{3.8}$$

i.e., the derivatives in front of $\Lambda(\mathbf{x})$ commute. In the theory of partial differential equations this is referred to as the *integrability condition* for the function $\Lambda(\mathbf{x})$. In general, a function $\Lambda(\mathbf{x})$ which satisfies (3.8) in a simply-connected domain can be defined uniquely in this domain. If this domain is multiply connected, there is more than one path along which to continue the function $\Lambda(\mathbf{x})$ from one spatial point to another and $\Lambda(\mathbf{x})$ becomes multi-valued. This happens, for example, if (3.8) is nonzero on a closed line in three-dimensional space in which case the set of paths between two given points decomposes into equivalence classes, depending on how often the closed line is being encircled. Each of these paths allows for another continuation of $\Lambda(\mathbf{x})$. By (3.8), such functions are *not* allowed in gauge transformations (3.6).

Let us now see how we can construct the partition function of fluctuating free magnetic fields in the presence of external currents $j_i(\mathbf{x})$. If we follow the formalism already developed in Chapter 1 we have to calculate the path integral

$$Z_0[j] = \int \mathscr{D}A_i(\mathbf{x})\, e^{-(1/T)\int d^3x[(1/2\mu)(\nabla \times \mathbf{A})^2 - \mathbf{j}\cdot\mathbf{A}]}. \tag{3.9}$$

The gradient term in the exponent can be partially integrated and rewritten as

$$\int d^3x(\nabla \times \mathbf{A})^2 = \int d^3x A_i(-\partial^2\delta_{ij} + \partial_i\partial_j)A_j. \tag{3.10}$$

Hence we can identify the functional matrix $D(\mathbf{x}, \mathbf{x}')$ defined in (1.17), (2.46) as

$$D(\mathbf{x}, \mathbf{x}')_{ij} = (-\partial^2\delta_{ij} + \partial_i\partial_j)\cdot\delta^{(3)}(\mathbf{x} - \mathbf{x}'). \tag{3.11}$$

Remembering now formula (1.47) it appears, at first sight, that the partition function (3.9) should simply be equal to

$$Z_0[j] = (\det D_{ij})^{-1/2}\, e^{(\mu/2T)\int d^3x d^3x' j_i(\mathbf{x}) D(\mathbf{x},\mathbf{x}')^{-1}_{ij} j_i(\mathbf{x}')}. \tag{3.12}$$

Unfortunately, this expression is meaningless since the inverse of the functional matrix does not exist.

In order to see this explicitly we diagonalize the functional part (i.e., the $\mathbf{x}$-space part) of the matrix $D(\mathbf{x}, \mathbf{x}')_{ij}$ by going to the Fourier transform (compare (2.52))

$$D_{ij}(q) = \int d^3x e^{iq\cdot x} D_{ij}(\mathbf{x}) = q^2\delta_{ij} - q_iq_j. \tag{3.13}$$

For every momentum q, this matrix has obviously an eigenvector with zero eigenvalue, namely q_i. This prevents us from inverting the matrix D_{ij}. In addition, when trying to form the inverse determinant of the functional matrix D_{ij} in (3.12), we encounter an infinite number of infinities, one for every momentum q.

The difficulty can be resolved using the fact that not only the magnetic field energy (3.3), but also the coupling to the source $\mathbf{j}$ and thus the entire integrand of (3.9) is gauge invariant. Indeed, if we change $\mathbf{A}$ according

to (3.6), the coupling to the external current picks up a term $\int d^3x j_i(\mathbf{x})$ $\partial_i \Lambda(\mathbf{x})$. By a partial integration this is equal to $-\int d^3x \partial_i j_i(\mathbf{x}) \Lambda(\mathbf{x})$ and this expression vanishes identically, due to current conservation (3.1). As a consequence, if we perform the sum over all A_i fluctuations via the path integrals $\Pi_{\mathbf{x}} \int dA_1 dA_2 dA_3(\mathbf{x})$, not all of the degrees of freedom describe different physical field configurations. Certain combined fluctuations are unphysical since they merely correspond to gauge transformations, under which the exponent is invariant. Since a path integral is a product of infinitely many integrals from minus to plus infinity, the gauge invariance of the integral produces an infinite product of infinite factors. This is precisely the origin of the infinities encountered when trying to calculate the inverse determinant of the functional matrix $D_{ij}(\mathbf{x}, \mathbf{x}')$.

It is now clear how this infinity can be avoided. All we have to do is to restrict the path integral to *physical* field fluctuations. This is easily achieved. In momentum space, the gauge degrees of freedom of $A_i(\mathbf{q})$ have the form $iq_i\Lambda(\mathbf{q})$ which are precisely the eigenvectors with zero eigenvalues. In order to avoid them, we take the path integral in momentum space,

$$\int \mathscr{D} A_i \equiv \prod_q \int \frac{dA_1(\mathbf{q})}{\sqrt{2\pi\mu T/a^3}} \frac{dA_2(\mathbf{q})}{\sqrt{2\pi\mu T/a^3}} \frac{dA_3(\mathbf{q})}{\sqrt{2\pi\mu T/a^3}}, \tag{3.14}$$

and separate off the unphysical gauge-like component proportional to $iq_i\Lambda(\mathbf{q})$:

$$dA_i^L(\mathbf{q}) = iq_i d\Lambda(\mathbf{q}) \equiv \sqrt{q^2}\, i\hat{\mathbf{q}}_i d\Lambda(\mathbf{q}). \tag{3.15}$$

The superscript L indicates that the component is longitudinal to $\mathbf{q}$. The remaining components may be chosen such that they are polarized *transverse* to A_i, i.e., to satisfy

$$q_i A_i^T(\mathbf{q}) = 0. \tag{3.16}$$

There are two possible transverse components, say, $A_i^{(1)}(\mathbf{q})$, $A_i^{(2)}(\mathbf{q})$. Hence the physical path integral may be defined as

$$\int \mathscr{D} A_i^T = \prod_q \int \frac{dA^{(1)}(\mathbf{q})}{\sqrt{2\pi\mu T/a^3}} \frac{dA^{(2)}(\mathbf{q})}{\sqrt{2\pi\mu T/a^3}}. \tag{3.17}$$

The transverse components have a certain freedom of choice in the two-dimensional plane orthogonal to $\mathbf{q}$. Therefore the form (3.17) is not very explicit. There is a way of writing the integration measure so that this freedom disappears. For this we keep all three $A_i(\mathbf{q})$ integrations but eliminate the unphysical part by a δ-function

$$\int \mathscr{D}A_i^T = \prod_{\mathbf{q}} \left[\left(\prod_i \int \frac{dA_i(\mathbf{q})}{\sqrt{2\pi\mu T/a^3}} \right) \sqrt{\frac{2\pi\mu T\mathbf{q}^2}{a^2}}\, \delta(q_i A_i(\mathbf{q})) \right]. \tag{3.18}$$

We can verify that this has the desired property. The factors $\Pi_{\mathbf{q}} \sqrt{\mathbf{q}^2}$ properly compensate for the momenta inside the δ-functions: for example, if $\mathbf{q}$ points in the z-direction, then

$$\delta(q_i A_i) = \delta(q_3 A_3) = \frac{1}{q_3}\delta(A_3) = \frac{1}{\sqrt{q^2}}\delta(A_3) \tag{3.19}$$

and this eliminates the longitudinal $A_3(\mathbf{q})$ component while leaving only the transverse physical components pointing in the 1 and 2 directions:

$$\prod_{\mathbf{q}} \int \frac{dA_1(\mathbf{q})\, dA_2(\mathbf{q})}{2\pi\mu T/a^3}.$$

The measure (3.18) can be written in $\mathbf{x}$-space as follows:

$$\int \mathscr{D}A^T(\mathbf{x}) = \left(\prod_{\mathbf{x}} \sqrt{2\pi\mu T/a^3} \right) \det(-\partial^2)^{1/2} \int \mathscr{D}A_i \prod_{\mathbf{x}} \delta(\partial_i A_i(\mathbf{x})). \tag{3.20}$$

The factor in front is often omitted since it is a trivial constant which disappears when calculating correlation functions via the general rule (1.10). In this section we shall, however, retain it for greater clarity.

Using (3.20), the partition function now takes the form

$$Z_0[\mathbf{j}] = \left(\prod_{\mathbf{x}} \sqrt{2\pi\mu T/a^3} \right) \det(-\partial^2)^{1/2} \int \mathscr{D}A_i(\mathbf{x}) \prod_{\mathbf{x}} \delta(\partial_i A_i(\mathbf{x})) \times \exp\left\{ -\frac{1}{T} \int d^3x \left[\frac{1}{2\mu} (\nabla \times \mathbf{A})^2 - \mathbf{j}\cdot\mathbf{A} \right] \right\}. \tag{3.21}$$

Now, there is no longer any problem in evaluating the path integral. For

practical purposes it is useful to choose an exponential representation for the δ-functions, namely,

$$\delta(\xi) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi\varepsilon}} e^{-(1/\varepsilon)\xi^2}, \tag{3.22}$$

where ε is an infinitesimal number. This is zero for $\xi \gg \varepsilon$ and singular inside the small interval $\xi \leq \varepsilon$ so that $\int d\xi \delta(\xi) = 1$. Setting $\varepsilon \equiv 2\mu T \cdot \lambda/a^3$, Eq. (3.21) becomes

$$Z_0[\mathbf{j}] = \left(\prod_{\mathbf{x}} \frac{1}{\sqrt{\lambda}}\right) \det(-\partial^2)^{1/2} \int \mathscr{D}A_i(\mathbf{x}) \times \exp\left\{-\frac{1}{T}\int d^3x \left[\frac{1}{2\mu} A_i(-\partial^2\delta_{ij} + \partial_i\partial_j - \frac{1}{\lambda}\partial_i\partial_j)A_j - j_i A_i\right]\right\}. \tag{3.23}$$

Thanks to the δ-functional, the functional matrix between the A_i fields has been changed from (3.11) to

$$D(\mathbf{x}, \mathbf{x}')_{ij} = \left(-\partial^2\delta_{ij} + \partial_i\partial_j - \frac{1}{\lambda}\partial_i\partial_j\right)\delta(\mathbf{x} - \mathbf{x}'), \tag{3.24}$$

or, in momentum space, to

$$D(\mathbf{q})_{ij} = (\mathbf{q}^2\delta_{ij} - q_i q_i) + \frac{1}{\lambda} q_i q_j. \tag{3.25}$$

We now see that the eigenvectors $i\hat{\mathbf{q}}_i\Lambda(\mathbf{q})$ which previously had a vanishing eigenvalue now have a very large one, $\mathbf{q}^2/\lambda$. The two transverse components, on the other hand, have the same eigenvalues $\mathbf{q}^2$ as before in (3.13). As a consequence, the functional integration gives the product

$$\prod_q \frac{1}{\sqrt{\mathbf{q}^2 \cdot \mathbf{q}^2 \cdot (\mathbf{q}^2/\lambda)}}.$$

In $\mathbf{x}$-space, this is equal to

$$\prod_{\mathbf{x}} \sqrt{\lambda} \det(-\partial^2)^{-3/2}.$$

As long as λ is finite, this is nonsingular. For the same reason, the matrix D can be inverted with the result

$$D^{-1}(q)_{ij} = \frac{1}{\mathbf{q}^2}(\delta_{ij} - q_i q_j/q^2) + \frac{\lambda}{\mathbf{q}^2} q_i q_j/q^2. \tag{3.26}$$

The second term can be dropped because of the infinitesimal size of λ. Putting everything together, we arrive at the partition function

$$Z_0[\mathbf{j}] = e^{-\mathrm{tr}\log(-\partial^2)}\, e^{(\mu/2T)\int d^3x d^3x' j_i(\mathbf{x}) D^{-1}(\mathbf{x}, \mathbf{x}')_{ij} j_j(\mathbf{x}')}. \tag{3.27}$$

The infinitesimal parameter λ has disappeared. Equation (3.27) is the proper generalization of Eq. (1.68) to gauge fields.

Notice that the exponent $-\mathrm{tr}\log(-\partial^2)$ carries a factor 2 with respect to a single scalar field in (1.47) [which in the massless case would be $-(1/2)\,\mathrm{tr}\log(-\partial^2)$]. This factor accounts for the two physical transverse degrees of freedom carried by the vector potential A_i. The third unphysical component does not contribute at all.

The current piece in the exponent of (3.27) can be simplified by observing that the law of current conservation (3.1) amounts, in momentum space, to the constraint $q_i j_i(q) = 0$. Then, in the Fourier representation

$$\int d^3x d^3x'\, j_i(\mathbf{x})\, D^{-1}(\mathbf{x}, \mathbf{x}')_{ij} j_j(\mathbf{x}) \equiv \sum_{q,\, q'} j_i^*(\mathbf{q})\, D^{-1}(\mathbf{q})_{ij} j_j(\mathbf{q}),$$

with $D^{-1}(\mathbf{q})_{ij}$ given by (3.26); the terms proportional to $q_i q_j$ can be dropped. Hence we can write $Z[\mathbf{j}]$ also in the form

$$Z[\mathbf{j}] = e^{-\mathrm{tr}\log(-\partial^2)}\, e^{(\mu/2T)\int d^3x d^3x' j_i(\mathbf{x}) G_0(\mathbf{x}, \mathbf{x}') j_i(\mathbf{x})}, \tag{3.28}$$

where

$$G_0(\mathbf{x}, \mathbf{x}') = (-\partial^2)^{-1} = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{1}{\mathbf{q}^2} \tag{3.29}$$

is simply the correlation function of a massless scalar field. This function is easily calculated

$$G_0(\mathbf{x}, \mathbf{x}') = \frac{2\pi}{(2\pi)^3} \int_0^\infty dq \int_{-1}^1 d\cos\theta\, e^{iqR\cos\theta} = \frac{1}{2\pi^2 R} \int_0^\infty \frac{dq}{q} \sin qR = \frac{1}{4\pi R}, \tag{3.30}$$

where $R = |\mathbf{x} - \mathbf{x}'|$ is the distance between the points $\mathbf{x}$ and $\mathbf{x}'$. It coincides with the standard *Coulomb Green function* of electrostatics.

3.2. GAUGE-FIXING FACTORS

It should be realized now that due to the gauge invariance of the exponential in (3.21), the δ-functional,

$$\Phi[\mathbf{A}] = \left(\prod_{\mathbf{x}} \sqrt{2\pi\mu T/a^3}\right) \det(-\partial^2)^{1/2} \prod_{\mathbf{x}} \delta(\partial_i A_i(\mathbf{x})), \tag{3.31}$$

which was inserted into the path integral (3.21), is not the only possible constraint leading to the final expression (3.28). In general, we can choose a variety of other so-called *gauge-fixing factors*. For example, we may observe that, because of current conservation, the final result (3.28) does not depend on the parameter $\lambda \propto \varepsilon$, so that we do not really have to take the limit $\lambda \to 0$, as required by (3.22). The functional (3.23) gives the *same result* for *all* λ. Thus, instead of (3.31) we could have chosen the gauge-fixing factor

$$\Phi[\mathbf{A}] = \left(\prod_{\mathbf{x}} \frac{1}{\sqrt{\lambda}}\right) \det(-\partial^2)^{1/2} e^{-(1/(2\mu T\lambda))\int d^3x (\partial_i A_i(\mathbf{x}))^2}, \tag{3.32}$$

with arbitrary λ.

For $\lambda = 1$, the path integral has the particularly simple form

$$Z_0[\mathbf{j}] = \det(-\partial^2)^{1/2} \int \mathscr{D} A_i \, e^{-(1/T)\int d^3x [1/2\mu (\partial_i A_j)^2 - j_i A_i]}. \tag{3.33}$$

Another possibility is to choose a gauge-fixing factor

$$\Phi[\mathbf{A}] = \left(\prod_{\mathbf{x}} \sqrt{2\pi\mu T/a^3}\right) \left(\det \frac{\partial_3^2}{\partial^2}\right)^{1/2} \prod_{\mathbf{x}} \delta(A_3), \tag{3.34}$$

which simply eliminates one of the components of the vector potential. This choice violates manifest rotational invariance. Nevertheless, due to the gauge invariance of Z, the partition function is the same as before. Indeed, using this $\Phi[\mathbf{A}]$ instead of (3.31), the partition function (3.21) takes the form

$$Z[\mathbf{j}] = \left(\det\frac{\partial_3^2}{\partial^2}\right)^{1/2}\prod_{\mathbf{x}}\left[\int\frac{dA_1(\mathbf{x})\,dA_2(\mathbf{x})}{2\pi\mu T/a^3}\right]e^{-(1/T)\int d^3x[(1/2\mu)(\partial\times\mathbf{A}^\perp)^2-\mathbf{j}\cdot\mathbf{A}^\perp]}, \quad (3.35)$$

where the orthogonal sign $\perp$ as a superscript indicates that only the components A_1 and A_2 are present. In Fourier space, the matrix $D_{ij}(\mathbf{q})$ has now only 2×2 entries and can be written as

$$D_{ij}(\mathbf{q}) = \mathbf{q}^2\delta_{ij}^\perp - q_i^\perp q_j^\perp = \mathbf{q}^2\left(\delta_{ij}^\perp - \frac{q_i^\perp q_j^\perp}{q^{\perp 2}}\right) + q_3^2\frac{q_i^\perp q_j^\perp}{q^{\perp 2}},$$

where $q^{\perp 2}\equiv q_1^2+q_2^2$ and

$$\delta_{ij}^\perp \equiv \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

This result is immediately inverted to

$$D^{-1}(q)_{ij} = \frac{1}{\mathbf{q}^2}\left(\delta_{ij}^\perp - \frac{q_i^\perp q_i^\perp}{q^{\perp 2}}\right) + \frac{1}{q_3^2}\frac{q_i^\perp q_j^\perp}{q^{\perp 2}} = \frac{1}{\mathbf{q}^2}\left(\delta_{ij}^\perp + \frac{q_i^\perp q_j^\perp}{q_3^2}\right).$$

The 2×2 matrix $D_{ij}(\mathbf{q})$ has two eigenvectors in the 1-2 plane, one parallel to $q_i^\perp$ with eigenvalue q_3^2 and one orthogonal with eigenvalue $\mathbf{q}^2$. The product of the square roots of these eigenvalues results in a determinant $\det(\partial^2\partial_3^2)$. Hence the partition function (3.35) becomes

$$Z_0[\mathbf{j}] = e^{-\mathrm{tr}\log(-\partial^2)}\,e^{(\mu/(2T))\Sigma_{\mathbf{q}} j_i^\perp(\mathbf{q})^* D^{-1}(\mathbf{q})_{ij} j_j^\perp(\mathbf{q})}. \quad (3.36)$$

We now make use of current conservation which, in momentum space, reads

$$q^\perp j^\perp(\mathbf{q}) + q_3 j_3(\mathbf{q}) = 0. \quad (3.37)$$

Inserting this into (3.36) we find, using the above explicit form of $D^{-1}(\mathbf{q})_{ij}$,

$$j_i^\perp(\mathbf{q})^\dagger D^{-1}(\mathbf{q})_{ij} j_j^\perp(\mathbf{q}) = \frac{1}{\mathbf{q}^2}\left(|j_i^\perp(\mathbf{q})|^2 + \frac{1}{q_3^2}|q_i^\perp j_i^\perp(\mathbf{q})|^2\right) = \frac{1}{\mathbf{q}^2}|j_i(\mathbf{q})|^2. \quad (3.38)$$

Hence we arrive again at

$$Z_0[\mathbf{j}] = e^{-\mathrm{tr}\log(-\partial^2)}\, e^{(\mu/(2T))\Sigma_{\mathbf{q}}(1/\mathbf{q}^2)|j_i(\mathbf{q})|^2},$$

in agreement with the previous result (3.28).

From these examples it is clear that *any* gauge dependent functional $\Phi[\mathbf{A}]$ can be used equally well to fix the gauge. The only condition is that it is normalized in the following way:

$$\int \mathscr{D}\Lambda(\mathbf{x})\,\Phi[\mathbf{A} + \nabla\Lambda] = \det(-\partial^2)^{-1/2}. \tag{3.39}$$

Indeed, consider an arbitrary gauge invariant path integral whose gauge is fixed by such a factor:

$$Z_0 = \int \mathscr{D}A(\mathbf{x})\,\Phi[\mathbf{A}]\, e^{-(1/T)E[\mathbf{A}]}. \tag{3.40}$$

The measure of the path integral can be split into a transverse and a longitudinal part as

$$\int \mathscr{D}A = \int \mathscr{D}A^T \int \mathscr{D}A^L.$$

The total A_i field can always be written as a gauge transformed version of a purely transverse field,

$$A_i(\mathbf{x}) = A_i^T(\mathbf{x}) + \partial_i\Lambda(\mathbf{x}),$$

and the component of A_i along the momentum direction q_i can be identified as

$$A^L(\mathbf{q}) = i\sqrt{\mathbf{q}^2}\,\Lambda(\mathbf{q}).$$

We shall choose the positive square root if q_i points in the upper half-sphere and negative if it points in the opposite direction. This definition has the advantage that the property $\Lambda(\mathbf{q}) = \Lambda^*(-\mathbf{q})$, reflecting the reality of $\Lambda(\mathbf{x})$, is shared also by $A^L(\mathbf{q})$,

$$A^L(\mathbf{q}) = A^L(-\mathbf{q})^*,$$

so that also $A^L(\mathbf{x})$ is real. If we now form the path integral

$$\int \mathscr{D}A^L \equiv \prod_{\mathbf{q}} \left[i\sqrt{\mathbf{q}^2} \right] \prod_{\mathbf{q}} \left[\int d\Lambda(\mathbf{q}) \right]$$

then every factor $i\sqrt{\mathbf{q}^2}$ has a complex conjugate partner so that we may just as well write

$$\int \mathscr{D}A^L = \prod_{\mathbf{q}} \left[\sqrt{\mathbf{q}^2} \right] \prod_{\mathbf{q}} \left[\int d\Lambda(\mathbf{q}) \right]$$

or, in x-space,

$$\int \mathscr{D}A^L(\mathbf{x}) = \det(-\boldsymbol{\partial}^2)^{1/2} \int \mathscr{D}\Lambda(\mathbf{x}).$$

Inserting this into (3.40) and writing $A(\mathbf{x}) = A^T(\mathbf{x}) + \partial_i \Lambda\ (\mathbf{x})$, we find

$$Z_0 = \int \mathscr{D}A^T \det(-\boldsymbol{\partial}^2)^{1/2} \int \mathscr{D}\Lambda \Phi[A_i^T + \partial_i \Lambda] e^{-(1/T)E[A^T]},$$

where we have used the fact that $E[A] = E[A^T]$, due to gauge invariance. This allows us to take advantage of (3.39) and integrate out the Λ field such that Z becomes

$$Z_0 = \int \mathscr{D}A^T e^{-(1/T)E[A^T]},$$

as it should. This has been achieved through an *arbitrary* gauge-fixing factor $\Phi[A]$ which merely had to satisfy the normalization condition (3.39).

It must be realized that the partition function with conserved external currents cannot be used directly to generate the correlation functions of the vector potential. This is due to the fact that a conserved current has only two independent components and thus its functional derivatives $\delta/\delta j_i(\mathbf{x})$ are not independent. If we forget this fact, a naive differentiation of the expression (3.28) would give the correlation function

$$\langle A_i(\mathbf{x}) A_j(\mathbf{x}') \rangle = Z_0^{-1}[\mathbf{j}](\mu T)^2 \frac{\delta}{\delta j_i(\mathbf{x})} \frac{\delta}{\delta j_j(\mathbf{x}')} Z_0[\mathbf{j}] \Big|_{j=0} = G_0(\mathbf{x}, \mathbf{x}')\, \delta_{ij}. \quad (3.41)$$

A proper direct calculation of the correlation function, however, leads to

$$\langle A_i(\mathbf{x}) A_j(\mathbf{x}')\rangle = D^{-1}(\mathbf{x}, \mathbf{x}')_{ij} = G_0(\mathbf{x}, \mathbf{x}')_{ij}, \tag{3.42}$$

with

$$G_0(\mathbf{x}, \mathbf{x}')_{ij} = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \frac{1}{\mathbf{q}^2}\left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2}\right),$$

$$G_0(\mathbf{x}, \mathbf{x}')_{ij} = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \frac{1}{\mathbf{q}^2}\left(\delta_{ij} - (1-\lambda)\frac{q_i q_j}{\mathbf{q}^2}\right),$$

$$G_0(\mathbf{x}, \mathbf{x}')_{ij} = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \frac{1}{\mathbf{q}^2}\left(\delta_{ij}^{\perp} - \frac{q_i^{\perp} q_j^{\perp}}{q_3^2}\right), \tag{3.43}$$

for the three choices of gauge-fixing factors (3.31), (3.32) and (3.34), respectively. Thus, contrary to the case for the partition function, the correlation functions are *gauge dependent.*

The first choice of gauge is called *transverse gauge*, the second choice for $\lambda = 1$ is the *Feynman gauge*, and the third choice is known as the *axial gauge*.

It is possible to derive the correlation functions (3.43) of the vector potential from a generating functional $Z_0[\mathbf{j}]$ but only if the partition function is calculated for external currents which do *not* satisfy the current conservation law. In that case $Z_0[\mathbf{j}]$ is found to be

$$Z_0[\mathbf{j}] = e^{-\mathrm{tr}\log(-\partial^2)} e^{(\mu/2T)\int dx\, dx'\, j_i(\mathbf{x}) G_0(\mathbf{x},\mathbf{x}')_{ij} j_j(\mathbf{x}')},$$

where $G_0(\mathbf{x}, \mathbf{x}')_{ij}$ are the gauge dependent functions (3.43).

3.3. CHARGED SCALAR FIELDS

So far, the sources of the magnetic field have been external. In nature there exist many charged systems, whose motion gives rise to internal conserved currents. The simplest field theory which does describe such systems consists of two scalar fields $\varphi_1(\mathbf{x})$, $\varphi_2(\mathbf{x})$ coupled in such a way that the energy is invariant under rotations:

$$(\varphi_1, \varphi_2) \to (\varphi_1', \varphi_2') = (\varphi_1 \cos\alpha - \varphi_2 \sin\alpha,\ \varphi_1 \sin\alpha + \varphi_2 \cos\alpha). \tag{3.44}$$

Up to quartic interactions, this leads to the energy expression

$$E = \int d^3x \left\{\frac{1}{2}[(\partial\varphi_1)^2 + (\partial\varphi_2)^2] + \frac{m^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{g}{4}(\varphi_1^2 + \varphi_2^2)^2\right\}. \tag{3.45}$$

Conventionally, one combines φ_1 and φ_2 into a single complex field

$$\varphi = \varphi_1 + i\varphi_2 \tag{3.46}$$

and rewrites (3.45) in the form

$$E = \int d^3x \left\{ \frac{1}{2}|\partial\varphi|^2 + \frac{m^2}{2}|\varphi|^2 + \frac{g}{4}|\varphi|^4 \right\}. \tag{3.47}$$

The rotational invariance under (3.44) now corresponds to an invariance under changes of the phase

$$\varphi(\mathbf{x}) \to e^{i\alpha}\varphi(\mathbf{x}). \tag{3.48}$$

One often speaks of a "$U(1)$ invariance" ($U(1)$ is the group of "unitary matrices in one dimension" which are trivial phases). We shall see later that this invariance is the reason for the conservation law of the internal currents carried by the complex fields φ. Notice that the non-derivative terms have actually a larger invariance group. They can be transformed by a generalization of (3.48) in which φ, instead of being a constant, is an arbitrary function of $\mathbf{x}$. In that case one speaks of a *local* $U(1)$ invariance. The derivation term is the only one which is not invariant under such local $U(1)$ transformations. It changes according to the law

$$\partial_i \varphi(\mathbf{x}) \to e^{i\alpha(\mathbf{x})}(\partial_i + i\partial_i \alpha(\mathbf{x}))\,\varphi(\mathbf{x}). \tag{3.49}$$

This additive change by an arbitrary function of $\mathbf{x}$ is strongly reminiscent of the change of the vector potential of magnetism under gauge transformations

$$A_i(\mathbf{x}) \to A_i(x) + \partial_i \Lambda(\mathbf{x}).$$

This suggests the possibility of combining both fields, $\varphi(\mathbf{x})$ and $A_i(\mathbf{x})$, and constructing a coupled field theory of magnetic and scalar fields which has complete invariance under simultaneous local gauge transformations and phase changes. This would guarantee the independence of all physical phenomena of such a system under local gauge transformations.

There is no problem in finding a simple interaction energy which achieves this goal. All we have to do is to replace the derivative $\partial_i\varphi(\mathbf{x})$ in (3.47) by what is called a *covariant derivative*,

$$\partial_i \varphi(\mathbf{x}) \to D_i \varphi(\mathbf{x}) \equiv (\partial_i - iqA_i)\,\varphi(\mathbf{x}), \tag{3.50}$$

where q is defined as the *charge* of the field. Obviously, this modified derivative $D_i\varphi_i(\mathbf{x})$ can be made gauge covariant if every gauge transformation $A_i(\mathbf{x}) \to A_i(\mathbf{x}) + \partial_i \Lambda(\mathbf{x})$ is accompanied by a local phase rotation $\varphi(\mathbf{x}) \to e^{i\alpha(\mathbf{x})}\varphi(\mathbf{x})$ with $\alpha(\mathbf{x}) = q\Lambda(\mathbf{x})$. Then $D_i\varphi$ changes like φ,

$$D_i \varphi(\mathbf{x}) \to e^{iq\Lambda(\mathbf{x})} D_i \varphi(\mathbf{x}). \tag{3.51}$$

In this way we arrive at the gauge invariant energy of the coupled fields,

$$E[\varphi, \mathbf{A}] = \int d^3x \left\{ \frac{1}{2} |(\partial_i - iqA_i)\,\varphi|^2 + \frac{m^2}{2}|\varphi|^2 + \frac{g}{4}|\varphi|^4 + \frac{1}{2\mu}(\partial \times \mathbf{A})^2 \right\}. \tag{3.52}$$

A field energy of this type was first applied to physical phenomena by *Ginzburg and Landau* in 1950. It governs the magnetic properties of superconductors in the neighbourhood of the critical temperature. This will be discussed in more detail in Part II, Section 3.4. In quantum field theory, the same field energy is referred to as *scalar QED* in three dimensions (sometimes also as *abelian Higgs model*).

3.4. FEYNMAN DIAGRAMS

The partition function of this system is

$$Z = \int \mathscr{D}\varphi\, \mathscr{D}\varphi^\dagger \mathscr{D}A\, \Phi[\mathbf{A}]\, e^{-(1/T)E[\varphi, \varphi^\dagger, A]}, \tag{3.53}$$

where $\mathscr{D}\varphi\, \mathscr{D}\varphi^\dagger$ denotes the path integral over real and imaginary parts, $\mathscr{D}\varphi_1\, \mathscr{D}\varphi_2$, and $\Phi[\mathbf{A}]$ is any one of the gauge-fixing factors (3.31), (3.32) or (3.34). In order to calculate correlation functions we have to introduce a source for each of the fields A_i, φ_1, φ_2, multiplying the integrand of (3.53) by

$$e^{(1/T)\int d^3x (j_i A_i + \eta_1 \varphi_1 + \eta_2 \varphi_2)}. \tag{3.54}$$

For $q = 0$, $g = 0$, the partition function decouples and factorizes into a partition function of a φ field and that of an A_i field,

$$Z_0 = Z_\varphi Z_A. \tag{3.55}$$

Each of the factors can be calculated using the previously described methods [see (1.47), (3.28)] so that

$$Z_0[\mathbf{j},\ \eta_1,\ \eta_2] = (\det G_0)^{-1}(\det G_0^m)^{-1} \times e^{(\mu/2T)\int d^3x d^3x' j_i(\mathbf{x}) G_0(\mathbf{x},\mathbf{x}')_{ij} j_j(\mathbf{x}) + (1/2T)\int d^3x d^3x' \eta_a(\mathbf{x}) G_0^m(\mathbf{x},\mathbf{x}')\eta_a(\mathbf{x}')}, \tag{3.56}$$

where $G_0(\mathbf{x},\ \mathbf{x}')_{ij}$ is one of the massless Green functions (3.43) and $G_0^m = (-\partial^2 + m^2)^{-1}$ the free correlation function of a scalar field of mass m. In complex field notation it is preferable to define complex sources

$$\eta(\mathbf{x}) = \tfrac{1}{2}(\eta_1 + i\eta_2)(\mathbf{x}),$$

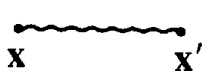
so that the φ source term reads

$$e^{(1/T)\int d^3x(\eta^*\varphi + \varphi^*\eta)},$$

and the second factor in (3.56) becomes

$$e^{(2/T)\int d^3x d^3x' \eta^*(\mathbf{x}) G_0^m(\mathbf{x},\mathbf{x}')\eta(\mathbf{x}')}. \tag{3.57}$$

The free correlation functions can then be obtained by differentiating $Z_0[\mathbf{j},\ \eta,\ \eta^\dagger]$ with respect to j_i, η, η^*, where j_i is completely arbitrary and is not conserved. This gives for A_i the correlation functions (3.41). They will be represented graphically by a wavy line connecting the positions $\mathbf{x}$ and $\mathbf{x}'$:



$$= \frac{\mu}{T} G_0(\mathbf{x},\ \mathbf{x}')_{ij}. \tag{3.58}$$

Differentiations with respect to η, η^*, on the other hand, gives the correlation functions of the complex field

$$\langle \varphi(\mathbf{x})\,\varphi^*(\mathbf{x}')\rangle = Z^{-1}\frac{\delta^2}{\delta\eta^*(\mathbf{x})\,\delta\eta(\mathbf{x}')} Z[\eta]\Big|_{\eta=\eta^*=0} = \frac{2}{T} G_0^m(\mathbf{x},\ \mathbf{x}'),$$

$$\langle \varphi(\mathbf{x})\,\varphi(\mathbf{x}')\rangle = Z^{-1}\frac{\delta^2}{\delta\eta^*(\mathbf{x})\,\delta\eta^*(\mathbf{x}')} Z[\eta]\Big|_{\eta=\eta^*=0} = 0,$$

$$\langle \varphi^*(\mathbf{x})\,\varphi^*(\mathbf{x}')\rangle = Z^{-1}\frac{\delta^2}{\delta\eta(\mathbf{x})\,\delta\eta(\mathbf{x}')}Z[\eta]\Big|_{\eta=\eta^*=0} = 0. \tag{3.59}$$

A graphical representation which distinguishes $\varphi(\mathbf{x})$ and $\varphi^*(\mathbf{x})$ is obtained by assigning to each point $\mathbf{x}$ an arrow. An outgoing arrow represents $\varphi^*(\mathbf{x})$, an ingoing one, $\varphi(\mathbf{x})$. Only those points can be connected which allow for a consistent flow of arrows. The only nonzero correlation between two points is

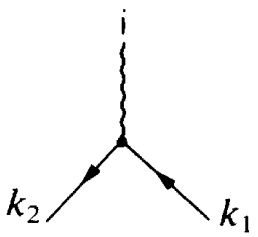
$$\mathbf{x} \longleftarrow \mathbf{x}' = \frac{2}{T}G_0^m(\mathbf{x},\,\mathbf{x}') \tag{3.60}$$

in accordance with (3.59).

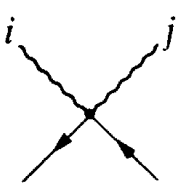
The perturbation series is obtained by expanding the exponential in (3.53) in powers of the coupling q, g. The complete interaction is

$$e^{-(1/T)\int d^3x\,[-(q/2i)(\varphi^*\partial_i\varphi - \partial_i\varphi^*\varphi)A_i + (q^2/2)A_i^2|\varphi|^2 + g/4|\varphi|^4]}. \tag{3.61}$$

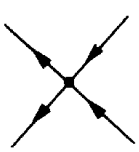
Expanding this expression in powers of q, g and performing the path integrals over the A_i and the fields φ one obtains all mixed correlation functions of A_i and φ. By a straightforward generalization of Wick's theorem to several independent fields, they are equal to all possible pair contractions and these can be represented by Feynman diagrams involving the lines (3.58) and (3.60). By the arrow rule (3.60), only equal numbers of fields φ and φ^* contribute. The lines meet at vertices given by (3.61) whose graphical representations are



$$\frac{i}{2T}q(k_2 + k_1)_i \tag{3.62}$$



$$\frac{1}{2T}q^2\delta_{ij} \tag{3.63}$$



$$\frac{1}{T}\frac{g}{4} \tag{3.64}$$

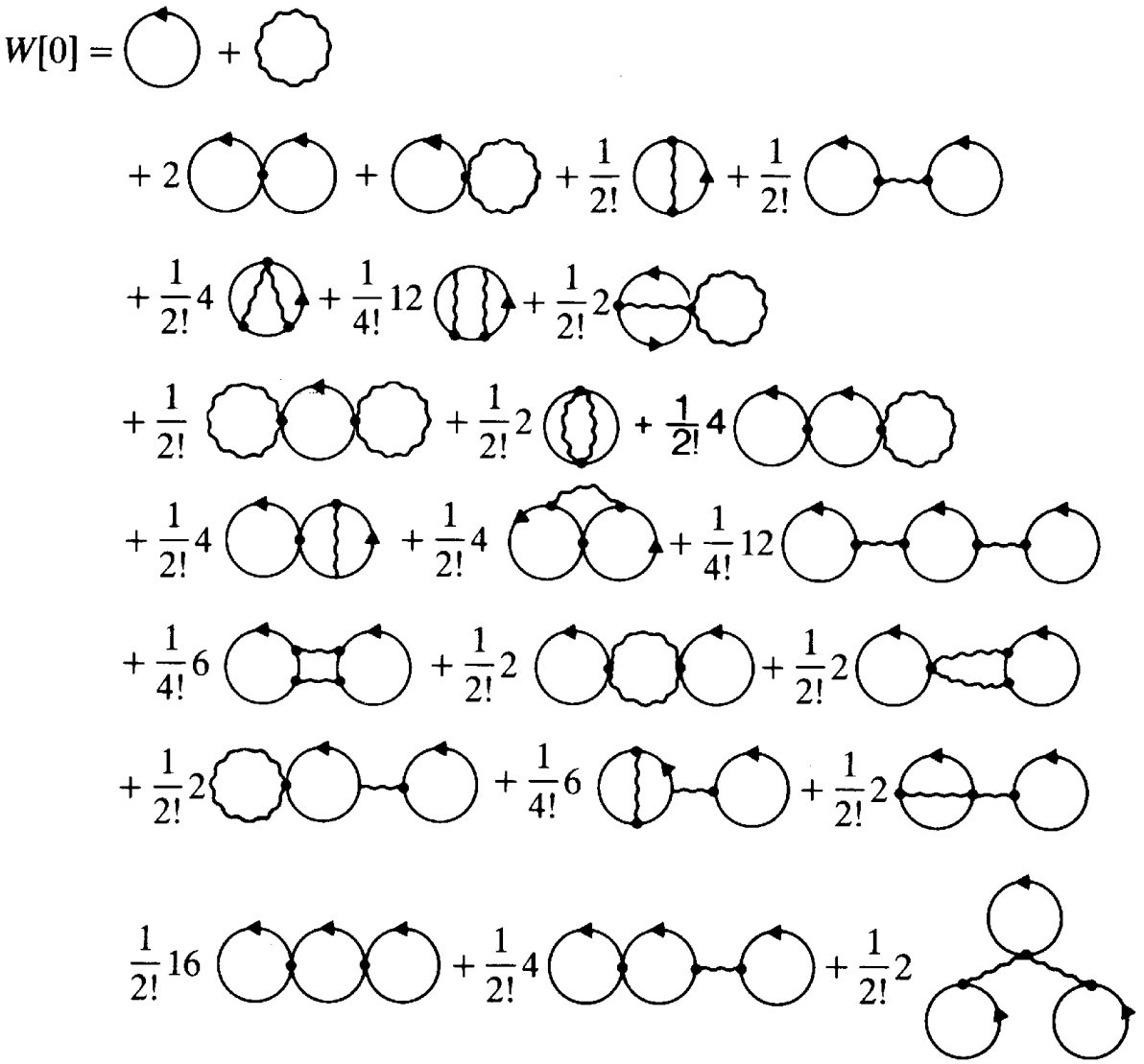
The momenta in (3.62) are, of course, the Fourier transforms of the derivatives in (3.61). The second graph is called *seagull graph* since it looks like a flying seagull.

Applying these rules and doing the proper counting of graphs we obtain for the connected vacuum graphs for $W[0]$, up to the two-loop level, as shown in Fig. 3.1. From these follows the partition function by exponentiation

$$Z[0] = e^{W[0]}. \tag{3.65}$$

The graphical expansions of self-energies and vertex functions can be found similarly.

FIG. 3.1. The vacuum graphs $W[0] = \log Z[0]$ of scalar quantum electrodynamics up to three loops. By differentiation with respect to the internal legs this gives the correlation functions of the scalar fields and the photon [recall Eq. (2.81)]. Differentiation with respect to the vertices gives the connected three- and four-point functions [cf. Eq. (2.83)].



3.5. NON-ABELIAN GAUGE THEORIES

The magnetic field and its vector potential have the virtue of transforming a complex scalar field theory, which was initially invariant under *global* $U(1)$ transformations,

$$\varphi(\mathbf{x}) \to e^{i\alpha}\varphi(\mathbf{x}), \tag{3.66}$$

into one which is invariant under *local* $U(1)$ transformations,

$$\varphi(\mathbf{x}) \to e^{i\alpha(\mathbf{x})}\varphi(\mathbf{x}). \tag{3.67}$$

This was achieved by

(i) changing the field derivatives $\partial_i \varphi(\mathbf{x})$ into covariant derivatives

$$D_i \varphi(\mathbf{x}) \equiv (\partial_i - iqA_i(\mathbf{x}))\,\varphi(\mathbf{x}), \tag{3.68}$$

(ii) requiring that whenever the field $\varphi(\mathbf{x})$ receives a local phase factor $e^{i\alpha(\mathbf{x})}$, the gauge field $A_i(\mathbf{x})$ transforms simultaneously via the *gauge transformation*

$$A_i(\mathbf{x}) \to A_i(\mathbf{x}) + \partial_i \Lambda(\mathbf{x}), \tag{3.69}$$

with

$$\Lambda(\mathbf{x}) = \frac{1}{q}\alpha(\mathbf{x}).$$

This gauge freedom implied that the three components of the gauge field $A_i(\mathbf{x})$ cannot be observables. The curl of $A_i(\mathbf{x})$, however,

$$F_{ij}(\mathbf{x}) \equiv \partial_i A_j(\mathbf{x}) - \partial_i A_i(\mathbf{x}) \equiv \varepsilon_{ijk} B_k \tag{3.70}$$

is gauge invariant and gives rise to the observable magnetic field energy

$$E_{\text{mag}} = \frac{1}{2}\int d^3x\, F_{ij}^2 = \frac{1}{2}\int d^3x\, B_k^2. \tag{3.71}$$

It was noticed many years ago by C.N. Yang and R.L. Mills in 1954 that this structure is merely the simplest example of a wide class of many more possible gauge theories. The group $U(1)$ can be replaced by an

arbitrary continuous Lie group G. The elements in the neighbourhood of the identity of such a group define a *Lie algebra* of matrices

$$[\ell_a, \ell_b] = if_{abc}\ell_c, \tag{3.72}$$

where the symbol $[A, B] = AB - BA$ denotes the standard *commutator* and f_{abc} are totally antisymmetric *structure constants* of the algebra. The group elements g far away from the identity are obtained by exponentiation,

$$g = e^{i\alpha_a\ell_a}. \tag{3.73}$$

In general, a group has many possible sets of matrices satisfying the same Lie algebra. These are referred to as different *representations* of the group.

One of the simplest nontrivial examples is the rotation group whose Lie algebra consists of the matrices $(\ell_a)_{bc} = (S_a)_{bc} \equiv -i\varepsilon_{abc}$, i.e., explicitly

$$S_1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.74}$$

They satisfy the Lie algebra

$$[S_a, S_b] = i\varepsilon_{abc}S_c, \tag{3.75}$$

i.e., the structure constants f_{abc} consist of the totally antisymmetric unit tensor ε_{abc}.

The Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.76}$$

give another representation of the same Lie algebra if we identify $\ell_a \equiv \sigma_a/2$.

In either representation, the group elements $g = e^{i\alpha_a\ell_a}$ are rotations by an angle $\alpha = \sqrt{\alpha_a^2}$ around an axis which points in the direction $\hat{\alpha}_a \equiv \alpha_a/\sqrt{\alpha_a^2}$.

For example, the group element

$$g = e^{i\alpha S_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \tag{3.77}$$

rotates vectors around the x-axis. With Pauli matrices, the same rotation of spin-1/2 objects is given by the 2×2 matrix

$$g = e^{i\alpha\sigma_1/2} = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}. \tag{3.78}$$

Both representations play an important role in many areas of theoretical physics. In nuclear physics, for example, pions and nucleons have fields which behave like

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \to e^{i\alpha_a S_a} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix},$$

$$\begin{pmatrix} p \\ n \end{pmatrix} \to e^{i\alpha_a \sigma_a/2} \begin{pmatrix} p \\ n \end{pmatrix},$$

under transformations called isotopic spin rotations, where p, n describe protons and neutrons and $\pi_0 = \pi_3$, $\pi_\pm = (1/\sqrt{2})(\pi_1 \pm i\pi_2)$ are the fields of the neutral and charged pions.

Obviously, the group elements $g = e^{i\alpha_a \ell_a}$ are a generalization of the phase transformations $e^{i\alpha}$ of the group $U(1)$ in (3.66) which, in turn, is a special case of a Lie group with only one nonzero element $\ell_a = 1$ and vanishing structure constants $f_{abc} = 0$. The general groups with $f_{abc} \neq 0$ are called *non-Abelian*.

We can now conceive of a field theory which is invariant under such a group of non-Abelian transformations g. Suppose $\boldsymbol{\varphi}(\mathbf{x})$ transforms according to a particular representation of the group, i.e., it is a field the number of whose components is equal to the dimensionality of the representation matrices g and has the transformation law

$$\boldsymbol{\varphi}(\mathbf{x}) \to g\boldsymbol{\varphi}(\mathbf{x}) = e^{i\alpha_a \ell_a}\boldsymbol{\varphi}(\mathbf{x}). \tag{3.79}$$

For example, $\boldsymbol{\varphi}(\mathbf{x})$ can be a real three-component *vector* field $\varphi_a(\mathbf{x})$ or a complex two-component *spinor* field $\varphi_\alpha(\mathbf{x})$ such that (3.79) describes rotations of the type (3.77) or (3.78). Then φ_a^2, $\varphi_\alpha^\dagger \varphi_\alpha$ are invariant under rotations and so are energy densities like

$$e(\mathbf{x}) = \frac{1}{2}(\boldsymbol{\partial}\varphi_a)^2 + \frac{m^2}{2}\varphi_a^2 + \frac{g}{4}(\varphi_a^2)^2, \tag{3.80}$$

or

$$e(\mathbf{x}) = \frac{1}{2}\boldsymbol{\partial}\varphi_\alpha^\dagger \cdot \boldsymbol{\partial}\varphi_\alpha + \frac{m^2}{2}\varphi_\alpha^\dagger \varphi_\alpha + \frac{g}{4}|\varphi_\alpha^\dagger \varphi_\alpha|^2. \tag{3.81}$$

These can be viewed as non-Abelian generalizations of the complex $|\varphi|^4$ theory in Eq. (3.47), which was invariant under global phase changes $\varphi(\mathbf{x}) \to e^{i\alpha}\varphi(\mathbf{x})$ only. There exist a great number of physical systems which are described by fields of this type and whose energy is invariant under non-Abelian transformation groups.

The question arises whether it is possible via the introduction of some gauge field, to generalize also the non-Abelian global invariance (3.79) to a *local* form in which α_a is an arbitrary function of $\mathbf{x}$. The result would be a non-Abelian generalization of electromagnetism. In order to answer this question let us first observe that given a globally invariant theory like (3.80), (3.81), local invariance is again disturbed by the derivative terms. These transform as follows

$$\partial_i \boldsymbol{\varphi}(\mathbf{x}) \to \partial_i g(\mathbf{x})\, \boldsymbol{\varphi}(\mathbf{x}) = g(\mathbf{x})(\partial_i + g^{-1}(\mathbf{x})\, \partial_i g(\mathbf{x}))\, \boldsymbol{\varphi}(\mathbf{x}), \tag{3.82}$$

which is the non-Abelian generalization of the local $U(1)$ transformation

$$\partial_i \varphi(\mathbf{x}) \to \partial_i e^{i\alpha(\mathbf{x})}\varphi(\mathbf{x}) = e^{i\alpha(\mathbf{x})}(\partial_i + i\partial_i \alpha(\mathbf{x}))\, \varphi(\mathbf{x}). \tag{3.83}$$

We now recall that in the $U(1)$ case, local invariance was achieved by introducing the gauge field so as to compensate for the additional derivative term:

$$D_i\varphi(\mathbf{x}) \equiv (\partial_i - iqA_i(\mathbf{x}))\, \varphi(\mathbf{x}) \to \left[\partial_i - iq\left(A_i(\mathbf{x}) + \frac{1}{q}\partial_i \alpha(\mathbf{x})\right)\right] e^{i\alpha(\mathbf{x})}\varphi(\mathbf{x})$$

$$= e^{-i\alpha(\mathbf{x})}\left[\partial_i + i\partial_i\alpha(\mathbf{x}) - iq\left(A_i(\mathbf{x}) + \frac{1}{q}\partial_i\alpha(\mathbf{x})\right)\right]\varphi(\mathbf{x})$$

$$= e^{-i\alpha(\mathbf{x})}D_i\varphi(\mathbf{x}). \tag{3.84}$$

It is easy to introduce such a field also in the non-Abelian case (3.82). We simply form

$$D_i\boldsymbol{\varphi}(\mathbf{x}) \equiv (\partial_i - iqA_i)\,\boldsymbol{\varphi}(\mathbf{x}), \tag{3.85}$$

where $A_i(\mathbf{x})$ is now a *matrix* field acting on the multicomponent field $\boldsymbol{\varphi}(\mathbf{x})$. In order to find the necessary transformation properties of $A_i(\mathbf{x})$, we perform the non-Abelian version of the transformation (3.84) and find

$$D_i\boldsymbol{\varphi}(\mathbf{x}) = (\partial_i - iqA_i(\mathbf{x}))\,\boldsymbol{\varphi}(\mathbf{x}) \to (\partial_i - iqA_i'(\mathbf{x}))\,g(\mathbf{x})\,\boldsymbol{\varphi}(\mathbf{x})$$

$$= g(\mathbf{x})[\partial_i + g^{-1}\partial_i g(\mathbf{x}) - iq\,g^{-1}A_i'(\mathbf{x})\,g(\mathbf{x})]\,\boldsymbol{\varphi}(\mathbf{x}). \tag{3.86}$$

Obviously, this can be made equal to

$$g(\mathbf{x})\,D_i\boldsymbol{\varphi}(\mathbf{x})$$

by postulating for $A_i(\mathbf{x})$ the transformation law

$$A_i(\mathbf{x}) \to A_i'(\mathbf{x}) = g(\mathbf{x})\,A_i(\mathbf{x})\,g^{-1}(\mathbf{x}) + \frac{1}{iq}(\partial_i \cdot g(\mathbf{x}))\,g^{-1}(\mathbf{x})$$

$$= g(\mathbf{x})\,A_i(\mathbf{x})\,g^{-1}(\mathbf{x}) - \frac{1}{iq}g(\mathbf{x})\,\partial_i g^{-1}(\mathbf{x}). \tag{3.87}$$

Then $D_i\boldsymbol{\varphi}(\mathbf{x})$ transforms just like $\boldsymbol{\varphi}(\mathbf{x})$ and represents a covariant derivative for non-Abelian gauge groups. The matrix field $A_i(\mathbf{x})$ is called a non-Abelian gauge field. Using $A_i(\mathbf{x})$ any globally invariant energy can be made locally invariant by merely replacing everywhere derivatives by covariant derivatives (3.85).

In order for the matrix field $A_i(\mathbf{x})$ to carry physical degrees of freedom there has to be a gauge invariant energy also for it. In other words, there must be an analogue to the magnetic field

$$F_{ij} = \partial_i A_j - \partial_j A_i \equiv \varepsilon_{ijk}B_k$$

from which one can construct a gauge invariant magnetic energy corresponding to $\frac{1}{2}\int d^3x\, F_{ij}^2$. Precisely the same curl operation as in the Abelian case would not lead to a useful quantity since this would have a very complicated transformation behavior:

$$\partial_i A_j - \partial_j A_i \to \partial_i(gA_i g^{-1}) - \partial_j(gA_i g^{-1}) - \frac{1}{iq}[\partial_i(g\partial_j g^{-1}) - \partial_j(g\partial_i g^{-1})]. \tag{3.88}$$

In fact, there exists no non-Abelian analogue of F_{ij} which has the same property of being *invariant* under gauge transformations. This is, however, not necessary. For the purpose of constructing an invariant field energy which is quadratic in F_{ij} a *covariant* object is sufficient. It is given by the so-called *covariant curl* which reads

$$F_{ij}(\mathbf{x}) = \partial_i A_j(\mathbf{x}) - \partial_j A_i(\mathbf{x}) - iq[A_i(\mathbf{x}), A_j(\mathbf{x})]. \tag{3.89}$$

Since A_i are matrices, the commutator is nontrivial. Under local gauge transformations, $F_{ij}(\mathbf{x})$ transforms in the following simple way

$$F_{ij}(\mathbf{x}) \to F'_{ij}(\mathbf{x}) = g(\mathbf{x})\, F_{ij}(\mathbf{x})\, g^{-1}(\mathbf{x}). \tag{3.90}$$

This follows directly from inserting (3.87) into (3.89),

$$\begin{aligned} F_{ij}(\mathbf{x}) \to{}& \partial_i(gA_j g^{-1}) - \partial_j(gA_i g^{-1}) - \frac{1}{iq}(\partial_i(g\partial_j g^{-1}) - \partial_j(g\partial_i g^{-1})) \\ & - iqg[A_i, A_j]g^{-1} + [g\partial_i g^{-1}, gA_j g^{-1}] + [gA_i g^{-1}, g\partial_j g^{-1}] \\ & - \frac{1}{iq}[g\partial_i g^{-1}, g\partial_j g^{-1}] \\ ={}& gF_{ij}g^{-1} + \{(\partial_i g)A_j g^{-1} + gA_j(\partial_i g^{-1}) - (i \leftrightarrow j)\} \\ & + \{g(\partial_i g^{-1})gA_j g^{-1} - gA_j(\partial_i g^{-1}) - (i \leftrightarrow j)\} \\ & - \frac{1}{iq}\{\partial_i(g\partial_j g^{-1}) - \partial_j(g\partial_i g^{-1}) + [g\partial_i g^{-1}, g\partial_j g^{-1}]\}. \end{aligned} \tag{3.91}$$

Using

$$gg^{-1} = 1,$$

$$g\partial_i g^{-1} = -(\partial_i g)g^{-1},$$

we see that the first two curly brackets cancel each other, while the last one vanishes identically. Notice that the last curly bracket contains the covariant curl (3.89) of the field

$$\Delta A_i \equiv \frac{1}{iq}(\partial_i g)\, g^{-1} = -\frac{1}{iq} g \partial_i g^{-1}, \tag{3.92}$$

i.e., it can be written as $1/iq$ times

$$F_{ij}^{\Delta A} = \partial_i \Delta A_j - \partial_j \Delta A_i - iq[\Delta A_i, \Delta A_j].$$

A field ΔA_i of the type (3.92), i.e., a field whose covariant curl vanishes identically is generally referred to as a *pure gauge field*. It may be viewed as the non-Abelian generalization of the Abelian pure gauge field

$$\Delta A_i = \frac{1}{q}\partial_i \alpha, \tag{3.93}$$

whose magnetic field vanishes.

Now that we are in possession of a covariant curl of the gauge field it is trivial to find a generalization of the usual magnetic energy

$$E_{\text{mag}} = \frac{1}{4}\int d^3x\, F_{ij}^2 = \frac{1}{2}\int d^3x\, B_k^2 \tag{3.94}$$

to the non-Abelian case. We simply form

$$E = \frac{1}{4}\int d^3x\, \text{tr}(F_{ij}^2), \tag{3.95}$$

where tr denotes the trace in the matrix space of the group representation. This expression is manifestly invariant under (3.90), and hence under local gauge transformations of the non-Abelian gauge field:

$$A_i(\mathbf{x}) \to g(\mathbf{x}) A_i(\mathbf{x}) g^{-1}(\mathbf{x}) - \frac{1}{iq} g(\mathbf{x})\, \partial_i g^{-1}(\mathbf{x}).$$

Together with the original $\varphi(\mathbf{x})$ energy, this leads to the total field energy

$$E[\varphi, \varphi^\dagger, \mathbf{A}] = \int d^3x \left[\frac{1}{2} |(\partial_i - iqA_i)\varphi|^2 + \frac{m^2}{2} \varphi^\dagger \cdot \varphi + \frac{g}{4} (\varphi^\dagger \cdot \varphi)^2 + \frac{1}{4} \operatorname{tr} F_{ij}^2 \right]. \tag{3.96}$$

This is the desired construction which generalizes the $U(1)$ invariant gauge field theory (3.52) to arbitrary non-Abelian groups. It is invariant under local non-Abelian gauge transformations

$$\varphi(\mathbf{x}) \to 2g(\mathbf{x})\varphi(\mathbf{x}) = e^{i\alpha_a(\mathbf{x})\ell_a}\varphi(\mathbf{x}), \tag{3.97}$$

$$A_i(\mathbf{x}) \to g(\mathbf{x}) A_i(\mathbf{x}) g^{-1}(\mathbf{x}) - \frac{1}{iq} g(\mathbf{x})\, \partial_i g^{-1}(\mathbf{x}). \tag{3.98}$$

An important aspect of this construction is that several field theories which are globally invariant under *different* representations of a group can be made simultaneously locally invariant by one and the *same* gauge field. Consider, for example, both the isovector and isospinor fields (3.80) and (3.81). The covariant derivative of the fields φ_a and φ_α are

$$D_i\varphi_a = [(\partial_i - iqA_i^v)\varphi]_a, \tag{3.99}$$

$$D_i\varphi_\alpha = [(\partial_i - iqA_i^s)\varphi]_\alpha, \tag{3.100}$$

where A_i^v and A_i^s are 3×3 and 2×2 matrices, respectively, transforming according to (3.87):

$$A_i^{v,s}(\mathbf{x}) \to g^{v,s}(\mathbf{x}) A_i^{v,s}(\mathbf{x}) g^{v,s}(\mathbf{x}) - \frac{1}{iq} g^{v,s}(\mathbf{x})\, \partial_i g^{v,s\,-1}(\mathbf{x}), \tag{3.101}$$

with $g^{v,s}(\mathbf{x})$ being once a 3×3 representation $e^{i\alpha_a(\mathbf{x})S_a}$ and once a 2×2 representation $e^{i\alpha_a\sigma_a/2}$ of isospin rotations. We now observe that the additional piece can be written in either case as

$$-\frac{1}{iq} g(\mathbf{x})\, \partial_i g^{-1}(\mathbf{x}) = -\frac{1}{iq} e^{i\alpha_a(\mathbf{x})\ell_a}\, \partial_i e^{-i\alpha_a(\mathbf{x})\ell_a} = -\frac{1}{iq} \sum_a \ell_a \Delta_{ai}(\mathbf{x}), \tag{3.102}$$

where ℓ_a are the elements of the Lie algebra and Δ_{ai} are functions of the rotation angles α_a which are *independent* of the particular representation. For infinitesimal angles $\Delta_{ai} = -\partial_i\alpha_a$ and this statement is obvious. For

finite angles it follows from the fact that any finite group element can be written as a product of infinitesimal ones

$$g(\mathbf{x}) = g_1(\mathbf{x}) \ \ldots \ g_n(\mathbf{x}).$$

The derivative in $g\partial_i g^{-1}$ can then be executed one by one, using the product rule:

$$\begin{aligned} g\partial_i g^{-1} &= \sum_m g_1 g_2 \cdots g_m \cdots g_n g_n^{-1} \cdots \partial_i g_m^{-1} \cdots g_2 g_1 \\ &= \sum_m g_1 g_2 \cdots g_m \partial_i g_m^{-1} \cdots g_2^{-1} g_1^{-1} \\ &= \sum_m g_1 g_2 \cdots g_{m-1} \ell_a g_{m-1}^{-1} \cdots g_2^{-1} g_1^{-1} \Delta_{ai}^m(\mathbf{x}). \end{aligned} \tag{3.103}$$

The calculation of $g_{m-1}\ell_a g_{m-1}^{-1}$ is straightforward using the fact that $g(\mathbf{x})$ has the form $e^{i\alpha_a \ell_a}$ and applying the well-known Baker-Hausdorff formula

$$e^{iA} B e^{-iA} = B + i[A, B] + \frac{i^2}{2!}[A, [A, B]] + \ldots$$

Then

$$g\ell_a g^{-1} = \ell_a + i\alpha_c[\ell_c, \ell_a] + \frac{i^2}{2!}\alpha_d \alpha_c [\ell_d, [\ell_c, \ell_a]] + \ldots$$

i.e., $g\ell_a g^{-1}$ can be expressed as a series involving only commutators of the matrices ℓ_a. But the matrices are elements of a Lie algebra whose basic property is that the commutator of two elements is a linear combination of matrices ℓ_a. Hence we can write

$$g\ell_a g^{-1} = \ell_b G_{ba}. \tag{3.104}$$

The matrices G_{ba} follow the same multiplication rules as the group elements g. They form the so-called *adjoint representation* of the group. Because of (3.104) we can expand

$$g_{m-1}\ell_a g_{m-1}^{-1} \Delta_{ai}^m \equiv \ell_a \Delta_{ai}^{m-1},$$

where Δ_{ai}^{m-1} is another set of functions of the rotation angles. By going through the same steps for all $g_1, \ldots, g_{m-2}$ in (3.103) we indeed arrive at

the result (3.102) for the finite matrices $g = e^{i\alpha_a \ell_a}$. That this result is independent of the particular representation is an obvious consequence of the fact that during the calculation we had to evaluate only *commutators* of ℓ_a matrices which are the *same* for *all* representations of a given group.

Considering the form of (3.102) we conclude that the compensating gauge fields (3.99), (3.100) can be chosen to be linear combinations

$$A_i = A_{ai}\ell_a . \tag{3.105}$$

Only ℓ_a depends on the representation. The components A_{ai} transform according to the law

$$A_{ai} \to A'_{ai} = G_{ab} A_{bi} - \frac{1}{iq}\Delta_{ai}[\mathbf{A}]. \tag{3.106}$$

Since $\Delta_{ai}[\mathbf{A}]$ was obtained using only commutators $[\ell_a, \ell_b]$, this transformation law contains no more reference to a particular representation of the fields $\boldsymbol{\varphi}(\mathbf{x})$. Thus, one and the same compensating field $A_{ai}(\mathbf{x})$ can indeed be used to form covariant derivatives for fields $\boldsymbol{\varphi}(\mathbf{x})$ of *any* representation. In our examples of isovector and spinor fields, these are

$$\begin{aligned} D_i\varphi &= (\partial_i - iqS_a A_{ai})\,\varphi(\mathbf{x}), \\ D_i\varphi &= (\partial_i - iq\frac{\sigma_a}{2}A_{ai})\,\varphi(\mathbf{x}). \end{aligned} \tag{3.107}$$

It is useful to write down the gauge transformations for infinitesimal group elements

$$g(\mathbf{x}) \sim 1 + i\alpha_a(\mathbf{x})\,\ell_a .$$

Then Eqs. (3.97), (3.98) become

$$\delta\boldsymbol{\varphi}(\mathbf{x}) \equiv \boldsymbol{\varphi}'(\mathbf{x}) - \boldsymbol{\varphi}(\mathbf{x}) = i\alpha_a \ell_a \boldsymbol{\varphi}(\mathbf{x}), \tag{3.108a}$$

$$\delta A_i(\mathbf{x}) = A'_i(\mathbf{x}) - A_i(\mathbf{x}) = i\alpha_a(\mathbf{x})[\ell_a, A_i(\mathbf{x})] + \frac{1}{q}\partial_i\alpha_a\ell_a , \tag{3.108b}$$

where ℓ_a in φ and in A_i are, in general, different representations of the same group. Expanding A_i and δA_i in terms of the Lie algebra elements and using the structure constants (3.72), the second relation amounts to

$$\delta A_{bi} = \alpha_a f_{abc} A_{ci} + \frac{1}{q} \partial_i \alpha_b . \tag{3.109}$$

The first term is of the same form as the transformation law of the φ_a field $\varphi_b \to (e^{i\alpha_a \ell_a})_{bc} \varphi_c$ if we replace φ_a by the vector A_{ai} and ℓ_a by the matrix $(-if_a)$, whose matrix elements are $(\ell_a)_{bc} = (-if_a)_{bc} = -if_{abc}$. Indeed, it is well known that the structure constants of any group form a representation of the Lie algebra of that group and satisfy the commutation rules (3.72), i.e., in matrix notation

$$[-if_a, -if_b] = if_{abc}(-if_c). \tag{3.110}$$

They are referred to as the *adjoint representation*. The second term in (3.109) is the analogue of $(1/q)\,\partial_i \alpha$ in the Abelian case.

Thus, infinitesimally a non-Abelian gauge field transforms according to the adjoint representation of the group plus the gradient of the transformation angle. This gradient is eliminated when forming the covariant curl (3.89) of $F_{ij}(\mathbf{x})$. If we also expand $F_{ij}(\mathbf{x})$ in terms of the representation matrices ℓ_a we find the infinitesimal version of (3.90),

$$\delta F_{bij}(\mathbf{x}) = \alpha_a f_{abc} F_{cij}(\mathbf{x})$$

or, written in analogy with (3.108a) in matrix form,

$$\delta F_{ij}(\mathbf{x}) = i\alpha_a(-if_a)\, F_{ij}(\mathbf{x}).$$

This shows that $F_{ij}(\mathbf{x})$ transforms covariantly under the adjoint representation (i.e., with no additional gradient term).

An important consequence of the use of one and the same gauge field to ensure local invariance is the so-called *universality* of its coupling. In each covariant derivative there appears *one and the same* charge parameter q. All differences between particles arise from the representation matrices ℓ_a. They completely specify the couplings to the gauge field A_{ai}. The specification is unique if the group has no Abelian invariant subgroup (i.e., if it is semi-simple). Because of this universality property, non-Abelian gauge theories have begun playing an important role in the physics of elementary particles. Forced by the need of bringing order into an increasing number of different particles and couplings, people have tried many different larger and larger groups with different choices of fields A_{ai}. After years of trial and error this has led to the development of a unified theory of weak and electromagnetic interactions which has, up

to now, passed many impressive experimental tests. Its predictions are so reliable that they recently led to the discovery of new elementary particles associated with gauge fields, such as the intermediate vector bosons W and Z.

Non-Abelian gauge fields have the peculiar property that the magnetic energy alone, $\frac{1}{4}\int d^3x \operatorname{tr} F_{ij}^2$, is a *non-trivial interacting theory* due to the presence of the commutator $[A_i, A_j]$ in F_{ij}. In fact, if the invariance group is chosen to be $SU(3)$, this pure gauge theory seems to be the theory which describes the forces between the fundamental constituents of matter, the quarks. The exploration of such non-Abelian gauge theories is only in its beginnings, but decisive progress has been made in understanding some of its properties.

3.6. NON-ABELIAN FIELD THEORIES INVOLVING THE SPACE GROUP

The non-Abelian theories discussed in the last section were restricted to what are called *internal degrees of freedom*, such as isotopic spin. They were completely independent of the space in which physics takes place. An important set of invariances, however, involves the homogeneity and isotropy of space itself (including the time dimension in quantum theories). Any field theory which is to describe physical phenomena in an otherwise empty space has to be invariant under the Euclidean space group which comprises rotations and translations. These translations in space can be parametrized by

$$x_i \to x_i' = x_i + a_i .$$

The rotations are

$$x_i \to x_i' = R_{ij} x_j = (e^{i\alpha_a S_a})_{ij} x_j , \tag{3.111}$$

with $R^T R = 1$, and where S_a are the same Lie algebra matrices of spin one as given before in (3.74) for isospin rotations. The coordinates x_i' are understood to be the transformed coordinates of the same space point (i.e., of the same physical point) in the new coordinate frame (this convention is called the *passive* point of view).

A field is called a *scalar* if, under transformations of the coordinate frame, its numerical values at the same space point remain the same. Since the same space point is given once by x_i and once by $x_i' = R_{ij} x_j + a_i$, this implies that

$$\varphi'(\mathbf{x}) = \varphi'(R\mathbf{x} + \mathbf{a}) = \varphi(\mathbf{x}). \tag{3.112}$$

When comparing the fields before and after the transformation at the *same values* of coordinates (i.e., associated with different space points), one finds the change

$$\varphi(\mathbf{x}) \to \varphi'(\mathbf{x}) = \varphi(R^{-1}(\mathbf{x} - \mathbf{a})). \tag{3.113}$$

A *vector* field $v_i(\mathbf{x})$ is defined by its having the same size and orientation with respect to the space points before and after the coordinate transformation. It is specified by the components with respect to the basic vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, i.e.,

$$\mathbf{v}(\mathbf{x}) = \sum_{i=1}^{3} \mathbf{e}_i v_i(\mathbf{x}). \tag{3.114}$$

After the transformation, the basis vectors point into new directions

$$\mathbf{e}_i' = R_{ij}\mathbf{e}_j = \mathbf{e}_j R_{ji}^T = \mathbf{e}_j R_{ji}^{-1}. \tag{3.115}$$

If we therefore specify the same vector $\mathbf{v}$ with respect to the new transformed coordinate system, its components are

$$v_i' = R_{ij} v_j.$$

Then $\mathbf{v} = \mathbf{e}_i' v_i' = \mathbf{e}_i v_i$. By comparing the same vector at the same point in different coordinate frames we arrive at the basic property

$$v_i'(R\mathbf{x} + \mathbf{a}) = R_{ij} v_j(\mathbf{x}), \tag{3.116}$$

or the transformation law

$$v_i(\mathbf{x}) \to v_i'(\mathbf{x}) = R_{ij} v(R^{-1}(\mathbf{x} - \mathbf{a})). \tag{3.117}$$

The scalar and vector transformation laws (3.113), (3.117) can be generalized to higher tensor fields of arbitrary rank n:

$$t_{i_1 \ldots i_n}(\mathbf{x}) \to t'_{i_1 \ldots n}(\mathbf{x}) = R_{i_1 j_1} \ldots R_{i_n j_n} t_{j_1 \ldots j_n}(R^{-1}(\mathbf{x} - \mathbf{a})), \tag{3.118}$$

or to higher fields in which the 3×3 rotation matrices are replaced by the

2×2 spinor representation $\sigma_a/2$, just as in the previously described case of isospin. Transformation laws of this type involving the spatial coordinates $\mathbf{x}$ are referred to as *external*.

The construction of theories which are invariant under global external rotations can proceed very similarly to the previous case of internal symmetries. The only difference is that vector indices of fields can be contracted with vector indices of derivatives to form further invariants. For example, if $v_i(\mathbf{x})$ is a vector field, $\partial_i v_i(\mathbf{x})$ is a scalar. This follows directly from

$$v_i'(\mathbf{x}') = R_{ij}\, v_j(\mathbf{x})$$

and $x_i' = R_{ij} x_j + a_i$ so that

$$\partial_j = \partial_i' R_{ij}. \tag{3.119}$$

Hence ∂_i transforms like

$$\partial_i' = R_{ij}\, \partial_j, \tag{3.120}$$

just as the vector v_i itself and $\partial_i v_i$ is indeed a scalar field.

$$\partial_i' v_i'(\mathbf{x}') = R_{ij}\, \partial_j R_{ik}\, v_k(\mathbf{x}) = (R^T R)_{jk}\, \partial_j v_k(\mathbf{x}) = \partial_i v_i(\mathbf{x}). \tag{3.121}$$

If one wants to take a theory which is invariant under global spatial rotations and construct from it a locally invariant one, the direct involvement of the derivative in symmetry transformations necessitates a slightly different gauge field. Since the neighborhood of each point must be treated differently, the local rotations can only be defined *differentially* by

$$dx_i' = R_{ij}(\mathbf{x})\, dx_j. \tag{3.122}$$

A vector field is characterized by the local transformation law

$$v_i'(\mathbf{x}') = R_{ij}(\mathbf{x})\, v_j(\mathbf{x}). \tag{3.123}$$

This generalizes readily to tensor and spinor fields, e.g.,

$$t_{i_1 \ldots i_n}(\mathbf{x}') = R_{i_1 j_1}(\mathbf{x}) \ldots R_{i_n j_n}(\mathbf{x})\, t_{j_1 \ldots j_n}(\mathbf{x}). \tag{3.124}$$

Consider now a derivative $\partial_i v_j(\mathbf{x})$. From (3.122), we see that Eq. (3.120) still reads

$$\partial_i' = R_{ij}(\mathbf{x})\,\partial_j. \tag{3.125}$$

Hence the derivatives in the different frames are related by

$$\begin{aligned}\partial_i'\,\mathbf{v}'(\mathbf{x}') &= R_{ik}(\mathbf{x})\,\partial_k R(\mathbf{x})\,\mathbf{v}(\mathbf{x}) \\ &= R_{ik}(\mathbf{x})R(\mathbf{x})[\partial_k + R^{-1}(\mathbf{x})\,\partial_k R(\mathbf{x})]\,\mathbf{v}(\mathbf{x}).\end{aligned} \tag{3.126}$$

This is quite similar to the case of internal symmetry (3.82). The only difference is the additional rotation of the vector index k by $R_{ik}(\mathbf{x})$. We can therefore easily find a gauge field which makes $\partial_i v_j(\mathbf{x})$ a proper tensor field. All we have to do is to form

$$D_i\,\mathbf{v}(\mathbf{x}) \equiv (\partial_i - iqA_i(\mathbf{x}))\,\mathbf{v}(\mathbf{x}), \tag{3.127}$$

where the 3×3 matrices $A_i(\mathbf{x})$ transform according to the law

$$\begin{aligned}A_i(\mathbf{x}) \to A_i'(\mathbf{x}') &= R_{ik}(\mathbf{x})\left[R(x)\,A_k(\mathbf{x})\,R^{-1}(\mathbf{x}) + \frac{1}{iq}\,\partial_k R(\mathbf{x})\,R^{-1}(\mathbf{x})\right] \\ &= R_{ik}(\mathbf{x})\left[R(\mathbf{x})\,A_k(\mathbf{x})\,R^{-1}(\mathbf{x}) - \frac{1}{iq}R(\mathbf{x})\,\partial_k R^{-1}(\mathbf{x})\right].\end{aligned} \tag{3.128}$$

Then $D_i\mathbf{v}(\mathbf{x})$ transforms as follows

$$\begin{aligned}D_i'\mathbf{v}'(\mathbf{x}') &= R_{ik}(\mathbf{x})\,R(\mathbf{x})[\partial_k + R^{-1}(\mathbf{x})\,\partial_k \mathrm{R}(\mathbf{x})]\,\mathbf{v}(\mathbf{x}) - iA_i'(\mathbf{x}')\,\mathbf{v}'(\mathbf{x}') \\ &= R_{ik}(\mathbf{x})\,R(\mathbf{x})\,D_k\mathbf{v}(\mathbf{x}),\end{aligned} \tag{3.129}$$

in agreement with the general tensor law (3.124). Hence $D_i\mathbf{v}$ is a covariant derivative of the vector field.

This covariant derivative can be generalized to arbitrary representations of the rotation group in precisely the same way as in the case of internal symmetries [see Eq. (3.107)]. We merely have to write

$$A_i = A_{ai}S_a, \tag{3.130}$$

and exchange the 3×3 matrices S_a of the rotation group by any other

representation L_a. For example, a tensor of rank n transforming according to the direct product representation, has a covariant derivative $D_i = \partial_i - A_{ai} L_a$ with

$$L_a = S_a \underbrace{\times 1 \times \ldots \times}_{n \text{ factors}} 1 + 1 \times S_a \times \ldots \times 1 + \ldots + 1 \times 1 \times \ldots \times S_a, \tag{3.131}$$

where 1 denotes the 3×3 unit matrices

$$1 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us now see which invariant energy is associated with the gauge field itself. Consider the covariant curl in the 3×3 matrix representation

$$F_{ij} = \partial_i A_j - \partial_j A_i - iq[A_i, A_j]. \tag{3.132}$$

By going through the same steps as in (3.91) we find that the field components in different frames are related by

$$F'_{ij}(\mathbf{x}') = R_{ik}(\mathbf{x}) R_{j\ell}(\mathbf{x}) R(\mathbf{x}) F_{k\ell}(\mathbf{x}) R^{-1}(\mathbf{x}). \tag{3.133}$$

Therefore we can again form an invariant energy

$$E = \frac{1}{4} \int d^3x \, \mathrm{tr}(F_{ij}^2(\mathbf{x})). \tag{3.134}$$

In contrast to the internal symmetry case, however, this is now not the most general quadratic energy. Since the indices ij and the matrix indices transform with the same rotation matrices, also *mixed contractions* result in locally gauge invariant objects. Explicitly, if we write F_{ij} as $(F_{ij})_{k\ell}$, we can form, for example,

$$(F_{ij})_{k\ell}(F_{kj})_{i\ell}, \; (F_{ij})_{k\ell}(F_{k\ell})_{ij}, \; ((F_{ij})_{ij})^2. \tag{3.135}$$

Notice also that there now exists a nontrivial linear invariant $(F_{ij})_{ij}$. All these can appear in the pure gauge field energy and it depends on the physical system which one is actually treating.

With the successful construction of gauge theories invariant under local rotations the question arises whether it is possible to extend the pro-

cedure to include also local translations and thus arrive at theories which are invariant under a local version of the entire Euclidean space group. This is no longer a straightforward task. This becomes apparent from the following argument. If we want to define a local version of the general Euclidean transformation

$$x_i \rightarrow x_i' = R_{ij} x_j + a_i,$$

we have to do this again differentially, and the appropriate form is obviously

$$dx_i \rightarrow dx_i' = R_{ij}(\mathbf{x})\, dx_j + \frac{\partial a_i(\mathbf{x})}{\partial x_j}\, dx_j = \left(R_{ij}(\mathbf{x}) + \frac{\partial a_i(\mathbf{x})}{\partial x_j} \right) dx_j. \quad (3.136)$$

This transformation law has a peculiar feature: The local rotations cannot be distinguished from the local translations. The matrix $R_{ij}(\mathbf{x})$ can be absorbed into $(\delta a_i(\mathbf{x}))/\delta x_j$ and all we really see is the joint transformation

$$dx_i' = \frac{\partial x_i'}{\partial x_j} dx_j. \quad (3.137)$$

This is a general coordinate transformation. Thus it appears to be quite easy to construct a theory which is invariant under the local Euclidean space group. We only have to make sure that the field energy is invariant under general coordinate transformations.

This invariance is the fundamental principle of Einstein's theory of gravitation. There it is postulated for the four-dimensional space-time group rather than the three-dimensional space group. The construction rules are the same and are well known.

It is, however, not generally true that a theory which is invariant under general coordinate transformations displays automatically invariance under local rotations. As soon as a system contains intrinsic angular momentum which is due to spin and not to orbital motion, additional fields are necessary to distinguish local rotations from general coordinate transformations. These are known as *dreibein* (in gravity *vierbein*) fields. Only with these fields can one find a consistent locally invariant theory for fields with any spin. It is then possible to interprete these dreibein fields as the gauge fields associated with local translations.

The construction of such theories is so interesting in its own right that it

is preferable to present it in a separate part of the book, Part IV. That part is principally dedicated to the relation between the geometry of defect configurations and gravitation. After having worked out that relation, we shall have developed enough theoretical background material for the construction of locally translation and rotation invariant theories involving spin (see Part IV).

Let us end this section by recording the infinitesimal versions of the transformation laws of the fields $\varphi(\mathbf{x})$ and $A_i(\mathbf{x})$ under transformations involving the space group. Consider first global rotations. For small angles, the rotation matrix is

$$R_{ij} = 1 + i\alpha_a (S_a)_{ij} + \ldots = 1 + \alpha_a \varepsilon_{aij} + \ldots, \tag{3.138}$$

such that $x_i' = R_{ij} x_j$ amounts to

$$\delta x_i = x_i' - x_i = x_i + \alpha_a \varepsilon_{aij} x_j - x_i = -(\boldsymbol{\alpha} \times \mathbf{x})_i,$$

which is the well-known expression for small rotations of a vector $\mathbf{x}$ around a rotation axis $\boldsymbol{\alpha}$. Inserting this into (3.112) we find that a scalar field transforms as follows:

$$\begin{aligned} \delta\varphi(\mathbf{x}) &\equiv \varphi'(\mathbf{x}) - \varphi(\mathbf{x}) = \varphi(R^{-1}\mathbf{x}) - \varphi(\mathbf{x}) \\ &= \varphi(x_i + \alpha_a \varepsilon_{aji} x_j) - \varphi(x_i) = \alpha_a \varepsilon_{aji} x_j \partial_i \varphi(\mathbf{x}). \end{aligned} \tag{3.139}$$

Identifying $(1/i)\,\partial_i$ with the quantum mechanical momentum operator $\hat{p}_i$, this is recognized to have the same general form as that for internal symmetries (see (3.108))

$$\delta\varphi(\mathbf{x}) = i\alpha_a \widehat{L}_a \varphi(\mathbf{x}), \tag{3.140}$$

where $\widehat{L}_a$ is now a functional matrix

$$\widehat{L}_a = \varepsilon_{aji} x_j \frac{1}{i} \partial_i = (\mathbf{x} \times \hat{\mathbf{p}})_a, \tag{3.141}$$

which is just the differential operator of angular momentum, the well-known generator of rotations on fields. It satisfies the same algebra as any other representation matrices ℓ_a of the rotation group (compare (3.75))

$$[\widehat{L}_a, \widehat{L}_b] = i\varepsilon_{abc} \widehat{L}_c. \tag{3.142}$$

If translations are included, the scalar field transforms as follows

$$\begin{aligned}\delta\varphi(\mathbf{x}) \equiv \varphi'(\mathbf{x}) - \varphi(\mathbf{x}) &= \varphi(R^{-1}(\mathbf{x}-\mathbf{a})) - \varphi(\mathbf{x}) \\ &= \varphi(x_i + \alpha_a \varepsilon_{aji} x_j - a_i) - \varphi(\mathbf{x}) \\ &= (-a_i \partial_i + \alpha_a \varepsilon_{aji} x_j \partial_i)\,\varphi(\mathbf{x}) \\ &= i(-\mathbf{a}\cdot\hat{\mathbf{p}} + \boldsymbol{\alpha}\cdot(\mathbf{x}\times\hat{\mathbf{p}}))\,\varphi(\mathbf{x}) \\ &= i(-\mathbf{a}\cdot\hat{\mathbf{p}} + \boldsymbol{\alpha}\cdot\hat{\mathbf{L}})\,\varphi(\mathbf{x}). \end{aligned} \tag{3.143}$$

In addition to the differential operator of angular momentum, there appears now also the differential operator of momentum itself, which is the generator of translations.

It is now straightforward to write down the infinitesimal transformation law for a vector field. Using (3.117), the analoque of Eq. (3.143) becomes

$$\begin{aligned}\delta v_i(\mathbf{x}) = v_i'(\mathbf{x}) - v_i(\mathbf{x}) &= R_{ij} v_j(R^{-1}(\mathbf{x}-\mathbf{a})) - v_i(\mathbf{x}) \\ &= (\delta_{ij} + \alpha_a \varepsilon_{aij})\, v_j(x_k + \alpha_a \varepsilon_{a\ell k} x_\ell - a_k) - v_i(\mathbf{x}) \\ &= i\{-a_\ell \hat{p}_\ell \delta_{ij} + \alpha_a((\mathbf{x}\times\hat{\mathbf{p}})_a \delta_{ij} - i\varepsilon_{aij})\}\, v_j(\mathbf{x}) \\ &= i\{-\mathbf{a}\cdot\hat{\mathbf{p}}\delta_{ij} + \boldsymbol{\alpha}\cdot(\hat{\mathbf{L}}\delta_{ij} + \mathbf{S}_{ij})\}\, v_j(\mathbf{x}). \end{aligned} \tag{3.144}$$

The differential operators $\hat{\mathbf{p}}$, $\hat{\mathbf{L}} = \mathbf{x}\times\hat{\mathbf{p}}$ may be viewed as infinitesimal matrices acting on the space coordinates as "indices." Then, in a vector notation which omits both "indices" i and $\mathbf{x}$

$$\delta\mathbf{v}(\mathbf{x}) = i\{-\mathbf{a}\cdot\hat{\mathbf{p}} + \boldsymbol{\alpha}\cdot(\hat{\mathbf{L}} + \mathbf{S})\}\,\mathbf{v}(\mathbf{x}). \tag{3.145}$$

The combination

$$\mathbf{J} = \hat{\mathbf{L}} + \mathbf{S} \tag{3.146}$$

is known in Schrödinger's quantum mechanics at the *total angular momentum* operator.

The transformation law (3.145) can immediately be generalized to arbitrary tensors and spinors. All we have to do is replace S_a by the corresponding representation matrices L_a of the rotation group (see (3.131)).

In the form (3.143), (3.145) these infinitesimal transformation laws are

unchanged when going to the local versions (3.137), (3.138) in which $a_i(\mathbf{x})$, $\alpha_a(\mathbf{x})$ depend upon $\mathbf{x}$.

For the gauge field itself, the infinitesimal form of (3.128) reads [compare with (3.108b) for internal symmetry transformations]

$$\begin{aligned}\delta A_i(\mathbf{x}) &= A_i'(\mathbf{x}) - A_i(\mathbf{x}) \\ &= i\alpha_a(\mathbf{x})\, S_{aij} A_j(\mathbf{x}) + \alpha_a(\mathbf{x})[\ell_a, A_i(\mathbf{x})] + \frac{1}{q}\,\partial_i \alpha_a(\mathbf{x})\,\ell_a, \end{aligned} \tag{3.147}$$

where ℓ_a form an arbitrary representation of the rotation group. Expanding $A_i(\mathbf{x})$ in terms of the matrices ℓ_a, this becomes

$$\delta A_{ai} = \alpha_c(\mathbf{x})(\varepsilon_{cij} A_{aj} + \varepsilon_{cab} A_{bi}) + \frac{1}{q}\,\partial_i \alpha_a. \tag{3.148}$$

It differs from the purely internal gauge transformation (3.109) only by the first term by which also the spatial index i is rotated.

NOTES AND REFERENCES

For non-Abelian gauge theories see the textbooks quoted in Chapter 1. Also relevant are the book by

V.N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel Dordrecht, 1983)

and the review article by

E.S. Abers and B.W. Lee, *Phys. Reports* **C9** (1973) 1.

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C.N. Yang and R.L. Mills, *Phys. Rev.* **96** (1954) 191.

Its application to weak and electromagnetic interactions is described in

J.C. Taylor, *Gauge Theory of Weak Interactions* (Cambridge Univ. Press, Cambridge, 1976).

See also the papers by

G.'t Hooft, *Nucl. Phys.* **B79** (1974) 276,

A.M. Polyakov, *JETP Lett.* **20** (1974) 194

for relevant properties of such theories.

CHAPTER FOUR

LOW TEMPERATURE EXPANSION AROUND EXTREMA

4.1. EXTREMAL FIELD CONFIGURATIONS

In many cases, the perturbation expansion in terms of free fields is not very effective in producing even an approximate insight into the physics of a system. This happens, in particular, if the field configuration $\phi \equiv 0$ is not a minimum of the energy. In this case, even small fluctuations show huge effects. They drive the system to the proper minimum. If this lies far away from the $\phi \equiv 0$ configuration, the perturbation series to finite order can never succeed in describing this process. An example which illustrates this situation is provided by the energy

$$E = \int d^3x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 - \frac{g}{3}\phi^3 + \frac{\varepsilon g}{4}\phi^4 \right], \tag{4.1}$$

where ε is some small positive number. This type of field energy will later play an important role in the context of first order phase transitions. The minimum of this energy lies at the field origin $\phi = 0$ only, if $g < 4\varepsilon m^2$. For a stronger coupling g, a new pair of extrema forms with a minimum and a maximum at

$$\phi_{\min,\max} = \frac{1}{2\varepsilon}\left(1 \pm \sqrt{1 - \frac{4m^2\varepsilon}{g}}\right), \tag{4.2}$$

respectively. At these places, the energy density has a value

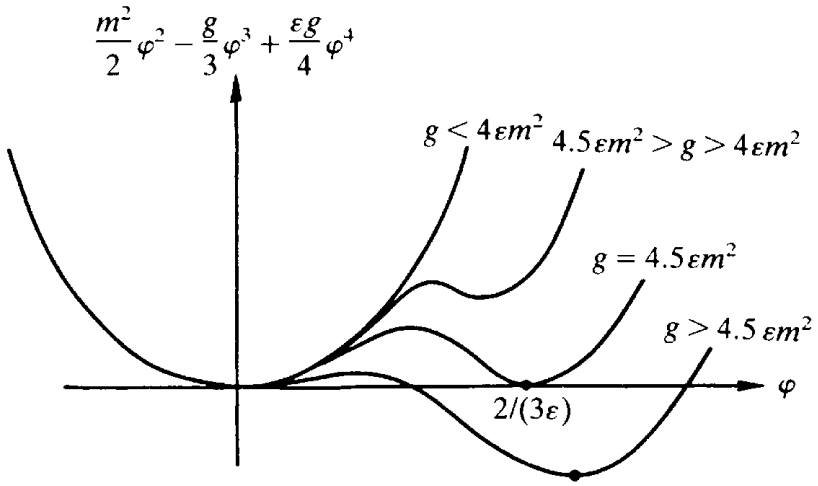
$$-\frac{1}{12g}\phi^2_{\min,\max}\left(\phi_{\min,\max}-\frac{3m^2}{g}\right). \tag{4.3}$$

For $g > \frac{9}{2}\varepsilon m^2$, the field at the new minimum is larger than $\frac{2}{3}\varepsilon$ while $3m^2/g$ is smaller than $\frac{2}{3}\varepsilon$ so that the energy density at $\phi_{\min}$ drops below that at the field origin. Inspection of the situation in Fig. 4.1 makes it obvious that for such couplings, the potential $(g/3)\,\phi^3 + (\varepsilon g/4)\,\phi^4$ can no longer be considered as a small perturbation — not even if both ε and g are small. No finite order Taylor series in g can carry the fluctuations from $\phi = 0$ to the new absolute minimum $\phi_{\min}$. This is precisely the situation for which the low-temperature expansion is required, which will be discussed now.

For very low temperatures, a classical system in equilibrium is always frozen into the absolute minimum. As the temperature is increased, fluctuations carry the field further and further away from it. Hence, in the low temperature limit, the partition function may well be approximated by calculating it at the minimum and taking the fluctuations around it into account as small corrections. The temperature T may be used as a small parameter and the corrections can be organized as a power series in T, similar to the previous expansion in g.

In general, there may be more than one field configuration of minimal energy and the discussion will depend on the way in which these minima are situated in the functional $\phi(\mathbf{x})$ space. If there is an infinite number of

FIG. 4.1. The energy of constant field configurations which has two extrema, one at the origin and one at a remote place in field space, separated by a barrier. Perturbation theory can never describe the fluctuations across the barrier.



minima which are degenerate and, moreover, continuously connected, they give rise to particular long wavelength excitations called *Nambu-Goldstone modes*. These result from local fluctuations between the minima. If the minima are disjoint and separated by a barrier, there exists a communication between them only via *macroscopic fluctuations* across the barrier. These, in turn, will be dominated by space-dependent *extremal* field configurations. In fact with respect to almost all directions in the functional space, these extrema are really local minima of the energy functional. There is usually only one line in functional space where the energy decreases when going away from the extremal point along the two opposite functional directions leading to two minima; in all other directions the energy increases. Hence such extremal macroscopic fluctuations constitute saddle points in functional space.

As an example, consider the everyday phenomenon of boiling overheated water. The lowest energy state consists of vapor. Nevertheless, the system can be caught for a long time in the metastable liquid state. As long as fluctuations around this state are infinitesimal, there can be no transition into the vapor phase. From time to time, however, a vapor bubble forms with a certain critical radius. It is determined by the balance of surface tension with volume energy. This is the type of extremal macroscopic fluctuations constituting a functional saddle point between two different local minima: The bubble can expand and transform the whole liquid into the vapor phase. The bubble can also shrink back to zero and leave the system in the liquid phase. So, in this example, the radius of the bubble characterizes the line in functional space leading to the two minima, vapor or liquid. On this line there is an extremal point, the *critical radius*, from which the energy can be lowered in two ways by letting the bubble grow to infinite size (vapor phase) or shrink to zero (liquid phase).

Each of these different extrema (absolute minimum, local minimum, extremal solution connecting minima of equal height, extremal saddle-point solution carrying the local minimum toward a proper minimum) are of particular physical relevance as contributions to the partition function. It turns out that at low temperatures the path integral in the partition function can be approximated quite well by summing over all such extremal solutions and calculating small fluctuation connections around these. The proof is based on a generalization of the saddle-point approximation for simple integrals. We discuss this topic in the next section.

4.2. SADDLE-POINT APPROXIMATION OF INTEGRALS

Consider an integral of the type appearing in partition functions

$$Z = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi T}} e^{-(1/T)f(x)}. \tag{4.4}$$

Suppose $f(x)$ has an extremum at $x = x_e$. We shall, for simplicity, assume this to be the absolute and only minimum such that $f(x)$ increases monotonously to both sides of this point. Later on other extrema will be considered. Close to x_e, we can expand $f(x)$ as

$$f(x) = f(x_e) + \frac{1}{2!} f^{(2)}(x_e)(x - x_e)^2 + \frac{1}{3!} f^{(3)}(x_e)(x - x_e)^3 + \ldots$$

and have

$$Z = e^{-(1/T)f(x_e)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi T}} \times \exp\left\{ -\frac{1}{T}\left[\frac{1}{2!} f^{(2)}(x_e)(x - x_e)^2 + \frac{1}{3!} f^{(3)}(x_e)(x - x_e)^3 + \ldots \right] \right\}. \tag{4.5}$$

We can now convince ourselves that for small temperatures, the quadratic term dominates such that Z becomes approximately

$$Z \underset{T \to 0}{\longrightarrow} e^{-f(x_e)/T} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi T}} e^{-(1/T)(1/2)f^{(2)} \cdot (x - x_e)^2} = e^{-f(x_e)/T} \frac{1}{\sqrt{f^{(2)}}} \equiv Z^e. \tag{4.6}$$

The reason for this dominance lies in the fact that the $f^{(2)}$ term by itself limits fluctuations of x away from x_e to a size of the order of $\sqrt{T}$:

$$\langle (x - x_e)^2 \rangle = \int \frac{dx}{\sqrt{2\pi T}} (x - x_e)^2 e^{-(1/T)(1/2)f^{(2)} \cdot (x - x_e)^2} \left[\int \frac{dx}{\sqrt{2\pi T}} e^{-(1/T)(1/2)f^{(2)} \cdot (x - x_e)^2} \right]^{-1}$$

$$= -\sqrt{f^{(2)}}\, 2T \frac{\partial}{\partial f^{(2)}} \frac{1}{\sqrt{f^{(2)}}} = \frac{T}{f^{(2)}}. \tag{4.7}$$

Therefore, the higher order terms in (4.5), $(x - x_e)^3$, $(x - x_e)^4$, ... are all of order $(\sqrt{T/f^{(2)}})^3$, $(\sqrt{T/f^{(2)}})^4$, ... It is useful to absorb these factors into x; define $y = \sqrt{1/T}\,(x - x_e)$, and rewrite (4.5) in the form

$$Z = e^{-f(x_e)/T} \int \frac{dy}{\sqrt{2\pi}} e^{-(1/2!)f^{(2)}y^2} e^{-(1/3!)\sqrt{T}f^{(3)}y^3 - (1/4!)Tf^{(4)}y^4 - \cdots}. \tag{4.8}$$

We can now expand the second exponential as a power series in $\sqrt{T}$, using the well-known integrals

$$\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-(1/2!)f^{(2)}y^2} y^n = \begin{cases} \dfrac{(n-1)!!}{(f^{(2)})^{(n+1)/2}} & n = \text{even}, \\ 0 & n = \text{odd}. \end{cases} \tag{4.9}$$

Since only even powers of y contribute, the resulting series for Z contains only even powers of $\sqrt{T}$ and has the form

$$Z = Z^e \sum_{n=0}^{\infty} T^n z^{(n)} = Z^e(1 + Tz^{(1)} + T^{(2)}z^{(2)} + \ldots), \tag{4.10a}$$

where the coefficients are given by

$$z^{(1)} = -\frac{3!!}{4!}\frac{f^{(4)}}{f^{(2)^2}} + \frac{1}{2}\frac{5!!}{3!^2}\frac{f^{(3)^2}}{f^{(2)^3}}, \tag{4.10b}$$

$$z^{(2)} = \frac{5!!}{6!}\frac{f^{(6)}}{f^{(2)^3}} + \frac{1}{2!}\frac{7!!}{3!5!}\frac{f^{(3)}f^{(5)}}{f^{(2)^4}} + \frac{1}{2!}\frac{7!!}{4!^2}\frac{f^{(4)^2}}{f^{(2)^4}}$$
$$-\frac{3}{3!}\frac{9!!}{3!^2 4!}\frac{f^{(3)^2}f^{(4)}}{f^{(2)^5}} - \frac{1}{4!}\frac{11!!}{3!^4}\frac{f^{(3)4}}{f^{(2)^6}}. \tag{4.10c}$$

Notice that, algebraically, the problem of evaluating (4.10) is precisely the same as that arising in the perturbation expansion of the vacuum contribution (2.25). This becomes most obvious by deriving the series with the technique of external currents used there. For this we define the auxiliary integral in the presence of a source j as

$$Z^e[j] = e^{-f(x_e)/T} \int \frac{dy}{\sqrt{2\pi}} e^{-(1/2!)f^{(2)}y^2 + jy}. \tag{4.11}$$

This can be integrated to give

$$Z^e[j] = e^{-f(x_e)/T} \frac{1}{\sqrt{f^{(2)}}} e^{(1/2)j[f^{(2)}]^{-1}j}. \tag{4.12}$$

Thanks to the external currents we can remove the interaction terms from the integral in the form of derivatives, just as in (2.12), and arrive at the following differential representation of the full partition function

$$Z = e^{-(1/3!)\sqrt{T}f^{(3)}(d/dj)^3 - (1/4!)Tf^{(4)}(d/dj)^4 - \cdots} Z^e[j]. \tag{4.13}$$

This leads again to the expansion (4.10).

4.3. SADDLE-POINT APPROXIMATION FOR FLUCTUATING FIELDS

This method can immediately be generalized to the partition function of a field in D dimensions

$$Z = \prod_{\ell} \int \frac{d\phi(\mathbf{x}_\ell)}{\sqrt{2\pi T/a^D}} e^{-(1/T)E[\phi]}, \tag{4.14}$$

i.e., the single variable x is replaced by infinitely many variables ϕ, one for every lattice point $\mathbf{x}_e$. For $T \to 0$ this is dominated by such field configurations which in the grated version satisfy

$$\left.\frac{\partial E[\phi]}{\partial \phi(\mathbf{x}_\ell)}\right|_{\phi(\mathbf{x}_\ell) = \phi_e(\mathbf{x}_\ell)} = 0. \tag{4.15}$$

Multiplying this by a^D (in D dimensions) and taking the continuum limit $a \to 0$, this becomes the functional equation

$$\left.\frac{\delta E}{\delta \phi(\mathbf{x})}\right|_{\phi(\mathbf{x}) = \phi_e(\mathbf{x})} = 0. \tag{4.16}$$

Expansion around this functional minimum gives, for $D = 3$,

$$\begin{aligned} E = E[\phi_e] &+ \frac{1}{2!}\int d^3x_1 d^3x_2 (\phi - \phi_e)(\mathbf{x}_1)\, D^e(\mathbf{x}_1, \mathbf{x}_2)(\phi - \phi_e)(\mathbf{x}_2) \\ &+ \frac{1}{3!}\int d^3x_1 d^3x_2 d^3x_3 V_3^e(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(\phi - \phi_e)(\mathbf{x}_1)(\phi - \phi_e)(\mathbf{x}_2) \\ &\times (\phi - \phi_e)(\mathbf{x}_3) + \ldots, \end{aligned} \tag{4.17}$$

where the coefficient functions are

$$D^e(\mathbf{x}_1, \mathbf{x}_2) = \frac{\delta^2 E[\phi]}{\delta\phi(\mathbf{x}_1)\,\delta\phi(\mathbf{x}_2)}\bigg|_{\phi_e(\mathbf{x})}, \tag{4.18}$$

$$V_n^e(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{\delta^n E[\phi]}{\delta\phi(\mathbf{x}_1)\ldots i\delta\phi(\mathbf{x}_n)}\bigg|_{\phi_e(\mathbf{x})}. \tag{4.19}$$

Then the partition function becomes, with $\delta\phi(\mathbf{x}) \equiv \phi(\mathbf{x}) - \phi_e(\mathbf{x})$,

$$\begin{aligned} Z = e^{-(1/T)E[\phi_e]} \prod_\ell \left[\int \frac{d\delta\phi(\mathbf{x}_\ell)}{\sqrt{2\pi T/a^3}}\right] e^{-(1/2T)\int d^3x_1 d^3x_2 \delta\phi(\mathbf{x}_1) D^e(\mathbf{x}_1,\mathbf{x}_2)\delta\phi(\mathbf{x}_2)} \\ \times \exp\left\{-\frac{1}{3!T}\int d^3x_1 d^3x_2 d^3x_3 V_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\,\delta\phi(\mathbf{x}_1)\,\delta\phi(\mathbf{x}_2)\,\delta\phi(\mathbf{x}_3)\right. \\ \left. -\frac{1}{4!T}\int d^3x_1 d^3x_2 d^3x_3 d^3x_4 V_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\,\delta\phi(\mathbf{x}_1)\ldots\delta\phi(\mathbf{x}_4) - \ldots\right. \end{aligned} \tag{4.20}$$

The first functional integral accounts for the quadratic fluctuations around $\phi_e(\mathbf{x})$ and gives $(\det D^e)^{-1/2}$. Neglecting, for the moment, the higher terms the deviations of $\phi(\mathbf{x})$ from $\phi_e(\mathbf{x})$ have the correlation function

$$\langle(\phi(\mathbf{x}_1) - \phi_e(\mathbf{x}_1))(\phi(\mathbf{x}_2) - \phi_e(\mathbf{x}_2))\rangle = T(D^e)^{-1}(\mathbf{x}_1, \mathbf{x}_2), \tag{4.21}$$

just as derived in Eq. (1.67) [compare also (4.7)]. We may again absorb the factor T into $\delta\phi$ and define $\phi' \equiv \sqrt{1/T}\,\delta\phi$ such that Z has a form analogous to (4.8):

$$\begin{aligned} Z = e^{-(1/T)E[\phi_e]} \prod_\ell \int \frac{d\phi'(\mathbf{x}_\ell)}{\sqrt{2\pi/a^3}} e^{-(1/2)\int dx_1 dx_2 \phi'(\mathbf{x}_1) D^e(\mathbf{x}_1,\mathbf{x}_2)\phi'(\mathbf{x}_2)} \\ \times \exp\left\{-\frac{\sqrt{T}}{3!}\int d^3x_1 d^3x_2 d^3x_3 V_3^e(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\,\phi'(\mathbf{x}_1)\,\phi'(\mathbf{x}_2)\,\phi'(\mathbf{x}_3)\right. \\ \left. -\frac{T}{4!}\int d^3x_1 d^3x_2 d^3x_3 d^3x_4 V_4^e(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\,\phi'(\mathbf{x}_1)\ldots\phi'(\mathbf{x}_4) - \ldots\right\}. \end{aligned} \tag{4.22}$$

This can be evaluated after a power series expansion of the second exponential leading to a series of type (4.10).

Alternatively, we may introduce an external source $\int dx\, j(x)\phi(x)$, remove the interaction terms from the functional integral by means of a functional derivative, and write in analogy with (4.13),

$$Z = \exp\left\{ -\frac{1}{3!}\sqrt{T}\int d^3x_1 d^3x_2 d^3x_3 V_3^e(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\frac{\delta}{\delta j(\mathbf{x}_1)}\frac{\delta}{\delta j(\mathbf{x}_2)}\frac{\delta}{\delta j(\mathbf{x}_3)} \right.$$
$$\left. -\frac{1}{4!}T\int d^3x_1 d^3x_2 d^3x_3 d^3x_4 V_4^e(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_4)} - \cdots \right\}$$
$$\times e^{-(1/T)E[\phi_e]}\prod_\ell\left(\int\frac{d\phi'(\mathbf{x}_\ell)}{\sqrt{2\pi/a^3}}\right) e^{-(1/2)\int d^3x_1 d^3x_2 \phi'(\mathbf{x}_1)D^e(\mathbf{x}_1,\mathbf{x}_2)\phi'(\mathbf{x}_2)} e^{\int d^3x j\phi'(\mathbf{x})}, \quad (4.23)$$

where the functional integral over $d\phi'$ can be performed, just as in Eq. (2.12), giving for the second factor [compare (4.12)]

$$Z^e[j] \equiv e^{-(1/T)E[\phi_e]}(\det D^e)^{-1/2} e^{(1/2)\int d^3x_1 d^3x_2 j(\mathbf{x}_1)(D^e)^{-1}(\mathbf{x}_1,\mathbf{x}_2)j(\mathbf{x}_2)}. \quad (4.24)$$

Expanding the exponential in (4.23) in powers of V_3, V_4, ... we arrive at the same type of perturbation series as was discussed before in Chapter 2. The main difference lies in the presence of interaction terms of higher powers in the fields.

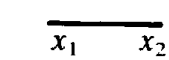
It is again useful to keep track of all arising terms by employing Feynman diagrams. These are now composed of vertices of all types,


$= -\frac{\sqrt{T}}{3!}V_3^e(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3),$

$= -\frac{T}{4!}V_4^e(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4),$


$= -\frac{T^{3/2}}{5!}V_5^e(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5), \quad (4.25)$

and lines describing the quadratic fluctuations around the extremum


$= (D^e)^{-1}(\mathbf{x}_1, \mathbf{x}_2) \equiv G_e(\mathbf{x}_1, \mathbf{x}_2). \quad (4.26)$

For zero external current, the partition function consists of the following diagrams (with the total power of t marked explicitly)

$$\frac{Z[0]}{Z^e[0]} = 1 + \left[3 + \frac{1}{2}\left(6 + 9\right)\right] T$$

$$+ \left[15 + \left(45 + 60\right) + \frac{1}{2}\left(9 + 72 + 24\right)\right]$$

$$+ \frac{3}{3!}\left(27 + 18 + 108 + 216 + 216 + 108 + 252\right)$$

$$+ \frac{1}{4!}\left(648 + 108 + 324 + 243 + 1944\right)$$

$$+ \left(3888 + 1944 + 1296\right)\Big] T^2 + \ldots \tag{4.27}$$

They have the same form as the vacuum diagrams calculated in Section 2.2 in the ϕ^4 theory, except for the larger variety of vertices. For this reason the counting of diagrams is somewhat tedious and left to the Appendix.

As a cross check we may compare the number of diagrams of each type with the result (4.10) for a simple integral and see that they add up correctly.

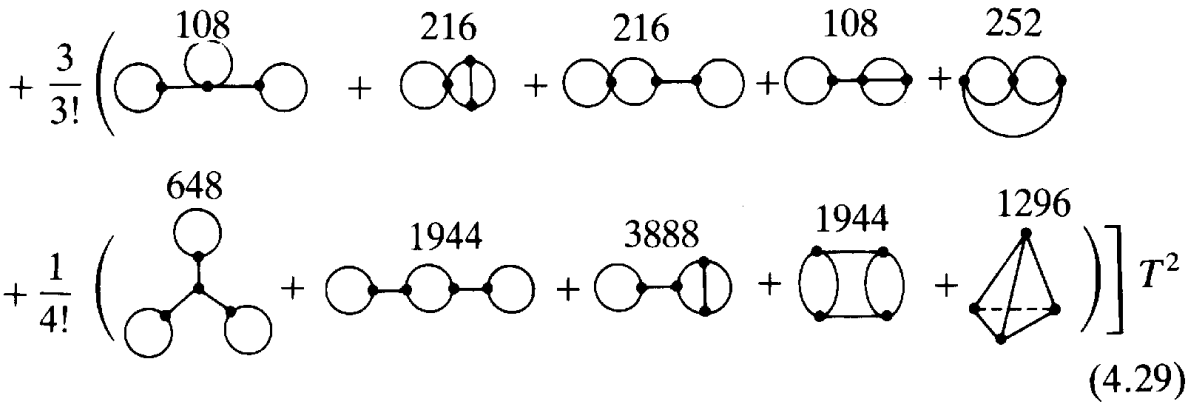
Just as in the perturbation expansion of the last chapter we can easily verify that the present graphical series can be obtained as an exponential of only connected graphs,

$$Z[0] = e^{(1/T)W[0]}, \tag{4.28}$$

where

$$\frac{1}{T}W[0] = -\frac{1}{T}E[0] + \frac{1}{2} T^0 + \left[3 + \frac{1}{2}\left(6 + 9\right)\right] T$$

$$+ \left[15 + \left(45 + 60\right) + \frac{1}{2}\left(72 + 24\right)\right]$$



(4.29)

The first single bubble graph stands for [cf. Eq. (2.27)]

$$= -\mathrm{tr}\log D^e. \tag{4.30}$$

4.4. CONNECTED CORRELATION FUNCTIONS

The Feynman rules given in the last section can be used to calculate all higher correlation functions. We have seen before in Chapter 2 in the ordinary perturbation expansion of the ϕ^4 theory that these can be decomposed into products of disconnected pieces and that this saves a lot of work when collecting all graphs. The present case is more general, since there are interactions of arbitrary powers. Therefore the perturbation series yields a much richer set of disconnected diagrams than the simple ϕ^4 theory.

The question arises whether there exists a simple general composition rule which describes how disconnected correlation functions are built from connected ones. For the vacuum contribution this was the case. There we observed that all disconnected diagrams could be simply obtained from an exponentiation of only the connected ones. It would be useful to find a similar composition rule for correlation functions. In Chapter 2 we saw that for a ϕ^4 theory calculated in perturbation theory around $\phi = 0$, the two-point function was connected [see Eq. (2.36)], i.e.,

$$G^{(2)} = G_c^{(2)}, \tag{4.31}$$

while the four-point function had two pieces [see Eq. (2.43)]

$$G^{(4)} = G_c^{(4)} + 3G_c^{(2)}G_c^{(2)}, \tag{4.32}$$

with the factor 3 indicating the three different permutations of the pairs of arguments.

In the general perturbation theory around an extremum, neither relation is true any longer. First of all, even to lowest order, an extremum can lie at any nonzero field $\phi_e(\mathbf{x}) \neq 0$. Hence, the one-point function

$$G^{(1)} = \langle \phi_e(\mathbf{x}) \rangle \tag{4.33}$$

is not zero as it was before. It is easy to see that the two-point function is now disconnected as well. Indeed, let us rewrite $G^{(2)}(\mathbf{x}_1, \mathbf{x}_2)$ as follows:

$$\begin{aligned} G_e(\mathbf{x}_1, \mathbf{x}_2) &= \langle [\phi_e(\mathbf{x}_1) + \phi(\mathbf{x}_1) - \phi_e(\mathbf{x}_1)][\phi_e(\mathbf{x}_2) + \phi(\mathbf{x}_2) - \phi_e(\mathbf{x}_2)] \rangle \\ &= \phi_e(\mathbf{x}_1)\,\phi_e(\mathbf{x}_2) + \langle [\phi(\mathbf{x}_1) - \phi_e(\mathbf{x}_1)][\phi(\mathbf{x}_2) - \phi_e(\mathbf{x}_2)] \rangle \\ &\quad - (\langle \phi(\mathbf{x}_2) - \phi_e(\mathbf{x}_1) \rangle\, \phi_e(\mathbf{x}_2) + (1 \leftrightarrow 2)). \end{aligned} \tag{4.34}$$

But to this order, $\phi_e(\mathbf{x}) = G^{(1)}(\mathbf{x})$ and $\langle \phi(\mathbf{x}_i) - \phi_e(\mathbf{x}_i) \rangle = 0$ such that

$$G^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = G^{(1)}(\mathbf{x}_1)\, G^{(1)}(\mathbf{x}_2) + T(D^e)^{-1}(\mathbf{x}_1, \mathbf{x}_2). \tag{4.35}$$

The first term is the disconnected piece. The second order term is the correlation function of small fluctuations around the extremum. It was represented by a line, i.e., by a connected diagram.

Note that (4.35) has the same connectedness as the two-point function of a free field theory in the presence of a nonzero external current j, Eq. (1.64).

When going to higher orders in the perturbation series we can see that the same decomposition can always be made. Odd powers in the field give rise to graphs of the type

(4.36)

which start at one point only and whose sum contributes to $G^{(1)}(\mathbf{x})$. These are always connected, i.e., $G^{(1)}(\mathbf{x}) = G_c^{(1)}(\mathbf{x})$.

The point is now that a decomposition of the type (4.35) survives to all orders of perturbation theory. When calculating the diagrams contributing to the two-point function they can always be decomposed into two types. One consists of a direct product of graphs of the form (4.36), the other of connected diagrams. Therefore one can write

$$G^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = G_c^{(1)}(\mathbf{x}_1)\, G_c^{(1)}(\mathbf{x}_2) + G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2), \tag{4.37}$$

where $G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2)$ collects *all connected* diagrams with two external lines.

In the general expansion around a local extremum there exist also three-point functions. These possess the following connectedness property:

$$\begin{aligned} G^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = {} & G_c^{(1)}(\mathbf{x}_1)\, G_c^{(1)}(\mathbf{x}_2)\, G_c^{(1)}(\mathbf{x}_3) \\ & + (G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2)\, G_c^{(1)}(\mathbf{x}_3) + 2 \text{ cyclic perm}) \\ & + G_c^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3). \end{aligned} \tag{4.38}$$

The four-point functions have the decomposition

$$\begin{aligned} G^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = {} & G_c^{(1)}(\mathbf{x}_1)\, G_c^{(1)}(\mathbf{x}_2)\, G_c^{(1)}(\mathbf{x}_3)\, G_c^{(1)}(\mathbf{x}_4) \\ & + (G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2)\, G_c^{(1)}(\mathbf{x}_3)\, G_c^{(1)}(\mathbf{x}_4) + 5 \text{ perm}) \\ & + (G_c^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\, G_c^{(1)}(\mathbf{x}_4) + 3 \text{ perm}) \\ & + (G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2)\, G_c^{(2)}(\mathbf{x}_3, \mathbf{x}_4) + 2 \text{ perm}) \\ & + G_c^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4). \end{aligned} \tag{4.39}$$

This is a direct generalization of the relation (2.42) found for the perturbation expansion of the ϕ^4 theory around $\phi = 0$.

The proof of these relations as well as the extension to higher n point functions can be given on the basis of functional differential equations. We refer the reader to the literature quoted in the references. Here we shall be content with merely stating a simple rule which is a direct extension of the exponentiation rule (4.28): Given the functional $Z[j]$ which generates all correlation functions via

$$G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = Z^{-1}[j] \frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_1)} Z[j] \Bigg|_{j=0}. \tag{4.40}$$

If we then define the generating functional $W[j] = \log Z[j]$ [compare $W[0]$ in Eq. (4.28)] this can be used to calculate all connected Green functions by

$$G_c^{(n)} = \frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_n)} W[j]. \tag{4.41}$$

From this rule it is very simple to derive explicitly the connectedness structure of all higher correlation functions by differentiation. Abbreviating the operation $\delta/\delta j$ by a prime, we calculate

$$\begin{aligned}
G^{(1)} &= e^{-W}(e^{W})' = W' = G_c^{(1)},\\
G^{(2)} &= e^{-W}(e^{W})'' = e^{-W}(W'e^{W})' = W'' + W'^2 = G_c^{(2)} + G_c^{(1)2}, \qquad (4.42)\\
G^{(3)} &= e^{-W}(e^{W})''' = e^{-W}((W'' + W'^2)e^{W})' = W''' + 3W''W' + W'^3\\
&= G_c^{(3)} + 3G_c^{(2)}G_c^{(1)} + G_c^{(1)3},\\
G^{(4)} &= e^{-W}(e^{W})'''' = e^{-W}((W''' + 3W''W' + W'^3)e^{W})'\\
&= W'''' + 4W'''W' + 3W''^2 + 6W''W'^2 + W'^4\\
&= G_c^{(4)} + 4G_c^{(3)}G_c^{(1)} + 3G_c^{(2)2} + 6G_c^{(2)}G_c^{(1)2} + G_c^{(1)4}.
\end{aligned}$$

It is obvious that there exists a simple recursion relation

$$G^{(n)} = G^{(n-1)'} + G^{(n-1)}G_c^{(1)}. \qquad (4.43)$$

This allows us to express $G^{(n)}$ in terms of the connected contents of $G^{(n-1)}$ by using the fact that

$$G_c^{(n)'} = G_c^{(n+1)} \qquad (4.44)$$

[remember $G_c^{(n)} \equiv W^{(n)}$]. The expansion of $G^{(1)}$ into its connected parts is often called the *cumulant expansion.*

It is easy to invert this expansion. From (4.43) we see that

$$\begin{aligned}
G_c^{(1)} &= G^{(1)},\\
G_c^{(2)} &= G^{(2)} - G^{(1)2},\\
G_c^{(3)} &= G^{(3)} - 3G^{(2)}G^{(1)} + 2G^{(1)3},\\
G_c^{(4)} &= G^{(4)} - 4G^{(3)}G^{(1)} + 12G^{(2)}G^{(1)2} - 3G^{(2)2} - 6G^{(1)4}. \qquad (4.45)
\end{aligned}$$

Each equation can be obtained from the one above by differentiation [since $G_c^{(n+1)} = G_c^{(n)'}$] where the derivatives on the right-hand side satisfy the recursion relation

$$\begin{aligned}
G^{(n)'} &= (Z^{-1}Z^{(n)})' = (Z^{-1}Z^{(n+1)}) - (Z^{-1}Z^{(n)})(Z^{-1}Z')\\
&= G^{(n+1)} - G^{(n)}G^{(1)}. \qquad (4.46)
\end{aligned}$$

4.5. SYMMETRY PROPERTIES OF EXTREMAL FIELD CONFIGURATIONS: CURRENTS AND CONSERVATION LAWS

In extremal field configurations an important role is played by *local conservation laws*. They are usually stated in the form of a vanishing divergence of some vectorial or tensorial field quantity

$$\partial_i j_i(\mathbf{x}) = 0, \quad \partial_i j_{ij}(\mathbf{x}) = 0, \quad \ldots \tag{4.47}$$

The best known conservation law of this type is that of electric currents $j_i(\mathbf{x})$, and for this reason any such conserved field quantities are generally called *currents*.

The appearance of conservation laws can be linked with the existence of continuous symmetries in physical systems. The most important symmetries in our present context will be internal symmetries, such as invariance under phase transformations of fields or under more general linear group transformations, and external symmetries which relate to the geometry of space, such as translational and rotational symmetry. The general theorem concerning this matter is due to E. Noether. It states: Suppose a field theory is local, i.e., the field energy is a space integral over a function of the fields and its derivatives in the following form

$$E = \int d^D x\, e(\varphi(\mathbf{x}), \partial\varphi(\mathbf{x})). \tag{4.48}$$

Suppose further that the total energy is invariant, up to a pure surface term, under some continuous set of symmetry transformations,

$$\delta_s \varphi(\mathbf{x}) = \varepsilon M \varphi(\mathbf{x}), \tag{4.49}$$

where M is a matrix acting on the components of $\varphi(\mathbf{x})$ (which may include the space-time variables, i.e., M may contain $\mathbf{x}$ and ∂). This assumption means that the energy density can change at most by a pure gradient term

$$\delta_s e(\varphi, \partial\varphi) = \varepsilon \left(\frac{\partial e}{\partial \varphi(\mathbf{x})} M\varphi(\mathbf{x}) + \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} M \partial_i \varphi(\mathbf{x}) \right) = \varepsilon \partial_i \Lambda_i(\varphi(\mathbf{x}), \partial\varphi(\mathbf{x})), \tag{4.50}$$

where $\Lambda_i(\varphi(\mathbf{x}), \partial\varphi(\mathbf{x}))$ is an arbitrary function of the fields and the derivatives.[a] Then we can define a *canonical Noether current*

$$j_i(\mathbf{x}) = \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} M\varphi(\mathbf{x}) - \Lambda_i(\varphi, \partial\varphi). \tag{4.51}$$

This current is conserved for any field configuration which extremizes the energy functional

$$\partial_i j_i(\mathbf{x}) = 0. \tag{4.52}$$

The proof of this theorem is based on the fact that extremal field configurations satisfy the Euler-Lagrange equations

$$\frac{\partial e}{\partial \varphi(\mathbf{x})} - \partial_i \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} = 0. \tag{4.53}$$

These follow directly from the relation $\delta E = 0$ for arbitrary $\delta\varphi(\mathbf{x})$ vanishing at the boundary, since

$$\delta E = \int d^D x \left(\frac{\partial e}{\partial \varphi(\mathbf{x})} \delta\varphi(\mathbf{x}) + \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} \partial_i \delta\varphi(\mathbf{x}) \right) = 0 \tag{4.54}$$

becomes, after a partial integration,

$$\delta E = \int d^D x \left(\frac{\partial e}{\partial \varphi(\mathbf{x})} - \partial_i \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} \right) \delta\varphi(\mathbf{x}) + \int d^D x \, \partial_i \left(\frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} \delta\varphi(\mathbf{x}) \right),$$

and the second term can be converted into a pure surface term

$$\int dS \left(\frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} \delta\varphi(\mathbf{x}) \right),$$

which vanishes together with $\delta\varphi(\mathbf{x}) = 0$. Using these equations we calculate directly

$$\partial_i j_i(\mathbf{x}) = \left(\partial_i \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} \right) M\varphi(\mathbf{x}) + \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} M \partial_i \varphi(\mathbf{x}) - \partial_i \Lambda_i$$

[a]Notice that $\Lambda_i(\varphi, \partial\varphi)$ may contain further suppressed indices contracted with those in ε. For example, ε could be a vector such that Λ_i is really a tensor, i.e., $\varepsilon \partial_i \Lambda_i = \varepsilon_i \partial_i \Lambda_{ij}$.

$$= \frac{\partial e}{\partial \varphi(\mathbf{x})} M\varphi(\mathbf{x}) + \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} M\partial_i \varphi(\mathbf{x}) - \partial_i \Lambda_i$$

and this vanishes due to the symmetry assumption (4.50).

Another proof which makes the working principle of the theorem more transparent goes as follows: Consider a transformation of the type (4.49) but modify it in such a way that ε, instead of being a constant, becomes a smooth function of $\mathbf{x}$, i.e., let

$$\delta\varphi(\mathbf{x}) = \varepsilon(\mathbf{x}) M\varphi(\mathbf{x}). \tag{4.55}$$

Certainly, this transformation is in general not a symmetry of the energy. If we, however consider only such $\varepsilon(\mathbf{x})$ which vanish at the surface, then since $\varphi(\mathbf{x})$ is an extremal field configuration by assumption, the variation of E vanishes,

$$\delta E = 0. \tag{4.56}$$

Working out the functional derivative we calculate, as in (4.54),

$$\delta E = \int d^D x \left(\frac{\partial e}{\partial \varepsilon(\mathbf{x})} - \partial_i \frac{\partial e}{\partial \partial_i \varepsilon(\mathbf{x})} \right) \varepsilon(\mathbf{x}) + \int dS_i \left(\frac{\partial e}{\partial \partial_i \varepsilon(\mathbf{x})} \varepsilon(\mathbf{x}) \right). \tag{4.57}$$

Since $\varepsilon(\mathbf{x})$ vanishes at the surface, the last term disappears. Thus we find that the derivatives of the energy with respect to $\varepsilon(\mathbf{x})$ and $\partial\varepsilon(\mathbf{x})$ satisfy once more an equation of the Euler-Lagrange type

$$\left[\frac{\partial e}{\partial \varepsilon(\mathbf{x})} - \partial_i \frac{\partial e}{\partial \partial_i \varepsilon(\mathbf{x})} \right] \Bigg|_{\varepsilon(\mathbf{x}) = 0} = 0. \tag{4.58}$$

Let us now impose the condition that for constant $\varepsilon(\mathbf{x}) \equiv \varepsilon$, $\delta\varphi(\mathbf{x})$ is a symmetry transformation, i.e., that $\delta_s e(\mathbf{x})$ is a pure gradient,

$$\begin{aligned} \delta_s e|_{\varepsilon(\mathbf{x}) = \text{const.}} &= [e(\varphi + \delta_s\varphi,\ \partial\varphi + \partial\delta_s\varphi) - e(\varphi,\ \partial\varphi)]|_{\varepsilon(\mathbf{x}) = \text{const}} \cdot \\ &= \varepsilon \partial_i \Lambda_i(\varphi,\ \partial\varphi). \end{aligned} \tag{4.59}$$

When expanding the difference in the first line in powers of $\varepsilon(\mathbf{x})$, $\partial\varepsilon(\mathbf{x})$, we see that, to linear order and for constant ε, only $\partial e/\partial\varepsilon(\mathbf{x})$ survives. Hence the symmetry condition can also be formulated for $\mathbf{x}$ dependent functions $\varepsilon(\mathbf{x})$ which vanish at the surface as

$$\left.\frac{\partial e}{\partial \varepsilon(\mathbf{x})}\right|_{\varepsilon(\mathbf{x})=0} = \partial_i \Lambda_i . \tag{4.60}$$

Combining this with the Euler-Lagrange equation (4.58) we see that the current

$$j_i(\mathbf{x}) = \frac{\partial e}{\partial \partial_i \varepsilon(\mathbf{x})} - \Lambda_i(\mathbf{x}) \tag{4.61}$$

is conserved. It is easily verified that this current is the same as the Noether current introduced in Eq. (4.51).

The advantage of this derivation of Noether's theorem is that it can readily be extended to energy densities which depend on higher derivatives, i.e., to

$$E[\varphi] = \int d^D x\, e(\varphi(\mathbf{x}), \partial_i \varphi(\mathbf{x}), \partial_{i_1}\partial_{i_2}\varphi(\mathbf{x}), \ldots, \partial_{i_1}\partial_{i_2}\ldots\partial_{i_N}\varphi(\mathbf{x})). \tag{4.62}$$

By varying this energy with respect to $\delta\varphi(\mathbf{x})$ we find the generalized Euler-Lagrange equation (cf. Eq. (4.41))

$$\frac{\partial e}{\partial \varphi(\mathbf{x})} - \partial_i \frac{\partial e}{\partial \partial_i \varphi(\mathbf{x})} + \partial_{i_1}\partial_{i_2} \frac{\partial e}{\partial \partial_{i_1 i_2}\varphi(\mathbf{x})} + \ldots + (-)^N \partial_{i_1}\ldots\partial_{i_N} \frac{\partial e}{\partial \partial_{i_1}\ldots\partial_{i_N}\varphi(\mathbf{x})} = 0. \tag{4.63}$$

The assumption that the general transformation (4.54) be a symmetry of the system implies again

$$\left.\frac{\partial e}{\partial \varepsilon(\mathbf{x})}\right|_{\varepsilon(\mathbf{x})=0} = \partial_i \Lambda_i . \tag{4.64}$$

Thus we can always define a generalized Noether current

$$j_i(\mathbf{x}) = \frac{\partial e}{\partial \partial_i \varepsilon(\mathbf{x})} - \partial_{i_1} \frac{\partial e}{\partial \partial_i \partial_{i_1} \varepsilon(\mathbf{x})} + \ldots + (-)^{N-1} \partial_{i_1}\ldots\partial_{i_{N-1}} \frac{\partial^N e}{\partial \partial_i \partial_{i_1}\ldots\partial_{i_{N-1}}\varepsilon(\mathbf{x})} - \Lambda_i , \tag{4.65}$$

which is conserved

$$\partial_i j_i(\mathbf{x}) = 0. \tag{4.66}$$

The reader will have noticed that in the latter general formulation the particular form of the symmetry transformation (4.49) is completely irrelevant. The form (4.49) was chosen for the sake of discussion only since it is the most commonly encountered one. In fact, in this text we shall often have to deal with another form in which the fields transform by an additive piece, which does not involve the field itself, i.e.

$$\delta_s \varphi(\mathbf{x}) = \varepsilon v \tag{4.67}$$

where v is a constant field with as many components as $\varphi(\mathbf{x})$. Still, if $\delta_s e = \varepsilon \partial_i \Lambda_i$, there is the Noether current (4.65) which is conserved.

For fluctuating fields, Noether's theorem makes no direct statement and the consequences of the symmetry are somewhat harder to evaluate. It is possible to derive the so-called "Ward identities" for correlation functions which have to be satisfied to all orders in perturbation theory. We shall not go into this since there still is a more efficient method of respecting symmetries of fluctuating field theories. It is based on the construction of the so-called effective energy functional. This will be presented in the next chapter.

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H. Kleinert, *Fortschr. Phys.* **30** (1982) 187.

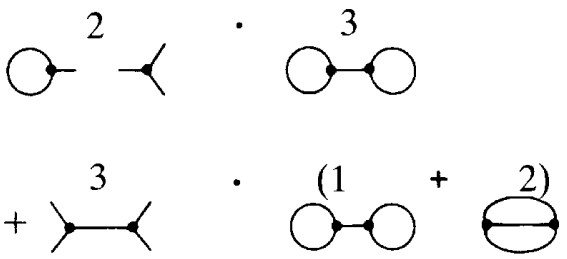
APPENDIX 4A. VACUUM DIAGRAMS

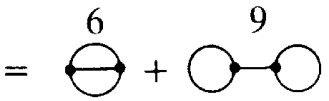
The diagrams can be constructed following Wick's rule. For this we draw all vertices appearing in the power series expansion of $Z[0]$, choose one vertex to start with and connect its first leg successively with all free legs. After having exhausted all choices, we take the second leg and proceed in the same way. We use the notation of Eq. (4.10).

For the $f^{(4)}/f^{(2)^2}$ term there are $3!! = 3$ possibilities, all associated with the graph

(4A.1)

For $f^{(3)^2}/f^{(2)^3}$ there are $5!! = 15$ possibilities given by

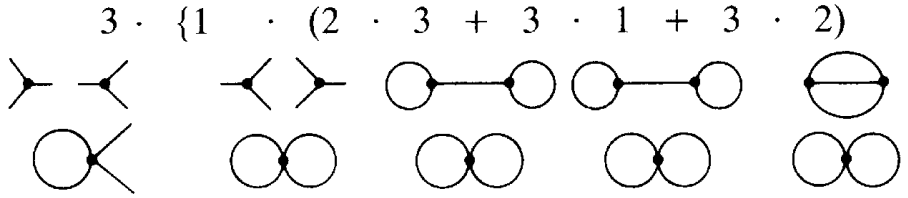


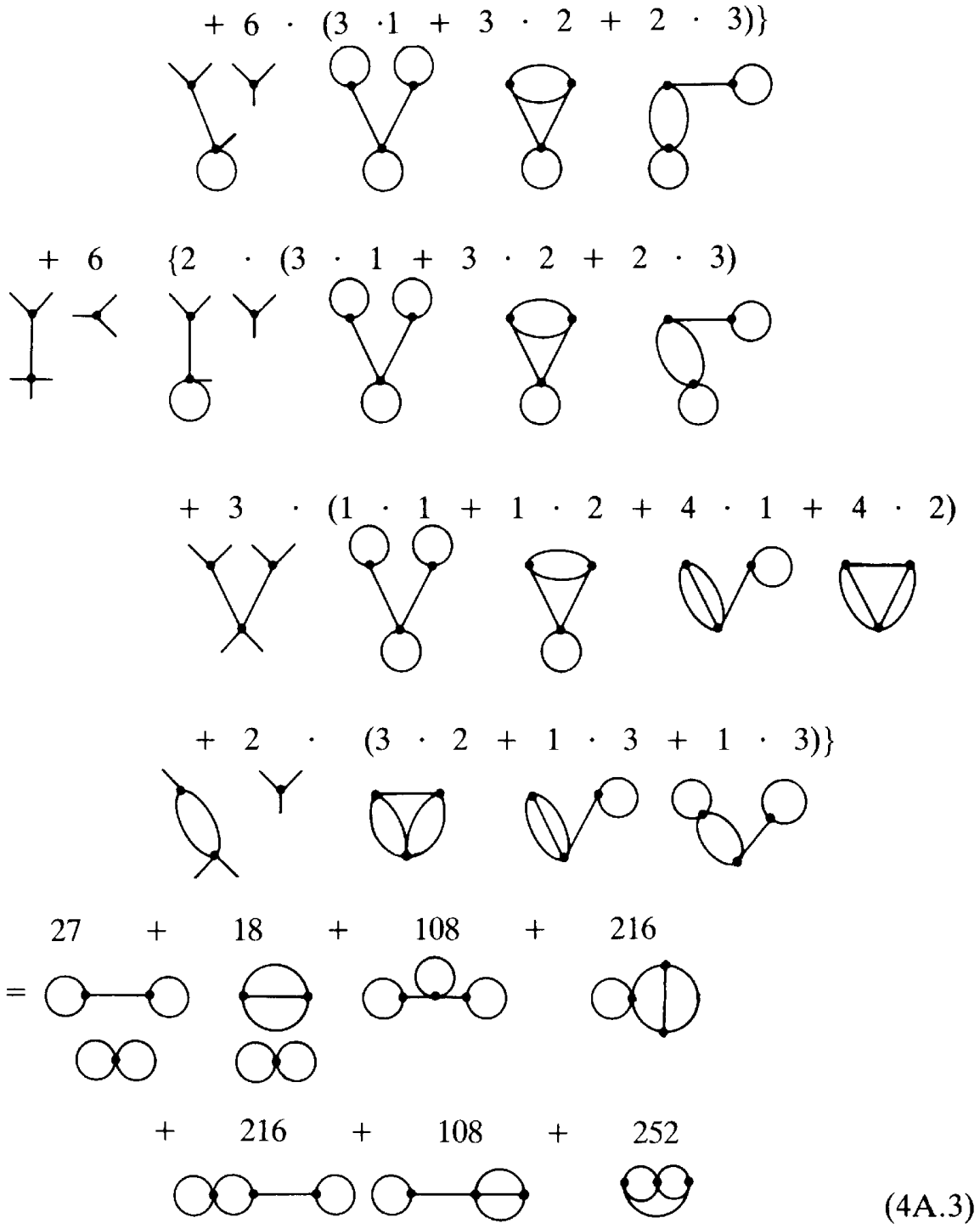


(4A.2)

Our graphical notation is the following: In each column we display all vertices to be connected. From left to right we execute successively all possible types of contractions and write the multiplicity factor on top. Parentheses collect graphs with the same number of contracted legs.

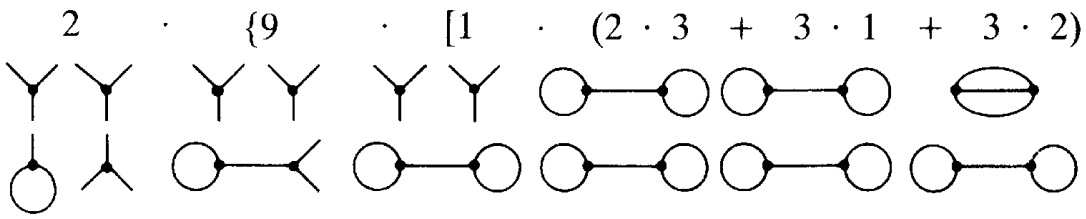
For $f^{(3)^2}f^{(4)}/f^{(2)^5}$ there are $9!! = 945$ possibilities which are constructed as follows:

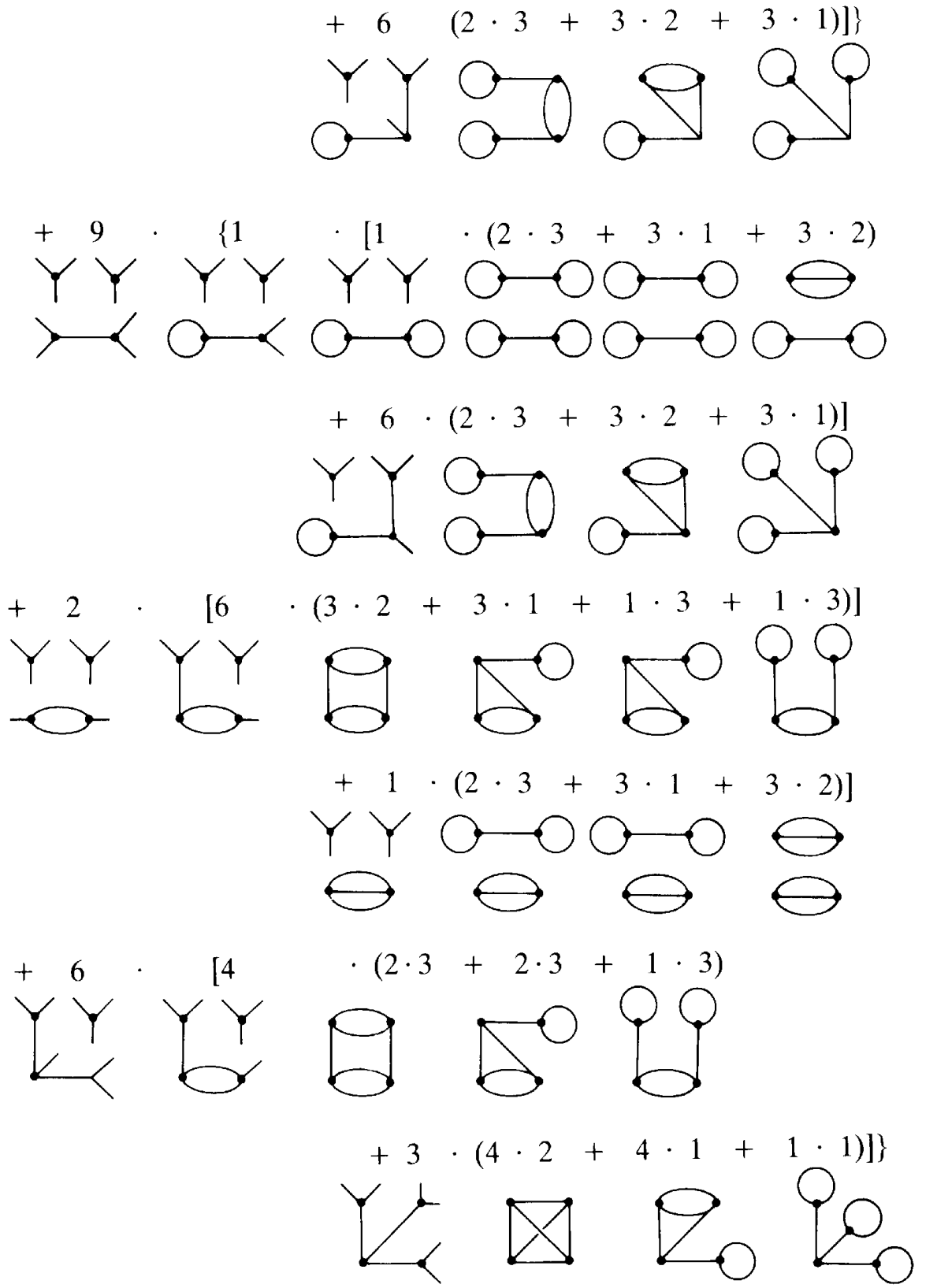




(4A.3)

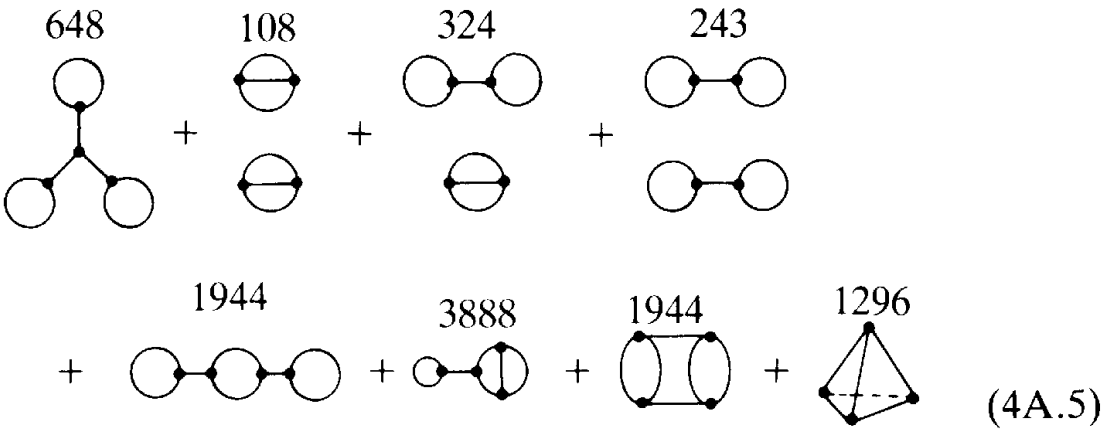
For $f^{(3)^4}/f^{(2)^6}$, finally, there are 11!! = 10,395 possibilities:





(4A.4)

These may be collected as follows:



(4A.5)

CHAPTER FIVE

EFFECTIVE ENERGY

5.1. EXTREMAL PRINCIPLE

The saddle-point expansion described in Chapter 4 can be made more powerful by what may be considered as a self-consistent extension of it. Recall that to lowest approximation one calculates the partition function by inserting a field configuration which extremizes the energy. At that level, the field expectation at the extremum is $\langle \phi(\mathbf{x}) \rangle = \phi_e(\mathbf{x})$. Then one performs fluctuation corrections, after which the expectation will be different, say

$$\langle \phi(\mathbf{x}) \rangle \equiv \Phi(\mathbf{x}) \equiv \phi_e(\mathbf{x}) + \delta\phi_e(\mathbf{x}). \tag{5.1}$$

The question arises whether one can improve upon the procedure by studying fluctuations right away around the proper *final* field expectation $\Phi(\mathbf{x})$. Moreover, one may wonder whether it is possible to construct an improved version of the energy functional, to be called $\Gamma[\Phi]$, which incorporates the effect of fluctuations and serves to calculate directly the full field expectation $\Phi(\mathbf{x})$ rather than $\phi_e(\mathbf{x})$ from an extremal principle:

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(\mathbf{x})} = 0. \tag{5.2}$$

Then, the field expectation $\Phi(\mathbf{x})$ would play the same role with respect to the functional $\Gamma[\Phi]$ as the field ϕ_e did with respect to the original energy E. The consequences of the symmetries of a system would then be just as easy to find as for the solutions of the field equations in Section 4.4. A functional with these pleasant proporties does indeed exist and is called *effective energy*.

Consider the partition function

$$Z[j] = e^{(1/T)W[j]} \tag{5.3}$$

and recall that $W[j]$ is the generating functional of all connected n-point functions which are given by the functional derivations $(1/T)\,T^n\,(\delta^n/\delta j^n)\,W[j]$. In particular, the one-point function, which is the expectation of the fluctuating field, coincides with the first derivative

$$\Phi(\mathbf{x}) = \langle \phi(\mathbf{x}) \rangle = Z^{-1} T \frac{\delta}{\delta j(\mathbf{x})} Z = \frac{\delta W[j]}{\delta j(\mathbf{x})}. \tag{5.4}$$

In order to shorten formulas, we shall from now on indicate functional differentiation by a simple subscript, i.e.

$$\frac{\delta}{\delta j(\mathbf{x})} W[j] \equiv W_j(\mathbf{x})$$

It is then straightforward to write an extremal principle for determining the field expectation $\Phi(\mathbf{x})$. Consider the functional constructed as a Legendre transform of $W[j]$:

$$\Gamma[\Phi] = -W[j] + W_j j = -W[j] + \Phi j. \tag{5.5}$$

Here, multiplication is to be understood in the functional sense, i.e.

$$\Phi j \equiv \int dx\, \Phi(\mathbf{x})\, j(\mathbf{x}).$$

By this very construction, $\Gamma[\Phi]$ satisfies the inverse relation of (5.4), i.e.,

$$\Gamma_\Phi(\mathbf{x}) \equiv \frac{\delta}{\delta \Phi} \Gamma[\Phi] = -W_j j_\Phi + j + \Phi \cdot j_\Phi = j. \tag{5.6}$$

In terms of $\Gamma[\Phi]$, the partition function is given by

$$Z[j] = e^{(1/T)W[j]} = e^{-(1/T)(\Gamma[\Phi] - j\Phi)}, \tag{5.7}$$

to be evaluated at the field configuration $\Phi(\mathbf{x})$ which extremizes the exponent

$$\frac{\delta}{\delta\Phi}\left(\Gamma[\Phi] - j\Phi\right) = 0. \tag{5.8}$$

The physical situation usually corresponds to $j \equiv 0$ so that we find that $\Gamma[\Phi]$ is indeed extremal for the physical field expectation $\Phi(\mathbf{x})$.

Of particular interest will be the effective energy for a constant field, $\Phi(\mathbf{x}) \equiv$ constant. Then since the energy density is constant over all space, the spatial integration gives a volume factor V and we may define

$$v(\Phi) \equiv \frac{1}{V}\Gamma[\Phi]. \tag{5.9}$$

This quantity is called the *effective potential*. Notice that unlike $\Gamma[\Phi]$, v is no longer a functional but just an ordinary function of $\Phi \equiv$ constant.

5.2. TREE GRAPHS AND VERTEX FUNCTIONS

Apart from providing an extremal principle for fluctuating systems, the effective energy plays another important role: It can be used to organize efficiently the perturbation theory of a system around the full self-consistent extremal field configuration Φ. We have seen in the analysis of Chapter 2 that the perturbative calculation of $Z[j]$ involves, to each order, only a few really new amplitudes. A large portion of the diagrams is a rather trivial composition of simpler diagrams which already appeared at a lower order. The composition law was rather trivial: By going from $Z[j]$ to $W[j] = T\log Z[j]$, the calculation could be restricted to connected diagrams only, with the exponentiation restoring all disconnected pieces. We have also realized the crucial role of the other fundamental diagrammatic building blocks called one-particle irreducible graphs.

It turns out that the effective energy efficiently organizes the sum of all connected Feynman graphs into its one-particle irreducible content. Moreover, it permits a simple algebraic formulation of the composition

rules. Thus $\Gamma[\Phi]$ is really effective in two ways and fully deserves its name.

Suppose for a moment $\Gamma[\Phi]$ were a known functional. Let us demonstrate that it does contain all information necessary for recovering all the connected correlation functions.

First, by construction, the one-point correlation function $\Phi(\mathbf{x}) \equiv \langle \phi(\mathbf{x}) \rangle$ can be found by the simple extremization $\delta\Gamma[\Phi]/\delta\Phi = j(\mathbf{x})$ with j to be set equal to zero at the end.

Second, the connected two-point function is given by

$$G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = TW_{jj}(\mathbf{x}_1, \mathbf{x}_2) = T\Phi_j(\mathbf{x}_1, \mathbf{x}_2).$$

But the functional derivative Φ_j can be calculated as the inverse functional of the functional derivative $(\delta j(\mathbf{x}_1)/\delta\Phi(\mathbf{x}_2)) \equiv j_\Phi(\mathbf{x}_1, \mathbf{x}_2)$ which, in turn, is $\Gamma_{\Phi\Phi}(\mathbf{x}_1, \mathbf{x}_2)$, due to (5.6). Therefore, we find

$$G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = Tj_\Phi^{-1}(\mathbf{x}_1, \mathbf{x}_2) = T\Gamma_{\Phi\Phi}^{-1}(\mathbf{x}_1, \mathbf{x}_2). \tag{5.10}$$

The notation $\Gamma_{\Phi\Phi}^{-1}(\mathbf{x}_1, \mathbf{x}_2)$ denotes the inverse functional, as defined by

$$\int dx_2 \Gamma_{\Phi\Phi}(\mathbf{x}_1, \mathbf{x}_2)\, \Gamma_{\Phi\Phi}^{-1}(\mathbf{x}_2, \mathbf{x}_3) \equiv \delta(\mathbf{x}_1, \mathbf{x}_3). \tag{5.11}$$

Proceeding in the same way, the full connected three-point function can be calculated. The result is

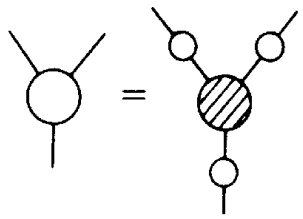
$$\begin{aligned} G_c^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= T^2 W_{jjj} = T^2(\Gamma_{\Phi\Phi})^{-1} \\ &= -T^2\Gamma_{\Phi\Phi}^{-2}\Gamma_{\Phi\Phi\Phi}\Phi_j \\ &= -T^2\Gamma_{\Phi\Phi}^{-3}\Gamma_{\Phi\Phi\Phi} \\ &= -G_c^{(2)3}\frac{1}{T}\Gamma_{\Phi\Phi\Phi}. \end{aligned} \tag{5.12}$$

The product is to be read as the functional contraction

$$-\int dx_1' dx_2' dx_3' G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_1') G_c^{(2)}(\mathbf{x}_2, \mathbf{x}_2') G_c^{(2)}(\mathbf{x}_3, \mathbf{x}_3') \frac{1}{T}\Gamma_{\Phi\Phi\Phi}(\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_3'), \tag{5.13}$$

i.e., the third derivative of $\Gamma[\Phi]$ is to be contracted with the full

connected two-point functions at each spatial argument. We may represent this contraction graphically as follows

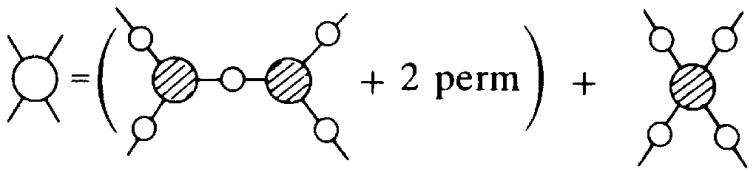


(5.14)

An open circle with n legs stands for a full connected n-point function and the shaded circle with n legs denotes the n-th derivative of $-(1/T)\,\Gamma[\Phi]$. Continuing this process we find

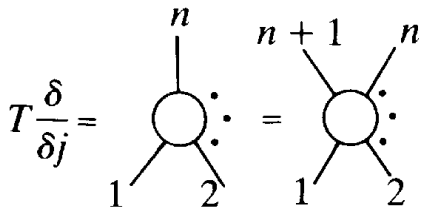
$$\begin{aligned} G_c^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= T^3 W_{jjjj} \\ &= -T^3(\Gamma^{-3}_{\Phi\Phi}\Gamma_{\Phi\Phi\Phi})_j \\ &= 3T^3\Gamma^{-4}_{\Phi\Phi}\Gamma_{\Phi\Phi\Phi}\Phi_j\Gamma_{\Phi\Phi\Phi} - T^3\Gamma^{-3}_{\Phi\Phi}\Gamma_{\Phi\Phi\Phi\Phi}\Phi_j \\ &= 3T^3\Gamma^{-5}_{\Phi\Phi}\Gamma^2_{\Phi\Phi\Phi} - T^3\Gamma^{-4}_{\Phi\Phi}\Gamma_{\Phi\Phi\Phi\Phi} \\ &= 3G_c^{(2)^5}\left(\frac{1}{T}\Gamma_{\Phi\Phi\Phi}\right)^2 - G_c^{(2)^4}\frac{1}{T}\Gamma_{\Phi\Phi\Phi\Phi}. \end{aligned} \tag{5.15}$$

This may be represented as

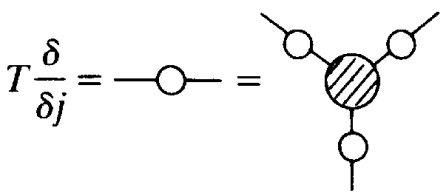


(5.16)

The graphical procedure of deriving higher graphs is illustrated in Fig. 5.1. For $n \geq 2$, one leg is added to an open circle by $T(\delta/\delta j)$, i.e.,



For $n = 2$, the same operation produces a connected three-point function, due to (5.12), i.e.,



(5.17)

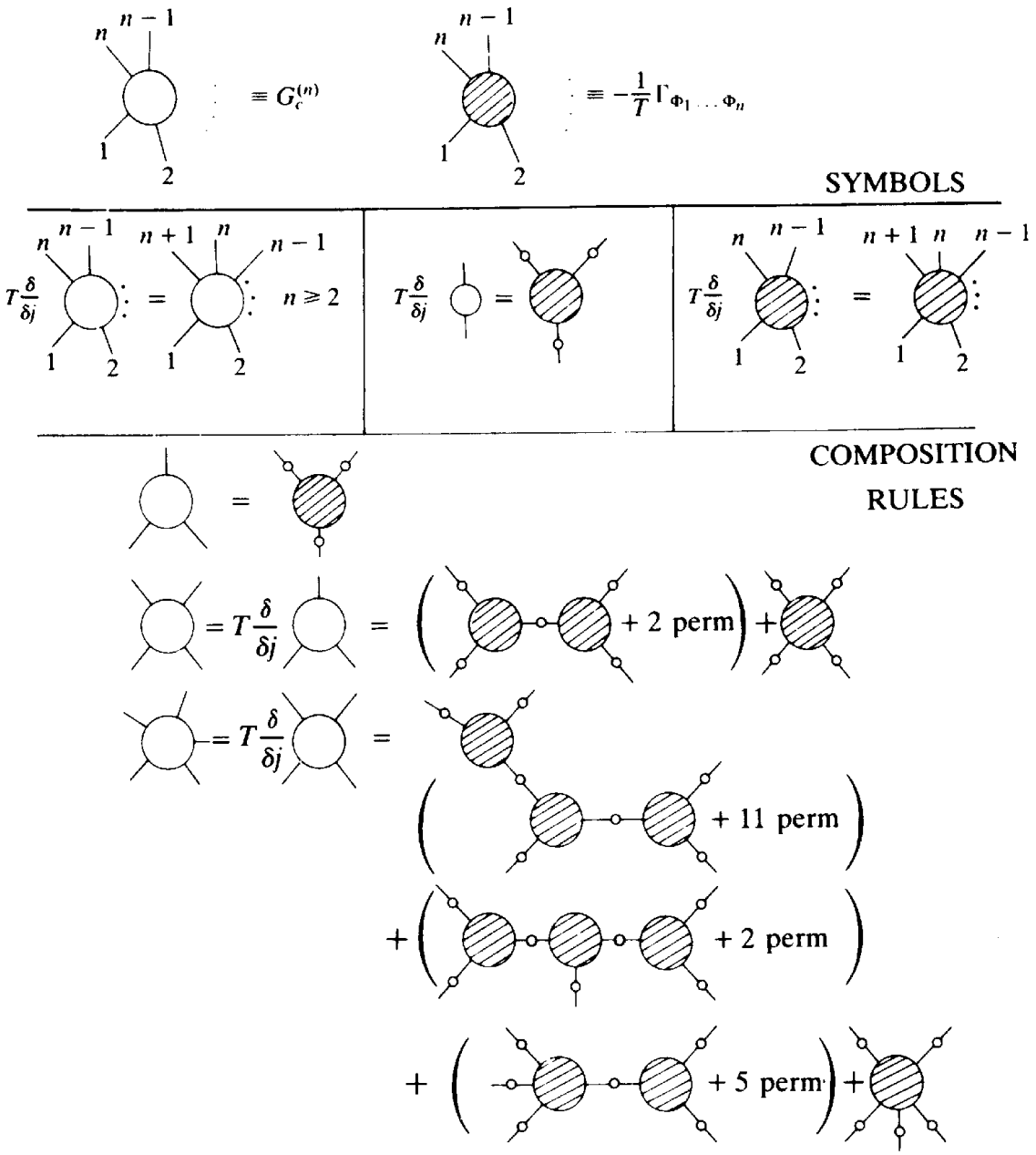
When applied to a shaded circle, one finally has

$$T\frac{\delta}{\delta j}\left(-\frac{1}{T}\Gamma_{\underbrace{\Phi\ldots\Phi}_{n}}\right) = -\Gamma_{\underbrace{\Phi\ldots\Phi\Phi}_{n+1}}\Phi_j = -\Gamma_{\Phi\ldots\Phi\Phi}\Gamma^{-1}_{\Phi\Phi} = -\frac{1}{T}\Gamma_{\Phi\ldots\Phi\Phi}G_c^{(2)}, \tag{5.18}$$

which amounts to adding one leg with an open circle

$$T\frac{\delta}{\delta j} = \quad n \quad \vdots \quad = \quad n+1 \quad n \quad \vdots \tag{5.19}$$
$$\qquad\qquad 1 \quad 2 \qquad\qquad 1 \quad 2$$

FIG. 5.1. Graphical differentiation rules for the construction of all connected n-point functions from vertex functions and connected two-point functions. The rules are explained after Eq. (5.16).



This construction shows that all correlation functions can indeed be obtained from the effective energy $\Gamma[\Phi]$. What one has to do is:

(i) calculate the functional derivative and extract the connected two-point functions $G_c^{(2)} = T\Gamma^{-1}_{\Phi\Phi}$ as well as the n-point vertex functions $-(1/T)\,\Gamma_{\Phi\Phi\ldots\Phi}$ with $n \geq 3$;

(ii) use these elements and compose all possible one-particle *reducible* graphs containing *no loops*. Because of their appearance, these are called *tree graphs*. Since these tree graphs display directly all possible one-particle reducible structures, it is suggestive to assume that the vertex functions $(1/T)\,\Gamma_{\Phi\Phi\ldots\Phi}$ with more than three derivatives allow for no further reduction, i.e., they are all one-particle irreducible. This can indeed be proved and we shall see examples in the next section. Notice that the second derivative plays a special role which we shall soon understand better.

5.3. GRAPHICAL CONTENT OF THE EFFECTIVE ENERGY

So far we have convinced ourselves of the usefulness and power of the effective energy $\Gamma[\Phi]$. Thus it is desirable to find an efficient way of calculating it. For this we start out with the path integral representation for the generating functional

$$Z[j] = e^{(1/T)W[j]} = e^{-(1/T)(\Gamma[\Phi] - j\Phi)} = \int \mathscr{D}\phi e^{-(1/T)(E[\phi] - j\phi)}. \tag{5.20}$$

For $T \to 0$, this expression is straightforward to calculate since the path integral is dominated by the extremum $\phi_e(\mathbf{x}) = \phi_e[j](\mathbf{x})$ which satisfies

$$\frac{\delta}{\delta\phi(\mathbf{x})} E[\phi]\bigg|_{\Phi = \Phi_e[j](\mathbf{x})} = j(\mathbf{x}). \tag{5.21}$$

In terms of $\phi_e(\mathbf{x}) = \phi_e[j](\mathbf{x})$, we can therefore write

$$-W[j] = \Gamma[\Phi] - j\Phi = E[\phi_e] - j\phi_e. \tag{5.22}$$

In order to use this equation for identifying $\Gamma[\Phi]$ we have to find the connection between the field $\Phi(\mathbf{x})$ and $\phi_e(\mathbf{x})$. This is obtained by differentiating $W[j]$ with respect to j, i.e.,

$$\Phi(\mathbf{x}) = \frac{\delta}{\delta j(\mathbf{x})} W[j] = -E_\phi[\phi_e]\frac{\delta \phi_e}{\delta j} + \phi_e + j\frac{\delta \phi_e}{\delta j} = \phi_e(\mathbf{x}).$$

Hence we see that, to this approximation, the field expectation coincides with the extremal solution. Inserting this back into (5.22) we find

$$\Gamma[\Phi] = E[\Phi], \tag{5.23}$$

i.e., the effective energy is simply the original energy functional itself, only that the fluctuating field variable is replaced by the non-fluctuating field expectation $\Phi(\mathbf{x})$. This approximation to $\Gamma[\Phi]$ is called the *mean field approximation*.

It will be useful to separate out this lowest order content in $\Gamma[\Phi]$ and write

$$\Gamma[\Phi] = E[\Phi] + \tilde{\Gamma}[\Phi], \tag{5.24}$$

where $\tilde{\Gamma}[\Phi]$ contains all the fluctuation corrections. Inserting this back into (5.20) we have

$$Z[j] = \exp\left\{-\frac{1}{T}(E[\Phi] - j\Phi)\right\} \exp\left\{-\frac{1}{T}\tilde{\Gamma}[\Phi]\right\},$$

with $\tilde{\Gamma}[\Phi]$ given by the path integral over $\phi'(\mathbf{x}) \equiv \phi(\mathbf{x}) - \Phi(\mathbf{x})$,

$$e^{-(1/T)\tilde{\Gamma}[\Phi]} = \int \mathscr{D}\phi' e^{(-1/T)(E[\phi' + \Phi] - E[\Phi] - j\phi')}. \tag{5.25}$$

The left-hand side depends only on Φ, the right-hand side on Φ and j. Using

$$j = \Gamma_\Phi[\Phi] = E_\Phi[\Phi] + \tilde{\Gamma}_\Phi[\Phi], \tag{5.26}$$

the equation takes a pure Φ form

$$e^{(-1/T)\tilde{\Gamma}[\Phi]} = \int \mathscr{D}\phi' e^{(-1/T)(E[\phi' + \Phi] - E[\Phi] - E_\Phi[\Phi]\phi' - \tilde{j}[\Phi]\phi')}, \tag{5.27}$$

where we have used the notation

$$\tilde{j}[\Phi] = \tilde{\Gamma}_\Phi[\Phi]. \tag{5.28}$$

The energy in the exponent can be written as $\Delta E \equiv E[\phi' + \Phi] - E[\Phi] - E_\Phi[\Phi]\phi'$. We now expand

$$\Delta E = \frac{1}{2}\phi' E_{\Phi\Phi}[\Phi]\phi' + \frac{1}{3!}E_{\Phi\Phi\Phi}\phi'\phi'\phi' + \frac{1}{4!}E_{\Phi\ldots\Phi}\phi'\ldots\phi' + \ldots$$

$$\equiv \frac{1}{2}\phi' E_{\Phi\Phi}[\Phi]\phi' + \tilde{E}[\Phi, \phi'], \tag{5.29}$$

and we see that the expression (5.25) is nothing but the Φ dependent partition function

$$\tilde{Z}[\tilde{j}] = e^{(1/T)\tilde{W}[\tilde{j}]} = \int \mathscr{D}\phi' e^{-(1/T)((1/2)\phi' E_{\Phi\Phi}\phi' + \tilde{E}[\Phi,\phi'] - \tilde{j}\phi')} \tag{5.30}$$

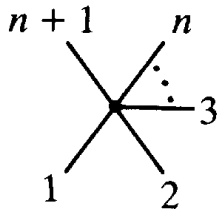
of an auxiliary energy in which the expansion coefficients of the fluctuating field ϕ' depend on Φ. It can therefore be calculated by the methods described in the last chapter as

$$e^{(-1/T)\tilde{E}[\Phi, T\delta/\delta\tilde{j}]}\int \mathscr{D}\phi' e^{(-1/T)((1/2)\phi' E_{\Phi\Phi}\phi' - \tilde{j}\phi')} = e^{(-1/2)\mathrm{tr}\log E_{\Phi\Phi}} e^{-(1/T)\tilde{E}[\Phi, T\delta/\delta\tilde{j}]} e^{(1/2T)\tilde{j}E^{-1}_{\Phi\Phi}\tilde{j}}. \tag{5.31}$$

Thus we know that $T(\delta/\delta\tilde{j})^n(1/T)\,\tilde{W}[\tilde{j}]$ are the connected n-point functions constructed from Feynman diagrams involving legs

$$\overline{\qquad}_{x_1 \quad x_2} = TE^{-1}_{\Phi\Phi}(\mathbf{x}_1, \mathbf{x}_2) \equiv G_\Phi(\mathbf{x}, \mathbf{x}_2) \tag{5.32a}$$

and vertices



$$= -\frac{1}{T}\frac{1}{n!}V_n^\Phi(\mathbf{x}_1, \ldots, \mathbf{x}_n) = -\frac{1}{T}\frac{1}{n!}\underbrace{E_{\Phi\Phi\ldots\Phi}}_{n}. \tag{5.32b}$$

The full functional $\tilde{W}[\tilde{j}]$ is the sum of all these diagrams with each free leg contracted with a current $\tilde{j}$:

$$\tilde{W}[\tilde{j}] = \sum_n \frac{1}{n!}\int dx_1 \ldots dx_n \tilde{j}(\mathbf{x}_1)\ldots\tilde{j}(\mathbf{x}_n)\,\tilde{W}_{jj\ldots j}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

$$= \sum_n \frac{1}{n!}\int dx_1 \ldots dx_n \tilde{j}(\mathbf{x}_1)\ldots\tilde{j}(\mathbf{x}_n)\,G_\Phi^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n). \tag{5.33}$$

The $j = 0$ piece, $\tilde{W}[0]$, consists of all vacuum diagrams. It has the same graphical expansion as that displayed in Eq. (4.29) with G_e, V_n^e replaced by G_Φ, V_n^Φ.

Now, what formula (5.27) teaches us is that the fluctuation correction to the effective energy coincides with $\tilde{W}[\tilde{j}]$ evaluated at a specific value $\tilde{j} = \tilde{j}[\Phi] = \tilde{\Gamma}_\Phi[\Phi]$. This value can be characterized in more physical terms as follows: It is an external current chosen in such a way that the expectation of the fluctuating field in the path integral (5.26) vanishes identically. That this is so can most easily be inferred from the fact that

$$\Phi(\mathbf{x}) = \langle \phi(\mathbf{x}) \rangle = \langle \phi \rangle + \langle \phi' \rangle = \Phi(\mathbf{x}) + \langle \phi' \rangle$$

implies

$$\langle \phi' \rangle = 0,$$

since the current $\tilde{j}[\Phi] = \tilde{\Gamma}_\Phi$ was chosen to make the field expectation $\langle \phi \rangle$ equal to Φ. This property tremendously simplifies the calculation of $\tilde{W}[\tilde{j}]$ over that of $\tilde{W}[0]$. It is possible to prove that it has the effect of removing from the graphical expansion (4.29) precisely all those graphs which are one-particle reducible.

In order to see this it is useful to consider the partition function (5.30) as a field theory in ϕ' for an *arbitrary* external current $\tilde{j}$. The field expectation of $\langle \phi' \rangle$ is given by

$$\langle \phi'(\mathbf{x}) \rangle = \frac{\delta}{\delta \tilde{j}(\mathbf{x})} \tilde{W}[\tilde{j}].$$

Let us denote this expectation by $\Phi'(\mathbf{x})$. It is now convenient to introduce an auxiliary effective energy $\tilde{\Gamma}[\Phi']$ via the Legendre transform of this system by

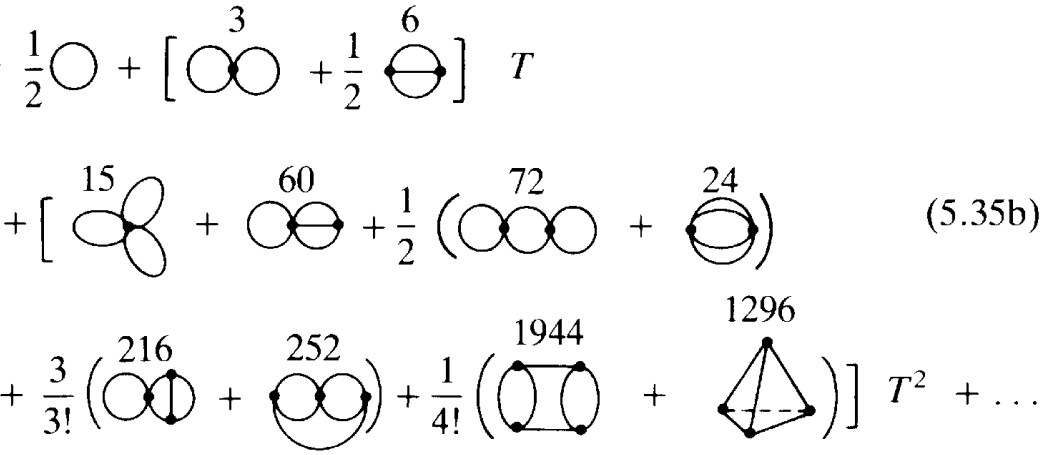
$$\tilde{\Gamma}'[\Phi'] \equiv -\tilde{W}[\tilde{j}] + \Phi' \tilde{j}. \tag{5.34}$$

As discussed before, the current $\tilde{j} = \tilde{j}[\tilde{\Phi}]$ has the property that Φ' vanishes, so that $\tilde{\Gamma}'[0]$ coincides with $-\tilde{W}[\tilde{j}]$. But this is equal to $\tilde{\Gamma}[\Phi]$, by definition [see (5.27)]. Hence the auxiliary effective energy coincides with the fluctuation correction to the proper effective energy if its field argument Φ' is set equal to zero,

$$\tilde{\Gamma}[\Phi] = \tilde{\Gamma}'[0] \tag{5.35a}$$

and this, in turn, is equal to $-\tilde{W}[\tilde{j}[\Phi]]$. It is this identity which leads to the simplification of the graphical expansion of $-\tilde{W}[\tilde{j}]$ at $\tilde{j}=\tilde{j}[\Phi]$. It reduces the proof to a completely general theorem on the effective actions $\Gamma[\Phi]$ of any theory which says that at $\Phi\equiv 0$, $\Gamma[\Phi]$ consists precisely of all one-particle irreducible vacuum graphs of that theory.

Let us, for a moment, accept this theorem. Then, in the present case, the partition function is given by (5.30) with lines and vertices shown in (5.32a) and (5.32b). Taking the expansion (4.29) and dropping all one-particle reducible graphs, we arrive at the following series

$\frac{1}{T}\tilde{\Gamma}[\Phi] =$ 

(5.35b)

This series is referred to as the loop expansion of the effective action. The explicit power of T in each graph of $(1/T)\,\tilde{\Gamma}[\Phi]$ coincides with the number of loops [remember to add the zero loop part $(1/T)\,E[\Phi]$ when going to the full $(1/T)\,\Gamma[\Phi]$, according to (5.24)].

5.4. THE ONE-PARTICLE IRREDUCIBILITY OF $\Gamma[0]$

Let us now convince ourselves that an effective energy at zero field $\Gamma[0]$ is indeed the sum of all one-particle irreducible vacuum diagrams. For this, consider an arbitrary partition function of the same type as $\tilde{Z}[\tilde{j}]$ in (5.30):

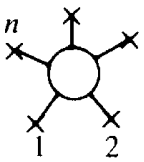
$$e^{(1/T)W[j]} = e^{-1/T(\Gamma[\Phi]-j\Phi)} = \int \mathscr{D}\phi e^{-(1/T)(\phi D\phi + (1/3!)V_3\phi^3 + (1/4!)V_4\phi^4 + \ldots - j\phi)}. \tag{5.36}$$

Notice that there is no linear term in the energy, just as in $\tilde{Z}[\tilde{j}]$. We have to show that if j has a value j_0 with the property $\Phi = \langle\phi\rangle = W_j = 0$

for $j = j_0$, the functional $W[j_0] = -\Gamma[0]$ collects all one-particle irreducible vacuum graphs. Let us calculate all graphs contributing to $(1/T)\,W(j)$ up to order T^2 by using the expansion (5.33) for $j = j_0$.

To zeroth order, these are all the vacuum graphs of $(1/T)\,W[0]$ displayed in Eq. (4.29). To higher orders we have to find the Feynman diagrams in which connected n-point functions are coupled to the external current j_0. If the coupling to $(1/T)\,j_0$ is denoted by a cross, the additional terms can be represented as follows

$$\frac{1}{T}(W[j_0] - W[0]) = \sum_{n=1}^{\infty} \tag{5.37}$$

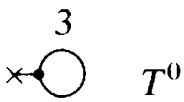


The sum contains a trivial term of order j_0^2, namely, the free field expression

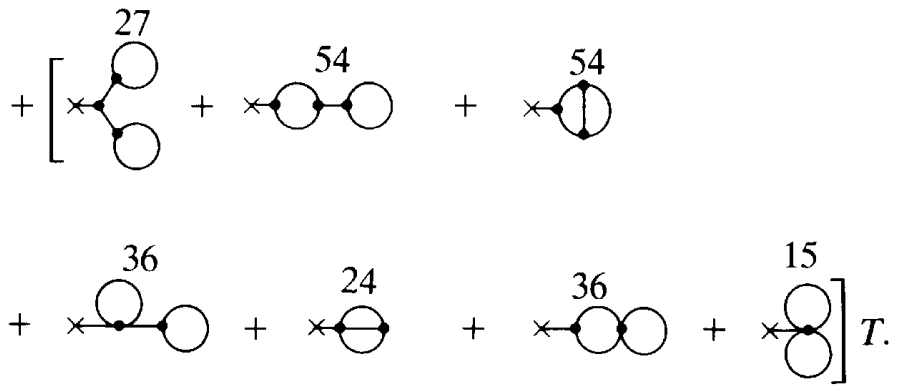
$$\frac{1}{T}W_0[j_0] = \frac{1}{2T}j_0\,\mathrm{D}^{-1}j_0 = \frac{1}{2} \tag{5.38}$$

All other diagrams in (5.37) require careful counting which is done in Appendix 5A. Here we only give the results.

First, there are the terms linear in j_0, which are called *tadpole diagrams* (alluding to the biological shape of the first graph)

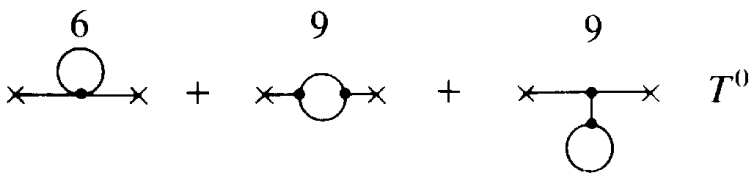


(5.39)



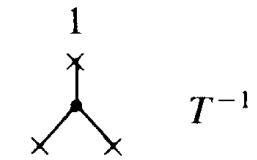
$$+ O(T^2). \tag{5.40}$$

The interaction terms quadratic in j_0 collect all connected two-point functions



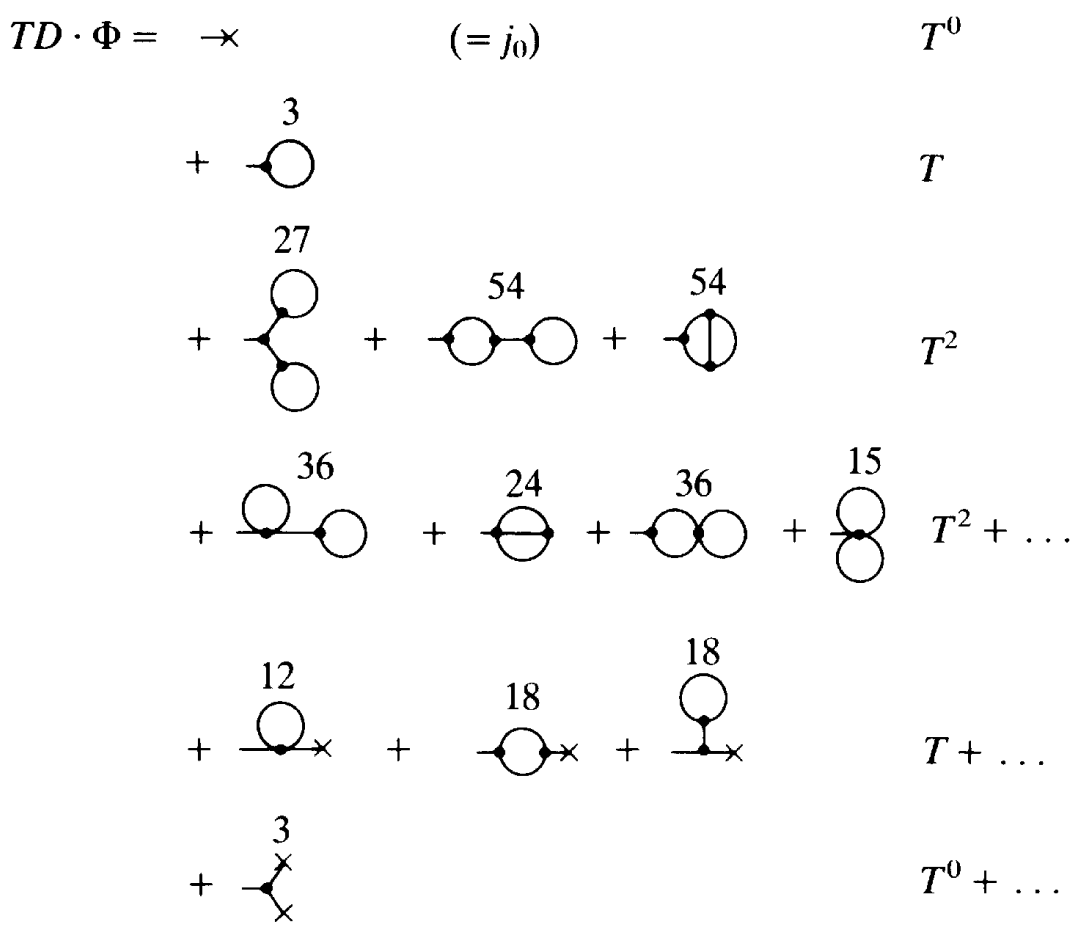
(5.41)

The cubic terms start out with



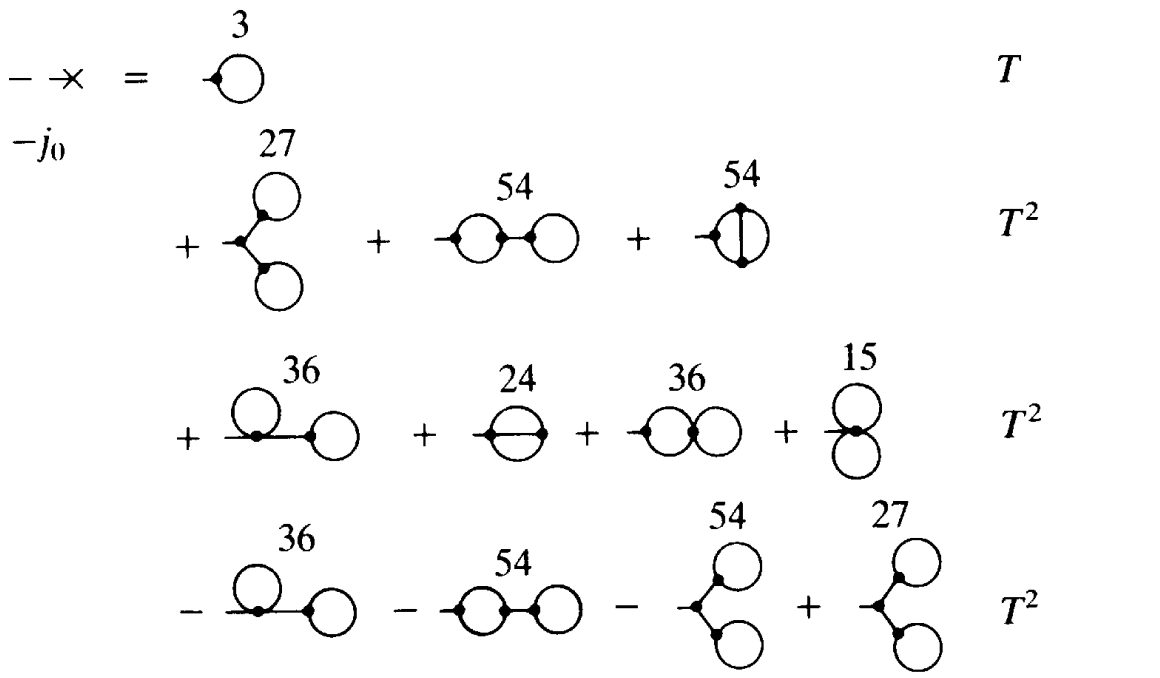
(5.42)

Collecting all these diagrams, we can calculate the field expectation $T\Phi$ in the presence of the source j_0 by forming the derivative $\delta/\delta j$ which amounts to multiplying each diagram by the number it crosses and dropping one of the crosses. If we further multiply the equation by the free Green function D, which amounts to amputating the free log (leaving only a short trunk),



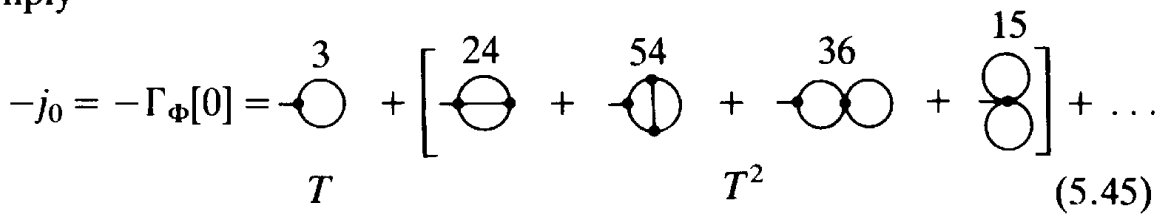
(5.43)

We can now see that if we want to make this field expectation vanish, the current is of order T. Thus we solve the condition $\Phi = 0$ iteratively for j_0 as follows:



$$+O(T^3). \tag{5.44}$$

It is gratifying to note that all one-particle reducible graphs cancel and j_0 is simply



$$+ \ldots \tag{5.45}$$

This shows that the functional derivative of $\Gamma[\Phi]$ at $\Phi = 0$ collects only one-particle irreducible diagrams.

The result may be extended to the function $(1/T)\,\Gamma[0] = (-1/T) \cdot W[j_0]$ itself. All we have to do is insert (5.45) into (5.37) which reads explicitly

$$W[j] = \frac{1}{T}\left(W[0] + W_j j + \frac{1}{2!} W_{jj} j^2 + \frac{1}{3!} W_{jjj} j^3 + \ldots \right)\Bigg|_{j = j_0}. \tag{5.46}$$

The calculation can be simplified by noting that $W_j[j_0] = \Phi = 0$ so that we can subtract from (5.46) the series

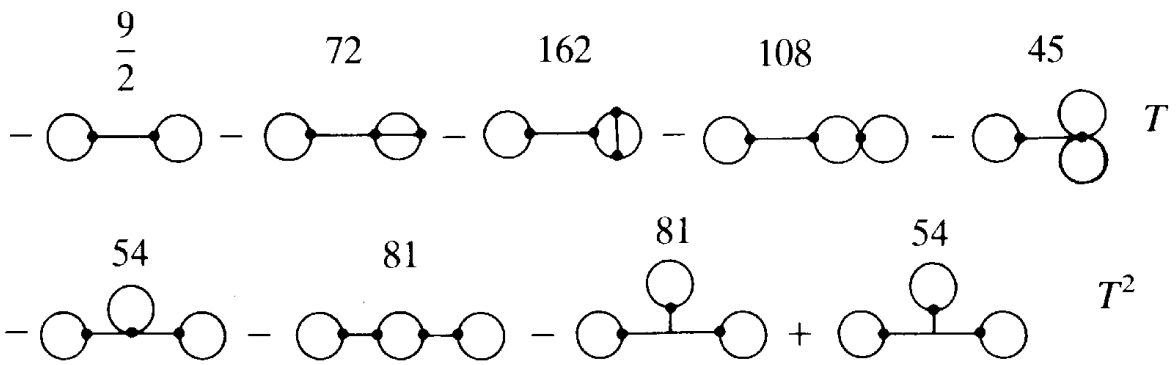
$$0 = W_j j_0 + W_{jj} j_0^2 + \frac{3}{3!} W_{jjj} j_0^3 + \ldots \tag{5.47}$$

leaving the expression

$$\frac{1}{T}W[j_0] = \frac{1}{T}\left(W[0] - \frac{1}{2!}W_{jj}j_0^2 - \frac{2}{3!}W_{jjj}j_0^3 - \frac{3}{4!}W_{jjjj}j_0^4 - \dots\right). \tag{5.48}$$

In this way we arrive at the graphical expansion

$$-\frac{1}{T}\Gamma[0] = \frac{1}{T}W[j_0] = \frac{1}{T}W[0]$$



(5.49)

Comparing with Eq. (4.29) we see that the additional terms have the property of cancelling all one-particle reducible graphs in $(1/T)\,W[0]$. Assuming that this cancellation persists to all orders we have therefore shown that $(1/T)\,W[j_0]$ or, equivalently, the effective energy $(-1/T)\,\Gamma[0]$, is indeed the sum of all one-particle irreducible vacuum graphs.

5.5. PROPER SELF-ENERGY

At this stage it is also easy to demonstrate what we have observed in Chapter 2 for the special case of the ϕ^4 theory, namely, that the full two-point function in a perturbation expansion can be obtained from a geometric sum of one-particle irreducible self-energy graphs. Remembering the decomposition

$$\Gamma[\Phi] = E[\Phi] + \tilde{\Gamma}[\Phi], \tag{5.50}$$

and the definition

$$G_\Phi = TE^{-1}_{\Phi\Phi}, \tag{5.51}$$

we see that the full two-point function is

$$G = T\Gamma^{-1}_{\Phi\Phi} = (G_\Phi^{-1} + T\bar{\Gamma}_{\Phi\Phi})^{-1}$$
$$= G_\Phi - G_\Phi T\bar{\Gamma}_{\Phi\Phi}G_\Phi + G_\Phi T\bar{\Gamma}_{\Phi\Phi}G_\Phi T\bar{\Gamma}_{\Phi\Phi}G_\Phi + \ldots \qquad (5.52)$$

Thus $T\bar{\Gamma}_{\Phi\Phi}$ plays the same role as the proper self-energy graphs $-\Pi$ in Eq. (2.37). From what we have just learned, $\bar{\Gamma}[\Phi]$ contains all one-particle irreducible vacuum graphs involving the two-point correlation function G_Φ and vertices

$$V_n^\Phi = \frac{\delta^n E[\Phi]}{\delta\Phi \ldots \delta\Phi}.$$

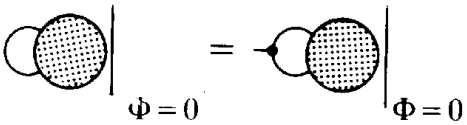
The correlation function contains the Φ field in the form

$$G_\Phi^{-1} = -\partial^2 + E_{\Phi\Phi}[\Phi]$$
$$= -\partial^2 + V_2 + V_3\Phi + \frac{1}{2!}V_4\Phi^2 + \frac{1}{3!}V_5\Phi^3 + \ldots, \qquad (5.53)$$

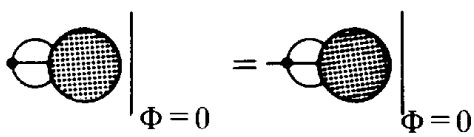
where V_n are the functional derivatives of $E[\Phi]$ at zero field, i.e., $V_n = (\delta^n E[\Phi]/\delta\Phi \ldots \delta\Phi)$. The vertices have an explicit Φ dependence

$$V_3^\Phi = V_3 + V_4\Phi + \frac{1}{2!}V_5\Phi^2 + \ldots,$$
$$V_4^\Phi = V_4 + V_5\Phi + \frac{1}{2!}V_6\Phi^2 + \ldots \qquad (5.54)$$

Now differentiating G_Φ with respect to Φ and setting $\Phi = 0$ has the effect of inserting a three-point vertex into a line of a vacuum diagram

$$\frac{\delta}{\delta\Phi}$$  $$= $$ $$\Big|_{\Phi=0}$$ (5.55)

with all the diagrams on the right-hand side constructed from G_0 and $V_n(x_1, \ldots, x_n)$. Differentiating a vertex V_n^Φ and setting $\Phi = 0$ replaces V_n^Φ by V_{n+1} with one line emerging from it. For example,

$$\frac{\delta}{\delta\Phi}$$  $$= $$ $$\Big|_{\Phi=0}$$ (5.56)

It is obvious that, with $\Gamma[\Phi]$ containing only one-particle irreducible vacuum diagrams involving G_Φ, V_n^Φ, the result of this operation renders again one-particle irreducible diagrams involving G_0, V_n. But then, performing two derivatives with respect to Φ we see that $T\bar{\Gamma}_{\Phi\Phi}[0]$ in the geometric series (5.52) at $\Phi = 0$ containly coincides with the sum of all one-particle irreducible amputated two-point functions, i.e., with the proper self-energy graphs $-\Pi$. Thus we have proved that the repetition of proper self-energy graphs in the two point-function is indeed governed by the simple geometric series (2.33).

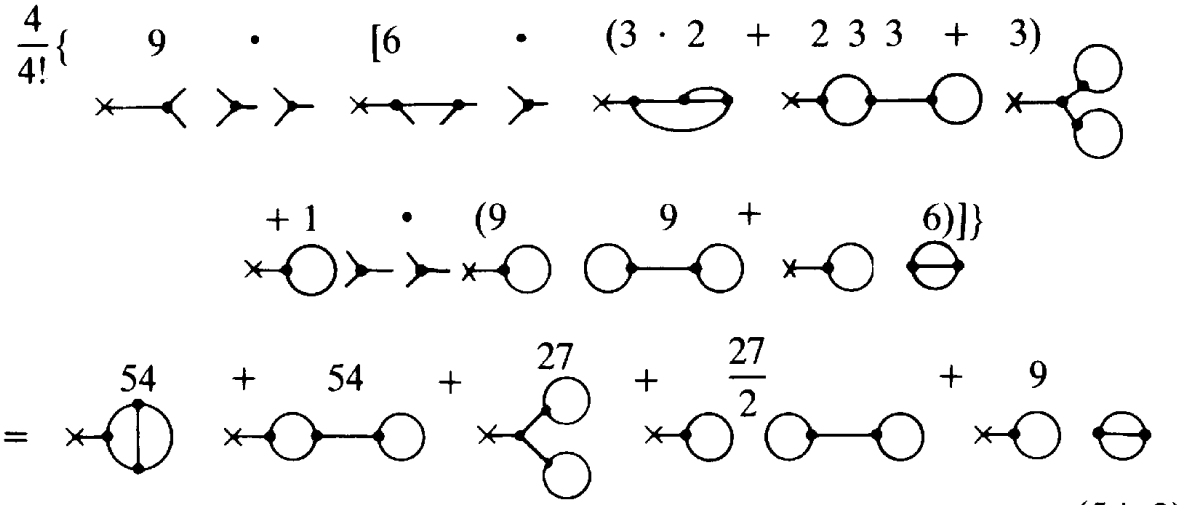
APPENDIX 5A. GRAPHS CONTRIBUTING TO $(1/T)\,W[j]$

1. Terms linear in j

(a) There is the correlation function

$$\frac{1}{4!}4\left(\frac{j}{T}\right)\left(-\frac{V_3}{3!T}\right)^3\langle\phi\phi^3\phi^3\phi^3\rangle. \tag{5A.1}$$

This gives the $9!! = 945$ Wick contractions[b]



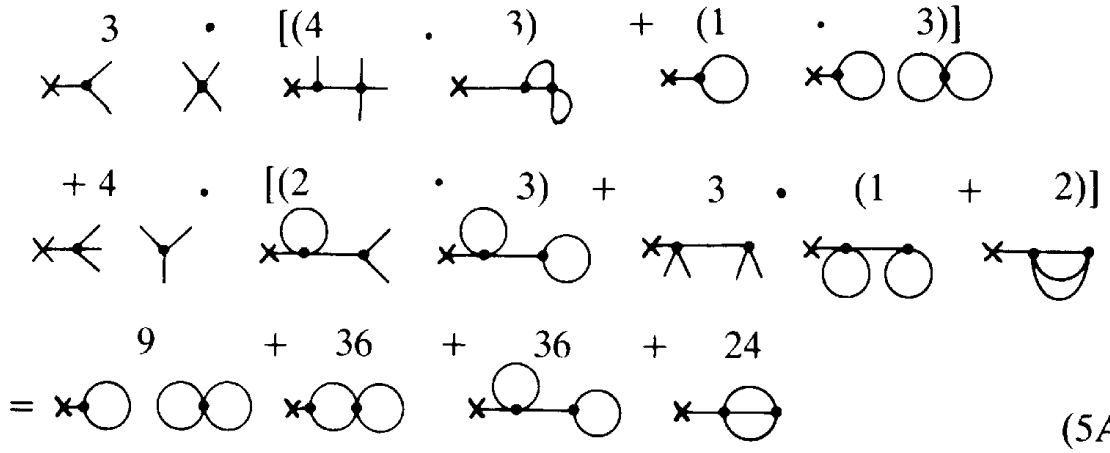
(5A.2)

(b) There is the correlation function

$$\frac{1}{3!}3!\left(\frac{j}{T}\right)\left(-\frac{V_3}{3!T}\right)\left(-\frac{V_4}{4!T}\right)\langle\phi\phi^3\phi^4\rangle. \tag{5A.3}$$

[b]For our graphical notations, see Appendix 4A.

This gives the 7!! = 105 Wick contractions



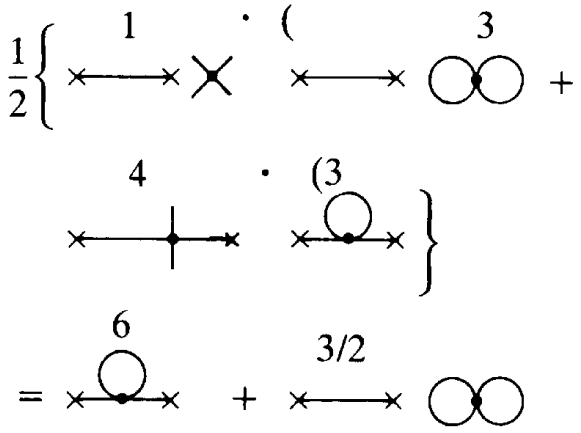
(5A.4)

2. *Terms quadratic in j*

(a) The correlation function

$$\frac{1}{3!}3\left(\frac{j}{T}\right)^2\left(-\frac{V_4}{4!T}\right)\langle\phi\phi\phi^4\rangle \tag{5A.5}$$

has 5!! = 15 Wick contractions

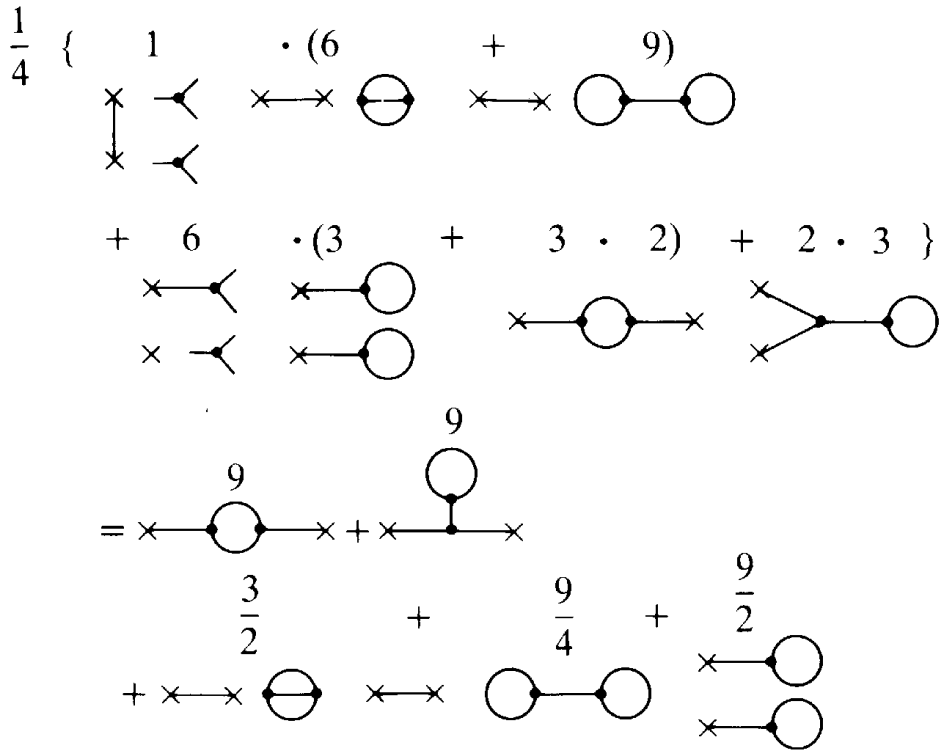


(5A.6)

(b) The correlation function

$$\frac{1}{4!}\frac{4!}{2!2!}\left(\frac{j}{T}\right)^2\left(-\frac{V_3}{3!T}\right)^2\langle\phi\phi\phi^3\phi^3\rangle \tag{5A.7}$$

has 7!! = 105 Wick contractions which are



(5A.8)

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The effective action was first studied by
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P. Martin and C. de Dominicis, *J. Math. Phys.* **5** (1964) 14, 31,
and later by
G. Jona-Lasinio, *Nuovo Cimento* **34** (1964) 1790,
H. Dahmen, G. Jona-Lasinio, *Nuovo Cimento* **52A** (1967) 807,
A.N. Vasileo and A.K. Kazanskii, *Teor. Math. Phys.* **12** (1972) 875,
J.M. Cornwall, R. Jackiw, and E. Tomboulis, *Phys. Rev.* **10** (1974) 2428,
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CHAPTER SIX

FEYNMAN DIAGRAMS AND RANDOM WALKS

6.1. GENERAL REMARKS

The expansion of the effective action into loop diagrams has a rather direct physical meaning. It is a reflection of the quantum mechanical duality of particles and fields. Duality implies that the Schrödinger theory for an arbitrary number N of particles can be reformulated as a second quantized field theory. Both formulations describe one and the same physics. They differ only in the physical quantities to be treated as canonical quantum mechanical variables. In the N-body Schrödinger equation, these are the N particle orbits $x_i(t)$, $i = 1, 2, \ldots, N$ and the canonical momenta $p_i(t)$. Field theory replaces these by a single complex field $\psi(\mathbf{x}, t)$. This single field can describe *any* number of particle orbits in one stroke. This is possible since a field corresponds to an *infinite* number of generalized coordinates. If we decompose it into real and imaginary parts, $\psi(\mathbf{x}, t) = \psi_1(\mathbf{x}, t) + i\psi_2(\mathbf{x}, t)$, the real part may be viewed as a set of Lagrange coordinates $q_x(t)$, one for every space point $\mathbf{x}$. These can generate an integer number of excitations, i.e., "quanta" of the harmonic oscillator type. The quantum number N has the physical interpretation of counting the number of particles at each space point. An N-particle state is obtained by applying the creation operator, $\psi^\dagger(\mathbf{x}, t)$, N times to the ground state $|0\rangle$ which contains no quanta, i.e., to the vacuum state.

The proof of the equivalence is straightforward and can be found in any elementary textbook on quantum field theory.[a] What is usually not so transparent in these standard proofs and what will play an important role in the development to come, is the simple physical way in which the quantized field is capable of accounting for the particle orbits. The crucial observations are contained in Feynman's original work on quantum theory, where particle orbits play a central role.[b] From the beginning, Feynman had always viewed field propagators as a summary of the fluctuation history of particle orbits. In his view, the diagrammatic rules for constructing the amplitudes of perturbation theory in configuration space are direct pictures of what actually happens to individual particles when running along their fluctuating orbits.

From our discussion in Section 1.7 we know that all quantum mechanical properties have a statistical analogue. Thus, the equivalence of quantized N particle orbits and a simple quantum field must exist also in statistical physics. All we have to do is continue the time to imaginary values $t = -i\tau$ and this automatically changes quantum fluctuations of particle orbits into thermal fluctuations. As we shall soon see in detail, the thermal analogue of a free quantum fluctuating particle orbit is a random walk. Therefore, as a thermal analogue of the duality in quantum physics, fluctuating fields in a thermal partition function are equivalent to a grand canonical ensemble of particles performing random walks in a thermal environment.

It turns out that such ensembles are not precisely the systems which we want to study. We shall mostly be interested in ensembles of defect lines. These will be considered, to a certain approximation, as *random chains* rather than random walks. A random chain consists of a number of links of equal length, say a, which can freely rotate around the joints. Let us label the joints successively by the integer numbers $n = 0, 1, 2, 3, \ldots, N + 1$. If the chain is fully extended, the n-th joint lies a distance $s = na$ away from the zeroth link. The quantity s is called the length parameter of the chain. Let the position of the joints be given by a function $\mathbf{x}(s)$ (see Fig. 6.1).

It is now easy to see that random chains and particle orbits are very similar. Orbits are characterized by functions $\mathbf{x}(t)$. By interpreting s as a "pseudotime," a chain may be viewed as the orbit of a single particle as a

[a]See, for instance, Schweber's book quoted in the references of Chapter 1.

[b]See Feynman's books quoted in the references of Chapter 1. The original paper was published in *Phys. Rev.* **76** (1949) 749.

function of the pseudotime s. Correspondingly, an ensemble of N chains looks the same as a long exposure photograph tracing the orbits of a gas of N mass points (see Fig. 6.2). It is this analogy which will make it possible to describe the ensemble of defect lines by the same type of field theory as the ensemble of particles in Brownian motion.

FIG. 6.1. A random chain looks like the orbit of a free fluctuating particle in quantum theory traced out as a function of time.

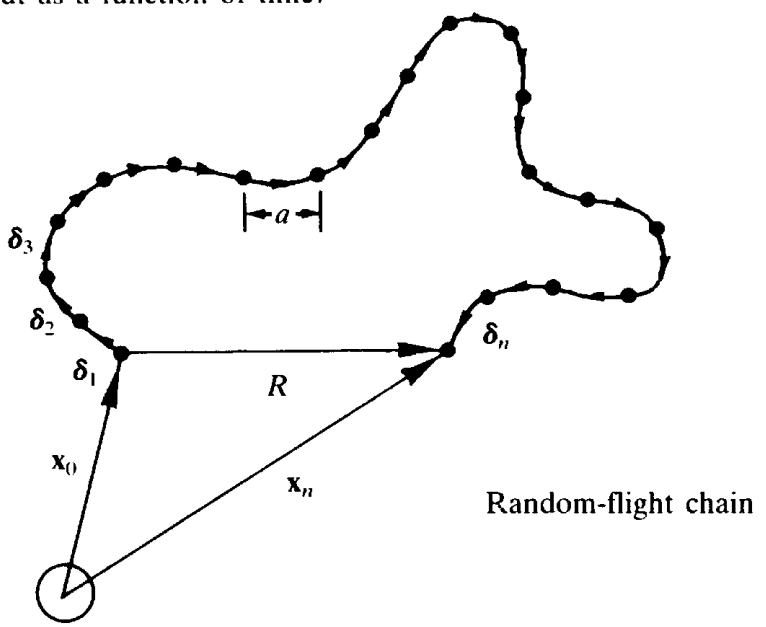
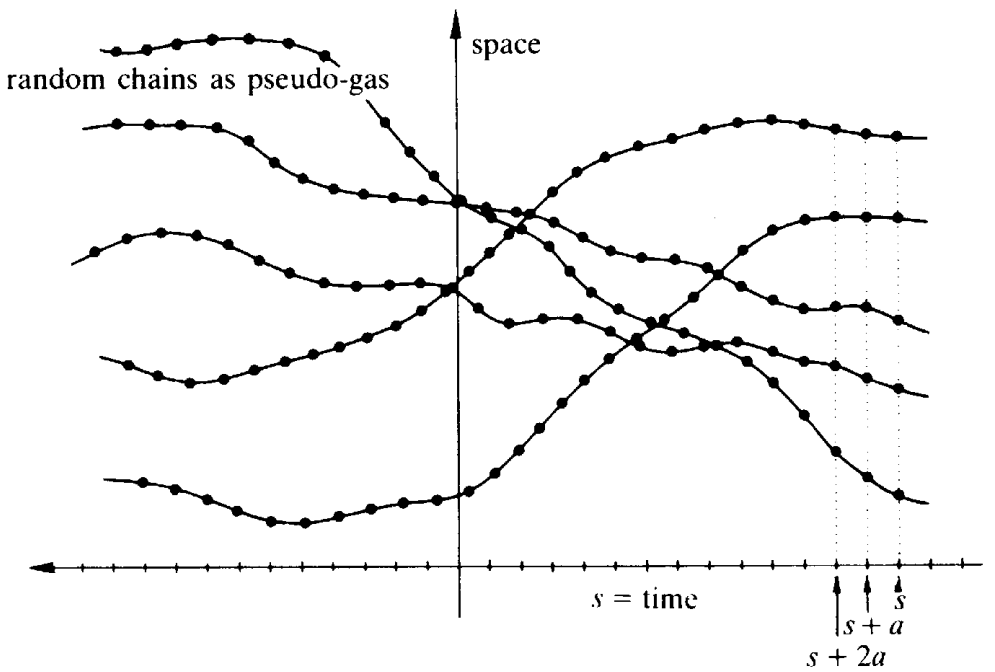


FIG. 6.2. If the positions of atoms in an ensemble of random chains are plotted against the length parameter $s = na$, where a is the length of the chain links, the resulting picture is indistinguishable from a plot of the positions of a gas of free particles as a function of time.



There is one place, however, where the close analogy between random walks and random chains is not perfect. The two systems have quite different properties when it comes to interactions. Consider, for example, a chain where two joints with numbers n and n' have a crossover point. At that point they usually have a repulsive short-range interaction. This may be so strong that crossovers are ruled out completely ("excluded volume effect in polymers"). The repulsive interaction is *independent* of the link numbers n and n' and thus on the distance $s - s'$ of the two links along the chain. When interpreted as pseudotime of a random walk, this would mean that a particle can never get to a point where it had been before.[c] Such non-instantaneous interactions are quite uncommon. Usually, a particle performing a random walk has no memory where it was at earlier times. The interaction is nonzero only for zero time distance $t - t'$ along the orbit.

Fortunately, there is no problem in incorporating this difference into a field theory of random chains when constructing it in analogy with random walks. Not only is there no problem, the s independence of the chain interactions turns out to be even a virtue. It can be exploited to simplify the field theory by eliminating the s variable altogether and treating immediately ensembles of chains of *arbitrary* length with some distribution $e^{-\text{const.}\cdot s}$. Such ensembles may be described by a field theory which does not contain any pseudotime variable s. Its fields depend only on the space variable $\mathbf{x}$.

In Chapter 1 we saw that field fluctuations in continuous space can be properly studied only by decomposing the space into a fine lattice of particle separation a. A lattice will also be the proper scenario for understanding the correspondence between fields and random walks or chains.

6.2. SINGLE RANDOM WALK

We shall work in a general D-dimensional space and specialize to $D = 1$ to 4 in the final formulas. For simplicity, let us consider only hypercubic lattices where the atoms are situated at positions

$$\mathbf{x}_\mathbf{n} = (n_1, n_2, n_3, \ldots, n_D)a = \mathbf{n}a, \tag{6.1}$$

[c] An everyday example of a random walk which tries to avoid places visited earlier is that of a traveling salesman.

with integer numbers $n_i = 0, \pm 1, \pm 2, \ldots$ The subscript $\mathbf{n}$ of $\mathbf{x}_\mathbf{n}$ will be dropped, for brevity, and we shall write summation symbols $\Sigma_\mathbf{x}$ instead of $\Sigma_\mathbf{n}$.

A useful concept for the discussion to come is that of *oriented link vectors* of the lattice, which are the D vectors pointing from $\mathbf{x}$ to the nearest neighbors, i.e.,

$$\begin{aligned}
\mathbf{x} + \mathbf{i} &= (n_1 + 1, n_2, n_3, \ldots, n_D)a \quad i = 1, \\
&= (n_1, n_2 + 1, n_3, \ldots, n_D)a \quad i = 2, \\
&\vdots \\
&= (n_1, n_2, n_3, \ldots, n_D + 1)a \quad i = D,
\end{aligned}$$

along the *positive* directions of the D coordinate axes. Each point $\mathbf{x}$ has $2D$ nearest neighbors situated at $\mathbf{x} \pm \mathbf{i}$ where $\mathbf{i}$ are the D unit vectors[d]

$$\begin{aligned}
\mathbf{i} &= (1, 0, 0, \ldots, 0)a \quad i = 1, \\
&= (0, 1, 0, \ldots, 0)a \quad i = 2, \\
&\vdots \\
&= (0, 0, 0, \ldots, 1)a \quad i = D.
\end{aligned} \tag{6.2}$$

Consider now a random walk beginning at a certain point $\mathbf{y}$. In order to study its statistical behavior we introduce the quantity $P(\mathbf{x}, \mathbf{y}, \mathbf{n})$. It measures the probability for the random walk to arrive at another point $\mathbf{x}$ after n steps. As long as there are no interactions, this probability can be calculated from a simple recursion formula. First of all, the probability of remaining at $\mathbf{y}$ after no step is trivially equal to unity

$$P(\mathbf{y}, \mathbf{y}, 0) = 1. \tag{6.3}$$

Suppose the particle executing random walk has arrived at a certain place $\mathbf{x}$. Then it is obvious that it has had $2D$ possibilities, all of equal probability $1/2D$, of arriving at this place from any of the $2D$ neighboring sites, jumping along the links $\pm\mathbf{i}$. At each of these it had previously arrived with probabilities $P(\mathbf{x} \pm \mathbf{i}, \mathbf{y}, n - 1)$. Therefore, we find the recursion formula

[d]When written as vectors $\mathbf{i}$, the links look the same as the basis vectors $\mathbf{a}_i$ of the lattice. They are, however, different objects, conceptually, since they are distinguishable according to the place $\mathbf{x}$ where they are attached. To be precise, one should really write $\mathbf{i}(\mathbf{x})$.

$$P(\mathbf{x}, \mathbf{y}, n) = \frac{1}{2D} \sum_{\pm \mathbf{i}} P(\mathbf{x} \pm \mathbf{i}, \mathbf{y}, n-1), \quad n \geq 1, \tag{6.4}$$

or, after subtracting $P(\mathbf{x}, \mathbf{y}, n-1)$ on both sides,

$$P(\mathbf{x}, \mathbf{y}, n) - P(\mathbf{x}, \mathbf{y}, n-1)$$

$$= \frac{1}{2D} \sum_{\pm \mathbf{i}} (P(\mathbf{x} + \mathbf{i}, \mathbf{y}, n-1) - P(\mathbf{x}, \mathbf{y}, n-1)), \quad n \geq 1. \tag{6.5}$$

This can be cast in a more convenient form by introducing a symbol which is the lattice equivalent of a derivative. Let $f(\mathbf{x})$ be an arbitrary function on the lattice. The lattice derivative is then defined as

$$\nabla_{\mathbf{i}} f(\mathbf{x}) = \frac{1}{a}(f(\mathbf{x} + \mathbf{i}) - f(\mathbf{x})). \tag{6.6}$$

In the limit of zero lattice spacing this obviously reduces to the ordinary derivative

$$\lim_{a \to 0} \nabla_{\mathbf{i}} f(\mathbf{x}) = \mathbf{i} \cdot \partial f(\mathbf{x}) = \partial_i f(\mathbf{x}). \tag{6.7}$$

Here, the subscript $\mathbf{i}$ denotes the directions along which the link vector points. By this direct correspondence we shall also write ∇_i instead of $\nabla_{\mathbf{i}}$. It must be realized that, on a lattice, there are other possibilities of defining the equivalent of a derivative. Consider, for example, the object

$$\bar{\nabla}_i f(\mathbf{x}) \equiv -\nabla_{-i} f(\mathbf{x} - \mathbf{i}) = \frac{1}{a}(f(\mathbf{x}) - f(\mathbf{x} - \mathbf{i})). \tag{6.8}$$

Also this reduces to a proper derivative in the limit $a \to 0$. In difference equations on lattices, both operations ∇_i and $\bar{\nabla}_i$ are useful and often appear together.

Difference equations may involve lattice analogues of all higher derivatives. Consider, for example, the second difference of a lattice function

$$\frac{1}{a^2}[(f(\mathbf{x} + \mathbf{i}) - f(\mathbf{x})) - (f(\mathbf{x}) - f(\mathbf{x} - \mathbf{i}))]. \tag{6.9}$$

It can be written as

$$\frac{1}{a}(\nabla_i f(\mathbf{x}) - \nabla_{-i} f(\mathbf{x})), \tag{6.10}$$

or in the form

$$\begin{aligned}\bar{\nabla}_i \nabla_i f(\mathbf{x}) &= \frac{1}{a}(\bar{\nabla}_i f(\mathbf{x}+\mathbf{i}) - \bar{\nabla}_i f(\mathbf{x})) \\ &= \frac{1}{a^2}[(f(\mathbf{x}+\mathbf{i}) - f(\mathbf{x}+\mathbf{i}-\mathbf{i})) - (f(\mathbf{x}) - f(\mathbf{x}-\mathbf{i}))].\end{aligned} \tag{6.11}$$

If summed over all oriented links, this operation is seen to be the lattice analogue of the Laplace operator, to which it reduces in the limit $a \to 0$, i.e.,

$$\lim_{a\to 0} \sum_i \bar{\nabla}_i \nabla_i f(\mathbf{x}) \equiv \lim_{a\to 0} \bar{\nabla} \cdot \nabla f(\mathbf{x}) = \partial^2 f(\mathbf{x}). \tag{6.12}$$

In the following, it will be useful to label the number of steps not by the integers n but by the length parameter $s = na$ which the random walk has covered. Correspondingly we shall introduce the symbol ∇_s, $\bar{\nabla}_s$ to denote lattice differentiation in this variable:

$$\begin{aligned}\nabla_s P(\mathbf{x}, \mathbf{y}, s) &= \frac{1}{a}(P(\mathbf{x}, \mathbf{y}, s+a) - P(\mathbf{x}, \mathbf{y}, s)) = \bar{\nabla}_s P(\mathbf{x}, \mathbf{y}, s+a) \\ \bar{\nabla}_s P(\mathbf{x}, \mathbf{y}, s) &= \frac{1}{a}(P(\mathbf{x}, \mathbf{y}, s) - P(\mathbf{x}, \mathbf{y}, s-a)) = \nabla_s P(\mathbf{x}, \mathbf{y}, s-a).\end{aligned} \tag{6.13}$$

Then it is possible to rewrite (6.5) in the suggestive form

$$\bar{\nabla}_s P(\mathbf{x}, \mathbf{y}, s) = \frac{a}{2D} \sum_i \bar{\nabla}_i \nabla_i P(\mathbf{x}, \mathbf{y}, s-a), \quad n \geq 1. \tag{6.14}$$

This is a partial difference equation which has to be solved together with the initial condition (6.3).

In order to go about this task it is useful to observe that Eq. (6.4) is the difference form of the well-known diffusion equation which governs Brownian motion, i.e.,

$$\partial_t P(\mathbf{x}, \mathbf{y}, t) = \frac{a^2}{2D\bar{\tau}} \partial^2 P(\mathbf{x}, \mathbf{y}, t), \quad t > 0. \tag{6.15}$$

Here t is the time and $\bar{\tau}$ the mean time between collisions. The factor $a^2/(2D\bar{\tau})$ is the standard diffusion constant. Equation (6.15) has to be solved with the initial condition

$$P(\mathbf{x}, \mathbf{y}, 0) = \delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{6.16}$$

The close correspondence of (6.14) with (6.15) and of (6.3) with (6.16) is, of course, no accident since Brownian motion is a continuous form of random walk. Moreover, in the light of the equivalence of thermal and quantum fluctuations discussed in Section 1.7, the substitution $t \to it$, brings (6.15) to the quantum mechanical propagation law

$$-i\partial_t G(\mathbf{x}, \mathbf{y}, t) = \frac{1}{2M} \partial^2 G(\mathbf{x}, \mathbf{y}, t), \quad t > 0. \tag{6.17}$$

The quantity $G(\mathbf{x}, \mathbf{y}, t)$ is the Green function of the Schrödinger equation

$$i\partial_t \psi(\mathbf{x}, t) = -\frac{1}{2M} \partial^2 \psi(\mathbf{x}, t) \tag{6.18}$$

and the probability *amplitude* for a particle to run from $\mathbf{y}$ to $\mathbf{x}$ in a time interval t. For $t = 0$ the amplitude is a δ-function

$$G(\mathbf{x}, \mathbf{y}, 0) = \delta^{(D)}(\mathbf{x} - \mathbf{y}), \tag{6.19}$$

since the particle cannot have moved at all. Thus, the probability equation (6.14) may also be viewed as a lattice version of the Schrödinger equation with an effective particle mass

$$M = \frac{D}{a} \tag{6.20}$$

and with time continued to imaginary values $t = -is$. Since we are quite familiar with solving (6.15) and (6.16), we shall make use of the same methods in solving the difference equation (6.14).

Let us recall that (6.17) is solved directly by the Fourier ansatz

$$G(\mathbf{x}, \mathbf{y}, t) = \int \frac{d^D k}{(2\pi)^D} \int \frac{dE}{2\pi} e^{-iEt + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} G(\mathbf{k}, E) \tag{6.21}$$

with

$$G(\mathbf{k}, E) = 2\pi\delta(E - \mathbf{k}^2/2M). \tag{6.22}$$

Integrating over E gives

$$G(\mathbf{x}, \mathbf{y}, t) = \int \frac{d^D k}{(2\pi)^D} e^{-i(k^2/2M)t - i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{6.23}$$

and a quadratic completion in the exponent leads to

$$G(\mathbf{x}, \mathbf{y}, t) = \left(\int \frac{d^D k}{(2\mu)^D} e^{-i(t/2M)(\mathbf{k} - M(\mathbf{x}-\mathbf{y})/t)^2} \right) e^{i(M/2)(\mathbf{x}-\mathbf{y})^2/t}. \tag{6.24}$$

Now the integrals over $\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \ldots \int_{-\infty}^{\infty} dk_D$ can be performed one by one. Shifting the integration variable from k_i to

$$k_i' = k_i - M\frac{x_i - y_i}{t}, \tag{6.25}$$

we can use the Fresnel formula

$$\int_{-\infty}^{\infty} \frac{dk'}{2\pi} e^{-i(a/2)k'^2} = \frac{1}{\sqrt{2\pi i a}} \tag{6.26}$$

and (6.24) yields the well-known result

$$G(\mathbf{x}, \mathbf{y}, t) = \frac{1}{(2\pi i t/M)^{D/2}} e^{i(M/2)(\mathbf{x}-\mathbf{y})^2/t}. \tag{6.27}$$

Let us now adapt this procedure to the lattice and solve the difference equation (6.14). For this we have to calculate Fourier transforms on the lattice. This requires the introduction of a reciprocal lattice basis. It consists of a set of vectors $\mathbf{c}_i$ which are orthogonal to all basis vectors $\mathbf{a}_i$, i.e.

$$\mathbf{c}_i \cdot \mathbf{a}_j = \delta_{ij}. \tag{6.28}$$

In three dimensions

$$\mathbf{c}_i = \frac{\mathbf{a}_j \times \mathbf{a}_k}{\mathbf{a}_i \cdot (\mathbf{a}_j \times \mathbf{a}_k)} \quad i,\ j,\ k, \text{ cyclic}. \tag{6.29}$$

In the present simple cubic case

$$\begin{aligned} \mathbf{c}_1 &= (1, 0, 0, \ldots, 0)\, a^{-1}, \\ \mathbf{c}_2 &= (0, 1, 0, \ldots, 0)\, a^{-1}, \ \ldots, \\ \mathbf{c}_D &= (0, 0, 0, \ldots, 1)\, a^{-1}. \end{aligned} \tag{6.30}$$

We shall suppose that the lattice has the shape of a cube with length L and a total number of N atoms, i.e., $N = L^D$. Then we can Fourier analyze each function on the lattice as

$$f(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{N}} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k}), \tag{6.31}$$

where the N discrete $\mathbf{k}$ vectors are given by

$$k_i = \frac{2\pi m_i}{L}, \quad m_i = 0, \ldots, L-1. \tag{6.32}$$

Alternatively we may shift these vectors by half a period and obtain an equivalent range which is known as the *Brillouin zone*. At the end we always shall take the thermodynamic limit $N \to \infty$.

The inverse of the expansion (6.31) is

$$f(\mathbf{k}) = \sum_{\mathbf{x}} \frac{1}{\sqrt{N}} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}). \tag{6.33}$$

As a mnemonic device, the two Fourier expansions can be thought of as

being generated by the completeness relations of exponentials in **k** and **x** space:

$$\sum_{\mathbf{x}} \frac{1}{N} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = \delta_{\mathbf{k},\mathbf{k}'}, \tag{6.34}$$

$$\sum_{\mathbf{k}} \frac{1}{N} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \delta_{\mathbf{x},\mathbf{x}'}. \tag{6.35}$$

For example,

$$\begin{aligned} f(\mathbf{x}) &= \sum_{\mathbf{x}'} \delta_{\mathbf{x},\mathbf{x}'} f(\mathbf{x}') \\ &= \sum_{\mathbf{k}} \frac{1}{\sqrt{N}} \left(\sum_{\mathbf{x}'} \frac{1}{\sqrt{N}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} f(\mathbf{x}') \right) \\ &= \sum_{\mathbf{k}} \frac{1}{\sqrt{N}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\sum_{\mathbf{x}'} \frac{1}{\sqrt{N}} e^{-i\mathbf{k}\cdot\mathbf{x}'} f(\mathbf{x}') \right) = \sum_{\mathbf{k}} \frac{1}{\sqrt{N}} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k}). \end{aligned} \tag{6.36}$$

In the Fourier decomposition on the lattice, difference operators ∇_i, $\bar{\nabla}_i$ become diagonal just as ordinary derivatives would be in a standard integral:

$$\begin{aligned} \nabla_i f(\mathbf{x}) &\equiv \frac{1}{a}(f(\mathbf{x}+\mathbf{i}) - f(\mathbf{x})) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \frac{1}{a}(e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{i})} - e^{i\mathbf{k}\cdot\mathbf{x}}) f(\mathbf{k}) \\ &= \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \frac{1}{a}(e^{i\mathbf{k}\cdot\mathbf{i}} - 1)\, e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k}), \end{aligned} \tag{6.37}$$

$$\begin{aligned} \bar{\nabla}_i f(\mathbf{x}) &= \frac{1}{a}(f(\mathbf{x}) - f(\mathbf{x}-\mathbf{i})) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \frac{1}{a}(e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{i})}) f(\mathbf{k}) \\ &= \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \frac{1}{a}(1 - e^{-i\mathbf{k}\cdot\mathbf{i}})\, e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k}). \end{aligned} \tag{6.38}$$

It is useful to denote the eigenvalues by

$$\begin{aligned} K_i &\equiv \frac{1}{ai}(e^{i\mathbf{k}\cdot\mathbf{i}} - 1) = \frac{1}{ai}(e^{ik_i a} - 1), \\ \bar{K}_i &\equiv \frac{1}{ai}(1 - e^{-i\mathbf{k}\cdot\mathbf{i}}) = \frac{1}{ai}(1 - e^{-ik_i a}). \end{aligned} \tag{6.39}$$

Then the Fourier expansions on the lattice will look very similar to their continuous limits. In fact, in this limit, both eigenvalues reduce to the usual k_i vectors since

$$\begin{Bmatrix} K_i \\ \bar{K}_i \end{Bmatrix} = \pm\frac{1}{ai}(e^{\pm ik_i a} - 1) \xrightarrow[a\to 0]{} \mathbf{k}\cdot\mathbf{i}/a = k_i. \tag{6.40}$$

It is now possible to solve (6.14) by a Fourier ansatz analogous to (6.23):

$$P(\mathbf{x}, \mathbf{y}, s) = \frac{1}{N}\sum_{\mathbf{k}} (P(\mathbf{k}))^{s/a} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \tag{6.41}$$

Inserting this into (6.14) we find for each $\mathbf{k}$

$$\begin{aligned}\frac{1}{a}(P(\mathbf{k}) - 1) &= -\frac{a}{2D}\sum_i \bar{K}_i K_i = \frac{1}{2Da}\sum_i (e^{ik_i a} - 1)(1 - e^{-ik_i a}) \\ &= \frac{1}{Da}\sum_i (\cos k_i a - 1).\end{aligned}$$

Thus the ansatz (6.41) is a solution of the difference equation (6.14) if we choose

$$P(\mathbf{k}) = 1 - \frac{a^2}{2D}\sum_i \bar{K}_i K_i = \frac{1}{D}\sum_i \cos k_i a. \tag{6.42}$$

In the limit of zero lattice spacing this reduces to

$$P(\mathbf{k}) \xrightarrow[a\to 0]{} 1 - \frac{a^2}{2D}k^2 \sim e^{-(a^2/2D)k^2}, \tag{6.43}$$

so that (6.42) becomes

$$P(\mathbf{x}, \mathbf{y}, s) \to \frac{1}{N}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) - (a/2D)k^2 s}, \tag{6.44}$$

which is the same as the Schrödinger Green function (6.23) for a particle of mass

$$M = D/a \tag{6.45}$$

"moving" along the imaginary time axis $t = -is$ [apart from the lattice normalization which is due to the initial condition on the lattice being (6.3) rather than (6.16) in the continuum].

It should be noted that the total probability of finding a particle anywhere in space after $n = s/a$ steps, $\Sigma_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}, n)$, always equals unity, as it should. Before any step (i.e., for $n = 0$), this is trival due to the initial condition (6.3) which implies

$$\sum_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}, 0) = 1.$$

After $n > 0$ steps this is still fulfilled as we can see from (6.42)

$$\sum_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}, n) = \frac{1}{N} \sum_{\mathbf{k}} (P(\mathbf{k}))^{s/a} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} = (P(\mathbf{0}))^{s/a} = 1.$$

Incidentally, the same statement holds in quantum mechanics for the probability *amplitude* $G(\mathbf{x}, \mathbf{y}, t)$ with the continuous normalization (6.19). Notice that in this case the probability density is given by the absolute square of the Green function, i.e.,

$$|G(\mathbf{x}, \mathbf{y}, t)|^2 = \frac{1}{(2\pi t/M)^D},$$

which is the probability for finding a particle at $\mathbf{x}$ after having been localized at $\mathbf{y}$. Due to the uncertainly relation, this is uniform in space and not normalizable![e]

6.3. MATERIAL RANDOM CHAINS IN THERMAL EQUILIBRIUM

So far we have dealt only with the stochastic aspects of free random walks or chains. Let us now turn to their thermal properties. As explained in the beginning, we shall be interested in physical systems containing vortex lines or defect lines. In thermal equilibrium these can be created and destroyed. The probability that this happens will depend on their energy.

[e]The time derivative of this probability density balances with of a radial particle current flowing away from $\mathbf{y}$.

In a first approximation these objects will be considered as random chains. Let ε be the energy per link. Then a chain with n links carries a Boltzmann factor $e^{-n\varepsilon/T}$. Suppose the system contains a single chain of fluctuating length ns. This chain has, at each time, a total number of $(2D)^n$ possible configurations in space. The endpoints are distributed according to the probability $P(\mathbf{x}, \mathbf{y}, n)$. Hence, in thermal equilibrium, we can calculate the partition function of a single chain of arbitrary length ns with endpoints at $\mathbf{x}$ and $\mathbf{y}$, as follows:

$$P(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{\infty} (2D)^n e^{-(\varepsilon/T)n} P(\mathbf{x}, \mathbf{y}, n) = \sum_{n=0}^{\infty} e^{-(\varepsilon/T - \log 2D)n} P(\mathbf{x}, \mathbf{y}, n). \quad (6.46)$$

The chain configurations contained in this sum are *oriented*, i.e., those chains which run through the same sequence of points in opposite directions, are counted separately. Inserting the Fourier decomposition (6.42), the sum can be performed as follows:

$$\begin{aligned} P(\mathbf{x}, \mathbf{y}) &= \frac{1}{N} \sum_{\mathbf{k}} \sum_{n=0} P(\mathbf{k})^n \mathrm{e}^{-(\varepsilon/T - \log 2D)n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \frac{1}{N} \sum_{\mathbf{k}} \sum_{n=0} (2DP(\mathbf{k}))^n e^{-(\varepsilon/T)n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \frac{1}{N} \sum_{\mathbf{k}} \frac{1}{1 - 2DP(\mathbf{k})\, e^{-\varepsilon/T}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \end{aligned} \quad (6.47)$$

It is useful to rewrite this expression in another way:

$$P(\mathbf{x}, \mathbf{y}) = \frac{1}{a^2} e^{\varepsilon/T} \frac{1}{N} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\frac{1}{a^2}[(e^{\varepsilon/T} - 2D) + 2D(1 - P(\mathbf{k}))]} \quad (6.48a)$$

and introduce the mass parameter

$$m^2 = \frac{1}{a^2}(e^{\varepsilon/T} - 2D). \quad (6.49)$$

Then, remembering (6.42), $P(\mathbf{x}, \mathbf{y})$ takes the form

$$P(\mathbf{x}, \mathbf{y}) = \left(m^2 + \frac{2D}{a^2}\right) \frac{1}{N} \sum_{k} \frac{1}{m^2 + \bar{K}_i K_i} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \quad (6.48b)$$

Applying now the difference operator $m^2 - \bar{\nabla}_i \nabla_i$ and using (6.33)–(6.40), we see that $P(\mathbf{x}, \mathbf{y})$ satisfies a simple difference equation

$$(m^2 - \bar{\nabla}_i \nabla_i) P(\mathbf{x}, \mathbf{y}) = \left(m^2 + \frac{2D}{a^2}\right) \delta_{\mathbf{x},\mathbf{y}}.$$

If we remove the factor $m^2 + 2D/a^2$, the quantity

$$G_0(\mathbf{x} - \mathbf{y}) \equiv G_0(\mathbf{x}, \mathbf{y}) = \frac{1}{m^2 + \frac{2D}{a^2}} P(\mathbf{x}, \mathbf{y}) \tag{6.50}$$

satisfies

$$(m^2 - \bar{\nabla}_i \nabla_i) G_0(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x},\mathbf{y}}. \tag{6.51}$$

Thus $G_0(\mathbf{x}, \mathbf{y})$ is the lattice analogue of the Green function of a scalar field of mass m [compare (2.59)]. It will therefore be referred to as the free massive lattice Green function.

Actually, the defect and vortex lines to be studied later will always form closed loops. Let us, therefore, look at closed random chains. The probability that a random chain returns to its starting point after n steps is

$$P(\mathbf{x}, \mathbf{x}, n) = \frac{1}{N} \sum_{\mathbf{k}} (P(\mathbf{k}))^n. \tag{6.52}$$

The partition function of a single, closed oriented random chain is obtained from this as follows. An arbitrary chain of n links has $(2D)^n$ possible configurations. Among these, a fraction $P(\mathbf{x}, \mathbf{y}, n)/n$ is closed with one link sitting on the point $\mathbf{x}$. The factor $1/n$ accounts for the fact that cyclically permuted chains (i.e., those differing only by which of the n links can run through the point $\mathbf{x}$), are really indistinguishable. Using the same Boltzmann factor as in (6.46) for the open chain we find the thermal canonical partition functions of a single closed oriented random chain of any length to be

$$Z_1 = \sum_{\mathbf{x},n} \frac{1}{n} P(\mathbf{x}, \mathbf{x}, n)\, e^{-(\varepsilon/T - \log 2D)/n}. \tag{6.53}$$

The smallest closed chain has two links and it is physically obvious that only chains with an even link number can contribute to the sum. Formally, this can be seen directly by writing Z_1 in the more explicit form

$$Z_1 = \sum_{\mathbf{k}} \sum_n \frac{1}{n} (P(\mathbf{k}))^n e^{-(\varepsilon/T - \log 2D)n} = \sum_{\mathbf{k}} \sum_n \frac{1}{n} (2DP(\mathbf{k}))^n e^{-(\varepsilon/T)n}.$$

The probability $P(\mathbf{k})$ of (6.42) consists of D terms $\cos k_i a$, each of which covers the interval $(-1, 1)$ in a completely symmetric manner. To every value of k_i with $\cos k_i a > 0$ there is a value $k_i = \pi/a - k_i$ with $\cos k_i a < 0$. Therefore, with respect to sums over k_i, $P(\mathbf{k})$ behaves just like an odd function in $\mathbf{k}$ space and only even powers of n can survive.

Notice that $P^2(\mathbf{k})$ is a quantity between 0 and 1 and reaches 1 only at the origin in $\mathbf{k}$ space. Thus, in order that the sum converge, it is necessary and sufficient that

$$\frac{\varepsilon}{T} - \log 2D > 0.$$

The sum can easily be performed explicitly if we recall the well-known formula

$$-\log(1 - x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n, \quad |x| < 1.$$

Hence, Z_1 can be written as

$$Z_1 = -\sum_{\mathbf{k}} \log(1 - P(\mathbf{k}) e^{-(\varepsilon/T - \log 2D)}) = -\sum_{\mathbf{k}} \log(1 - 2DP(\mathbf{k}) e^{-\varepsilon/T}). \quad (6.54)$$

Alternatively, we can incorporate the fact that only even powers of $P(\mathbf{k})$ contribute and use the formula

$$-\frac{1}{2} \log(1 - x^2) = \sum \frac{1}{2n} x^{2n}$$

to rewrite Z_1 as

$$Z_1 = -\frac{1}{2} \sum_{\mathbf{k}} \log(1 - (2DP(\mathbf{k}))^2 e^{-2(\varepsilon/T)}).$$

6.4. GRAND CANONICAL ENSEMBLE OF CLOSED ORIENTED RANDOM CHAINS AND DISORDER FIELDS

From the previous results it is straightforward to go to a grand canonical ensemble of any number of closed random chains by a simple exponentiation:

$$Z = e^{Z_1}. \tag{6.55}$$

The expansion

$$Z = 1 + Z_1 + \frac{1}{2!}Z_1^2 + \frac{1}{3!}Z_1^3 + \dots$$

displays the no loop, one loop, two loops, etc. content in this sum. Using (6.54), the right-hand side in (6.55) can also be written explicitly as

$$Z = \prod_{\mathbf{k}} G_0(\mathbf{k}), \tag{6.56}$$

where

$$\begin{aligned} G_0^{-1}(\mathbf{k}) &= 1 - e^{-\varepsilon/T}2DP(\mathbf{k}) \\ &= \frac{1}{m^2 + 2D/a^2}\left(m^2 + 2D/a^2 - \frac{2D}{a^2}\left(1 - \frac{a^2}{2D}\bar{\mathbf{K}}\cdot\mathbf{K}\right)\right) \\ &= \frac{1}{m^2 + 2D/a^2}(m^2 + \bar{\mathbf{K}}\cdot\mathbf{K}). \end{aligned}$$

The result (6.56) forms the basis for a disorder field theory of fluctuating random loops. This field is introduced in the following, obvious way: Each factor in the infinite product (6.56) can be generated by a Gaussian integral over two real variables

$$G_0(\mathbf{k}) = \int \frac{d\varphi_1(\mathbf{k})}{\sqrt{2\pi}} \frac{d\varphi_2(\mathbf{k})}{\sqrt{2\pi}} e^{-1/2(\varphi_1(\mathbf{k})G_0^{-1}(\mathbf{k})\varphi_1(\mathbf{k}) + \varphi_2(\mathbf{k})G_0^{-1}(\mathbf{k})\varphi_2(\mathbf{k}))}.$$

Thus the partition function (6.56) becomes the product of integrals:

$$Z = \prod_{\mathbf{k}} \left[\int \frac{d\varphi_1(\mathbf{k})}{\sqrt{2\pi}} \frac{d\varphi_2(\mathbf{k})}{\sqrt{2\pi}} \right] e^{-1/2\Sigma_{\mathbf{k}}(\varphi_1(\mathbf{k})G_0^{-1}(\mathbf{k})\varphi_1(\mathbf{k}) + \varphi_2(\mathbf{k})G_0^{-1}(\mathbf{k})\varphi_2(\mathbf{k}))}. \tag{6.57}$$

Such a product of integrals can be viewed as the Fourier transformed path integral of two real fields

$$Z = \int \mathscr{D}\varphi_1(\mathbf{x})\,\mathscr{D}\varphi_2(\mathbf{x})\, e^{-1/2\Sigma_{\mathbf{x}}[\varphi_1(\mathbf{x})G_0^{-1}(\partial/i)\varphi_1(\mathbf{x}) + (1\leftrightarrow 2)]}, \tag{6.58}$$

where the measure of integration is defined as

$$\int \mathscr{D}\varphi_{1,2}(\mathbf{x}) = \prod_{\mathbf{x}} \int_{-\infty}^{\infty} \frac{d\varphi_{1,2}(\mathbf{x})}{\sqrt{2\pi}}.$$

The equality of (6.57) and (6.58) follows from expanding the complex field $\varphi(\mathbf{x}) \equiv \varphi_1(\mathbf{x}) + i\varphi_2(\mathbf{x})$ into a Fourier series

$$\varphi(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{N}} e^{i\mathbf{k}\cdot\mathbf{x}} \varphi(\mathbf{k})$$

and observing that the real and imaginary parts of $\varphi(\mathbf{k}) = \varphi_1(\mathbf{k}) + i\varphi_2(\mathbf{k})$ are related to the fields $\varphi_1(\mathbf{x})$, $\varphi_2(\mathbf{x})$ by a transformation

$$\begin{pmatrix} \varphi_1(\mathbf{x}) \\ \varphi_2(\mathbf{x}) \end{pmatrix} = \sum_{\mathbf{k}} M_{\mathbf{x},\mathbf{k}} \begin{pmatrix} \varphi_1(\mathbf{k}) \\ \varphi_2(\mathbf{k}) \end{pmatrix},$$

with a matrix

$$M_{\mathbf{x},\mathbf{k}} = \frac{1}{\sqrt{N}} \begin{pmatrix} \cos \mathbf{k}\cdot\mathbf{x} & -\sin \mathbf{k}\cdot\mathbf{x} \\ \sin \mathbf{k}\cdot\mathbf{x} & \cos \mathbf{k}\cdot\mathbf{x} \end{pmatrix}.$$

This matrix is obviously orthogonal (due to (6.35), (6.36)):

$$\sum_{\mathbf{k}} M_{\mathbf{x},\mathbf{k}} M_{\mathbf{x}',\mathbf{k}} = \frac{1}{N} \sum_{\mathbf{k}}$$

$$\times \begin{pmatrix} \cos \mathbf{k}\cdot\mathbf{x} \cos \mathbf{k}\cdot\mathbf{x}' + \sin \mathbf{k}\cdot\mathbf{x} \sin \mathbf{k}\cdot\mathbf{x}' & \cos \mathbf{k}\cdot\mathbf{x} \sin \mathbf{k}\cdot\mathbf{x}' - \sin \mathbf{k}\cdot\mathbf{x} \cos \mathbf{k}\cdot\mathbf{x}' \\ \sin \mathbf{k}\cdot\mathbf{x} \cos \mathbf{k}\cdot\mathbf{x}' - \cos \mathbf{k}\cdot\mathbf{x} \sin \mathbf{k}\cdot\mathbf{x}' & \cos \mathbf{k}\cdot\mathbf{x} \cos \mathbf{k}\cdot\mathbf{x}' + \sin \mathbf{k}\cdot\mathbf{x} \sin \mathbf{k}\cdot\mathbf{x}' \end{pmatrix}$$

$$= \frac{1}{N} \sum_{\mathbf{k}} \begin{pmatrix} \cos \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') & -\sin \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') \\ \sin \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') & \cos \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta_{\mathbf{x},\mathbf{x}'}.$$

Hence $\det M = 1$ and the product of the $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$ integrals is equal to the product of $\varphi_1(\mathbf{k})$ and $\varphi_2(\mathbf{k})$ integrals:

$$\prod_{\mathbf{x}} \int \frac{d\varphi_1(\mathbf{x})}{\sqrt{2\pi}} \frac{d\varphi_2(\mathbf{x})}{\sqrt{2\pi}} = \prod_{\mathbf{k}} \int \frac{d\varphi_1(\mathbf{k})}{\sqrt{2\pi}} \frac{d\varphi_2(\mathbf{k})}{\sqrt{2\pi}}.$$

The exponents of (6.57) and (6.58) are trivially equal:

$$\sum_{\mathbf{k}} (\varphi_1(\mathbf{k}) G_0^{-1}(\mathbf{k}) \varphi_1(\mathbf{k}) + \varphi_2(\mathbf{k}) G_0^{-1}(\mathbf{k}) \varphi_2(\mathbf{k}))$$

$$= \sum_{\mathbf{x}} \left(\varphi_1(\mathbf{x}) G_0^{-1}(-i\partial) \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) G_0^{-1}(-i\partial) \varphi_2(\mathbf{x}) \right).$$

In the following we shall always use the complex field notation such that the partition function of all closed oriented random chains of all lengths is given by the free complex disorder field theory

$$Z = \int \mathscr{D}\varphi(\mathbf{x}) \mathscr{D}\varphi^{\dagger}(\mathbf{x})\, e^{-1/2 \Sigma_{\mathbf{x}} \varphi^{\dagger}(\mathbf{x}) G_0^{-1}(\partial/i) \varphi(\mathbf{x})}. \tag{6.59}$$

This is the central result of this section. The sum over all fluctuating chains has been turned into a sum over all fluctuating field configurations of a single complex field. The energy which governs the field fluctuations is

$$\omega^2(\mathbf{k}) = T G_0^{-1}(\mathbf{k}), \tag{6.60}$$

for each eigenmode of momentum $\mathbf{k}$. Under the condition for which we performed the sum over n, $\varepsilon/T > \log 2D$, the quantity $G_0^{-1}(\mathbf{k})$ is always positive and the ϕ fluctuations are stable. The largest fluctuations are those with $\mathbf{k} = 0$ which have energy

$$\omega^2(\mathbf{0}) = T G_0^{-1}(\mathbf{0}) = T(1 - 2De^{-\varepsilon/T}). \tag{6.61}$$

6.5. CRITICAL REGION

From formula (6.61) we see that the field fluctuations are always stable in a sufficiently cold environment. It is useful to calculate the temperature, at which $\varepsilon/T - \log 2D$ and hence $\omega^2(\mathbf{0})$ changes sign; we shall call it the *critical temperature* of random chains

$$T_c = \frac{\varepsilon}{\log 2D}. \tag{6.62}$$

Obviously, at the critical temperature, the long range fluctuations of the field destabilize and the partition function ceases to exist. The temperature region close to the critical temperature is called the *critical region*. In this region it is possible to simplify the partition function somewhat by neglecting all higher $\mathbf{k}$ modes which are not in danger of destabilizing. For these modes we can approximate

$$G_0^{-1}(\mathbf{k}) \sim G_0^{-1}(\mathbf{k})_{\mathrm{cr}} \frac{a^2}{2D}\mathbf{k}^2 + \frac{\varepsilon}{T} - \log 2D.$$

Thus, in the critical region, we can work with a simple partition function

$$Z_{\mathrm{cr}} = \int \mathscr{D}\varphi(\mathbf{k})\mathscr{D}\varphi^{\dagger}(\mathbf{k})\, e^{-(1/2)\Sigma_{\mathbf{k}}\varphi^{\dagger}(\mathbf{k})((a^2/2D)\mathbf{k}^2 + \varepsilon/T - \log 2D)\varphi(\mathbf{k})} \approx Z, \qquad (6.63)$$

where the momenta $\mathbf{k}$ are understood to cover only a small sphere in momentum space, centered around the origin of the Brillouin zone.

In $\mathbf{x}$ space, this becomes

$$Z_{\mathrm{cr}} = \int \mathscr{D}\varphi(\mathbf{x})\, \mathscr{D}\varphi^{\dagger}(\mathbf{x})\, e^{-(1/2)\Sigma_{\mathbf{x}}\varphi^{\dagger}(\mathbf{x})(-(a^2/2D)\partial^2 + \varepsilon/T - \log 2D)\varphi(\mathbf{x})}. \qquad (6.64)$$

Since we only consider small momenta in (6.63), the fields $\varphi(\mathbf{x})$ are restricted to smooth configurations. As a consequence, the lattice structure of space in the sum becomes irrelevant. We can, therefore, go over to a fully continuous description by introducing a new (dimensionful field)

$$\varphi(\mathbf{x}) \to \sqrt{a^D \frac{2D}{a^2}}\,\varphi(\mathbf{x}) \qquad (6.65)$$

and rewrite the sums over $\mathbf{x}$ as integrals, $\Sigma_{\mathbf{x}} \to (1/a^D)\int d^D x$. Then the exponent becomes

$$-\frac{1}{2}\int d^D x\, \varphi^{\dagger}(\mathbf{x})(-\partial^2 + m^2)\,\varphi(\mathbf{x})$$

and the partition function coincides with that of a continuous field theory of the type discussed in Chapter 2:

$$Z_{\mathrm{cr}} = \int \mathscr{D}\varphi(\mathbf{x})\mathscr{D}\varphi^{\dagger}(\mathbf{x})\, e^{-(1/2)\int d^{D}x\varphi^{\dagger}(\mathbf{x})(-\partial^{2}+m^{2})\varphi(\mathbf{x})}. \tag{6.66}$$

This is a standard free field theory of a two-scalar field of mass

$$m_{\mathrm{cr}}^{2} = \left(\frac{\varepsilon}{T} - \log 2D\right)\frac{2D}{a^{2}} \approx m^{2}, \tag{6.67}$$

which have been combined to a single complex field ϕ. This form of the partition function will be very useful for our further discussion. Notice that the only place where the temperature enters is in the term ε/T in m^2. The other parts are temperature independent and thus purely entropic.

6.6. CLOSED RANDOM CHAINS AND LOOP DIAGRAMS

We are now ready for a very simple but rather important observation. According to the general discussion in Chapter 2, the free field partition function (6.56) can be written as

$$Z = e^{(1/T)W[0]},$$

where $(1/T)W[0]$ has the graphical representation

$$\frac{1}{T}W[0] = \frac{1}{2}\underset{1}{\bigcirc} + \frac{1}{2}\underset{2}{\bigcirc} = \bigcirc$$

with the oriented loop standing for the sum $-\frac{1}{2}\log G_0^{-1}(\mathbf{k})$ (one-half derived from the real part of ϕ and the other half from the imaginary part). On the other hand, according to (6.55), $(1/T)W[0]$ coincides with the partition function Z_1 of a single closed oriented random chain, of arbitrary length and shape. Thus we realize that the single loop diagram, which in the context of perturbation theory had been introduced purely as a device for remembering the zeroth order term, has a fundamental topological interpretation: It gives a direct pictorial representation of the random loop configurations summed in the partition function Z_1.

In fact, such a topological correspondence exists for all perturbation diagrams.

6.7. OPEN RANDOM CHAINS AND LINES IN FEYNMAN DIAGRAMS

The result of the last section suggests that not only the closed loops but also the open lines in Feynman diagrams, which represent the field correlation function, have a direct interpretation in terms of open random chains. In order to see this, consider the two-point functions of a complex free field:

$$G_{\phi\phi^\dagger}(\mathbf{x}_1, \mathbf{x}_2) = \langle \varphi(\mathbf{x}_1)\, \varphi^\dagger(\mathbf{x}_2)\rangle,$$

$$G_{\phi\phi}(\mathbf{x}_1, \mathbf{x}_2) = \langle \varphi(\mathbf{x}_1)\, \varphi(\mathbf{x}_2)\rangle,$$

$$G_{\phi^\dagger\phi^\dagger}(\mathbf{x}_1, \mathbf{x}_2) = \langle \varphi^\dagger(\mathbf{x}_1)\, \varphi^\dagger(\mathbf{x}_2)\rangle.$$

Focussing our attention upon the critical region, i.e., using the approximate Z_{cr} the correlation functions of φ_1 and φ_2 are

$$\langle \varphi_{1_2}(\mathbf{x}_1)\, \varphi_{1_2}(\mathbf{x}_2)\rangle = G_0(\mathbf{x}_1, \mathbf{x}_2) \equiv \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \frac{1}{\mathbf{k}^2 + m^2}.$$

Moreover, the $\varphi_1\varphi_2$ correlations vanishes, i.e.,

$$\langle \varphi_1(\mathbf{x}_1)\, \varphi_2(\mathbf{x}_2)\rangle = 0$$

(since both path integrands in

$$\int \mathscr{D}\varphi_1\, \mathscr{D}\, \varphi_2 \varphi_1(\mathbf{x}_1)\, \varphi_2(\mathbf{x}_2)\, e^{-(1/2)\Sigma_{\mathbf{x}}(\varphi_1 G_0^{-1}\varphi_1 + \varphi_2 G_0^{-1}\varphi_2)}$$

are odd functions in the field). Hence

$$G_{\phi\phi^\dagger}(\mathbf{x}_1, \mathbf{x}_2) = 2G_0(\mathbf{x}_1, \mathbf{x}_2),$$

$$G_{\phi\phi}(\mathbf{x}_1, \mathbf{x}_2) = 0,$$

$$G_{\phi^\dagger\phi^\dagger}(\mathbf{x}_1, \mathbf{x}_2) = 0.$$

In Chapter 3 we saw that in dealing with complex fields it was most convenient to introduce source terms of the type

$$E_{\mathrm{source}} = \int d^D x (\eta^\dagger(\mathbf{x})\, \varphi(\mathbf{x}) + \varphi^\dagger(\mathbf{x})\, \eta(\mathbf{x}))$$

and that there exists a simple generating functional

$$Z[\eta, \eta^\dagger] = \int \mathscr{D}\varphi(\mathbf{x})\, \mathscr{D}\varphi^\dagger(\mathbf{x})\, e^{-\int d^D x[(1/2)\varphi^\dagger(\mathbf{x})(-\partial^2 + m^2)\varphi(\mathbf{x}) - \eta^\dagger(\mathbf{x})\varphi(\mathbf{x}) - \varphi^\dagger(\mathbf{x})\eta(\mathbf{x})]}$$

$$= e^{-\mathrm{tr}\log(-\partial^2 + m^2)}\, e^{2\int d^3x\, d^3x'\, \eta^\dagger(\mathbf{x}) G_0(\mathbf{x}, \mathbf{x}')\eta(\mathbf{x}')}, \tag{6.68}$$

from which all Green functions can be derived via functional differentiation:

$$\begin{aligned}
&\langle \varphi(\mathbf{x}_1) \ldots \varphi(\mathbf{x}_n)\, \varphi^\dagger(\mathbf{y}_1) \ldots \varphi^\dagger(\mathbf{y}_m)\rangle \\
&\quad = G^{(n,m)}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_m) \\
&\quad = Z[\eta, \eta^\dagger]^{-1} \frac{\delta}{\delta\eta^\dagger(\mathbf{x}_2)} \cdots \frac{\delta}{\delta\eta^\dagger(\mathbf{x}_n)} \times \frac{\delta}{\delta\eta(\mathbf{y}_1)} \cdots \frac{\delta}{\delta\eta(\mathbf{y}_m)} Z[\eta, \eta^\dagger]\Big|_{\eta = \eta^\dagger = 0}
\end{aligned} \tag{6.69}$$

The result is the complex version of Wick's theorem which states that the correlation functions $G^{(n,m)}(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{y}_1, \ldots, \mathbf{y}_m)$ are given by the sum over all different pair contractions representing the correlation functions

$$\begin{aligned}
\overbrace{\varphi(\mathbf{x})\, \varphi^\dagger(\mathbf{y})} &= 2G_0(\mathbf{x}, \mathbf{y}), \\
\overbrace{\varphi(\mathbf{x})\, \varphi(\mathbf{y})} &= 0, \\
\overbrace{\varphi^\dagger(\mathbf{x})\, \varphi^\dagger(\mathbf{y})} &= 0.
\end{aligned}$$

These contractions were pictured by lines in Feynman graphs which can emerge from fields $\varphi(\mathbf{y})$ and disappear in fields $\varphi(\mathbf{x})$:

$$\begin{aligned}
\varphi(\mathbf{x}) &\triangleq \\
\varphi(\mathbf{y}) &\triangleq
\end{aligned} \quad = \quad \begin{aligned} x \bullet &\!\!\longleftarrow \\ \longleftarrow \bullet\, &y \end{aligned}$$

The correlation function of an arbitrary number of complex fields becomes equal to the sum of all oriented lines

$$\overbrace{\varphi(\mathbf{x})\, \varphi^\dagger(\mathbf{y})} \triangleq \qquad x \bullet \longleftarrow \bullet y$$

by which the fields $\varphi(\mathbf{x})$ and $\varphi^\dagger(\mathbf{y})$ can be connected.

These diagrams suggest, that the free field correlation functions have a direct interpretation in terms of open random chains. Indeed consider the two-point function as the simplest example. Rewriting it as an integral representation over the auxiliary variable s

$$G_0(\mathbf{x}, \mathbf{y}) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\mathbf{k}^2 + m^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

$$= \frac{1}{2M} \int_0^\infty ds\, e^{(-m^2/2M)s} \int \frac{d^D k}{(2\pi)^D} e^{(-k^2/2M)s + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \qquad (6.70)$$

where M is an arbitrary mass parameter, and using the identification (6.67), $m^2 = (\varepsilon/T - \log 2D)\,2D/a^2$, we find

$$G_0(\mathbf{x}, \mathbf{y}) = \frac{1}{2M} \int_0^\infty ds\, e^{-(D/Ma)(\varepsilon/T - \log 2D)s/a} \int \frac{d^D k}{(2\pi)^D} e^{-(k^2/2M)s + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

But this is recognized to be precisely the open chain partition function (6.47) in the continuum limit of (6.43) if we set $M = a/D$ and divide out a factor $a^2/2D = a/2M$ [compare also (6.50) and recall that in the critical regime $m^2 \ll 2D/a^2$]. Hence the oriented lines in the Feynman diagrams of a complex field theory are indeed the direct pictorial representations of the partition functions of open oriented random chains of any length s weighed by $\exp(-m^2/2M)s$.

6.8. FIELD FLUCTUATIONS VERSUS LINE FLUCTUATIONS

There exists yet another representation of the partition function which shows explicitly the fluctuating defect lines described by the field correlation functions. In order to derive it, we observe that the probability for a chain to run from $\mathbf{x}$ to $\mathbf{y}$ in n steps has a fundamental factorization property which may be stated as follows: Let m be an arbitrary number of steps smaller than or equal to n and $P(\mathbf{z}, \mathbf{y}, m)$ the probability that the particle has arrived at $\mathbf{z}$ after m steps. Then, $P(\mathbf{x}, \mathbf{y}, n)$ can always be split into a product of probabilities:

$$P(\mathbf{x}, \mathbf{y}, n) = \sum_{\mathbf{z}} P(\mathbf{x}, \mathbf{z}, n-m)\, P(\mathbf{z}, \mathbf{y}, m), \quad m \le n. \qquad (6.71)$$

By (6.41), this property is simply the $\mathbf{x}$-space version of the trivial equality of the Fourier components: $(P(\mathbf{k}))^n = P(\mathbf{k})^{n-m} \cdot P(\mathbf{k})^m$.

This result can be used to construct an explicit solution to the difference equation (6.4). We simply continue the factorization a large number of times until every factor refers to one step only, i.e.,

$$P(\mathbf{x}, \mathbf{y}, n) = \sum_{\mathbf{x}_1} \sum_{\mathbf{x}_2} \cdots \sum_{\mathbf{x}_{n-1}} P(\mathbf{x}, \mathbf{x}_{n-1}, 1) P(\mathbf{x}_{n-1}, \mathbf{x}_{n-2}, 1) \cdots P(\mathbf{x}_1, \mathbf{y}, 1). \tag{6.72}$$

The one-step probability, however, is $1/2D$ for all $2D$ neighbors, i.e.,

$$P(\mathbf{x}, \mathbf{y}, 1) \equiv \frac{1}{2D} H(\mathbf{x}, \mathbf{y}) = \frac{1}{2D} \sum_{\pm \mathbf{i}} \delta_{\mathbf{x}-\mathbf{y}, \mathbf{i}}. \tag{6.73}$$

The matrix $H(\mathbf{x}, \mathbf{y}) = \Sigma_{\pm \mathbf{i}} \delta_{\mathbf{x}-\mathbf{y}, \mathbf{i}}$ is often called the *hopping matrix*. Notice that it was this one-step probability which played the crucial role in deriving the difference equation (6.5). Obviously, that equation is just a special case of (6.71), namely,

$$P(\mathbf{x}, \mathbf{y}, n) = \sum_{\mathbf{z}} P(\mathbf{x}, \mathbf{z}, 1) P(\mathbf{z}, \mathbf{y}, n-1). \tag{6.74}$$

Let us now insert (6.73) into (6.72). Instead of the $n-1$ summation variables $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$ we may use the n position differences (see Fig. 6.1)

$$\boldsymbol{\delta}_\nu = \mathbf{x}_\nu - \mathbf{x}_{\nu-1}, \quad \nu = 1, \ldots, n, \tag{6.75}$$

and ensure the connection

$$\mathbf{x} - \mathbf{y} = \sum_{\nu=1}^{n} \boldsymbol{\delta}_\nu \tag{6.76}$$

by means of a Kronecker $\delta_{\mathbf{x}-\mathbf{y}, \sum_{\nu=1}^{n} \boldsymbol{\delta}_\nu}$. In this way we find the explicit formula

$$P(\mathbf{x}, \mathbf{y}, n) = \sum_{\boldsymbol{\delta}_1} \cdots \sum_{\boldsymbol{\delta}_n} \delta_{\mathbf{x}-\mathbf{y}, \sum_{\nu=1}^{n} \boldsymbol{\delta}_\nu} \prod_{\nu=1}^{n} \left(\frac{1}{2D} \sum_{\pm \mathbf{i}} \boldsymbol{\delta}_{\boldsymbol{\delta}_\nu, \mathbf{i}} \right). \tag{6.77}$$

This expresses the total end-to-end probability of a random chain of n links as a sum of all possible chain configurations connecting $\mathbf{x}$ and $\mathbf{y}$. The sum can be evaluated by representing the first Kronecker δ as a Fourier series

$$\delta_{\mathbf{x}-\mathbf{y}, \sum_{\nu=1}^{n} \boldsymbol{\delta}_\nu} = \frac{1}{N} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \left(\mathbf{x} - \mathbf{y} - \sum_{\nu=1}^{n} \boldsymbol{\delta}_\nu \right)} = \frac{1}{N} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{x}-\mathbf{y})} \prod_{\nu=1}^{n} e^{-i\mathbf{k} \cdot \boldsymbol{\delta}_\nu}. \tag{6.78}$$

Then (6.77) becomes

$$P(\mathbf{x}, \mathbf{y}, n) = \frac{1}{N}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \prod_{\nu=1}^{n} \left(\sum_{\boldsymbol{\delta}_\nu} \frac{1}{2D} \sum_{\pm \mathbf{i}} \delta_{\boldsymbol{\delta}_\nu, \mathbf{i}} e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_\nu} \right)$$

$$= \frac{1}{N}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} P(\mathbf{k}, n), \tag{6.79}$$

where

$$P(\mathbf{k}, n) = \left(\sum_{\boldsymbol{\delta}} \frac{1}{2D} \sum_{\pm \mathbf{i}} \delta_{\boldsymbol{\delta}, \mathbf{i}} e^{-i\mathbf{k}\cdot\boldsymbol{\delta}} \right)^n \equiv (P(\mathbf{k}, 1))^n \tag{6.80}$$

is the n-step probability in momentum space which decomposes into the product of n identical factors as a direct manifestation of the convolution form of the decomposition (6.72) (which is always diagonal in momentum space). The individual factors are the Fourier transforms of the one-step distribution (6.73),

$$P(\mathbf{k}, 1) = \frac{1}{2D} \sum_{\pm \mathbf{i}} \delta_{\boldsymbol{\delta}, \mathbf{i}} e^{-i\mathbf{k}\cdot\boldsymbol{\delta}} = \frac{1}{D} \sum_{\mathbf{i}} \cos \mathbf{k} \cdot \mathbf{i} = \frac{1}{D} \sum_{\mathbf{i}} \cos \mathbf{k}_{\mathbf{i}} \cdot \mathbf{a}. \tag{6.81}$$

These are recognized as the quantities $P(\mathbf{k})$ introduced in the ansatz (6.41) when trying to solve the difference equation (6.14). Here we have *derived* the form of the ansatz from the explicit decomposition (6.77).

We have argued in (6.63) that close to the temperature T_c the probability distribution (6.79) is dominated by the small $\mathbf{k}$ components of the Fourier representation. This approximation has led to a continuum approximation of the representation (6.41), (6.79) [namely, (6.44)].

It is possible to go to the continuum limit also in formula (6.72). Using once more the small a limit for $P(\mathbf{k}, n)$

$$P(\mathbf{k}, n) = (P(\mathbf{k}))^n \sim e^{-(s/a)(a^2/2D)\mathbf{k}^2}, \tag{6.82}$$

and the limiting $\mathbf{x}$-space probability

$$P(\mathbf{x}, \mathbf{y}, s) = \frac{1}{N}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-s(\mathbf{k}^2/2M)} = \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-s(\mathbf{k}^2/2M)} \tag{6.83}$$

we see that this has the factorization property

$$\int d^D z\, P(\mathbf{x}, \mathbf{z}, s')\, P(\mathbf{z}, \mathbf{y}, s) = P(\mathbf{x}, \mathbf{y}, s' + s) \tag{6.84}$$

for any s, s'. This is the continuous analogue of (6.72).

The same steps which led from (6.72) to (6.77) can now be seen to lead to a path-integral representation of fluctuating random walks. For this, we divide the pseudotimes into a large number of equal thin slices: $s = \nu\varepsilon$, $\nu = 1, 2, \ldots, n$.[f] We then apply (6.84) repeatedly to all these slices and find that the total probability can be obtained as a limit:

$$P(\mathbf{x}, \mathbf{y}, s) = \lim_{\varepsilon \to 0} \left(\prod_{\nu=1}^{n-1} \int d^D x_\nu \right) P(\mathbf{x}, \mathbf{x}_{n-1}, \varepsilon) \times P(\mathbf{x}_{n-1}, \mathbf{x}_{n-2}, \varepsilon) \ldots P(\mathbf{x}_2, \mathbf{x}_1, \varepsilon)\, P(\mathbf{x}_1, \mathbf{y}, \varepsilon), \tag{6.85}$$

where each factor is

$$P(\mathbf{x}_\nu, \mathbf{x}_{\nu-1}, \varepsilon) = \int \frac{d^D p_\nu}{(2\pi)^D} e^{i\mathbf{p}_\nu \cdot (\mathbf{x}_\nu - \mathbf{x}_{\nu-1}) - \varepsilon(\mathbf{p}_\nu^2/2M)} \tag{6.86}$$

and the endpoints $\mathbf{x}$ and $\mathbf{y}$ are identified with $\mathbf{x}_n$ and $\mathbf{x}_0$, respectively. Inserting this into (6.85) and collecting the exponents into a single expression, we arrive at the formula

$$P(\mathbf{x}, \mathbf{y}, s) = \prod_{\nu=1}^{n-1} \left(\int d^D x_\nu \frac{d^D p_\nu}{(2\pi)^D} \right) \int \frac{d^D p_\nu}{(2\pi)^D} e^{i \sum_{\nu=1}^{n} \mathbf{p}_\nu (\mathbf{x}_\nu - \mathbf{x}_{\nu-1}) - \varepsilon \sum_\nu \mathbf{p}_\nu^2/2M}. \tag{6.87a}$$

The exponent can be viewed as the discretized form of

$$\exp\left[\int_0^s ds' \left(i\mathbf{p}(s') \cdot \dot{\mathbf{x}}(s') - \frac{\mathbf{p}^2(s')}{2M} \right) \right] \tag{6.87b}$$

to which it reduces in the limit $\varepsilon \to 0$. In this limit, the products of the dx_ν and dp_ν integrals may be rewritten as a path integral in phase space, i.e.,

[f]Hopefully, the reader will not be confused by the two different uses of the quantity ε (one is for the energy in Eq. (6.46), and the other for the thickness of time slices).

$$\int \mathscr{D}^D x \frac{\mathscr{D}^D p}{(2\pi)^D} \equiv \prod_{\nu=1}^{n-1} \left(\int d^D x_\nu \int \frac{d^D p_\nu}{(2\pi)^D} \right) \int \frac{d^D p_n}{(2\pi)^D}, \tag{6.88}$$

and that the probability has the representation

$$P(\mathbf{x}, \mathbf{y}, s) = \int \mathscr{D}^D x \frac{\mathscr{D}^D p}{(2\pi)^D} e^{\int_0^s ds' (i\mathbf{p}(s') \cdot \dot{\mathbf{x}}(s') - (\mathbf{p}^2(s')/2M))}. \tag{6.89}$$

We saw earlier that the probability equals the Green function of the Schrödinger equation with time continued to negative imaginary values $t = -is$. Here we verify the same thing once more: Considering (6.89) for the special case $s' = s + \varepsilon (\varepsilon = \text{infinitesimal})$ we can derive precisely the differential equation (6.1) of Brownian motion since in this case, Eq. (6.84) reads

$$P(\mathbf{x}, \mathbf{y}, s + \varepsilon) = \int d^D z \int \frac{d^D p}{(2\pi)^D} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{z}) - \varepsilon(\mathbf{p}^2/2M)} P(\mathbf{z}, \mathbf{y}, s). \tag{6.90}$$

But the $\mathbf{p}^2$ part in the momentum integral may be brought outside the integral by introducing a derivative:

$$P(\mathbf{x}, \mathbf{y}, s + \varepsilon) = e^{-\varepsilon(1/2M)(-i\partial/\partial \mathbf{x})^2} \int d^D z \int \frac{d^D p}{(2\pi)^D} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{z})} P(\mathbf{z}, \mathbf{y}, s). \tag{6.91}$$

Now the momentum integration gives a δ-function which forces $\mathbf{z}$ to $\mathbf{x}$. Moreover, since ε is small we can expand $P(\mathbf{x}, \mathbf{y}, s + \varepsilon) \approx (1 - \varepsilon(1/2M)(-i\partial/\partial \mathbf{x})^2 P(\mathbf{x}, \mathbf{y}, s)$ and this amounts, in the limit $\varepsilon \to 0$, to the differential equation

$$-\frac{\partial}{\partial s} P(\mathbf{x}, \mathbf{y}, s) = \frac{1}{2M} \left(\frac{1}{i} \frac{\partial}{\partial \mathbf{x}} \right)^2 P(\mathbf{x}, \mathbf{y}, s), \tag{6.92}$$

which is indeed fulfilled by (6.83). Obviously, this derivation of the differential equation is nothing but the continuum version of the previous observation that (6.74) is the same as the difference equation (6.14).

In real time, the connection between (6.92) and (6.89) shows that the Green function of the Schrödinger equation has the path integral representation

$$G(\mathbf{x}, \mathbf{y}, t) = \int \mathscr{D}^D x \frac{\mathscr{D}^D p}{(2\pi)^D} e^{i\int_0^t dt'(\mathbf{p}\cdot\dot{\mathbf{x}} - (\mathbf{p}^2/2M))}. \tag{6.93}$$

Indeed, setting $t = -is$, this goes over into (6.89). In the quantum mechanical case, the exponent in the path integral is the classical action in the Hamiltonian form and the quantum mechanical *amplitude* is obtained from summing over all fluctuating paths in phase space with fixed endpoints $\mathbf{x}(s) = \mathbf{x}$, $\mathbf{x}(0) = \mathbf{y}$. The result (6.91) says that the *probability* distribution of stochastically fluctuating random lines is given by the same expression, only that t has been continued to $-is$. This is the same connection between the actions of quantum mechanics and statistical mechanics which was observed earlier in Section 1.7, in the context of partition function. Recall that an action continued in this way was called Euclidean and was denoted by A_E, i.e.,

$$e^{iA}\big|_{t=-is} \equiv e^{-A_E}.$$

In analogy to the quantum mechanical case it is sometimes useful to go from the Hamiltonian form to the Lagrangian form involving a pure $\mathbf{x}$ space path integral. This is obtained by integrating out the momentum variables in (6.87a), which is trivial after using the quadratic completion

$$e^{i\mathbf{p}_\nu\cdot(\mathbf{x}_\nu - \mathbf{x}_{\nu-1}) - \varepsilon\mathbf{p}_\nu^2/2M} = e^{-(\varepsilon/2M)(\mathbf{p}_\nu - iM(\mathbf{x}_\nu - \mathbf{x}_{\nu-1})/\varepsilon)^2} e^{-(M/2)(\mathbf{x}_\nu - \mathbf{x}_{\nu-1})^2/\varepsilon}$$

and Gauss' integral formula $\int (dp/\sqrt{2\pi})\, e^{-ap^2/2} = 1/\sqrt{a}$. The result is

$$P(\mathbf{x}_\nu, \mathbf{x}_{\nu-1}, \varepsilon) = \frac{1}{(2\pi\varepsilon/M)^{D/2}} e^{-(M/2)(\mathbf{x}_\nu - \mathbf{x}_{\nu-1})^2/\varepsilon}, \tag{6.94}$$

so that the total probability becomes

$$P(\mathbf{x}, \mathbf{y}, s) = \frac{1}{(2\pi\varepsilon/M)^{D/2}} \prod_{\nu=1}^{n} \int \frac{d^D x_\nu}{(2\pi\varepsilon/M)^{D/2}} e^{-(M/2)\Sigma_\nu(\mathbf{x}_\nu - \mathbf{x}_{\nu-1})^2/\varepsilon}. \tag{6.95}$$

This can be seen as the discretized version of the continuum expression

$$P(\mathbf{x}, \mathbf{y}, s) = \int \mathscr{D}^D x(s)\, e^{-(M/2)\int_0^s ds'\dot{\mathbf{x}}(s)^2}, \tag{6.96}$$

which is the thermal analogue of Feynman's quantum mechanical formula

$$G(\mathbf{x}, \mathbf{y}, t) = \int \mathscr{D}^D x(t) e^{i(M/2)\int_0^t dt' \dot{\mathbf{x}}^2(t')} \tag{6.97}$$

to which it reduces for $t = -is$. Just as in (6.87b), the exponent is again the action but in the Lagrangian form

$$A = \int_0^t dt' \, L(t'). \tag{6.98}$$

Notice that as we follow Feynman's convention the pure **x**-space path integral symbol $\mathscr{D}x$ is defined differently from the previous phase space symbol $\mathscr{D}x \, \mathscr{D}p/2\pi$. While that was just the product of individual integrals for each time slice, the integral $\mathscr{D}x$ is assumed to contain the factor $1/(2\pi\varepsilon/M)^{ND/2}$.

6.9. ENSEMBLES OF FLUCTUATING LINES IN THE CRITICAL REGION

Close to the critical temperature it is now possible to give a formula for the partition function of a grand canonical ensemble of closed loops on the basis of path integrals involving many fluctuating orbits rather than a single field. For this we consider once more the partition function of a single random line of any length s running from **y** to **x**. The Boltzmann factor is $\exp(-\varepsilon s/Ta)$ so that we have the continuum limit of (6.47)

$$\begin{aligned} P(\mathbf{x}, \mathbf{y}) &= \int_0^\infty d(s/a)(2D)^{s/a} e^{-\varepsilon s/Ta} P(\mathbf{x}, \mathbf{y}, s) \\ &= \int_0^\infty ds \, e^{-(\varepsilon/T - \log 2D)s/a} \int_{\mathbf{x}(0) = \mathbf{x}(s)} \mathscr{D}^D x(s) \frac{\mathscr{D}^D p(s)}{(2\pi)^D} e^{\int_0^s ds' (i\mathbf{p}\cdot\dot{\mathbf{x}} - \mathbf{p}^2/2m)} \\ &= \int_0^\infty ds \, e^{-(\varepsilon/T - \log 2D)s/a} \int_{\mathbf{x}(0) = \mathbf{x}(s)} \mathscr{D}^D x(s) \, e^{-(M/2)\int_0^s ds' \dot{\mathbf{x}}^2(s')}. \end{aligned} \tag{6.99}$$

The partition function of N random lines starting at $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(N)}$ and ending at $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}$, respectively, is obviously given by

$$P(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}; \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(N)}) = P(\mathbf{x}^{(1)} - \mathbf{y}^{(1)}) \ldots P(\mathbf{x}^{(N)} - \mathbf{y}^{(N)})$$

$$= \int_0^\infty ds^{(1)} \cdots \int_0^\infty ds^{(N)}\, \mathrm{e}^{-(\varepsilon/T - \log 2D)\sum_{i=1}^{N} s^{(i)}/a} \times \prod_{i=1}^{N} \left[\int_{\mathbf{x}^{(i)}(0)=\mathbf{x}^{(i)}(s)} \mathscr{D}^D x^{(i)} \frac{\mathscr{D}^D p^{(i)}}{(2\pi)^D} \right.$$

$$\left. \times\, e^{i\sum_{i=1}^{N}\int_0^{s^{(i)}} ds'^{(i)}(i\mathbf{p}^{(i)}\cdot\dot{\mathbf{x}}^{(i)} - \mathbf{p}^{(i)2}/2M)} \right], \tag{6.100}$$

where the path integral can, of course, also be written in the pure $\mathbf{x}$ form using the second version of (6.99). We can go to closed loops by equating $\mathbf{x}_1 = \mathbf{y}_1, \ldots, \mathbf{x}_N = \mathbf{y}_N$ and find

$$Z(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{x}_1, \ldots, \mathbf{x}_N)$$

$$= \prod_{i=1}^{N} \int_0^\infty ds^{(i)} e^{-(\varepsilon/T - \log 2D)s^{(i)}/a} \int_{\mathbf{x}^{(i)}(0)=\mathbf{x}^{(i)}(s)} \mathscr{D}^D x^{(i)} \frac{\mathscr{D}^D p^{(i)}}{(2\pi)^D} e^{\int_0^{s^{(i)}} ds'^{(i)}(i\mathbf{p}^{(i)}\cdot\dot{\mathbf{x}}^{(i)} - \mathbf{p}^{(i)2}/2M)}. \tag{6.101}$$

From this expression it is easy to derive the partition function of a grand canonical ensemble of fluctuating closed random loops. We merely have to sum over all $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}$, divide by $s^{(1)}, \ldots, s^{(N)}$ for cyclic invariance and divide by $N!$ to account for the indistinguishability of the loops. In addition, we have to insert the number of possible loops of lengths $s^{(i)}$, $(2D)^{\sum_{i=1}^{N} s^{(i)}/a}$. In this way we obtain the Hamiltonian path-integral representation

$$Z = \sum_N \frac{1}{N!} V^N \prod_{i=1}^{N} \int_0^\infty \frac{ds^{(i)}}{s^{(i)}} e^{-(\varepsilon/T - \log 2D)s^{(i)}/a}$$

$$\times \int_{\mathbf{x}^{(i)}(0)=\mathbf{x}^{(i)}(s)} \mathscr{D}^D x^{(i)} \frac{\mathscr{D}^D p^{(i)}}{(2\pi)^D} e^{\int_0^{s^{(i)}} ds'^{(i)}(i\mathbf{p}^{(i)}\cdot\dot{\mathbf{x}}^{(i)} - \mathbf{p}_i^{(i)2}/2M)}, \tag{6.102}$$

where V is the total volume coming from the diagonal sum over $\mathbf{x}^{(i)} = \mathbf{y}^{(i)}$. Alternatively, we may integrate out the p variables and arrive at the Lagrangian form

$$Z = \sum_N \frac{1}{N!} V^N \prod_{i=1}^{N} \int_0^\infty \frac{ds^{(i)}}{s^{(i)}} e^{-(\varepsilon/T - \log 2D)s^{(i)}/a} \int_{\mathbf{x}^{(i)}(0)=\mathbf{x}^{(i)}(s)} \mathscr{D}^D x\, e^{-(M/2)\int_0^{s^{(i)}} ds'^{(i)}\dot{\mathbf{x}}^2(s'^{(i)})}. \tag{6.103}$$

This complicated partition function describes exactly the same physics as

the simple field expression (6.66). It displays directly the fluctuations of any N orbits contained in the grand canonical ensemble.

The latter representation has been used by many authors to study ensembles of fluctuating line systems. It is obvious from the difference in complexity of the expressions (6.66) and (6.103) that disorder field theory is much more powerful and elegant in handling such systems. There are two places where work is saved: First, a Schrödinger wave function is easier to handle than a path integral involving a fluctuating orbit. Second, a fluctuating (second quantized) Schrödinger field is more economical in dealing with the N-body aspects than the set of Schrödinger equations for $N = 1, 2, 3, \ldots$ bodies.

6.10. GENERAL PROPERTIES OF LATTICE GREEN FUNCTIONS

For the applications to be covered it will be necessary to know the partition function of open random diagrams of any length $P(\mathbf{x}, \mathbf{y}) = (m^2 + 2D/a^2)\, G_0(\mathbf{x}, \mathbf{y})$ explicitly. Let us therefore calculate the lattice Green function $G_0(\mathbf{x}, \mathbf{y})$. Its Fourier representation is (recall (6.48b))

$$\begin{aligned} G_0(\mathbf{x}, \mathbf{y}) &= v_m(\mathbf{x} - \mathbf{y}) \\ &= \frac{1}{N} \sum_{\mathbf{k}} \frac{1}{m^2 + \bar{\mathbf{K}}_i \cdot \mathbf{K}_i} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \frac{1}{N} \sum_{\mathbf{k}} \frac{1}{m^2 + \dfrac{2}{a^2} \displaystyle\sum_{i=1}^{D} (1 - \cos k_i a)} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \end{aligned} \tag{6.104}$$

We have introduced the notation $v_m(\mathbf{x} - \mathbf{y})$ in order to indicate that $G_0(\mathbf{x}, \mathbf{y})$ is the lattice version of the Yukawa potential

$$v_m^Y(\mathbf{x} - \mathbf{y}) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{m^2 + \mathbf{k}^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \tag{6.105}$$

The momenta in the integral cover the Brillouin zone $|\mathbf{k}_i| < \pi/a$ (recall (6.32)). For infinite systems (thermodynamic limit), the sums can be replaced by integrals

$$\frac{1}{N}\sum_{\mathbf{k}} \to \prod_{i=1}^{D} \int_{-\pi}^{\pi} \frac{dk_i a}{2\pi} \tag{6.106}$$

and we have to calculate

$$v_m(\mathbf{x}) = a^D \left(\prod_{i=1}^{D} \int_{-\pi}^{\pi} \frac{dk_i}{2\pi} \right) \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{m^2 + 2D - \sum_{i=1}^{D} \cos k_i}. \tag{6.107}$$

In the following we shall mostly use natural units with $a = 1$ so that the factor a^D is absent. The cosine functions in the denominator are hard to integrate. For some purposes it will be advantageous to rewrite (6.107) in another form, using the formula

$$\int_0^\infty ds\, e^{-s\mu} = \frac{1}{\mu}, \tag{6.108}$$

namely as

$$v_m(\mathbf{x}) = \int_0^\infty ds \left(\prod_{i=1}^{D} \int_{-\pi}^{\pi} \frac{dk_i}{2\pi} \right) e^{-s\left[m^2 + 2\sum_{i=1}^{D}(1-\cos k_i)\right] + i\mathbf{k}\cdot\mathbf{x}}. \tag{6.109}$$

Remembering that in natural units the vector components x_i are all integer numbers [see (6.1)], the k_i integrations can trivially be performed using the formula

$$\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{2s\cos\alpha + i\alpha n} = I_n(2s),$$

where $I_n(x)$ are the modified Bessel functions of order n. In this way we arrive at

$$v_m(\mathbf{x}) = \int_0^\infty ds\, e^{-s(m^2+2D)} I_{x_1}(2s) \ldots I_{x_D}(2s). \tag{6.110}$$

The large distance behavior for $|\mathbf{x}| \gg 1$ is most conveniently extracted directly from (6.107) by observing that the exponential $e^{i\mathbf{k}\cdot\mathbf{x}}$ will allow only small k_i to contribute. In this limit, $1 - \cos k_i \sim k_i^2/2$ so that $v_m(\mathbf{x})$ reduces to the Yukawa potential (6.105):

$$v_m(\mathbf{x}) \xrightarrow[|\mathbf{x}| \gg 1]{} v_m^Y(\mathbf{x}) = \int_0^\infty ds \prod_{i=1}^{D} \left[\int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \right] e^{-s(m^2 + \mathbf{k}^2) + i\mathbf{k}\cdot\mathbf{x}}$$
$$= \int_0^\infty ds \frac{1}{\sqrt{4\pi s}^D} e^{-sm^2 - \mathbf{x}^2/4s}. \tag{6.111}$$

Setting $s = t|\mathbf{x}|/2m$, this becomes

$$v_m^Y(\mathbf{x}) = \frac{1}{(4\pi)^{D/2}} \left(\frac{2m}{|\mathbf{x}|} \right)^{(D-2)/2} \int_0^\infty dt\, t^{-D/2} e^{-(m/2)|\mathbf{x}|(t + 1/t)}. \tag{6.112}$$

This result is, of course, the same as if we had taken the limit of zero lattice spacing in (6.104). Notice that it is completely isotropic. All memory of the simple cubic structure is lost.

If $|\mathbf{x}|$ is so large that $|\mathbf{x}| \gg 1/m$, the integral may be approximated by the saddle-point method. It is extremal at $t = 1$ and behaves close to it as

$$-\frac{m|\mathbf{x}|}{2}\left(t + \frac{1}{t}\right) \sim -m|\mathbf{x}|\left(1 + \frac{1}{2}(t-1)^2 - \ldots\right). \tag{6.113}$$

Up to this order, the integral can be performed and we obtain

$$v_m(\mathbf{x}) \xrightarrow[|\mathbf{x}| \gg 1]{|\mathbf{x}| \gg 1/m} \frac{1}{2m} \left(\frac{m}{2\pi|\mathbf{x}|} \right)^{(D-1)/2} e^{-m|\mathbf{x}|}. \tag{6.114}$$

The massless case cannot be treated this way since then $|\mathbf{x}|m \equiv 0$. But then we can set $s = 1/4s'$ and integrate (6.111) directly with the result

$$v_0(\mathbf{x}) \xrightarrow[|\mathbf{x}| \gg 1]{} \frac{1}{4\pi^{D/2}} \int_0^\infty \frac{ds'}{s'} s'^{D/2 - 1} e^{-|\mathbf{x}|^2 s'}$$
$$\xrightarrow[|\mathbf{x}| \gg 1]{} \frac{1}{4\pi^{D/2}} \Gamma\left(\frac{D}{2} - 1\right) \frac{1}{|\mathbf{x}|^{D-2}}. \tag{6.115}$$

For $D = 3$, this reduces to the Coulomb potential (since $\Gamma(1/2) = \sqrt{\pi}$)

$$v_0(\mathbf{x}) \xrightarrow[|\mathbf{x}| \gg 1]{} \frac{1}{4\pi|\mathbf{x}|}. \tag{6.116}$$

For $D = 2$, this diverges. The subtracted expression $v_m'(\mathbf{x}) \equiv v_m(\mathbf{x}) -$

$v'_m(\mathbf{x}_0)$, however, does have a finite limit where $\mathbf{x}_0$ is an arbitrary distance far enough away to use the approximation. Setting $D = 2 + \varepsilon$ with a small number ε we may calculate

$$v'_m(\mathbf{x}) \xrightarrow[|\mathbf{x}| \gg 1]{} \frac{1}{4\pi^{1+\varepsilon/2}} \Gamma\left(\frac{\varepsilon}{2}\right)\left(\frac{1}{|\mathbf{x}|^{\varepsilon}} - \frac{1}{|\mathbf{x}_0|^{\varepsilon}}\right)$$

$$\xrightarrow[\varepsilon \to 0]{} \frac{1}{4\pi}\frac{2}{\varepsilon}(-\varepsilon \log(|\mathbf{x}|/|\mathbf{x}_0|)) = -\frac{1}{2\pi}\log(|\mathbf{x}|/|\mathbf{x}_0|). \tag{6.117}$$

Alternatively, a small mass m could have been used to keep the result finite. If the mass is arbitrary and $|\mathbf{x}|$ not necessarily much larger than $1/m$ [while still being much larger than the lattice space, $|\mathbf{x}| \gg 1$] we have to use the formula

$$\int_0^{\infty} \frac{dx}{x} x^{\nu} e^{-\alpha/x - \gamma x} = 2\left(\frac{\alpha}{\gamma}\right)^{\nu/2} K_{\nu}(2\sqrt{\alpha\gamma}), \tag{6.118}$$

where $K_{\nu}(z)$ are the modified Bessel functions, and integrate the Yukawa potential (6.112) to

$$v(\mathbf{x}) \xrightarrow[|\mathbf{x}| \gg 1]{} v^{Y}_m(\mathbf{x}) = \frac{1}{(4\pi)^{D/2}}\left(\frac{2m}{|\mathbf{x}|}\right)^{(D-2)/2} 2K_{D/2-1}(m|\mathbf{x}|). \tag{6.119}$$

For $|\mathbf{x}| \gg 1/m$, this reduces again to (6.114), due to $K_{\nu}(\mathbf{z}) \sim \sqrt{\pi/(2z)}\, e^{-z}$.

In the opposite limit $1/m \gg |\mathbf{x}| \gg 1$ we have to distinguish two cases. If $D > 2$, then $K_{\nu}(m|\mathbf{x}|) \sim \Gamma(\nu)/2(m|\mathbf{x}|/2)^{\nu}$ and (6.119) reduces to (6.115). For $D = 2$, however, $K_0(m|\mathbf{x}|)$ has the $m|\mathbf{x}| \ll 1$ limit $-\log(\frac{1}{2}m|\mathbf{x}|\, e^{\gamma})$, where $\gamma = 0.577\ldots$ is Euler's number, and we find

$$v_m(\mathbf{x}) \to -\frac{1}{2\pi}\left(\log\frac{m|\mathbf{x}|}{2} + \gamma\right). \tag{6.120}$$

6.11. CALCULATION OF LATTICE GREEN FUNCTIONS VIA THE HOPPING EXPANSION

The calculation of the full $v_m(\mathbf{x})$ could, in principle, proceed by integrating (6.107) or (6.110) numerically. For large masses or dimensions, however, Eq. (6.110) allows for a natural expansion in powers of $(m^2 + 2D)^{-1}$. This

obtained by expanding the Bessel functions in powers of s and combining the series to

$$I_{x_1}(2s)\ldots I_{x_D}(2s) = \sum_{n=0}^{\infty} \frac{H_n^{\mathbf{x}}}{n!} s^n. \tag{6.121}$$

The integration over s in (6.110) can now be done term by term and gives the desired expansion

$$v_m(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{H_n^{\mathbf{x}}}{(m^2 + 2D)^{n+1}}. \tag{6.122}$$

A quantity of special importance for future discussion is the value of the Green function at the origin:

$$v_m(\mathbf{0}) = \int_0^{\infty} ds\, e^{-(m^2+2D)s} (I_0(2s))^D. \tag{6.123}$$

Using

$$I_0(2s) = \sum_{k=\text{even}}^{\infty} c_k s^k = \sum_{k=\text{even}} \frac{1}{(k/2)!^2} s^k$$

and taking the D-th power

$$(I_0(2s))^D = \sum_{n=\text{even}} \frac{H_n^{\mathbf{0}}}{n!} s^n,$$

we find that the coefficients $H_n \equiv H_n^{\mathbf{0}}$ satisfy the recursion relation

$$H_0 = 1,$$

$$H_n = (n-1)! \sum_{k=2,4,\ldots}^{n} ((D+1)k - n)\, c_k \frac{H_{n-k}}{(n-k)!}, \tag{6.124}$$

which give $v_m(\mathbf{0})$ via the series

$$v_m(\mathbf{0}) = \sum_{n=0,2,4,\ldots} \frac{H_n}{(m^2 + 2D)^{n+1}}. \tag{6.125}$$

Explicitly, (6.124) is solved by

$$H_2 = ((D+1)2-2) = 2D = 2\binom{D}{1},$$

$$H_4 = 3!\left\{[(D+1)2-4]c_2\frac{H_2}{2!} + [(D+1)4-4]c_4 H_0\right\},$$

$$= 3!\left[(2D-2)D + 4D\frac{1}{4}\right] = 3\cdot(2D)(-1+2D) = 6\left[4\binom{D}{2} + \binom{D}{1}\right].$$

$$H_6 = 5!\left\{[(D+1)2-6]\frac{1}{4}D(2D-1) + [(D+1)4-6]\frac{1}{4}D\right.$$

$$\left. + [(D+1)6-6]\frac{1}{3!}2\right\}.$$

$$= 5(2D)(8-9(2D)+3(2D)^2) = 20\left[36\binom{D}{3} + 18\binom{D}{2} + \binom{D}{1}\right],$$

$$H_8 = 35(2D)(-33+41(2D)-18(2D)^2+3(2D)^3]$$

$$= 70\left[576\binom{D}{4} + 432\binom{D}{3} + 68\binom{D}{2} + \binom{D}{1}\right],$$

$$H_{10} = 63(2D)[912-1225(2D)+625(2D)^2-150(2D)^3+15(2D)^4]$$

$$= 252\left[14\,400\binom{D}{5} + 14\,400\binom{D}{4} + 3900\binom{D}{3} + 250\binom{D}{2} + \binom{D}{1}\right].$$

The coefficients of the binomials up to H_{10} are tabulated in Table 6.1.

In general, the coefficients C_{2n} are polynomials in $2D$ of the form

$$H_{2n} = a_n(2D)[b_{n,0}(2D)^0 + b_{n,1}(2D)^1 + \ldots + b_{n,n-1}(2D)^{n-1}], \qquad (6.126)$$

TABLE 6.1. The lowest 10 hopping coefficients expressed in terms of binomial numbers $\binom{D}{m}$, $H_n(D) = b\left[D + \sum_{m=2}^{\infty} h_n^{(m)}\binom{D}{m}\right]$.

n	b	$h_n^{(2)}$	$h_n^{(3)}$	$h_n^{(4)}$	$h_n^{(5)}$
2	2				
4	6	4			
6	20	18	36		
8	70	68	432	576	
10	252	250	3900	14400	14400

which are listed in Table 6.2 up to $2n = 30$. For $D = 1, 2, 3, 4$ they lead to the numbers shown in Table 6.3.

For the nearest neighbor positions $\mathbf{x} = \mathbf{i}$, the hopping expansion is very similar to that of $\mathbf{x} = \mathbf{0}$ since

$$v_m(\mathbf{1}) = \int_0^\infty ds\, e^{-(m^2+2D)s} I_1(2s)(I_0(2s))^{D-1}$$

and $I_1(2s)$ can be written as $\frac{1}{2}(d/ds) I_0(2s)$ so that the integral can be partially integrated as follows:

$$v_m(\mathbf{1}) = \frac{1}{2D}\int_0^\infty ds\, e^{-(m^2+2D)s} \frac{d}{ds}(I_0(2s))^D = -\frac{1}{2D} + \frac{m^2+2D}{2D} v_m(\mathbf{0}). \tag{6.127}$$

Actually, this result could have been anticipated. Writing (6.127) as

$$2D(v_m(\mathbf{1}) - v_m(\mathbf{0})) - m^2 v_m(\mathbf{0}) = 0$$

we see that the first two terms are just the lattice Laplacian $\Sigma_i \nabla_i \bar{\nabla}_i$ applied to $v_m(\mathbf{x})$ at the origin, since by cubic symmetry all $v_m(\pm\mathbf{i})$ are equal:

$$\sum_i \bar{\nabla}_i \nabla_i v(\mathbf{0}) = \sum_i (v_m(\mathbf{i}) - 2v_m(\mathbf{0}) + v_m(-\mathbf{i})) = 2D(v_m(\mathbf{1}) - v(\mathbf{0})).$$

Hence Eq. (6.127) is directly the difference equation for the Yukawa potential on the lattice [cf. Eq. (6.51)]

$$(\bar{\nabla} \cdot \nabla - m^2) v_m(\mathbf{x}) = 0$$

at $\mathbf{x} = 0$.

Thus, given $v_m(\mathbf{0})$, we know immediately $v_m(\mathbf{1})$ and don't need any expansion for it. The coefficients would be

$$H_n^{\mathbf{1}} = \frac{1}{2D} H_{n+1}, \quad n = 1, 3, 5, \ldots$$

For the next nearest neighbor position we use the recursion relation

TABLE 6.2. Expressions for H_n[number of closed random walks of length n on a simple cubic lattice as a function of spatial dimension D]. They determine the lattice Yukawa potential as $v_m(\mathbf{0}) = \sum_{n=0}^{\infty} \frac{H_n}{(2D + m^2)^{n+1}}$.

$$H_2 = 1\cdot(2D)\cdot[1]$$

$$H_4 = 3\cdot(2D)\cdot[-1 + 1\cdot(2D)^1]$$

$$H_6 = 5\cdot(2D)\cdot[8 - 9\cdot(2D)^1 + 3\cdot(2D)^2]$$

$$H_8 = 35\cdot(2D)\cdot[-33 + 41\cdot(2D)^1 - 18\cdot(2D)^2 + 3\cdot(2D)^3]$$

$$H_{10} = 63\cdot(2D)\cdot[912 - 1225\cdot(2D)^1 + 625\cdot(2D)^2 - 150\cdot(2D)^3 + 15\cdot(2D)^4]$$

$$H_{12} = 231\cdot(2D)\cdot[-18920 + 27041\cdot(2D)^1 - 15300\cdot(2D)^2 + 4425\cdot(2D)^3 - 675\cdot(2D)^4 + 45\cdot(2D)^5]$$

$$H_{14} = 429\cdot(2D)\cdot[1099200 - 1652672\cdot(2D)^1 + 1011017\cdot(2D)^2 - 330750\cdot(2D)^3 + 62475\cdot(2D)^4 - 6615\cdot(2D)^5 + 315\cdot(2D)^6]$$

$$H_{16} = 6435\cdot(2D)\cdot[-10643745 + 16695977\cdot(2D)^1 - 10867906\cdot(2D)^2 + 3893687\cdot(2D)^3 - 845250\cdot(2D)^4 + 113190\cdot(2D)^5 - 8820\cdot(2D)^6 + 315\cdot(2D)^7]$$

$$H_{18} = 12155\cdot(2D)\cdot[1060105760 - 1724185233\cdot(2D)^1 + 1181222841\cdot(2D)^2 - 454484898\cdot(2D)^3 + 109201743\cdot(2D)^4 - 17027010\cdot(2D)^5 + 1706670\cdot(2D)^6 - 102060\cdot(2D)^7 + 2835\cdot(2D)^8]$$

$$H_{20} = 46189\cdot(2D)\cdot[-65990956752 + 110756873497\cdot(2D)^1 - 79231797450\cdot(2D)^2 + 32313672375\cdot(2D)^3 - 8403001425\cdot(2D)^4 + 1463284305\cdot(2D)^5 - 172651500\cdot(2D)^6 + 13466250\cdot(2D)^7 - 637875\cdot(2D)^8 + 14175\cdot(2D)^9]$$

$$H_{22} = 88179\cdot(2D)\cdot[10041530647680 - 17326523250872\cdot(2D)^1 + 12865783650137\cdot(2D)^2 - 5510429987700\cdot(2D)^3 + 1528048599825\cdot(2D)^4 - 289906310925\cdot(2D)^5 + 38497560255\cdot(2D)^6 - 3567564000\cdot(2D)^7 + 222972750\cdot(2D)^8 - 8575875\cdot(2D)^9 + 155925\cdot(2D)^{10}]$$

$$H_{24} = 676039\cdot(2D)\cdot[-458391634516200 + 810709693368961\cdot(2D)^1 - 621981669884652\cdot(2D)^2 + 277832019888291\cdot(2D)^3 - 81301075515000\cdot(2D)^4 + 16533818621100\cdot(2D)^5 - 2405980255140\cdot(2D)^6 + 252624352365\cdot(2D)^7 - 18909804375\cdot(2D)^8 + 969073875\cdot(2D)^9 - 30873150\cdot(2D)^{10} + 467775\cdot(2D)^{11}]$$

$$H_{26} = 1300075\cdot(2D)\cdot[98920000589477760 - 178878576930128088\cdot(2D)^1 + 141272413116564457\cdot(2D)^2 - 65463436588230252\cdot(2D)^3 + 20059166239340943\cdot(2D)^4 - 4322912399863200\cdot(2D)^5 + 677380148822520\cdot(2D)^6 - 78339031931160\cdot(2D)^7 + 6683291559945\cdot(2D)^8 - 413057019375\cdot(2D)^9 + 17681739075\cdot(2D)^{10} - 474323850\cdot(2D)^{11} + 6081075\cdot(2D)^{12}]$$

$$H_{28} = 5014575\cdot(2D)\cdot[-12452264351578434240 + 22976094539255140896\cdot(2D)^1 - 18623444387478210553\cdot(2D)^2 + 8914580849622259903\cdot(2D)^3 - 2843376829775368887\cdot(2D)^4 + 643898702572416291\cdot(2D)^5 - 107318106477371790\cdot(2D)^6 + 13418935419069180\cdot(2D)^7 - 1266681583182015\cdot(2D)^8 + 89691404092290\cdot(2D)^9 - 4658703449400\cdot(2D)^{10} + 169149155175\cdot(2D)^{11} - 3873644775\cdot(2D)^{12} + 42567525\cdot(2D)^{13}]$$

$$H_{30} = 9694845\cdot(2D)\cdot[361734937 1670318316028 - 679860109524001 1924408\cdot(2D)^1 + 564170726293913 8797585\cdot(2D)^2 - 278011627031676 2620422\cdot(2D)^3 + 918710125430429 793824\cdot(2D)^4 - 217207261158372 580095\cdot(2D)^5 + 38159017944625923345\cdot(2D)^6 - 5091959684582520750\cdot(2D)^7 + 521573610655597500\cdot(2D)^8 - 41030465378277375\cdot(2D)^9 + 2453034619935900\cdot(2D)^{10} - 108655735938750\cdot(2D)^{11} + 3389439178125\cdot(2D)^{12} - 67043851875\cdot(2D)^{13} + 638512875\cdot(2D)^{14}]$$

TABLE 6.3. Hopping coefficients $H_n(D)$ in various dimensions.

n	$D = 1$	$D = 2$	$D = 3$	$D = 4$
0	1	1	1	1
2	2	4	6	8
4	6	36	90	168
6	20	400	1 860	5 120
8	70	4 900	44 730	190 120
10	252	63 504	1 172 556	7 939 002
12	924	853 776	32 496 156	357 713 664
14	3 432	11 778 624	936 369 720	16 993 726 464
16	12 870	165 636 900	27 770 358 330	839 358 285 480
18	48 620	2 363 904 400	842 090 474 940	42 714 450 658 880
20	184 756	34 134 779 536	25 989 269 017 140	2 225 741 588 095 168
22	705 432	497 634 306 624	813 689 707 488 840	118 227 198 981 126 144
24	2 704 156	7 312 459 672 336	25 780 447 171 287 900	6 380 762 273 973 278 464
26	10 400 600	108 172 480 360 000	825 043 888 527 957 000	349 019 710 593 278 412 800
28	40 116 600	1 609 341 595 560 000	26 630 804 377 937 061 000	19 310 744 204 362 333 900 800
30	155 117 520	24 061 445 010 950 400	865 978 374 333 905 289 360	1 079 054 103 459 778 710 405 120

$D = 1$: $\dfrac{n!}{(n/2)!^2}$

$D = 2$: $\dfrac{n!^2}{(n/2)!^4}$

$D = 3$: $\dfrac{n!}{(n/2)!^2} \sum_{k=0}^{n/2} \binom{n/2}{k}\binom{2k}{k}$

$D = 4$: $n! \sum_{k=0}^{n/2} \dfrac{(2k)!}{k!^4} \dfrac{(n-2k)!}{(n/2-k)!^4}$

$$I_2(2s) = I_0(2s) - \frac{1}{s} I_1(2s)$$

and calculate

$$\begin{aligned} v_m(\mathbf{2}) &= \int_0^\infty ds\, e^{-(m^2+2D)s} I_2(2s)(I_0(2s))^{D-1} \\ &= v_m(\mathbf{0}) - \int_0^\infty ds\, e^{-(m^2+2D)s} \frac{1}{s} I_1(2s)\, I_0(2s)^{D-1}. \end{aligned} \tag{6.128}$$

Expanding $v_m(\mathbf{0})$ and observing that, due to the factor $1/s$, the integral gives once more the series for $v_m(\mathbf{1})$ but with coefficients $H^{\mathbf{1}}_{n+1}/(n+1)$, we find the relation

$$H^{2\cdot\mathbf{1}}_n = H^{\mathbf{0}}_n - \frac{1}{n+1} H^{\mathbf{1}}_{n+1} = H_n - \frac{1}{2D(n+1)} H_{n+2}. \tag{6.129}$$

For the diagonal neighbor $\mathbf{x} = (1, 1, 0, \ldots, 0)$ we calculate, similarly

$$\begin{aligned} v_m(\mathbf{1}+\mathbf{2}) &= \int_0^\infty ds\, e^{-(m^2+2D)s} I_1(2s)^2 I_0(2s)^{D-2} \\ &= \frac{1}{2(D-1)} \int_0^\infty ds\, e^{-(m^2+2D)s} I_1(2s) \frac{d}{ds} (I_0(2s))^{D-1} \\ &= \frac{1}{2(D-1)} \int_0^\infty ds \Big[(m^2+2D)\, e^{-(m^2+2D)s} I_1(2s)\, I_0(2s)^{D-1} \\ &\quad - 2e^{-(m^2+2D)s} I_1'(2s)\, I_0(2s)^{D-1} \Big] \\ &= \frac{1}{2(D-1)} (m^2+2D)\, v_m(\mathbf{1}) - \frac{1}{(D-1)} \int_0^\infty ds\, e^{-(m^2+2D)s} \\ &\quad \times \left(I_0(2s) - \frac{1}{s} I_n(2s) \right) I_0(2s)^{D-1} \\ &= \frac{1}{2(D-1)} (m^2+2D)\, v_m(\mathbf{1}) - \frac{1}{(D-1)} v_m(\mathbf{0}) \\ &\quad + \frac{1}{(D-1)} \int_0^\infty ds\, e^{-(m^2+2D)s} \frac{1}{s} I_n(2s)\, I_0(2s)^{D-1}. \end{aligned}$$

Due to the factor $1/s$, the remaining integral again has the expansion of $v_m(\mathbf{1})$ with $H_n^{\mathbf{1}}$ replaced by $H_{n+1}^{\mathbf{1}}/(n+1)$. Thus we find the coefficients

$$H_n^{\mathbf{1}+\mathbf{2}} = \frac{1}{2(D-1)} H_{n+1}^{\mathbf{1}} - \frac{1}{(D-1)} H_n^{\mathbf{0}} + \frac{1}{(D-1)} \frac{1}{n+1} H_{n+1}^{\mathbf{1}}.$$

Inserting the previous result for $H_n^{\mathbf{1}}$ this becomes

$$H_n^{\mathbf{1}+\mathbf{2}} = \frac{1}{(D-1)} \left(\frac{n+3}{2(n+1)} \frac{H_{n+2}}{2D} - H_n \right). \tag{6.130}$$

It is useful to realize that the expansion coefficients $H_n^{\mathbf{x}}$ in (6.122) have a simple combinatorial meaning: They count the number of random walks of n steps which connect the origin which the lattice point $\mathbf{x}$. In order to see this we expand the propagator (6.106) in powers of $2\sum_{i=1}^{D} \cos k_i$ which by Eq. (6.42) is $2D$ times the one-step probability $P(\mathbf{k})$ in momentum space and, by Eq. (6.73) equal to the Fourier transform $H(\mathbf{k}) = 2DP(\mathbf{k})$ of the hopping matrix $H(\mathbf{x}, \mathbf{y}) = \Sigma_{\pm \mathbf{i}} \delta_{\mathbf{x}, \mathbf{y}+\mathbf{i}}$. The expansion is therefore

$$v_m(\mathbf{x}) = \sum_{n=0}^{\infty} \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{H(\mathbf{k})^n}{(m^2 + 2D)^{n+1}}.$$

Comparison with (6.122) shows that the coefficients $H_n^{\mathbf{x}}$ are the Fourier transforms of the hopping matrix to the n-th power $(H^n)(\mathbf{x}, 0)$, i.e.,

$$H_n^{\mathbf{x}} = \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{x}} (H^n)(\mathbf{x}, 0) = \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{x}} H(\mathbf{k})^n. \tag{6.131}$$

This is why the series (6.122) is called *hopping expansion*.

Using the multinomial expansion coefficients, the n-th power of the hopping matrix $H(\mathbf{k})$ can be written as follows

$$H(\mathbf{k})^n = \left(\sum_{i=-D}^{D} e^{ik_i} \right)^n$$

$$= \sum_{n_i,\ \sum_{i=-D}^{D} n_i = n} \frac{n!}{\prod_{i=-D}^{D} n_i!}\, e^{i\sum_{i=1}^{D}(n_i - n_{-i})k_i}.$$

Obviously, the coefficients give the total number of ways a random walk can arrive at a lattice point $\mathbf{x} = (n_1 - n_{-1}, n_2 - n_{-2}, \ldots, n_D - n_{-D})$ after n "hops." Applying now formula (6.131) these coefficients are seen to coincide with $H_n^{\mathbf{x}}$. Thus we arrive at the combinational formula for $H_n^{\mathbf{x}}$:

$$H_n^{\mathbf{x}} = \sum_{n_i,\ \sum_{i=1}^{D} n_i = \left(n + \sum_{i=1}^{D} |x_i|\right)/2} \frac{n!}{\prod_{i=1}^{D} n_i!(n_i - |x_i|)!} \tag{6.132}$$

In one dimension, there is no sum and the number of random walks of n steps from the origin to $\mathbf{x}$ is given by $(n \geq |\mathbf{x}|)$

$$H_n^{\mathbf{x}} = \frac{n!}{\left(\frac{n+x}{2}\right)!\left(\frac{n-x}{2}\right)!} = \binom{n}{\frac{n+x}{2}}, \qquad D = 1.$$

This implies that the potential $v_m(x)$ has the expansion

$$v_m(x) = \sum_{n + x \text{ even} > 0} \frac{n!}{\left(\frac{n+x}{2}\right)!\left(\frac{n-x}{2}\right)!} \frac{1}{(m^2+2)^{n+1}}.$$

This could, of course, have been derived directly using (6.110) together with the series

$$I_x(2s) = s^{|x|} \sum_{n=0,2,4}^{\infty} \frac{sn}{\left(\frac{n}{2}\right)!\left(\frac{n}{2} + |x|\right)!}$$

For $D > 1$, let us concentrate our attention first to closed loops with $\mathbf{x} = 0$ only. Then n must be even and we find the general formula

$$H_n \equiv H_n^{\mathbf{0}} = \sum_{\sum_{i=1}^{D} n_i = \frac{n}{2}} \frac{n!}{\prod_{i=1}^{D} (n_i!)^2}.$$

In $D = 2$ dimensions this amounts to

$$H_n = \sum_{n_1} \frac{n!}{(n_1!)^2 \left[\left(\frac{n}{2} - n_1\right)!\right]^2}.$$

Using the well-known sum

$$\sum_{k=0}^{m} \binom{m}{k}^2 = \binom{2m}{m} = \frac{(2m)!}{m!^2}$$

with $m = n/2$, this becomes

$$H_n = \frac{n!^2}{(n/2)!^4}, \quad D = 2. \tag{6.133}$$

Due to the simplicity of this expression it is again possible to sum up the series for $v_m(\mathbf{0})$

$$v_m(\mathbf{0}) = \frac{1}{m^2 + 4} \sum_{n=0,2,4,\ldots}^{\infty} \frac{1}{(m^2 + 4)^2} \frac{n!^2}{(n/2)!^4}.$$

For this we observe that the sum can be rewritten as

$$v_m(\mathbf{0}) = \frac{1}{m^2 + 4} \sum_{k=0}^{\infty} \frac{1}{(m^2 + 4)^{2k}} 4^{2k} \frac{(\frac{1}{2})_k (\frac{1}{2})_k}{(1)_k} \frac{1}{k!}$$

where $(a)_k$ denotes the product

$$(a)_k = a \cdot (a + 1) \ldots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}.$$

This follows directly from

$$\frac{(2k)!^2}{k!^4} = 4^k \frac{1 \cdot 2 \ldots 2k}{2^k \cdot 1 \cdot 2 \ldots k \cdot 2^k \cdot 1 \cdot 2 \ldots k} = 4^{2k} \frac{\left(\frac{1}{2}\frac{3}{2} \ldots \frac{2k-1}{2}\right)^2}{(1 \cdot 2 \ldots k)^2}$$

$$= 4^{2k} \left(\frac{\Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k + 1)}\right)^2 = \frac{(\frac{1}{2})_k (\frac{1}{2})_k}{(1)_k} \frac{1}{k!}.$$

But in the second form the series in recognized as that of a hypergeometric function:

$$v_m(\mathbf{0}) = \frac{1}{m^2+4} F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{1}{(1+m^2/4)^2}\right),$$

which, in turn, is expressible in terms of an elliptic integral

$$K(z) = \int_0^{\pi/2} d\theta \frac{1}{\sqrt{1 - z\sin^2\theta}}$$

as

$$v_m(\mathbf{0}) = \frac{1}{m^2+4} \frac{2}{\pi} K\left(\frac{1}{(1+m^2/4)^2}\right). \tag{6.134}$$

This may conveniently be used for numerical calculations since the elliptic integral possesses a very good polynomial approximation, i.e.,

$$K(z) = a_0 + a_1 z' + a_2 z'^2 - \log z' \,(b_0 + b_1 z' + b_2 z'^2) \tag{6.135}$$

where $z' \equiv 1 - z$ and

$$\begin{aligned} a_0 &= 1.3862944, \quad b_0 = 0.5, \\ a_1 &= 0.1119723, \quad b_1 = 0.1213478, \\ a_2 &= 0.0725296, \quad b_3 = 0.0288729, \end{aligned}$$

with an error of less than 3×10^{-5} for $z \in (0, 1)$.

In $D = 3$ dimensions, formula (6.132) leads to

$$H_n = \frac{n!}{(n/2)!^2} \sum_{k+n_1=n/2} \frac{(n/2)!^2}{k!^2 n_1!^2} \sum_{n_2+n_3=k} \frac{k!^2}{n_2!^2 n_3!^2}$$

$$- \frac{n!}{(n/2)!^2} \sum_{k=0}^{n/2} \binom{n/2}{k}^2 \binom{2k}{k}, \quad D = 3 \tag{6.136}$$

and in $D = 4$ dimensions, to

$$H_n = n! \sum_{k_1+k_2=n/2} \frac{1}{k_1!^2 k_2!^2} \sum_{n_1+n_2=k_1} \frac{k_1!^2}{n_1!^2 n_2!^2} \sum_{n_3+n_4=k_2} \frac{k_2!^2}{n_3!^2 n_4!^2}$$

$$= n! \sum_{k_1+k_2=n/2} \frac{1}{k_1!^2 k_2!^2} \binom{2k_1}{k_1} \binom{2k_2}{k_2}$$

$$= n! \sum_{k=0}^{n/2} \frac{1}{k!^4 \left(\frac{n}{2}-k\right)!^4} (2k)!(n-2k)!, \quad D = 4. \tag{6.137}$$

Using these sums for $D = 3, 4$ and the expressions in Table 6.1 for $D = 5$, 6, 7, 8 we can easily calculate $v_m(\mathbf{0})$ as shown in Table 6.4.

For completeness, we give some explicit formulas also for the $\mathbf{x} \neq 0$ hopping coefficients using the path-counting formula (6.132):

$$H_n^{(x_1,\ldots,x_D)} = \sum_{n_i,\ \sum_{i=1}^{D} n_i = \frac{n+\sum_{i=1}^{D}|x_i|}{2}} \frac{n!}{\prod_{i=1}^{D} n_i!(n_i-|x_i|)!}.$$

In $D = 2$ dimensions, this reduces to

$$H_n^{(x_1,x_2)} = n! \sum_{n_1+n_2=\frac{n+|x_1|+|x_2|}{2}} \frac{1}{n_1!n_2!} \frac{1}{(n_1-|x_1|)!(n_2-|x_2|)!}$$

TABLE 6.4. Massless lattice Green function at the origin, $v_0(\mathbf{0}) = -1/\bar{\nabla}\cdot\nabla(\mathbf{0})$, and the logarithm of the fluctuation determinant $\log\det(-\bar{\nabla}\cdot\nabla) = \mathrm{tr}\log(-\bar{\nabla}\cdot\nabla)$ in various dimensions.

D	$v(\mathbf{0})$	$\frac{1}{N}\mathrm{tr}\log(-\bar{\nabla}\cdot\nabla)$	$\frac{1}{N}\mathrm{tr}\log(-\bar{\nabla}\cdot\nabla)/2D)$
1	∞	0	−0.693 147 180 559 93
2	∞	1.166 243 615 897 6	−0.220 050 745 222 3
3	0.252 731 009 86	1.673 39	−0.118 37
4	0.154 933 39	1.999 71	−0.079 73
5	0.115 630 81	2.242 49	−0.060 10
6	0.093 080 28		
7	0.078 136 165		
8	0.067 415 44		

$$= \frac{n!}{\left(\frac{n+|x_1|+|x_2|}{2}\right)!\left(\frac{n-|x_1|-|x_2|}{2}\right)!} \times \sum_{n_1} \binom{\frac{n+|x_1|+|x_2|}{2}}{n_1} \binom{\frac{n-|x_1|-|x_2|}{2}}{n_1-|x_1|},$$

so that only a single sum remains which can be executed using the formula

$$\sum_{n=0}^{m} \binom{m}{k}\binom{m'}{k-\ell} = \binom{m+m'}{m-\ell},$$

with the result

$$H_n^{(x_1,x_2)} = \frac{n!^2}{\left(\frac{n+x_1+x_2}{2}\right)!\left(\frac{n+x_1-x_2}{2}\right)!\left(\frac{n-x_1+x_2}{2}\right)!\left(\frac{n-x_1-x_2}{2}\right)!}. \tag{6.138a}$$

In three dimensions, one obtains similarly

$$H_n^{\mathbf{x}} = n! \sum_{n_1+k=\frac{n+\sum|x_1|}{2}} \frac{1}{n_1!(n_1-|x_1|)!}\frac{1}{k}\frac{1}{!(k-|x_2|-|x_3|)!} \times \sum_{n_2=0}^{k} \binom{k}{n_2}\binom{k-|x_2|-|x_3|}{n_2-|x_2|}$$

$$= \frac{n!}{\left(\frac{n+\sum_1^3|x_i|}{2}\right)!\left(\frac{n-\sum_1^3|x_i|}{2}\right)!} \sum_{k=0}^{\frac{n+\sum_1^3|x_i|}{2}} \binom{\frac{n+\sum_1^3|x_i|}{2}}{k}\binom{\frac{n-\sum_1^3|x_i|}{2}}{k-|x_2|-|x_3|} \times \binom{2k-|x_2|-|x_3|}{k-|x_2|}, \tag{6.138b}$$

and in four dimensions

$$H_n^{\mathbf{x}} = n! \sum_{\substack{n_1,k,n_3,\ell \\ k+\ell=\left(n+\sum_1^4|x_i|\right)/2}} \frac{1}{k!\ell!(k-|x_1|-|x_2|)!(\ell-|x_1|-|x_2|)!}$$

$$\times \binom{k}{n_1}\binom{k-|x_1|-|x_2|}{n_1-|x_1|}\binom{\ell}{n_3}\binom{\ell-|x_3|-|x_4|}{n_3-|x_3|}$$

$$= \frac{n!}{\left(\frac{n+\sum_1^4|x_i|}{2}\right)!\left(\frac{n-\sum_1^4|x_i|}{2}\right)!} \sum_k \binom{\frac{n+\sum_1^4|x_i|}{2}}{k}\binom{\frac{n-\sum_1^4|x_i|}{2}}{k-|x_1|-|x_2|}$$

$$\times \binom{2k-|x_2|-|x_3|}{k-|x_1|}\binom{n+|x_1|+|x_2|-|x_3|-|x_4|-2k}{\frac{1}{2}(n+|x_1|+|x_2|+|x_3|+|x_4|)-k}. \tag{6.138c}$$

6.12. IMPROVING THE CONVERGENCE OF HOPPING EXPANSION

When evaluating $v(\mathbf{x})$ via the hopping expansion, we find that for $D = 2$, 3 the convergence is rather slow. Let us study this in some detail. For closed loops, the large n behavior of the hopping coefficients is found from (6.133), (6.136), (6.137), using Stirling's formula,

$$x! \to \sqrt{2\pi}\, x^{x+1/2} e^{-x}\left(1 + \frac{1}{12x} + \ldots\right),$$

as follows:

$$D = 2: \qquad H_n \sim \frac{1}{2\pi}\frac{1}{n} 4^{n+1}, \tag{6.139a}$$

$$D = 3: \qquad H_n \sim \frac{1}{2}\sqrt{\frac{3}{2\pi^3}}\frac{1}{n^{3/2}} 6^{n+1}, \tag{6.139b}$$

$$D = 4: \qquad H_n \sim \frac{1}{\pi^2}\frac{1}{n^2} 8^{n+1}. \tag{6.139c}$$

In general, the hopping expansion behaves asymptotically like

$\Sigma(1/n^{D/2})\{1/[1+(m^2/2D)^{n+1}]\}$. The $n^{-D/2}$ power is in correspondence with the large s behavior of the integral (6.110):

$$v_m(\mathbf{0}) \to \int_0^\infty ds\, e^{-sm^2} \frac{1}{(2\pi s)^{D/2}}.$$

For $m \neq 0$, the sum always converges. For $m = 0$ and $D = 1$ or $D = 2$, it diverges. In three dimensions, it converges, but very slowly. After summing n terms, the error is still of order $n^{-1/2}$ which even for $n = 10$ is not small [see Eq. (6.156) for an illustration]. In order to make use of the hopping expansion it is therefore essential to improve upon its convergence. This can be done by subtracting an appropriate $1/(m^2 + 2D)^n$ expansion which accounts explicitly for the small m behavior of $v_m(\mathbf{0})$. This behavior is known from the continuum limit of the Green function, i.e., from the Yukawa potential (6.105),

$$v_m(\mathbf{0}) \xrightarrow[m\to 0]{} v_m^Y(0) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{m^2 + \mathbf{k}^2}. \tag{6.140}$$

In D dimensions, we can write

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{m^2 + \mathbf{k}^2} = \frac{S_D}{(2\pi)^D} \frac{1}{2} \int_0^\infty dk^2 (k^2)^{D/2-1} \frac{1}{m^2 + \mathbf{k}^2}, \tag{6.141}$$

where

$$S_D = \frac{2\mu^{D/2}}{\Gamma(D/2)} \tag{6.142}$$

is the surface area of the D-dimensional unit sphere. Using the formula

$$\int_0^\infty dx\, x^\alpha (1+x)^{-\alpha-\beta} = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{6.143}$$

we arrive at the limiting behavior

$$v_m(\mathbf{0}) \to \frac{1}{2^D \pi^{D/2}} \Gamma(1 - D/2)(m^2)^{D/2-1}. \tag{6.144}$$

For $D \sim 2$ this gives

$$v_m(\mathbf{0}) \to \frac{1}{4\pi}\left(-\frac{1}{D/2-1} - \log m^2\right). \tag{6.145}$$

The pole at $D = 2$ comes from the high $\mathbf{k}$ divergence of the integral and is spurious since the momentum space is actually finite. Hence it disappears in a finite integral $\int (d^D k/(2\pi)^D)$. The small m behaviour, on the other hand, is due to the small-k region of the integral and hence is reliably given by (6.144). For $D = 3$ it implies

$$v_m(\mathbf{0}) \to \text{const.} - \frac{1}{4\pi} m \tag{6.146}$$

and for $D = 4$ (dropping again a spurious pole at $D = 4$)

$$v_m(\mathbf{0}) \to \text{const.} + \frac{1}{16\pi^2} m^2 \log m^2. \tag{6.147}$$

We shall now make use of this knowledge and set up a simple power series in $1/(m^2 + 2D)$ which has precisely this small m behavior. That series will be subtracted from (6.125) and thus improve the convergence.

Consider first the case of $D = 3$ dimensions. We write

$$m = \sqrt{m^2 + 6} - 6,$$

and expand the right-hand side as follows:

$$m = \sqrt{m^2 + 6} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-)^n \frac{6^n}{(m^2 + 6)^n}. \tag{6.148}$$

Since (6.125) contains only odd powers in $1/(m^2 + 6)$, we subtract a similar series for $\sqrt{m^2 + 6 + 6}$ and arrive at the expansion

$$\frac{\sqrt{6}}{4\pi} \frac{-\sqrt{m^2} + \sqrt{m^2 + 12}}{\sqrt{m^2 + 6}} = \frac{\sqrt{6}}{2\pi} \sum_{n=0,2,4,\ldots} \binom{\frac{1}{2}}{n+1} \frac{6^{n+1}}{(m^2 + 6)^{n+1}}. \tag{6.149}$$

The left-hand side has the same $m \to 0$ behavior, $-m/4\pi$, as $v_m(\mathbf{0})$ in

(6.146) and the series on the right-hand side has the same asymptotic behavior as the hopping expansion for $v_m(\mathbf{0})$. Indeed, writing the right-hand side in the form

$$\sum_{n=0,2,4,\ldots} \frac{\tilde{H}_n}{(m^2+6)^{n+1}}, \tag{6.150}$$

we see that

$$\tilde{H}_n = \frac{\sqrt{6}}{2\pi}\binom{\frac{1}{2}}{n+1} 6^{n+1} = \frac{\sqrt{6}}{2\pi}\frac{\frac{1}{2}\sqrt{\pi}}{(n+1)!\,(-\frac{1}{2}-n)!}6^{n+1}. \tag{6.151}$$

Using

$$(-\tfrac{1}{2}-n)! = \Gamma(\tfrac{1}{2}-n) = (-)^n\,\pi/\Gamma(\tfrac{1}{2}+n) = (-)^n\,\pi\frac{1}{(n-\frac{1}{2})(n-\frac{3}{2})\ldots(-\frac{1}{2})!}$$

$$= (-)^n\sqrt{\pi}\frac{4^n n!}{(2n)!},$$

$$\binom{\frac{1}{2}}{n+1} = \frac{1}{2}(-)^n\frac{(2^n)!}{2^{2n}n!^2(n+1)},$$

this becomes, for even n,

$$\tilde{H}_n = \frac{\sqrt{6}}{4\pi}\frac{(2n)!}{2^{2n}(n!)^2}\frac{1}{n+1}6^{n+1}. \tag{6.152}$$

Via Sterling's formula, this is seen to have the $n\to\infty$ asymptotic behaviour (6.139b), as expected.

Hence we arrive at the improved hopping expansion

$$v_m(\mathbf{0}) = \frac{\sqrt{6}}{4\pi}\frac{-\sqrt{m^2}+\sqrt{m^2+12}}{\sqrt{m^2+6}} + \sum_{n=0,2,4,\ldots}\frac{H_n-\tilde{H}_n}{(m^2+6)^{n+1}}. \tag{6.153}$$

This converges quite rapidly. If we take only the $n=0$ term we obtain for $m=0$

$$v_0(\mathbf{0}) = \frac{\sqrt{12}}{4\pi} - \frac{1}{6}\left(1-\frac{6\sqrt{6}}{4\pi}\right) \approx 0.247\,406\,9. \tag{6.154}$$

The true result is only ≈2% larger than this [see Table 6.4]:

$$v_0(\mathbf{0}) \approx 0.25273100986. \tag{6.155}$$

In Table 6.5 we have listed the coefficients $\bar{H}_n$ up to $n = 30$. So it is easy to calculate (6.153) to any desired accuracy.

Notice that in order to arrive at the accuracy (6.155) in the original unsubtracted hopping expansion (6.125) we would have had to add up terms up to $n = 1600$, since the error after n terms is $\sqrt{3/2\pi^3}\,1/n^{1/2} \sim 0.2/n^{1/2}$ (see (6.139b)). For example up to $n = 10$ the unsubtracted series merely gives

$$v_0(\mathbf{0}) = \frac{1}{6}\left(1 + \frac{1}{6} + \frac{5}{72} + \frac{1860}{36^3} + \frac{44\,730}{36^4} + \frac{1\,172\,556}{36^5}\right) = 0.2203, \tag{6.156}$$

which is still off by 13%, and even up to $n = 20$ one arrives only at 0.229.

The new series, on the other hand, converges like $0.02/n^{5/2}$ and the lowest ($n = 0$) approximation to the massive Green function at the origin

$$v_m^<(\mathbf{0}) = \frac{\sqrt{6}}{4\pi}\left(\frac{-\sqrt{m^2} + \sqrt{m^2+12}}{\sqrt{m^2+6}} - \frac{6}{\sqrt{m^2+6}}\right) + \frac{1}{n^2+6} \tag{6.157}$$

is quite acurrate. It has the correct behavior for $m \to 0$ and can be used down to $m = 0$, where $v_0^<(\mathbf{0}) = 0.2474069$ with a maximal error of 2%. We have introduced a superscript $<$ to indicate that (6.157) lies *lower* than the true $v_m(\mathbf{0})$ for all m, due to the positivity of all omitted terms $(H_n - \bar{H}_n)/(m^2 + 2D)^{n+1}$.

An even better approximation which lies slightly above the true $v_m(\mathbf{0})$ is obtained by using the known value of Table 6.4 and Eq. (6.155) and maximizing the $n = 2$ terms as follows

$$v_m^>(\mathbf{0}) = \frac{\sqrt{6}}{4\pi}\left(\frac{-\sqrt{m^2} + \sqrt{m^2+12}}{\sqrt{m^2+6}} - \frac{6}{\sqrt{m^2+6}}\right) + \frac{1}{n^2+6} + a\frac{6^3}{(m^2+6)^3}, \tag{6.158a}$$

where

$$a = v_0(\mathbf{0}) - v_0^<(\mathbf{0}) \approx 5.3241 \times 10^{-3}. \tag{6.158b}$$

TABLE 6.5. The asymptotic coefficients $\tilde{H}_n$ of the accelerated hopping expansion of s.c. lattice Green function $v_m(\mathbf{0})$ [see Eqs. (6.152), (6.163), (6.176)].

n	$D = 2$	$D = 3$	$D = 4$
0	0.318309886183791	1.169545201850514	0.810569469138702
2	3.395305452627100	5.262953408327314	8.646074337479490
4	32.594932345220165	82.891516181155191	166.004627279606205
6	372.513512516801883	1758.484307557363699	5059.188640902284335
8	4635.723711320201211	42863.054996710740175	188876.375927018615164
10	60685.837675464452218	1132753.644322164742626	7912203.093379107078854
12	821592.879298595660796	31564615.973515706001265	357063524.214031498943167
14	11392754.592940526496372	913119247.805275780750872	16975820122.632811835469434
16	160838888.370925079948775	27161940566.148846845791565	838805229588.915408340842604
18	2302535665.099559039266677	825580035628.99784491813837	42695676715449.822890050842
20	33331944866.203140377955707	25528114101699.511933218721	2225054695113727.91289864961
22	486936238045.40239856491812	800503135279972.64000865900	118200534001298431.420386762
24	7167701424028.323306875595	25395961966757132.004274707	6379676821830080671.863008
26	106188169244864.0489907496	813647550704180421.52157041	34897377486933774615 3.1902
28	1581837555647629.971172545	26288230902369671623.914289	193087361543468057378 94.74
30	23676536316790332.47174390	855554714754863280689.2620	107896387147214494686 7649.

For $m = 0$, $v_m^>(\mathbf{0})$ starts out with the correct value $v_0(\mathbf{0}) = 0.25273$ while for $m > 0$ it gives a slightly *larger* value than the omitted series

$$\sum_{n=2,4,6,\ldots}^{\infty} \frac{H_n - \tilde{H}_n}{(m^2+6)^{n+1}}. \tag{6.159}$$

For large m, the relative error is of order $10^{-3} \times (6/(m^2+6))^2$.

The same analysis can be applied to $D = 2$ dimensions. We shall do so, for completeness only since we are already in possession of the useful formula (6.134). In two dimensions, the small m behavior is $v_m(\mathbf{0}) \sim (1/4\pi)\log m^2$ and is simulated by the power series, near $D = 2$, of

$$-\frac{1}{4\pi}\frac{1}{\left(\frac{D}{2}-1\right)}[(m^2+4-4)^{D/2-1} - (m^2+4+4)^{D/2-1}]$$

$$= \frac{1}{4\pi}\frac{1}{\frac{D}{2}-1}(m^2+4)^{D/2-1}\,2\sum_{n=0,2,\ldots}\binom{\frac{D}{2}-1}{n+1}\frac{4^{n+1}}{(m^2+2D)^{n+1}}. \tag{6.160}$$

Using

$$\frac{1}{\Gamma\left(\frac{D}{2}-n-1\right)} = \Gamma\left(2-\frac{D}{2}+n\right)\frac{\sin\pi\left(\frac{D}{2}-1-n\right)}{\pi}$$

$$\sim \Gamma\left(2-\frac{D}{2}+n\right)\left(\frac{D}{2}-1\right), \tag{6.161}$$

for n = even and $D/2 \sim 1$, the series becomes, in the limit $D \to 2$,

$$-\frac{1}{4\pi}(\log m^2 - \log(m^2+8)) = \frac{1}{2\pi}\sum_{n=\text{even}}\frac{1}{n+1}\frac{4^{n+1}}{(m^2+4)^{n+1}}$$

$$= \sum \frac{\tilde{H}_n}{(m^2+4)^{n+1}} \tag{6.162}$$

with

$$\tilde{H}_n = \frac{1}{2\pi}\frac{1}{n+1}4^{n+1}. \tag{6.163}$$

Therefore we arrive at the improved series

$$v_m(\mathbf{0}) = -\frac{1}{4\pi}(\log m^2 - \log(m^2+8)) + \sum_{n=0,2,4,\ldots} \frac{H_n - \tilde{H}_n}{(m^2+4)^{n+1}}. \quad (6.164)$$

Table 6.5 shows the values of $\tilde{H}_n$. The lowest good approximation includes the $n = 2$ term and reads

$$v_m^<(\mathbf{0}) = -\frac{1}{4\pi}\left(\log m^2 - \log(m^2+8) + \frac{8}{m^2+4}\right) + \frac{1}{m^2+4} + \frac{4 - \frac{4^3}{6\pi}}{(m^2+4)^3}. \quad (6.165)$$

It has the correct small and large m^2 behavior. For zero mass, $v_m(\mathbf{0})$ is infinite as was observed before. The lowest m^2 terms are

$$v_m^<(\mathbf{0}) \underset{m^2\approx 0}{\approx} -\frac{1}{4\pi}\log m^2 + \frac{1}{4\pi}(\log 8 - 2) + \frac{1}{4} + \frac{4 - \frac{4^3}{6\pi}}{4^3} \sim -\frac{1}{4\pi}(\log m^2 - 3.40). \quad (6.166)$$

In a later calculation we shall find that this should really be equal to $(-1/4\pi)\log(m^2/32) \sim (-1/4\pi)(\log m^2 - 3.46)$ [see (6.204)] which shows that (6.165) is an excellent approximation. Having this information we may set up an even better approximation which is slightly larger than the true $v_m(\mathbf{0})$:

$$v_m^>(\mathbf{0}) = -\frac{1}{4\pi}\left(\log m^2 - \log(m^2+8) + \frac{8}{m^2+4}\right) + \frac{1}{m^2+4} + \frac{1}{m^2+4} + \left(\frac{2+\log 4}{4\pi} - \frac{1}{4}\right)\frac{4^3}{(m^2+4)^3}. \quad (6.167)$$

It starts out with the exact small m behavior and has, at large m^2, a small relative error of order $0.36 \times 4^2/(m^2+4)^2$.

In any number of dimensions D, we can expand the small m behavior of $v_m(\mathbf{0})$ as follows:

$$v_m(\mathbf{0}) \to \frac{1}{2^D \pi^{D/2}} \Gamma(1 - D/2)(m^2)^{D/2-1}$$

$$\sim \frac{(2D)^{D/2-1}}{(4\pi)^{D/2}\Gamma(D/2)}\Bigg\{\Gamma(D/2)\,\Gamma(1-D/2)$$
$$\times \frac{(m^2+2D-2D)^{D/2-1}-(m^2+2D+2D)^{D/2-1}}{(m^2+2D)^{D/2-1}}\Bigg\}$$
$$= -2\frac{(2D)^{D/2-1}}{(4\pi)^{D/2}\Gamma(D/2)}\Bigg\{\Gamma(D/2)\,\Gamma(1-D/2)$$
$$\times \sum_{n=0,2,4,\ldots} \frac{\Gamma(D/2)}{(n+1)!\,\Gamma(D/2-n-1)}\frac{(2D)^{n+1}}{(m^2+2D)^{n+1}}\Bigg\}. \quad (6.168)$$

Therefore we can identify

$$\tilde{H}_n = -2\frac{(2D)^{D/2-1}}{(4\pi)^{D/2}\Gamma(D/2)}\Bigg\{\Gamma(D/2)\,\Gamma(1-D/2)$$
$$\times \frac{1}{(n+1)!}\frac{\Gamma(D/2)}{\Gamma(D/2-n-1)}\Bigg\}(2D)^{n+1}. \quad (6.169)$$

For even D, Eq. (6.168) has to be regularized. For this purpose, we have written $\Gamma(1-D/2)$ as $(1/\Gamma(D/2))\,\Gamma(D/2)\,\Gamma(1-D/2)$ and proceed as follows. If the dimension D is close to an even number d, say $D = d+\varepsilon$, then $\Gamma(1-D/2) = \Gamma(1-d/2-\varepsilon/2)$ has a pole term

$$\frac{1}{1-\frac{d}{2}-\frac{\varepsilon}{2}}\frac{1}{2-\frac{d}{2}-\frac{\varepsilon}{2}}\cdots\left(-\frac{2}{\varepsilon}\right)\Gamma\left(1-\frac{\varepsilon}{2}\right)$$
$$= \frac{(-)^{d/2}}{\left(\frac{d}{2}-1\right)!}\frac{2}{\varepsilon}\left[1-\frac{\varepsilon}{2}\left(-\gamma+\sum_{k=1}^{d/2-1}\frac{1}{k}\right)\right],$$

where γ is Euler's number 0.577215 ... The term $\left(-\gamma+\sum_{k=1}^{d/2-1}\frac{1}{k}\right)$ $\equiv \psi(d/2)$ tends to enter the finite parts of (6.168). This can be avoided by taking the limit $\varepsilon\to 0$ in $\Gamma(D/2)\,\Gamma(1-D/2)$ only [since $\Gamma(D/2) = \Gamma(d/2+\varepsilon/2) = (d/2-1)!(1+(\varepsilon/2)\,\psi(d/2))$ has precisely the opposite factor]. Therefore, the limit $\varepsilon\to 0$ is simple if it is taken only *inside* the curly brackets on both sides of (6.168). Outside, ε is *set* equal to zero from the beginning.

Both curly brackets have a pole $((-)^{d/2})/(d/2-1)!)(2/\varepsilon)[(m^2)^{d/2-1} - (m^2+4D)^{d/2-1}]/(m^2+2D)^{d/2-1}$ which balances on both sides of the

equation and can be removed. On the right-hand side it appears in the curly bracket of $\tilde{H}_n$ for $n = 0, \ldots, d/2$ as follows:

$$\{\quad\} = (-)^{d/2}\frac{2}{\varepsilon}\frac{1}{(n+1)!}\frac{\Gamma\left(\frac{d}{2}+\frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{d}{2}-n-1+\frac{\varepsilon}{2}\right)}$$

$$= (-)^{d/2}\frac{2}{\varepsilon}\frac{1}{(n+1)!}\frac{\Gamma\left(\frac{d}{2}\right)\left(1+\frac{\varepsilon}{2}\psi\left(\frac{d}{2}\right)\right)}{\Gamma\left(\frac{d}{2}-n-1\right)\left(1+\frac{\varepsilon}{2}\psi\left(\frac{d}{2}-n-1\right)\right)}$$

$$= (-)^{d/2}\frac{2}{\varepsilon}\frac{1}{(n+1)!}\frac{\left(\frac{d}{2}-1\right)!}{\left(\frac{d}{2}-n-2\right)!}\left[1+\frac{\varepsilon}{2}\left(\psi\left(\frac{d}{2}\right)-\psi\left(\frac{d}{2}-n-1\right)\right)\right].$$

Removing the pole for D even and $n = 0, \ldots, D/2-2$, we have

$$\tilde{H}_n = 2\frac{(-2D)^{D/2-1}}{(4\pi)^{D/2}}\frac{1}{(n+1)!}\frac{1}{\left(\frac{D}{2}-n-2\right)!}$$

$$\times\left(\frac{1}{\frac{D}{2}-1}+\frac{1}{\frac{D}{2}-2}+\ldots+\frac{1}{\frac{D}{2}-n-1}\right)(2D)^{n+1}. \tag{6.170}$$

For $D = 4$ this becomes

$$\tilde{H}_0 = -\frac{8}{\pi^2}. \tag{6.171}$$

The higher coefficients with $n > D/2$ are all finite and can be calculated from (6.169) using

$$\frac{1}{\Gamma\left(\frac{D}{2}-n-1\right)} = \frac{\sin\pi\left(\frac{D}{2}-n-1\right)}{\pi}\Gamma\left(2-\frac{D}{2}+n\right) \tag{6.172}$$

and

$$\Gamma\left(\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)=-\frac{\pi}{\sin\pi\left(\frac{D}{2}-1\right)}.$$

Since n is even, the sine factors cancel and we arrive at

$$\tilde{H}_n=2\frac{(2D)^{D/2-1}}{(4\pi)^{D/2}}\frac{\left(1-\frac{D}{2}+n\right)!}{(n+1)!}(2D)^{n+1}, \tag{6.173}$$

so that for the special case $D=4$

$$\tilde{H}_n=\frac{1}{\pi^2}\frac{1}{n(n+1)}(2D)^{n+1}. \tag{6.174}$$

In the case of odd dimensions, there are no subtleties and formula (6.173) holds for all n.

Let us study the case $D=4$ in some more detail. The expansion reads explicitly

$$v_m(\mathbf{0})=\frac{1}{2\pi^2}(m^2\log m^2-(m^2+16)\log(m^2+16)+16\log(m^2+8))$$
$$\times(m^2+8)^{-1}+\frac{1+\frac{8}{\pi^2}}{m^2+8}+\sum_{n=2,4,\ldots}\frac{H_n-\tilde{H}_n}{(m^2+8)^{n+1}}. \tag{6.175}$$

Using the numbers $\tilde{H}_n$ shown in Table 6.5 we see that this converges extremely rapidly. Dropping the $n\geq 2$ terms of the sum we find

$$v_0(\mathbf{0})=-\frac{1}{\pi^2}(\log 2-1)+\frac{1}{8}\approx 0.156\,09. \tag{6.176}$$

Adding the term $n=2$

$$\frac{8-\frac{8^3}{6\pi^2}}{(m^2+8)^3}=-0.001\,26\frac{8^3}{(m^2+8)^3}, \tag{6.177}$$

this changes to 0.154 83 as compared to the correct value 0.154 93 which is reached after a few more steps. Since all omitted terms are positive, the expansion up to $n = 2$ is an excellent lower bound:

$$v_m^<(\mathbf{0}) = \frac{1}{2\pi^2}(m^2 \log m^2 - (m^2+16)\log(m^2+16) + 16\log(m^2+8) + 16)$$
$$\times (m^2+8)^{-1} + \frac{1}{m^2+8} + \frac{8 - \frac{8^3}{6\pi^2}}{(m^2+8)^3}. \tag{6.178}$$

Just as in (6.158), (6.167) we can use the correct value $v_0(\mathbf{0}) = 0.154\,93$ and find an extremely good upper bound

$$v_m^>(\mathbf{0}) = v_m^<(\mathbf{0}) + a\frac{8^3}{(m^2+8)^3}, \tag{6.179}$$

with $a = v_0(\mathbf{0}) - v_0^<(\mathbf{0}) = 0.0001$. Notice that the scale of the $m \to 0$ logarithmic behavior in (6.175) is given by

$$v_m(\mathbf{0}) \to \frac{1}{16\pi^2} m^2 \log(m^2/c), \tag{6.180}$$

with

$$c = 4e^{1+\pi^2/4} = 128.21, \tag{6.181}$$

which is quite close to the exact value $c = 120$.

In $D = 5$ dimensions, we find the approximation

$$v_m(\mathbf{0}) = \frac{10^{3/2}}{24\pi^2}\left(\frac{m^3 - (m^2+20)^{3/2} + 30\sqrt{m^2+10}}{(m^2+10)^{3/2}}\right)$$
$$+ \frac{1}{m^2+10} + \frac{1}{(m^2+10)^3}\left(10 - \frac{5\cdot 10^{7/2}}{96\cdot\pi^2}\right), \tag{6.182}$$

which, at $m = 0$, gives

$$v_0(\mathbf{0}) = 0.1162, \tag{6.183}$$

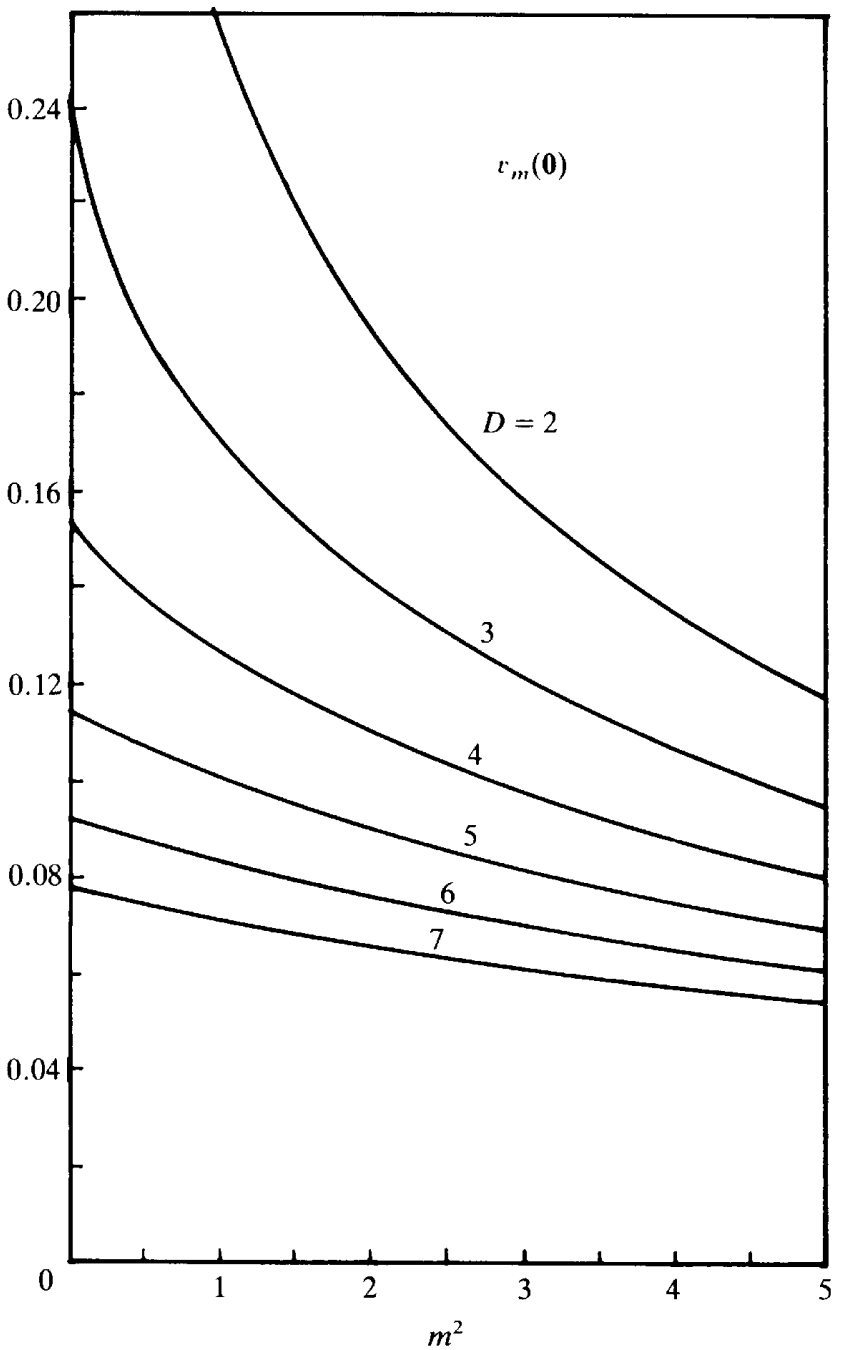
as compared to the exact value 0.11563. We may add $0.005\,342 \times 10^3/$

$(m^2 + 10)^3$ to get an even better approximation.

In more than 5 dimensions, the original series (6.125) converges rapidly enough to make the subtractions superfluous. We may use only five terms, with an error of less than 1%.

In Fig. 6.3 we have plotted $v_m(\mathbf{0})$ for various dimensions as a function of m^2.

FIG. 6.3. The lattice Green function $v_m(\mathbf{0})$ on a s.c. lattice for various dimensions, $D = 2$, 3, 4, 5 as a function of mass.



6.13. EXACT ONE- AND TWO-DIMENSIONAL COULOMB GREEN FUNCTION ON A SQUARE LATTICE

For $D = 1$ we can use the formula (6.110) together with the well known integral

$$\int_0^\infty dx e^{-\mu x} I_n(\alpha x) = \frac{1}{\sqrt{\mu^2 - \alpha^2}} \frac{\alpha^n}{(\mu + \sqrt{\mu^2 - \alpha^2})} = \frac{1}{\sqrt{\mu^2 - \alpha^2}} \frac{(\mu - \sqrt{\mu^2 - \alpha^2})^n}{\alpha},$$

to find directly

$$\begin{aligned} v_m(x) &= \frac{1}{\sqrt{(m^2+2)^2 - 4}} \left(\frac{2}{m^2 + 2 + \sqrt{(m^2+2)^2 + 4}} \right)^{|x|} \\ &= \frac{1}{\sqrt{(m^2+2)^2 - 4}} \left(\frac{m^2 + 2 - \sqrt{(m^2+2)^2 + 4}}{2} \right)^{|x|} \\ &= \frac{1}{2 \sinh \alpha} e^{-\alpha |x|}, \end{aligned} \tag{6.184}$$

with

$$\cosh \alpha = (m^2 = 2)/2, \qquad \sinh (\alpha/2) = m/2. \tag{6.185}$$

For $D = 2$ it is possible to calculate the zero-mass lattice Green function directly from (6.104),

$$v(\mathbf{x}) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi} \frac{1}{2 - \cos k_1 - \cos k_2} e^{i\mathbf{k}\cdot\mathbf{x}}$$

dropping the subscript m of $v_m(\mathbf{x})$ for $m = 0$. This expression diverges due to the singularity at $\mathbf{k} = 0$ (infrared), a divergence which was already observed in Eq. (6.117) and which appears in (6.146) for $m \to 0$. In order to obtain a finite expression we have to subtract the value of $v(\mathbf{x})$ at another place, say the origin, and form

$$\begin{aligned} v'(\mathbf{x}) \equiv v(\mathbf{x}) - v(\mathbf{0}) &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \frac{\cos \mathbf{k}\cdot\mathbf{x} - 1}{2 - \cos k_1 - \cos k_2} \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi} \frac{\cos k_1 x_1 \cos k_2 x_2 - 1}{2 - \cos k_1 - \cos k_2}. \end{aligned} \tag{6.186}$$

At the location of the nearest neighbor $\mathbf{x} = (1, 0)$, the value of $v'(\mathbf{x})$ is found trivially due to symmetry in k_1 and k_2:

$$v'(\mathbf{1}) = \frac{1}{2}\frac{1}{2}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\int_{-\pi}^{\pi}\frac{dk_2}{2\pi}\frac{(\cos k_1 - 1) + (\cos k_2 - 1)}{2 - \cos k_1 - \cos k_2} = -\frac{1}{4}.$$

In D dimensions the result would have been $-1/2D$ in accordance with the general result (6.127), which followed from the difference equation (6.51) for the massless Green function $G(\mathbf{x}) = v(\mathbf{x})$:

$$\begin{aligned}\bar{\nabla}_i\nabla_i v(\mathbf{x}) &= \sum_{i=1}^{D}[(v(\mathbf{x}+\mathbf{i}) - v(\mathbf{x})) + v(\mathbf{x}-\mathbf{i}) - v(\mathbf{x})] \\ &= \sum_{\pm\mathbf{i}}(v(\mathbf{x}+\mathbf{i}) - v(\mathbf{x})) = -\delta_{\mathbf{x},\mathbf{0}},\end{aligned} \tag{6.187}$$

implying that the $2D$ differences to the next neighbors add up to zero, except for those around the origin, which add up to minus one.

Coming back to the two-dimensional case, let us look at the values along the space diagonal

$$\mathbf{x}_d = n(1, 1). \tag{6.188}$$

Introducing new variables

$$\begin{aligned}p &= (k_1 + k_2)/2, \\ q &= (k_1 - k_2)/2,\end{aligned} \tag{6.189}$$

the integral (6.186) becomes

$$v'(\mathbf{x}_d) = \frac{1}{4}\int\frac{dp}{2\pi}\frac{dq}{\pi}\frac{\cos 2np - 1}{1 - \cos p\cos q}, \tag{6.190}$$

which is to be integrated over the diamond-shaped region $|p| + |q| < \pi$. Because of the symmetry of the integrand this region can be changed to $p \in (0, 2\pi)$, $q \in (p, p + \pi)$ and further to $p \in (0, 2\pi)$, $q \in (0, \pi)$ without altering its value. But then we may now use the formula

$$\int_0^{\pi}\frac{dp}{\pi}\frac{1}{a + b\cos q} = \frac{1}{\sqrt{a^2 - b^2}} \tag{6.191}$$

to rewrite (6.190) as

$$\begin{aligned}v'(\mathbf{x}_d) &= \frac{1}{4}\int_0^{2\pi}\frac{dp}{2\pi}\frac{\cos 2np - 1}{|\sin p|} = -\frac{1}{2}\int_0^{\pi}\frac{dp}{\pi}\frac{\sin^2 np}{\sin p} \\ &= -\frac{1}{2}\int_0^{\pi}\frac{dp}{\pi}\sum_{\ell=1}^{n}\sin(2\ell - 1)p = -\frac{1}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right).\end{aligned} \tag{6.192}$$

Explicitly, the values of $v'(\mathbf{x}_d)$ are 0, $-1/\pi$, $-4/3\pi$, $-23/15\pi$, for $n = 0$, 1, 2, 3, respectively. Once these are known it is easy to construct successively the remaining values by using the difference Eq. (6.187). The starting point is always the x-axis and from this one can work one's way up to the diagonal. The rest are obtained by symmetry. The value at $\mathbf{x} = (2, 0)$, for example, follows from

$$v'(0, 0) - v'(1, 0) + v'(2, 0) - v'(1, 0) + v'(1, 1) - v'(1, 0) + v'(-1, 1) - v'(1, 0) = 0 + \frac{1}{4} + v'(2, 0) + \frac{1}{4} - \frac{1}{\pi} + \frac{1}{4} - \frac{1}{\pi} + \frac{1}{4} = 0,$$

i.e., $v'(2, 0) = -1 + 2/\pi$. The value at $\mathbf{x} = (2, 1)$ follows from four times the value of its left neighbor minus the values on top, below, and to the left of this neighbor: $v'(2, 0) = 4(-1/\pi) - (\frac{1}{4} - 2/\pi) - (-\frac{1}{4}) - (-\frac{1}{4}) = \frac{1}{4} - (2/\pi)$. Similarly, we find the other values displayed in Tables 6.6 and 6.7.

It is easy to check that, in accordance with (6.187), each number is equal to the average of the horizontal and vertical apart from the origin, where the neighbors add up to minus one.

For large distances n, the sum (6.192) diverges like $-\log n$. Using the well-known formula

$$\sum_{k=1}^{n} \frac{1}{k} \to \gamma + \log n + O\left(\frac{1}{n}\right) \tag{6.193}$$

where $\gamma = 0.577\,216\,649\ldots$ is Euler's constant, we see that

$$\begin{aligned} &1 + \frac{1}{3} + \ldots + \frac{1}{2n-1} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2n-1} + \frac{1}{2n} + \ldots \\ &\quad - \frac{1}{2} \qquad - \frac{1}{4} \qquad\qquad\qquad - \frac{1}{2n} - \ldots \\ &\to \gamma + \log 2n - \frac{1}{2}(\gamma + \log n) \\ &= \frac{1}{2}\gamma + \frac{1}{2}\log 4n. \end{aligned} \tag{6.194}$$

Introducing the proper distance along the diagonal

$$|\mathbf{x}_d| \equiv \sqrt{2}\,n, \tag{6.195}$$

TABLE 6.6. Exact values of subtracted $D = 2$ Coulomb Green function on s.c. lattice: $-v'(\mathbf{x}) = -(v(\mathbf{x}) - v(\mathbf{0})) = -\int \frac{d^2k}{(2\pi)^2}(e^{i\mathbf{k}\cdot\mathbf{x}} - 1)\frac{1}{\bar{\mathbf{K}}\cdot\mathbf{K}}$.

x_2 \ x_1	0	1	2	3	4	5	6	7
0	0	$\frac{1}{4}$	$1-\frac{2}{\pi}$	$\frac{17}{4}-\frac{12}{\pi}$	$20-\frac{184}{3\pi}$	$\frac{401}{4}-\frac{940}{3\pi}$	$521-\frac{24526}{15\pi}$	$\frac{11073}{4}-\frac{130424}{15\pi}$
1		$\frac{1}{\pi}$	$-\frac{1}{4}+\frac{2}{\pi}$	$-2+\frac{23}{3\pi}$	$-\frac{49}{4}+\frac{40}{\pi}$	$-70+\frac{3323}{15\pi}$	$-\frac{1569}{4}+\frac{1234}{\pi}$	$-2188+\frac{721937}{3\cdot5\cdot7\pi}$
2			$\frac{4}{3\pi}$	$\frac{1}{4}+\frac{2}{3\pi}$	$3-\frac{118}{15\pi}$	$\frac{97}{4}-\frac{1118}{15\pi}$	$168-\frac{18412}{5\cdot7\pi}$	$\frac{4321}{4}-\frac{71230}{3\cdot7\pi}$
3				$\frac{23}{15\pi}$	$-\frac{1}{4}+\frac{36}{15\pi}$	$-4+\frac{499}{35\pi}$	$-\frac{161}{4}+\frac{13462}{3\cdot5\cdot7\pi}$	$-330+\frac{327143}{5\cdot7\cdot9\pi}$
4					$\frac{176}{105\pi}$	$\frac{1}{4}+\frac{20}{21\pi}$	$5-\frac{626}{5\cdot9\pi}$	$\frac{241}{4}-\frac{1312}{7\cdot\pi}$
5						$\frac{563}{315\pi}$	$-\frac{1}{4}+\frac{118}{5\cdot9\pi}$	$-6+\frac{14369}{7\cdot9\cdot11\pi}$
6							$\frac{6508}{5\cdot7\cdot9\cdot11\pi}$	$\frac{1}{4}+\frac{786}{7\cdot9\cdot11\pi}$
7								$\frac{88069}{5\cdot7\cdot9\cdot11\cdot13\pi}$

TABLE 6.7. Numerical values of subtracted $D = 2$ Coulomb potential $v'(\mathbf{x})$ on s.c. lattice.

	0	1	2	3	4	5	6	7	8	9	10
0	0.0000	0.2500	0.3634	0.4303	0.4770	0.5129	0.5421	0.5668	0.5881	0.6069	0.6237
1		0.3183	0.3866	0.4404	0.4824	0.5163	0.5444	0.5684	0.5893	0.6079	0.6245
2			0.4244	0.4622	0.4960	0.5253	0.5508	0.5732	0.5930	0.6108	0.6268
3				0.4881	0.5139	0.5382	0.5604	0.5805	0.5987	0.6154	0.6306
4					0.5335	0.5532	0.5720	0.5896	0.6061	0.6214	0.6356
5						0.5689	0.5847	0.6000	0.6146	0.6285	0.6416
6							0.5979	0.6110	0.6239	0.6364	0.6483
7								0.6223	0.6337	0.6448	0.6556
8									0.6436	0.6535	0.6633
9										0.6623	0.6711
10											0.6790

TABLE 6.7. (continued)

	11	12	13	14	15	16	17	18	19	20
0	0.6389	0.6527	0.6655	0.6773	0.6883	0.6986	0.7082	0.7173	0.7259	0.7341
1	0.6395	0.6533	0.6660	0.6777	0.6886	0.6989	0.7085	0.7176	0.7261	0.7343
2	0.6415	0.6549	0.6674	0.6789	0.6897	0.6998	0.7093	0.7183	0.7268	0.7349
3	0.6446	0.6576	0.6697	0.6809	0.6914	0.7013	0.7107	0.7195	0.7279	0.7359
4	0.6488	0.6612	0.6727	0.6836	0.6938	0.7034	0.7125	0.7212	0.7294	0.7372
5	0.6539	0.6656	0.6765	0.6869	0.6967	0.7060	0.7148	0.7233	0.7313	0.7389
6	0.6597	0.6706	0.6809	0.6908	0.7002	0.7091	0.7176	0.7257	0.7335	0.7410
7	0.6661	0.6762	0.6859	0.6951	0.7040	0.7126	0.7207	0.7286	0.7361	0.7433
8	0.6728	0.6821	0.6912	0.6999	0.7083	0.7164	0.7242	0.7317	0.7390	0.7459
9	0.6798	0.6884	0.6968	0.7049	0.7128	0.7205	0.7279	0.7351	0.7421	0.7488
10	0.6870	0.6948	0.7026	0.7102	0.7176	0.7249	0.7319	0.7388	0.7454	0.7519
11	0.6942	0.7014	0.7086	0.7157	0.7226	0.7294	0.7361	0.7426	0.7490	0.7552
12		0.7080	0.7147	0.7212	0.7277	0.7342	0.7405	0.7466	0.7527	0.7586
13			0.7208	0.7269	0.7330	0.7390	0.7449	0.7508	0.7565	0.7622
14				0.7326	0.7382	0.7439	0.7495	0.7550	0.7605	0.7659
15					0.7435	0.7488	0.7541	0.7593	0.7645	0.7697
16						0.7538	0.7588	0.7637	0.7686	0.7735
17							0.7634	0.7681	0.7728	0.7774
18								0.7725	0.7770	0.7814
19									0.7811	0.7853
20										0.7893

we arrive at the asymptotic behavior

$$v'(\mathbf{x}_d) \sim -\frac{1}{2\pi}\log(|\mathbf{x}_d|\, 2\sqrt{2}\, e^{\gamma}). \tag{6.196}$$

Let us compare this with the behavior along the x-axis. We have to integrate

$$\begin{aligned} v'(x) &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\int_{-\pi}^{\pi}\frac{dk_2}{2\pi}\frac{\cos k_1 x - 1}{2 - \cos k_1 - \cos k_2} \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\frac{\cos k_1 x - 1}{\sqrt{(2-\cos k_1)^2 - 1}} = \frac{1}{4}\int_0^{2\pi}\frac{dk_1}{2\pi}\frac{\cos k_1 x - 1}{\sin\dfrac{k_1}{2}\sqrt{1+\sin^2\dfrac{k_1}{2}}}. \end{aligned} \tag{6.197}$$

This can be split into two pieces:

$$v'(x) = \frac{1}{4}\int_0^{2\pi}\frac{dk_1}{2\pi}\frac{\cos k_1 x - 1}{\sin k_1/2} + \frac{1}{4}\int_0^{\pi}\frac{dk_1}{2\pi}\frac{\cos k_1 x - 1}{\sin k_1/2}\frac{1-\sqrt{1+\sin^2\dfrac{k_1}{2}}}{\sqrt{1+\sin^2\dfrac{k_1}{2}}}. \tag{6.198}$$

The first piece is just the diagonal propagator at (x, x), i.e., it behaves asymptotically as [compare (6.192), (6,196)]

$$-\frac{1}{2\pi}\log(4e^{\gamma}x). \tag{6.199}$$

The second piece is complicated. In the limit $x \to \infty$, however, the oscillating part due to $\cos k_1 x$ can be neglected, since the remaining integrand is a smooth function of k_1. In this limit, we may write it simply as

$$-\frac{1}{2\pi}\int_0^{\pi/2} dx \frac{1}{\sin x}\left(\frac{1}{\sqrt{1+\sin^2 x}} - 1\right). \tag{6.200}$$

With $y \equiv \sin x$ this becomes

$$-\frac{1}{2\pi}\int_{\varepsilon}^{1}\frac{dy}{y}\left(\frac{1}{\sqrt{1-y^4}}-\frac{1}{\sqrt{1-y^2}}\right), \tag{6.201}$$

where the parameter ε regulates the common singularity at the origin. The second integral is equal to

$$\int_{\varepsilon}^{1}\frac{dy}{y}\frac{1}{\sqrt{1-y^2}}=\log y-\log(1+\sqrt{1-y^2})\Big|_{\varepsilon}^{1}=-\log\varepsilon-\log 2.$$

In the first we can make the change of variables, $y^2\to y$ and obtain

$$\frac{1}{2}\int_{\varepsilon^2}^{1}\frac{dy}{y}\frac{1}{\sqrt{1-y^2}}=\frac{1}{2}(\log y-\log(1+\sqrt{1-y^2}))\Big|_{\varepsilon^2}^{1}=-\frac{1}{2}(\log\varepsilon^2-\log 2).$$

Thus, (6.201) gives $(1/2\pi)\log\sqrt{2}$. Adding this to (6.199) changes the factor 4 to $2\sqrt{2}$ so that the asymptotic behavior along the x axis becomes the same as that along the diagonal in Eq. (6.196). This is, of course, just a manifestation of the general formula (6.119) which tells us that the asymptotic behaviour is isotropic. Remarkably enough, the asymptotic form

$$v^{\text{'as}}(\mathbf{x})\sim-\frac{1}{2\pi}[\log|\mathbf{x}|+\log 2\sqrt{2}e^{\gamma})]=-\frac{1}{2\pi}(\log|\mathbf{x}|+1.616936) \tag{6.202}$$

is quite a good approximation to the exact Green function even down to the nearest neighbors. There the exact value is $-1/4$ while the asymptotic form gives -0.2573. There is an error of only 2.7%. The other values are compared in Table 6.8 whose entries show for each lattice place, from top to bottom the exact values, the asymptotic ones, and the error in %.

For $m\neq 0$, the calculation of (6.106) is again complicated. As an example, let us calculate the constant in the asymptotic behavior

$$v_m(\mathbf{0})\xrightarrow[m\to 0]{}-\frac{1}{4\pi}\log(m^2/\text{const.})+O(m^2).$$

This can be found by inserting a mass into (6.186) and going to the form (6.190):

$$v_m(\mathbf{0})=\frac{1}{4}\int_0^{2\pi}\frac{dp}{2\pi}\int_0^{\pi}\frac{dq}{\pi}\frac{1}{1-\cos p\cos q+\dfrac{m^2}{4}}.$$

Using (6.191) gives

$$v_m(\mathbf{0}) = \frac{1}{2\pi}\int_0^{\pi/2} dp \frac{1}{\sqrt{\sin^2 p + \frac{m^2}{2} + \frac{m^4}{16}}}. \tag{6.203}$$

For small m this may be evaluated by introducing a small mass parameter δ with $\pi/2 \gg \delta^2 \gg m^2$ and splitting the integral as follows:

$$v_m(\mathbf{0}) \sim \frac{1}{2\pi}\left\{\int_0^{\delta} dp \frac{1}{\sqrt{p^2 + \frac{m^2}{2}}} + \int_{\delta}^{\pi/2} dp\left(\frac{1}{\sin p} - \frac{1}{p}\right) + \log\frac{\pi}{2\delta}\right\}.$$

The first integral has the limit

$$\int_0^{\delta} dp \frac{1}{\sqrt{p^2 + \frac{m^2}{2}}} = \log(\delta\sqrt{2}/m + \sqrt{2\delta^2/m^2 + 1}) \to \log(2\sqrt{2}\,\delta/m),$$

TABLE 6.8. The top two values in each row give the $D = 2$ subtracted Coulomb Green function on the s.c. lattice $v'(x, y) = v(x, y) - v(0, 0)$, and the asymptotic form (6.202), $(-1/4\pi)\log(8e^{2\gamma}\mathbf{x}^2)$, respectively. The bottom values show the relative error of the asymptotic form.

y \ x	0	1	2	3
3	−0.4303 −0.4322 0.4%	−0.4404 −0.4406 0.05%	−0.4622 −0.4615 −0.15%	−0.4881 −0.4874 −0.14%
2	−0.3634 −0.3677 1.2%	−0.3866 −0.3854 −0.32%	−0.4244 −0.4228 −0.4%	−0.4622 −0.4615 −0.15%
1	−0.2500 −0.2573 2.9%	−0.3183 −0.3125 −1.8%	−0.3866 −0.3854 −0.32%	−0.4404 −0.4406 0.05%
0	0 ∞	−0.2500 −0.2573 2.92%	−0.3634 −0.3677 1.2%	−0.4303 −0.4322 0.44%

while the second gives

$$\log(2\tan\pi/4) - \log\pi/2$$

Together this shows that

$$v_m(\mathbf{0}) \to -\frac{1}{2\pi}\log\frac{m}{4\sqrt{2}} = -\frac{1}{4\pi}\log\frac{m^2}{32}.$$

For dimension higher than $D = 2$, we can use formula (6.107) or the expansion (6.122) to calculate $v_m(\mathbf{x})$.

In Table 6.9 we have listed the results for $v_0(\mathbf{x})$ in $D = 3$ dimensions and compared them with the asymptotic form $v_0(\mathbf{x}) \sim 1/(4\pi|\mathbf{x}|)$. Again, the agreement is surprisingly good even at the next neighbor place where $v_0(\mathbf{x}) \approx 0.0861$ and $1/(4\pi|\mathbf{1}|) \approx 0.080$. In Table 6.10 we have given the corresponding list for $D = 4$.

TABLE 6.9. Lattice propagator $v(i, j, k)$ ($D = 3$, s.c. lattice) and comparison with asymptotic form ($r = \sqrt{i^2 + j^2 + k^2}$, $v^{\text{as}} = 1/(4\pi r)$).

$i\ j\ k$	v	r	v^{as}	v/v^{as}
0 0 0	0.252731	0.0000	—	—
0 0 1	0.086064	1.0000	0.079577	1.0815
0 0 2	0.042889	2.0000	0.039789	1.0779
0 0 3	0.027545	3.0000	0.026526	1.0384
0 0 4	0.020289	4.0000	0.019894	1.0198
0 0 5	0.016101	5.0000	0.015915	1.0117
0 1 1	0.055191	1.4142	0.056270	0.9808
0 1 2	0.035932	2.2361	0.035588	1.0097
0 1 3	0.025523	3.1623	0.025165	1.0142
0 1 4	0.019522	4.1231	0.019300	1.0115
0 1 5	0.015738	5.0990	0.015606	1.0084
0 2 2	0.028055	2.8284	0.028135	0.9972
0 2 3	0.022075	3.6056	0.022071	1.0002
0 2 4	0.017843	4.4721	0.017794	1.0027
0 2 5	0.014830	5.3852	0.014777	1.0035
0 3 3	0.018715	4.2426	0.018757	0.9978
0 3 4	0.015899	5.0000	0.015915	0.9990
0 3 5	0.013652	5.8310	0.013647	1.0003
0 4 4	0.014045	5.6569	0.014067	0.9984
0 4 5	0.012416	6.4031	0.012428	0.9990
0 5 5	0.011242	7.0711	0.011254	0.9989

TABLE 6.9. (continued)

$i\ j\ k$	v	r	v^{as}	v/v^{as}
1 1 1	0.043 578	1.7321	0.045 944	0.9485
1 1 2	0.031 965	2.4495	0.032 487	0.9839
1 1 3	0.024 033	3.3166	0.023 994	1.0016
1 1 4	0.018 869	4.2426	0.018 757	1.0060
1 1 5	0.015 406	5.1962	0.015 315	1.0060
1 2 2	0.026 159	3.0000	0.026 526	0.9862
1 2 3	0.021 158	3.7417	0.021 268	0.9948
1 2 4	0.017 365	4.5826	0.017 365	1.0000
1 2 5	0.014 559	5.4772	0.014 529	1.0021
1 3 3	0.018 170	4.3589	0.018 256	0.9953
1 3 4	0.015 570	5.0990	0.015 606	0.9976
1 3 5	0.013 445	5.9161	0.013 451	0.9995
1 4 4	0.013 821	5.7446	0.013 853	0.9977
1 4 5	0.012 262	6.4807	0.012 279	0.9986
1 5 5	0.011 128	7.1414	0.011 143	0.9986
2 2 2	0.022 651	3.4641	0.022 972	0.9860
2 2 3	0.019 144	4.1231	0.019 300	0.9919
2 2 4	0.016 193	4.8990	0.016 244	0.9969
2 2 5	0.013 849	5.7446	0.013 853	0.9997
2 3 3	0.016 852	4.6904	0.016 966	0.9933
2 3 4	0.014 718	5.3852	0.014 777	0.9960
2 3 5	0.012 886	6.1644	0.012 909	0.9982
2 4 4	0.013 218	6.0000	0.013 263	0.9966
2 4 5	0.011 837	6.7082	0.011 863	0.9978
2 5 5	0.010 808	7.3485	0.010 829	0.9981
3 3 3	0.015 219	5.1962	0.015 315	0.9938
3 3 4	0.013 587	5.8310	0.013 647	0.9955
3 3 5	0.012 103	6.5574	0.012 135	0.9974
3 4 4	0.012 381	6.4031	0.012 428	0.9962
3 4 5	0.011 223	7.0711	0.011 254	0.9973
3 5 5	0.010 336	7.6811	0.010 360	0.9977
4 4 4	0.011 446	6.9282	0.011 486	0.9965
4 4 5	0.010 511	7.5498	0.010 540	0.9972
4 5 5	0.009 772	8.1240	0.009 795	0.9976
5 5 5	0.009 168	8.6603	0.009 189	0.9978

TABLE 6.10. Lattice propagator $v(i, j, k, l)$ ($D = 4$, s.c. lattice) and comparison with asymptotic form ($r = \sqrt{i^2 + j^2 + k^2 + l^2}$, $v^{as} = 1/(4\pi^2 r^2)$).

i j k l	v	r	v^{as}	v/v^{as}
0 0 0 0	0.154 933	0.0000	—	—
0 0 0 1	0.029 933	1.0000	0.025 330	1.1817
0 0 0 2	0.008 246	2.0000	0.006 333	1.3021
0 0 0 3	0.003 287	3.0000	0.002 814	1.1678
0 0 1 1	0.012 715	1.4142	0.012 665	1.0039
0 0 1 2	0.005 457	2.2361	0.005 066	1.0772
0 0 1 3	0.002 721	3.1623	0.002 533	1.0743
0 0 2 2	0.003 249	2.8284	0.003 166	1.0260
0 0 2 3	0.001 991	3.6056	0.001 948	1.0218
0 0 3 3	0.001 418	4.2426	0.001 407	1.0075
0 1 1 1	0.007 734	1.7321	0.008 443	0.9160
0 1 1 2	0.004 182	2.4495	0.004 222	0.9906
0 1 1 3	0.002 366	3.3166	0.002 303	1.0275
0 1 2 2	0.002 773	3.0000	0.002 814	0.9854
0 1 2 3	0.001 811	3.7417	0.001 809	1.0012
0 1 3 3	0.001 330	4.3589	0.001 333	0.9978
0 2 2 2	0.002 063	3.4641	0.002 111	0.9775
0 2 2 3	0.001 473	4.1231	0.001 490	0.9885
0 2 3 3	0.001 139	4.6904	0.001 151	0.9893
0 3 3 3	0.000 928	5.1962	0.000 938	0.9887
1 1 1 1	0.005 591	2.0000	0.006 333	0.8829
1 1 1 2	0.003 448	2.6458	0.003 619	0.9528
1 1 1 3	0.002 114	3.4641	0.002 111	1.0015
1 1 2 2	0.002 444	3.1623	0.002 533	0.9647
1 1 2 3	0.001 669	3.8730	0.001 689	0.9886
1 1 3 3	0.001 255	4.4721	0.001 267	0.9911
1 2 2 2	0.001 884	3.6056	0.001 948	0.9670
1 2 2 3	0.001 382	4.2426	0.001 407	0.9821
1 2 3 3	0.001 086	4.7958	0.001 101	0.9858
1 3 3 3	0.000 893	5.2915	0.000 905	0.9867
2 2 2 2	0.001 533	4.0000	0.001 583	0.9681
2 2 2 3	0.001 181	4.5826	0.001 206	0.9791
2 2 3 3	0.000 958	5.0990	0.000 974	0.9831
2 3 3 3	0.000 805	5.5678	0.000 817	0.9848
3 3 3 3	0.000 694	6.0000	0.000 704	0.9859

6.14. FLUCTUATION DETERMINANTS

In fluctuation studies on the lattice, not only the Green function itself but also its determinant, $\det(-\bar{\nabla}\cdot\nabla + m^2)$, will be required. Let us calculate also this quantity here. In one dimension, we may simply integrate Eq. (6.110) in m^2 and find directly

$$\begin{aligned}\frac{1}{N}\log\det(-\bar{\nabla}\cdot\nabla + m^2) &= \frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla + m^2)\\ &= \int^{m^2} dm'^2 v_{m'}(\mathbf{0})\\ &= \log\left(\frac{m^2+2}{2} + \sqrt{\frac{(m^2+2)^2-4}{4}}\right)\\ &= 2\sinh^{-1}(m/2) = \alpha, \qquad (6.204)\end{aligned}$$

where the constant of integration is chosen such that for large m^2, the right-hand side has the same asymptotic behavior as the left-hand side, namely, $\log m^2 + O(1/m^2)$ [i.e., with no $(m^2)^0$ term]. Inserting $m = 0$ we see that the trace log vanishes, i.e., the fluctuation determinant is equal to unity. This can also be found by performing the integral directly,

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\log(2-2\cos k) &= \log 2 + \int_{-\pi}^{\pi}\frac{dk}{2\pi}\log(1-\cos k)\\ &= \log 2 + \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^2}}(\log(1-x)+\log(1+x))\\ &= 0.\end{aligned}$$

The integral $\int_0^1 dx \frac{1}{\sqrt{1-x^2}}\log(1\pm x)$ can be found in most tables and has the value $(-\pi/2)\log 2 \pm 2G$ where $G = 0.915\,965\,594\ldots$ is Catalan's number $\sum_{k=0}^{\infty}(-)^k\frac{1}{(2k+1)^2}$.

In two dimensions, the $m = 0$ case can also be reduced to this standard integral. This goes as follows. In

$$\frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla) = \int_{-\pi}^{\pi}\frac{dk_1 dk_2}{(2\pi)^2}\log\left(2\sum_{i=1}^{2}(1-\cos k_i)\right),$$

we introduce again variables p and q, as in (6.190), and find

$$\log 4 + \int_0^{2\pi} \frac{dp}{2\pi} \int_0^{\pi} \frac{dq}{\pi} \log(1 - \cos p \cos q).$$

Now we use the formula

$$\int_0^{\pi} dx \log(a - b\cos x) = \pi \log[(a + \sqrt{a^2 - b^2})/2]$$

and arrive at

$$\log 4 + \int_0^{\pi} \frac{dp}{\pi} \log[(1 + \sin p)/2] = \log 2 + \int_0^{\pi/2} \frac{dp'}{\pi/2} \log(1 - \cos p')$$

$$= \log 2 + \frac{2}{\pi} \int_0^1 dx \frac{1}{\sqrt{1 - x^2}} \log(1 + x)$$

so that we have

$$\frac{1}{N} \operatorname{tr} \log(-\bar{\nabla} \cdot \nabla) = \frac{4}{\pi} G = 1.16625\ldots, \quad D = 2.$$

In general, we have to perform numerical calculations, for instance, via the hopping expansion in powers of $2 \sum_{i=1}^{D} \cos k_i$:

$$\frac{1}{N} \log \det(-\bar{\nabla} \cdot \nabla + m^2) = \int \frac{d^D k}{(2\pi)^D} \log\left(m^2 + 2 \sum_i (1 - \cos k_i) \right)$$

$$= \log(m^2 + 2D) - \int \frac{d^D k}{(2\pi)^D} \left(2 \sum_{i=1}^{D} \cos k_i \right)^n \frac{1}{n(m^2 + 2D)^n}$$

$$= \log(m^2 + 2D) - \sum_{n=2,4,\ldots}^{\infty} \frac{H_n}{n(m^2 + 2D)^n}. \qquad (6.205)$$

Using the H_n determined previously, this gives for $D = 2, 3, 4, 5, \ldots$ dimensions and for $m = 0$ (see also Table 6.4)

$$\frac{1}{N} \operatorname{tr} \log(-\bar{\nabla} \cdot \nabla) = 1.16625,\ 1.67339,\ 1.99971,\ 2.24249, \ldots \qquad (6.206)$$

respectively. For large D, this approaches $\log 2D$. It is convenient to remove this leading behavior and state the result in the form

$$\frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla/2D) = 0.22004,\ -0.11837,\ -0.07973,\ -0.06010,\ldots \tag{6.207}$$

The series (6.205) converges faster than that of $v_m(\mathbf{0})$ by one power of $1/n$, i.e., in D dimensions like $1/n^{D/2+1}$. The convergence can be improved by one more power using the same techniques as those employed for $v_m(\mathbf{0})$.

We can save work by simply integrating the equations (6.153), (6.164), (6.175). Thus, using

$$\int dx\left(\frac{x+a}{x+b}\right)^{1/2} = \sqrt{(x+a)(x+b)} + (a-b)\log[(\sqrt{x+a}+\sqrt{(x+b)})/2], \tag{6.208}$$

we are led to the following result in $D = 3$ dimensions:

$$\begin{aligned}\frac{1}{N}&\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla + m^2)\\ &= \frac{\sqrt{6}}{4\pi}\{-\sqrt{m^2(m^2+6)} + \sqrt{(m^2+12)(m^2+6)} - 6\\ &\quad + 6\log[(m+\sqrt{m^2+6})(\sqrt{m^2+12}+\sqrt{m^2+6})/4]\}\\ &\quad + \log(m^2+6)\left(1-\frac{6\sqrt{6}}{4\pi}\right) - \sum_{n=2,4,6\ldots,n}\frac{H_n - \tilde{H}_n}{n(m^2+6)^n}.\end{aligned} \tag{6.209}$$

We have again adjusted the constant of integration by the absence of the behavior $(m^2)^0$ for $m^2 \to \infty$. Taking only the $n = 2$ term of the series we obtain the maximizing approximation

$$\begin{aligned}\frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla + m^2)^{>} = \frac{\sqrt{6}}{4\pi}\Bigg\{&-m\sqrt{(m^2+6)} + \sqrt{(m^2+12)(m^2+6)} - 6\\ &+ 6\log\left[\frac{(\sqrt{m^2+6}+m)(\sqrt{m^2+12}+\sqrt{m^2+6})}{4(m^2+6)}\right]\Bigg\}\end{aligned}$$

$$+\log(m^2+6)-\frac{1}{2}\frac{6-\dfrac{6\sqrt{6}}{4\pi}\dfrac{6^2}{8}}{(m^2+6)^2}. \tag{6.210a}$$

For $m=0$ this gives

$$\begin{aligned}\frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla)^{>} &= \frac{6\sqrt{6}}{4\pi}\{\sqrt{2}-1)+\log[(1+\sqrt{2})/4]\}+\log 6-\frac{1}{2}\frac{6-\dfrac{6\sqrt{6}}{4\pi}\dfrac{6^2}{8}}{6^2}\\ &= \log 6+\frac{6\sqrt{6}}{4\pi}\left\{\left(\sqrt{2}-\frac{15}{16}\right)+\log(1+\sqrt{2})/4\right)\right]-\frac{1}{12}\\ &\approx \log 6-0.11632\end{aligned} \tag{6.210b}$$

in good agreement with the correct value $\log 6-0.11837$ [see (6.206b)]. This value can be reproduced by adding to (6.210a) the term $-(0.11837-0.11632)(6^2/(m^2+6)^2)$ and the resulting approximation is *smaller* than the exact expression.

In $D=2$ dimensions, a similar procedure gives

$$\begin{aligned}\frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla+m^2)^{>} &= -\frac{1}{4\pi}\{m^2[\log m^2-1]-(m^2+8)[\log(m^2+8)-1]\}\\ &\quad+\log(m^2+4)\left(1-\frac{8}{4\pi}\right)-\sum_{n=2,4,6,\ldots}\frac{H_n-\bar{H}_n}{n(m^2+4)^n}.\end{aligned} \tag{6.211}$$

Keeping the series only up to the $n=2$ term of the sum we have

$$\begin{aligned}&\frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla+m^2)^{>}\\ &\quad=-\frac{1}{4\pi}\{m^2\log m^2-(m^2+8)\log(m^2+8)+8\log(m^2+4)+8\}\\ &\qquad+\log(m^2+4)-\frac{1}{2(m^2+4)^2}\left(4-\frac{4^3}{6\pi}\right).\end{aligned} \tag{6.212}$$

For $m=0$ this becomes

$$\frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\nabla)^{>} = \log 4 - \frac{2}{\pi}(1-\log 2) - \frac{1}{32}\left(4 - \frac{4^3}{6\pi}\right)$$

$$= \log 4 - \frac{1}{\pi}\left(\frac{5}{3} - 2\log 2\right) - \frac{1}{8} \approx \log 4 - 0.214\,25, \tag{6.213}$$

again in good agreement with the correct value $\log 4 - 0.22004$ of (6.206b). Adding to (6.212) $-(0.220\,04 - 0.214\,25)(4^2/(m^2+4)^2)$ gives a maximizing approximation which is exact at $m = 0$.

In an arbitrary number of dimensions D, the logarithm behaves for $m \to 0$ like

$$\frac{1}{N}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla) \sim \int \frac{d^D k}{(2\pi)^D}\log(m^2+k^2) \sim -\frac{1}{(4\pi)^{D/2}}\Gamma\left(-\frac{D}{2}\right)(m^2)^{D/2}, \tag{6.214}$$

as can be found from (6.144) by integration. The following expression has the same $m \to 0$ behavior

$$\frac{(2D)^{D/2-1}}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}+1\right)}\left\{-\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(-\frac{D}{2}\right)\right.$$

$$\left.\times\frac{(m^2+2D-2D)^{D/2} - (m^2+2D+2D)^{D/2}}{(m^2+2D)^{D/2-1}}\right\}. \tag{6.215a}$$

This can be expanded in powers of $1/(m^2+2D)$, i.e.,

$$\frac{(2D)^{D/2-1}}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}+1\right)}\left\{2\sum_{n=1,3,5,\ldots}\frac{\Gamma^2\left(\frac{D}{2}+1\right)\Gamma\left(-\frac{D}{2}\right)}{n!\,\Gamma\left(\frac{D}{2}+1-n\right)}\frac{(2D)^n}{(m^2+2D)^{n-1}}\right\}$$

$$= -\sum_{n=0,2,4,\ldots}\frac{L_n}{(m^2+2D)^n}. \tag{6.215b}$$

For odd D or even D and $n > D/2 - 1$, the expansion coefficients L_n are found directly to be

$$L_n = \frac{(2D)^{D/2-1}}{(4\pi)^{D/2}}\, 2\, \frac{\Gamma\left(-\frac{D}{2}+n+1\right)}{(n+1)!}(2D)^{n+1} = \tilde{H}_n \frac{1}{-\frac{D}{2}+n+1}. \tag{6.216}$$

For even D we set $D = d + \varepsilon$ and take the limit $\varepsilon \to 0$ inside the curly bracket, just as we did with (6.168). Then (6.215a) becomes

$$\frac{(2d)^{d/2-1}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\left\{(-)^{d/2}\frac{2}{\varepsilon}\left[\frac{(m^2)^{d/2}-(m^2+4d)^{d/2}}{(m^2+2d)^{d/2-1}}\left(1-\frac{\varepsilon}{2}\log(m^2+2d)\right)\right.\right.$$

$$\left.\left.+\frac{\varepsilon}{2}\frac{(m^2)^{d/2}\log m^2-(m^2+4d)^{d/2}\log(m^2+4d)}{(m^2+2d)^{d/2-1}}\right]\right\}$$

$$= \text{pole term} + \frac{(2d)^{d/2-1}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}(-)^{d/2}$$

$$\times\{(m^2)^{d/2}\log m^2-(m^2+4d)^{d/2}\log(m^2+4d)$$

$$-[(m^2)^{d/2}-(m^2+4d)^{d/2}]\log(m^2+2d)\}/(m^2+2d)^{d/2-1} \tag{6.217}$$

The coefficients L_n for $n \geq D/2-2$ require subtraction with the pole terms cancelling on both sides of the equation. Explicitly, we have

$$L_n = -\frac{(2d)^{d/2-1}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\left\{2\left(-\frac{2}{\varepsilon}\right)\frac{\Gamma\left(\frac{d}{2}+1+\frac{\varepsilon}{2}\right)}{(n+1)!\,\Gamma\left(\frac{d}{2}+1-n+\frac{\varepsilon}{2}\right)}\right\}(2d)^{n+1}$$

$$= \frac{(2d)^{d/2-1}}{(4\pi)^{D/2}}\frac{4}{\varepsilon}\frac{1}{(n+1)!}\frac{1}{\left(\frac{d}{2}-n-1\right)!}$$

$$\times\left\{1+\frac{\varepsilon}{2}\left(\psi\left(\frac{d}{2}+1\right)-\psi\left(\frac{d}{2}-n\right)\right)\right\}(2d)^{n+1}, \tag{6.218}$$

where $\psi(n) \equiv \Gamma'(n)/\Gamma(n)$ is the usual psi function $-\gamma + \sum_{k=1}^{n-1} \frac{1}{k}$. Therefore, the finite part is

$$L_n = \frac{(2d)^{d/2-1}}{(4\pi)^{d/2}} 2 \frac{1}{(n+1)!} \frac{(2d)^{n+1}}{\left(\frac{d}{2}-n-1\right)!} \left(\frac{1}{\frac{d}{2}} + \frac{1}{\frac{d}{2}-1} + \ldots + \frac{1}{\frac{d}{2}-n}\right). \tag{6.219}$$

For $D = 4$ this leads to the improved series

$$\begin{aligned} &\frac{1}{N} \operatorname{tr} \log(-\bar{\nabla}\cdot\nabla + m^2) \\ &= \frac{8}{32\pi^2}[m^4 \log m^2 - (m^2+16)^2 \log(m^2+16) \\ &\quad - (m^4 - (m^2+16)^2) \log(m^2+8) + 16(m^2+8)]/(m^2+8) \\ &\quad + \log(m^2+8) - \sum_{n=2,4,6,\ldots} \frac{1}{(m^2+8)^n} \left(\frac{H_n}{n} - \frac{\bar{H}_n}{-\frac{D}{2}+n+1}\right). \end{aligned} \tag{6.220}$$

Keeping only the $n = 2$ term we arrive at the approximation

$$\begin{aligned} &\frac{1}{N} \operatorname{tr} \log(-\bar{\nabla}\cdot\nabla + m^2)^{<} \\ &= \frac{8}{32\pi^2}[m^4 \log m^2 - (m^2+16)^2 \log(m^2+16) \\ &\quad - (m^4 - (m^2+16)^2) \log(m^2+8) + 16(m^2+8)]/(m^2+8) \\ &\quad + \log(m^2+8) - \left(4 - \frac{8^3}{6\pi^2}\right) \Big/ (m^2+8)^2. \end{aligned} \tag{6.221}$$

For $m = 0$ this gives

$$\frac{1}{N} \operatorname{tr} \log(-\bar{\nabla}\cdot\nabla) = \log 8 + \frac{8}{\pi^2}\left(\frac{2}{3} - \log 2\right) - \frac{1}{16} = \log 8 - 0.083\,96, \tag{6.222}$$

which is to be compared with the correct value of $\log 8 - 0.07973$ (see (6.206a)). The next two corrections are

$$-\left(\frac{168}{4}-\frac{8^5}{60\pi^2}\right)\frac{1}{(m^2+8)^4}-\left(\frac{2560}{3}-\frac{8^7}{210\pi^2}\right)\frac{1}{(m^2+8)^6}$$

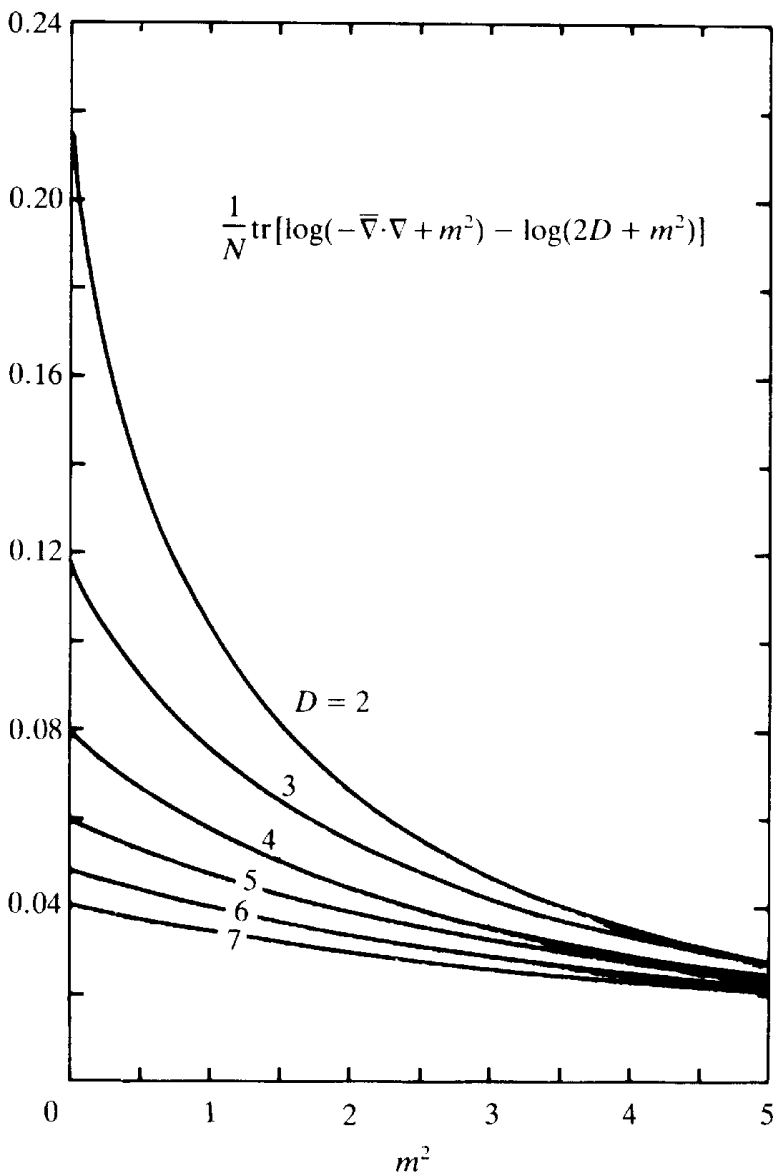
$$=0.003\,25\frac{8^4}{(m^2+8)^4}+6.047+10^{-4}\frac{8}{(m^2+8)^6} \tag{6.223}$$

so that the $m=0$ value becomes $\log 8-0.0801=1.9993$ and is correct up to 0.2%.

In more than 4 dimensions, the original series (6.205) converges fast enough to make the subtraction procedure superfluous. For example, in 5 dimensions, truncation after the third ($n=6$) term gives only a 1.5% error at the worst place $m=0$.

In Fig. 6.4 we have plotted the fluctuation determinant for various

FIG. 6.4. The fluctuation determinant $\int(d^Dk/(2\pi)^D)\log(\bar{\mathbf{K}}\cdot\mathbf{K}+m^2)$ on a simple cubic lattice as a function of m (with $\log(2D+m^2)$ subtracted).



dimensions as a function of m^2. In Appendix 6A we have given the results of a similar discussion for the triangular lattice, body-centered and face-centered cubic lattices.

6.15. FINITE-SIZE EFFECTS ON THE FLUCTUATION DETERMINANT AND QUANTUM STATISTICS OF A FREE FIELD

The results of the last sections are valid only in the limit of infinite systems. Under many circumstances, however, the systems are too small to justify this limit. This is true for *any* system showing quantum fluctuations at nonzero temperature. We have seen in Section 1.7 that these can be thought of as taking place in an imaginary time interval of length $\hbar/k_B T$ with all fields satisfying periodic boundary conditions $\varphi(\mathbf{x}, \tau) = \varphi(\mathbf{x}, \tau + \hbar/k_B T)$. Correspondingly, the random walks will have to be calculated in such a finite time interval. In addition, the systems may, of course, be finite in space.

There is a simple way of obtaining the finite size results from the calculations in infinite systems. All we have to do is impose upon the infinite system the periodic boundary conditions.

$$\varphi(x_i + L_i) = \varphi(x_i), \tag{6.224}$$

where x_i can now be space or time coordinates and L_i are the respective finite sizes. The effect of this condition is that the momentum integrals $\int (d^D k\, a^D/(2\pi)^D)$ become sums over discrete values:

$$\int \frac{d^D k\, a^D}{(2\pi)^D} \to \frac{1}{N} \sum_{\substack{k_i = 2\pi n_i/L_i \\ n_i = 0,1,\ldots,L_i - 1}}, \tag{6.225}$$

where $N = \sum_{i=1}^{D} L_i$ is the total number of sites in the finite system (the integrands remain the same as those for the infinite system).

As the most important example, consider the finite size version of the free massive Green function $v_m(\mathbf{x})$. If a superscript L records the finite size of the lattice, we can directly calculate

$$v_m^L(\mathbf{x}) = \frac{1}{N} \sum_{k_i = 2\pi n_i/L_i} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{\bar{\mathbf{K}}\cdot\mathbf{K} + m^2} = \frac{1}{N} \sum_{k_i = 2\pi n_i/L_i} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{\mathbf{x}'} e^{-i\mathbf{k}\cdot\mathbf{x}'} v_m(\mathbf{x}').$$

Performing the sum over n_i gives

$$\frac{1}{N}\sum_{k_i} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \sum_{\{\ell_i\}}\prod_i \delta_{x_i - x_i', \ell_i L_i}, \tag{6.226}$$

so that

$$v_m^L(\mathbf{x}) = \sum_{\{\ell_i\}} v_m(\ldots, x_i + \ell_i L_i, \ldots). \tag{6.227}$$

This result is physically clear: The set of random walks from the origin to the point x_i is augmented by those for the periodic recurrences $x_i + \ell_i L_i$.

From (6.227) it is obvious that the Green function $v_m^L(\mathbf{x})$ has the same $1/(m^2 + 2D)$ expansion as $v_m(\mathbf{x})$ with $C_n^{\mathbf{x}}$ becoming $C_n^{L\mathbf{x}} = \sum_{\{\ell_i\}} C_n^{(\ldots, x_i + \ell_i L_i, \ldots)}$.

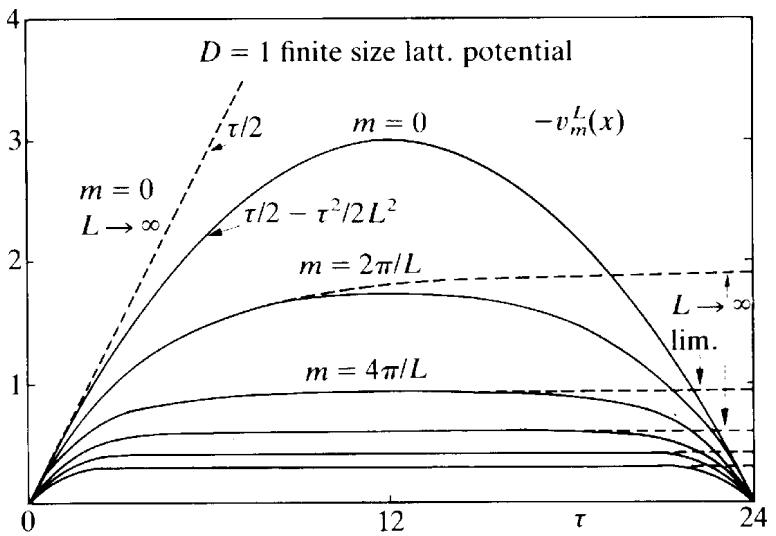
In one dimension, we can use Eq. (6.184) and do the sum directly, with the result

$$v_m^L(x) = \frac{1}{2\sinh\alpha}\sum_\ell e^{-\alpha|x+\ell L|} = \frac{1}{2\sinh\alpha}\frac{\cosh\alpha(x - L/2)}{\sinh\alpha L/2}, \quad 0 \le x \le L.$$

The potential is displayed in Fig. 6.5 for a lattice of size $L = 24$ and various masses $m = 0,\ 2\pi/L,\ 4\pi/L,\ \ldots$ Outside the interval $0 \le x \le L$, the potential is to be continued periodically.

The finite size correction to $\mathrm{tr}\log(-\bar{\nabla}\cdot\nabla + m^2)$ can be calculated in a similar way. We simply integrate the Green function (6.227) over the mass and obtain

FIG. 6.5. The finite size lattice potential $v_m^L(x)$ for $L = 24$ and various masses m. The dashed curves show the corresponding $L \to \infty$ limits.



$$\frac{1}{N}\mathrm{tr}\log^{L}(-\bar{\nabla}\cdot\nabla + m^2)^{>} = \int^{m^2} dm'^2 v_m^{L\prime}(\mathbf{0})$$

$$= \int^{m^2} dm'^2 \sum_{\{\ell_i\}} v_{m'}(\ldots, \ell_i L_i, \ldots)$$

$$= \sum_{\{\ell_i\}} \int^{m^2} dm'^2 \frac{1}{N}\sum_{\mathbf{k}} e^{ik_i\ell_i L_i} \frac{1}{\bar{\mathbf{K}}\cdot\mathbf{K} + m^2}.$$

The integration constant is adjusted in each line of the equation such that for $m^2 \to \infty$, the asymptotic behaviour is $\log m^2 + O(1/m^2)$ with no $(m^2)^0$ term. In fact, both sides behave like

$$\log(m^2 + 2D) + O\left(\frac{1}{(m^2 + 2D)^2}\right).$$

The absence of a term $1/(m^2 + 2D)$ follows from the fact that

$$\frac{1}{m^2 + 2D}\sum_{k_i} 2\cos k_i$$

vanishes, whether the system is infinite or finite.

Consider the special case $D = 1$. Using (6.184) we integrate

$$\int^{m^2} dm'^2 v_{m'}(x) = \begin{cases} -\log[(m^2 + 2 - \sqrt{(m^2+2)^2 - 4})/2] = \alpha & x = 0, \\ -\dfrac{1}{|x|}[(m^2 + 2 - \sqrt{(m^2+2)^2 - 4})/2]^{|x|} & x \neq 0. \end{cases} \tag{6.227'}$$

Performing now the sum over $x = \ell L$ we find the trace of the logarithm for a linear system with L lattice sites

$$\frac{1}{L}\mathrm{tr}\log^{L}(-\bar{\nabla}\cdot\nabla + m^2) = \alpha + \frac{2}{L}\log(1 - e^{-L\alpha}), \tag{6.228}$$

where

$$\alpha = -\log\{[(m^2 + 2) - \sqrt{(m^2+2)^2 - 4}]/2\},$$

$$= \cosh^{-1}[(m^2 + 2)/2]$$

$$= 2\sinh^{-1}(m/2). \tag{6.229}$$

This formula is of special importance since it governs the quantum fluctuations at nonzero temperature. According to the above remarks and the

general discussion in Section 1.6, this case is described by fields which are periodic on an imaginary time interval with period

$$L_\tau = \hbar\beta \equiv \hbar/k_B T.$$

Such fields have the Fourier decomposition

$$\varphi(\tau) = \frac{1}{\sqrt{N}} \sum_{\nu=0}^{N-1} e^{i\omega_\nu \tau} \varphi_\nu, \tag{6.230}$$

where ω_ν are the discrete frequencies corresponding to the momenta $k = 2\pi/L$, i.e.,

$$\omega_\nu = \frac{2\pi\nu}{\hbar\beta}. \tag{6.231}$$

They are commonly referred to as *Matsubura frequencies*. If the field is real, then the components φ_ν satisfy the condition $\varphi_\nu = \varphi^*_{-\nu}$. A free field has the energy

$$E = \int_0^{\hbar\beta} d\tau \left(\frac{1}{2}\dot{\varphi}^2(\tau) + \frac{\omega^2}{2}\varphi^2(\tau) \right) \tag{6.232}$$

and the Euclidean action reads (recall Eq. (1.84))

$$A_E = \int_0^{\hbar\beta} d\tau \left(-\pi(\tau)\, i\partial_\tau \varphi(\tau) + \frac{1}{2}\pi^2(\tau) + \frac{\omega^2}{2}\varphi^2(\tau) \right). \tag{6.233}$$

The quantum partition function is given by the path integral [recall Eq. (1.86)]

$$E = \prod_n \left[\int_{-\infty}^{\infty} d\varphi(\tau_n) \int_{-\infty}^{\infty} \frac{d\pi(\tau_n)}{2\pi} \right] \times \exp\left\{ -\frac{\varepsilon}{\hbar} \sum_n \left[-\frac{i}{\varepsilon}\pi(\tau_n)\nabla_i \varphi(\tau_n) + \frac{1}{2}\pi^2(\tau_n) + \frac{\omega^2}{2}\varphi(\tau_n) \right] \right\}, \tag{6.234}$$

where $\tau_n = (n/N)(\hbar/k_B T)$ are the grated times with spacing $\varepsilon = \hbar/Nk_B T = \hbar\beta/N$ introduced before in Eq. (1.87), and $\nabla_\tau \varphi(\tau_n)$ is the lattice derivative $\nabla_\tau \varphi(\tau_n) = \varphi(\tau_{n+1}) - \varphi(\tau_n)$. Performing the integrals over $\pi(\tau_n)$ we obtain the Lagrangian version of the partition function

$$Z = \prod_n \left[\int \frac{d\varphi(\tau_n)}{\sqrt{2\pi\varepsilon\hbar}} \right] e^{-(\varepsilon/2\hbar)\Sigma_n [(1/\varepsilon^2)(\nabla_\tau \varphi(\tau_n))^2 + \omega^2 \varphi^2(\tau_n)]}. \tag{6.235}$$

Because of translational invariance, the exponent is diagonalized by the Fourier decomposition

$$-\frac{1}{2\hbar\varepsilon}\sum_{\nu=0}^{N-1}(\bar{\Omega}_\nu\Omega_\nu+\varepsilon^2\omega^2)|\varphi_\nu|^2, \tag{6.236}$$

where

$$\Omega_\nu=\frac{1}{i}(e^{i\omega_\nu\varepsilon}-1),\quad \bar{\Omega}_\nu=\frac{1}{i}(1-e^{-i\omega_\nu\varepsilon}),\quad \bar{\Omega}_\nu\Omega_\nu=2-2\cos\omega_\nu\varepsilon, \tag{6.237}$$

are the temporal analogues of the lattice quantities **K**, $\bar{\mathbf{K}}$ defined in Eq. (6.39).

In order to perform the integration, we have to calculate the Jacobian. Decomposing the complex components φ_ν for $\nu\neq 0$ into real and imaginary parts, $\varphi_\nu=(1/\sqrt{2})(\varphi_\nu^{(1)}+i\varphi_\nu^{(2)})$, we see that

$$\varphi_\nu^{(1)}=\varphi_{N-\nu}^{(1)},\quad \varphi_\nu^{(2)}=\varphi_{N-\nu}^{(2)}, \tag{6.238}$$

and (6.230) can be written as

$$\varphi(\tau_n)=\frac{1}{\sqrt{N}}\left\{\varphi_0+\sum_{\nu=1}^{(N-1)/2}(\varphi_\nu^{(1)}\sqrt{2}\cos\omega_\nu\tau_n-\varphi_\nu^{(2)}\sqrt{2}\sin\omega_\nu\tau_n)\right\}, \tag{6.239}$$

if N is odd and

$$\varphi(\tau_n)=\frac{1}{\sqrt{N}}\left\{\varphi_0+\varphi_{N/2}\cos\omega_{N/2}\tau_n+\sum_{\nu=1}^{N/2-1}(\varphi_\nu^{(1)}\sqrt{2}\cos\omega_\nu\tau_n-\varphi_\nu^{(2)}\sqrt{2}\sin\omega_\nu\tau_n)\right\}, \tag{6.240}$$

if N is even. The inverse of this decomposition has precisely the same matrix form, namely,

$$\begin{aligned}\varphi_0&=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\varphi(\tau_n),\\ \varphi_\nu^{(1)}&=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\varphi(\tau_n)\sqrt{2}\cos\omega_\nu\tau_n,\\ \varphi_\nu^{(2)}&=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\varphi(\tau_n)\sqrt{2}\sin\omega_\nu\tau_n,\end{aligned} \tag{6.241}$$

if N is odd, with an additional equation

$$\varphi_{N/2}=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\varphi(\tau_n)(-)^n \tag{6.242}$$

if N is even. In either case, the inverse matrix is obviously just the transpose so that the Jacobian is equal to unity.

We can therefore write

$$\prod_n \int_{-\infty}^{\infty} \frac{d\varphi(\tau_n)}{\sqrt{2\pi\varepsilon\hbar}} = \int_{-\infty}^{\infty} \frac{d\varphi_0}{\sqrt{2\pi\varepsilon\hbar}} \prod_{\nu=1}^{(N-1)/2} \int_{-\infty}^{\infty} \frac{d\varphi_\nu^{(1)}\, d\varphi_\nu^{(2)}}{2\pi\varepsilon\hbar} \tag{6.243}$$

for $N =$ odd or

$$\prod_n \int_{-\infty}^{\infty} \frac{d\varphi(\tau_n)}{\sqrt{2\pi\varepsilon\hbar}} = \int_{-\infty}^{\infty} \frac{d\varphi_0}{\sqrt{2\pi\varepsilon\hbar}} \cdots \int_{-\infty}^{\infty} \frac{d\varphi_{N/2}}{\sqrt{2\pi\varepsilon\hbar}} \prod_{\nu=1}^{N/2-1} \int \frac{d\varphi_\nu^{(1)}\, d\varphi_\nu^{(2)}}{2\pi\varepsilon\hbar} \tag{6.244}$$

for $N =$ even.

Expressing the energy (6.236) in terms of these real components gives

$$-\frac{1}{2\hbar\varepsilon}\left[(\bar{\Omega}_0\Omega_0 + \varepsilon^2\omega^2)\,\varphi_0^2 + \sum_{\nu=1}^{(N-1)/2} (\bar{\Omega}_\nu\Omega_\nu + \varepsilon^2\omega^2)(\varphi_\nu^{(1)2} + \varphi_\nu^{(2)2})\right] \tag{6.245}$$

for odd N and

$$-\frac{1}{2\hbar\varepsilon}\left[(\bar{\Omega}_0\Omega_0 + \varepsilon^2\omega^2)\,\varphi_0^2 + (\bar{\Omega}_{N/2}\Omega_{N/2} + \varepsilon^2\omega^2)\,\varphi_{N/2}^2 + \sum_{\nu=1}^{N/2-1} (\bar{\Omega}_\nu\Omega_\nu + \varepsilon^2\omega^2)(\varphi_\nu^{(1)2} + \varphi_\nu^{(2)2})\right] \tag{6.246}$$

for even N.

Now the integrals can be performed and we find in either case

$$Z = \prod_{\nu=0}^{N-1} \frac{1}{\sqrt{\bar{\Omega}_\nu\Omega_\nu + \varepsilon^2\omega^2}} = e^{-(1/2)\sum_{\nu=0}^{N-1} \log(\bar{\Omega}_\nu\Omega_\nu + \varepsilon^2\omega^2)}. \tag{6.247}$$

At the end, we have to take the limit $\varepsilon \to 0$. Thus, the quantum statistical mechanics of a single real harmonic field follows directly from the trace of the logarithm which we have calculated above in Eq. (6.228).

Taking $m^2 = \varepsilon^2\omega^2$ and $L = \beta\hbar/\varepsilon$ we obtain

$$\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla + \varepsilon^2\omega^2) = \beta\hbar\omega_L + 2\log(1 - e^{-\beta\hbar\omega_L}), \tag{6.248}$$

where we have introduced $\omega_L \equiv \alpha/\varepsilon$, i.e.,

$$\varepsilon\omega_L \equiv -\log[(\varepsilon^2\omega^2 + 2 - \sqrt{(\varepsilon^2\omega^2 + 2)^2 - 4})/2], \quad \sinh(\varepsilon\omega_L/2) = \varepsilon\omega/2. \tag{6.249}$$

For $\varepsilon = 1$, $\beta\hbar/\varepsilon = L \to \infty$, this reduces to the infinite size result (6.204) as it should.

If we go to the limit $\varepsilon \to 0$ we obtain the fluctuation determinant of a free real quantum field at temperature $T = 1/(k_B\beta)$,

$$\frac{1}{2}\operatorname{tr}\log(-\bar{\nabla}\cdot\nabla + \varepsilon^2\omega^2) = \frac{\beta}{2}\hbar\omega + \log(1 - e^{-\beta\hbar\omega}). \tag{6.250}$$

This result is, of course, the same as what would have been obtained from a sum over all states of a harmonic oscillator of frequency ω:

$$Z = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\hbar\omega} = e^{-\beta\hbar\omega/2}\frac{1}{1 - e^{-\beta\hbar\omega/2}} = \frac{1}{2\sinh\dfrac{\beta\hbar\omega}{2}} = e^{-(\beta/2)\hbar\omega - \log(1 - e^{-\beta\hbar\omega})}. \tag{6.251}$$

Notice that a free massless field with $\omega = 0$ has no finite partition function. This is due to the fact that the integral over the mode φ_0 of zero Matsubara frequency is divergent. The remaining integrals, however, are all finite and we can separate out the infinity by rewriting Z as

$$Z = \int \frac{d\varphi_0}{\sqrt{2\pi\varepsilon\hbar}} \prod_{\nu>0} \frac{1}{\sqrt{\bar{\Omega}_\nu\Omega_\nu}}. \tag{6.252}$$

From the previous result it is easy to find the product $\Pi_{\nu>0} 1/\sqrt{\bar{\Omega}_\nu\Omega_\nu}$. All we have to do is take the finite expression $\Pi_\nu 1/\sqrt{\bar{\Omega}_\nu\Omega_\nu + \varepsilon^2\omega^2}$, multiply this by $\varepsilon\omega$ and take the limit $\omega \to 0$. This gives

$$\prod_{\nu>0} 1/\sqrt{\bar{\Omega}_\nu\Omega_\nu} = \lim_{\omega\to 0} \varepsilon\omega\, e^{-(\beta\hbar/2)(\omega_L + 2/\beta\hbar\log(1 - e^{-\beta\hbar\omega_L}))} = \frac{\varepsilon}{\beta\hbar}. \tag{6.253}$$

Hence, Z becomes

$$Z = \int \frac{d\varphi_0}{\sqrt{2\pi\hbar}} \frac{\sqrt{\varepsilon}}{\beta\hbar}. \tag{6.254}$$

It is useful to replace φ_0 by the average field $\bar{\varphi} = \langle\varphi(\tau_n)\rangle$ such that, according to (6.241),

$$\int d\varphi_0 = \sqrt{N}\int d\bar{\varphi} \tag{6.255}$$

and Z takes the form

$$Z = \int \frac{d\bar{\varphi}}{\sqrt{2\pi\beta\hbar^2}}. \tag{6.256}$$

This shows clearly that the divergence is proportional to the volume in field space.

This result can also be used to rephrase the integration measures of (6.243) and (6.244) as follows:

$$\prod_n \int_{-\infty}^{\infty} \frac{d\varphi(\tau_n)}{\sqrt{2\pi\varepsilon\hbar}} \equiv \int \mathcal{D}\varphi(\tau) = \int \frac{d\varphi_0}{\sqrt{2\pi\varepsilon\hbar}} \frac{\varepsilon}{\beta\hbar} \prod_{\nu\neq 0} \int \frac{d\varphi_\nu}{\sqrt{2\pi\varepsilon\hbar/\bar{\Omega}_\nu\Omega_\nu}}$$

$$= \int \frac{d\bar{\varphi}}{\sqrt{2\pi\beta\hbar^2}} \prod_{\nu\neq 0} \int \frac{d\varphi_\nu}{\sqrt{2\pi\varepsilon\hbar/\bar{\Omega}_\nu\Omega_\nu}} \tag{6.257}$$

where φ_ν are all real components with nonzero Matsubara frequency. This way of writing the measure is useful in direct calculations of fluctuation determinants. In order to see this, consider once more the partition function of the massive field (6.235), (6.326). Using the measure (6.257) we find immediately

$$Z = \int \frac{d\bar{\varphi}}{\sqrt{2\pi\beta\hbar^2}} e^{-(\beta/2)\omega^2\bar{\varphi}^2} e^{-(1/2)\sum_{\nu\neq 0}[\log(\bar{\Omega}_\nu\Omega_\nu+\varepsilon^2\omega^2)-\log\bar{\Omega}_\nu\Omega_\nu]}. \tag{6.258}$$

The subtraction of $\log\bar{\Omega}_\nu\Omega_\nu$ at each ν guarantees the finiteness of the sum. Since ε is very small, the nonzero contributions must come from the initial and, due to the invariance under $\nu \to N-\nu$, the final piece of the subtracted sum. With

$$\bar{\Omega}_\nu\Omega_\nu = \bar{\Omega}_{N-\nu}\Omega_{N-\nu} = 2 - 2\cos\omega_\nu\varepsilon \approx \omega_\nu^2\varepsilon^2, \tag{6.259}$$

this reads

$$-\frac{1}{2}\sum_{\nu\neq 0}\log\left(1+\frac{\omega^2}{\omega_\nu^2}\right),$$

with the sum running over $\nu > 0$ and $\nu < 0$. Thus we obtain the limit

$$Z \xrightarrow[\varepsilon\to 0]{} \int \frac{d\bar{\varphi}}{\sqrt{2\pi\beta\hbar^2}} e^{-(\beta/2)\omega^2\bar{\varphi}^2} e^{-(1/2)\sum_{\nu\neq 0}\log(1+\omega^2\beta^2\hbar^2/4\pi^2\nu^2)}. \tag{6.260}$$

The sum can be performed using the well-known expansion

$$\frac{\pi}{x}\coth \pi x = \sum_{\nu=-\infty}^{\infty} \frac{1}{\nu^2 + x^2} = \frac{2\pi}{x}\left(\frac{1}{2} + \frac{1}{e^{2\pi x} - 1}\right). \tag{6.261}$$

Taking out the $n = 0$ term and integrating over x^2 from zero to x^2 we have

$$\sum_{\nu \neq 0} \log\left(1 + \frac{x^2}{\nu^2}\right) = 2\pi x + 2\log(1 - e^{-2\pi x}) - 2\log(2\pi x)$$

$$= 2\log[\sinh(\pi x)/(\pi x)]. \tag{6.262}$$

The absence of an extra constant of integration can be verified by letting $x \to 0$ in which case the left-hand side goes to $\Sigma_{\nu \neq 0} x^2/\nu^2 = \pi^2/(3x^2)$ and so does the right-hand side. Identifying x with $\beta\hbar/2\pi$ we find

$$e^{-(1/2)\Sigma_{\nu \neq 0}\log(1+\omega^2/\omega_\nu^2)} = \beta\hbar\omega\, e^{-\beta\hbar\omega/2 - \log(1 - e^{-\beta\hbar\omega})}$$

$$= \frac{\beta\hbar\omega}{2\sinh\dfrac{\beta\hbar\omega}{2}}, \tag{6.263}$$

and the partition function becomes once more

$$Z = \int \frac{d\bar{\varphi}}{\sqrt{2\pi\beta\hbar^2}} e^{-(\beta/2)\omega^2\bar{\varphi}^2} \frac{\beta\hbar\omega}{2\sinh\dfrac{\beta\hbar\omega}{2}} = \frac{1}{2\sinh\dfrac{\beta\hbar\omega}{2}}$$

in agreement with (6.251).

6.16. SHORT-RANGE STERIC INTERACTIONS BETWEEN CHAINS VERSUS FIELD INTERACTIONS

Up to now we have seen that a grand canonical ensemble of independent random walks or chains can be described by means of a free disorder field theory. Certainly, line-like objects occurring in nature are never independent but always have interactions.

As discussed in the beginning of this chapter, for random walks, these interactions are instantaneous in time. For random chains, on the other hand, they are completely independent of the length parameters. Since

we are mainly interested in defect lines, which are chain-like objects, we shall only consider the latter case.

The interactions can be separated into two types: First, there is a short-range interaction which becomes active only at distances of the order of the thickness of the chain. This interaction will be referred to as *steric*. Second, there are long-range interactions. These may either be due to electric charges trapped on them, or to stress present in the surrounding matter. In the present section we shall focus our attention on the short-range steric interaction and see how it can be incorporated into the disorder field description of random chains. Thus, we shall consider an interaction energy of the form

$$E_{\text{int}} = \frac{1}{2} \sum_{i,j} V(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}), \tag{6.264a}$$

which only depends on the positions $\mathbf{x}_i$ of the chain elements and goes rapidly to zero for $|\mathbf{x}_i - \mathbf{x}_j| > a$. Let us confine our attention to the continuum limit for which the disorder field theory has the simplest form. Then the energy can be written as

$$E_{\text{int}} = \frac{1}{2} \sum_{i,j} \int \frac{ds^{(i)}}{a} \int \frac{ds^{(j)}}{a} V(\mathbf{x}^{(i)}(s^{(i)}), \mathbf{x}^{(i)}(s^{(j)})). \tag{6.264b}$$

It is straightforward to show that this interaction between chain elements is equivalent, in the disorder field theory, to a field interaction

$$E_{\text{int}} = \frac{D}{2a^2} \int d^3x\, d^3y\, \varphi^\dagger(\mathbf{x})\, \varphi(\mathbf{x})\, V(\mathbf{x}, \mathbf{y})\, \varphi^\dagger(\mathbf{y})\, \varphi(\mathbf{y}). \tag{6.265}$$

We shall prove this by starting out with the field theory. The partition function of interacting complex disorder fields is

$$Z = \int \mathscr{D}\phi\, \mathscr{D}\phi^\dagger\, e^{-\int d^3x (1/2)(|\partial\varphi|^2 + m^2/2|\varphi|^2) - E_{\text{int}}/T}. \tag{6.266}$$

Let us introduce a source field which is coupled to the composite field $\varphi^\dagger(\mathbf{x})\, \varphi(\mathbf{x})$, to be called *disorder density*,

$$\frac{1}{T} E_{\text{source}} = \frac{1}{2} \int d^3x\, K(\mathbf{x})\, \varphi^\dagger(\mathbf{x})\, \varphi(\mathbf{x}). \tag{6.267}$$

The source field $K(\mathbf{x})$ may be thought of as an additional $\mathbf{x}$ dependent chemical potential to the chain elements. The reason for introducing $K(\mathbf{x})$ is the fact that the interaction (6.265) involves only disorder densities. It can therefore be written in terms of functional derivatives with respect to $K(\mathbf{x})$ and removed from the functional integral [recall a similar procedure in (2.6)]. In this way, Z becomes

$$Z = \exp\left(-\frac{4D^2}{2Ta^4}\int d^3x\, d^3y\, \frac{\delta}{\delta K(\mathbf{x})}\, V(\mathbf{x}, \mathbf{y})\, \frac{\delta}{\delta K(\mathbf{y})}\right) Z[K]\Big|_{K=0}, \tag{6.268}$$

where $Z[K]$ is the generating functional

$$Z[K] = \int \mathscr{D}\varphi\, \mathscr{D}\varphi^{\dagger}\, e^{-(1/2)\int d^3x\,(|\partial\varphi|^2 + (m^2 + K(\mathbf{x}))|\varphi|^2)}. \tag{6.269}$$

This has the advantage of being quadratic in the field so that its properties can easily be studied. In particular, we can perform the ϕ integral and arrive at

$$Z[K] = \det(-\partial^2 + m^2 + K(\mathbf{x}))^{-1} \tag{6.270}$$

$$= e^{-\mathrm{tr}\log G_K^{-1}}, \tag{6.271}$$

where G_K denotes the Green function which solves the differential equation

$$(-\partial^2 + m^2 + K(\mathbf{x}))\, G_K(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \tag{6.272}$$

In analogy with the previous discussion on free random chains it is useful to introduce the probability $P_K(\mathbf{x}, \mathbf{y}, s)$ of finding a random walk of length s going from $\mathbf{x}$ to $\mathbf{y}$, but now in an external $\mathbf{x}$ dependent chemical potential $K(\mathbf{x})$. It is defined by the differential equation

$$-\partial_s P_K(\mathbf{x}, \mathbf{y}, s) = \frac{1}{2M}(-\partial^2 + K(\mathbf{x}))\, P_K(\mathbf{x}, \mathbf{y}, s). \tag{6.273}$$

This looks like the equation for a Schrödinger Green function in an external potential $(1/2M)\, K(\mathbf{x})$ when continued to imaginary time $t = -is$. Between P_K and G_K there obviously exists the same connection as in the $K(\mathbf{x}) = 0$ case:

$$G_K(\mathbf{x}, \mathbf{y}) = \frac{1}{2M}\int_0^\infty ds\, e^{(-s/a)((\varepsilon/T) - \log 2D)} P_K(\mathbf{x}, \mathbf{y}, s)$$

$$= \frac{1}{2M}\int_0^\infty ds\, e^{-s(m^2/2M)} P_K(\mathbf{x}, \mathbf{y}, s). \tag{6.274}$$

Indeed, applying (6.243) to this we find from (6.242):

$$(-\partial^2 + m^2 + K(\mathbf{x}))\, G_K(\mathbf{x}, \mathbf{y})$$

$$= \frac{1}{2M}\int_0^\infty ds\, e^{-s(m^2/2M)} (-\partial^2 + K(\mathbf{x}) - 2M\partial_s) P_K(\mathbf{x}, \mathbf{y}, s) = \delta(\mathbf{x} - \mathbf{y}). \tag{6.275}$$

The reason for introducing $P_K(\mathbf{x}, \mathbf{y}, s)$ is that it has a path-integral representation in terms of fluctuating chain configurations which is very similar to the $K(\mathbf{x}) = 0$ case, i.e.,

$$P_K(\mathbf{x}, \mathbf{y}, s) = \int \mathscr{D}x \exp\left\{ - \int_{\mathbf{x}(0) = \mathbf{x}}^{\mathbf{x}(s) = \mathbf{y}} ds \left(\frac{M}{2}\dot{\mathbf{x}}(s)^2 + \frac{1}{2M} K(\mathbf{x}(s)) \right) \right\}. \tag{6.276}$$

The additional source $K(\mathbf{x})$ is simply added to the "kinetic term" $(M/2)\dot{\mathbf{x}}(s)^2$ in the exponent, apart from the factor $1/2M$. That this is true is again a reflection of the particle-field duality. In the Schrödinger type equation (6.273), $(1/2M)\, K(\mathbf{x})$ accompanies the kinetic differential operator $-(1/2M)\,\partial^2$. In the path integral directly involving the fluctuating orbit $\mathbf{x}(s)$, $(1/2M)\, K(\mathbf{x}(s))$ appears at the corresponding place by accompanying the kinetic energy $(M/2)\dot{\mathbf{x}}^2$.

The formal proof of (6.276) is simple. All we have to do is go to the phase space version of this path integral

$$P_K(\mathbf{x}, \mathbf{y}, s) = \int \mathscr{D}x \frac{\mathscr{D}p}{2\pi} \exp \int_{\mathbf{x}(0) = \mathbf{x}}^{\mathbf{x}(s) = \mathbf{y}} ds \left(i\mathbf{p}(s) \cdot \dot{\mathbf{x}}(s) - \frac{\mathbf{p}^2(s)}{2M} - \frac{1}{2M} K(\mathbf{x}(s)) \right). \tag{6.277}$$

This satisfies the same factorization property (6.85) that we observed in the $K = 0$ case. Therefore, we can remove a factor associated with the last infinitesimal slice

$$P_K(\mathbf{x}, \mathbf{y}, \varepsilon) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x} - \mathbf{y}) - \varepsilon(\mathbf{p}^2/2M) - (\varepsilon/2M)K(\mathbf{x})}. \tag{6.278}$$

This can be rewritten as

$$\left[1 - \frac{\varepsilon}{2M}\left(-\partial^2 + K(\mathbf{x})\right)\right] \delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{6.279}$$

and we see that $P_K(\mathbf{x}, \mathbf{y}, s)$ satisfies the equation

$$P_K(\mathbf{x}, \mathbf{y}, s + a) = \left[1 - \frac{\varepsilon}{2M}(-\partial^2 + K(\mathbf{x}))\right] P_K(\mathbf{x}, \mathbf{y}, s). \tag{6.280}$$

In the limit $\varepsilon \to 0$, this indeed reduces to (6.273).

It is now easy to find what the field interaction means in terms of fluctuating random chains. In order to see this most clearly, let us introduce an external source $\eta(\mathbf{x})$ to the fields $\phi(\mathbf{x})$ and consider the partition function

$$Z[K, \eta, \eta^\dagger] = \int \mathscr{D}\varphi \, \mathscr{D}\varphi^\dagger \, e^{-\int d^3x [(1/2)|\partial\varphi|^2 + (1/2)(m^2 + K(\mathbf{x}))|\varphi|^2 - \eta^\dagger(\mathbf{x})\varphi(\mathbf{x}) - \varphi^\dagger(\mathbf{x})\eta(\mathbf{x})]}. \tag{6.281}$$

Completing squares and performing the φ integrals, we have

$$Z[K, \eta, \eta^\dagger] = e^{-\mathrm{tr}\log G_K^{-1} + 2\int d^3x d^3y \eta^\dagger(\mathbf{x}) G_K(\mathbf{x},\mathbf{y}) \eta(\mathbf{y})}. \tag{6.282}$$

Expanding this result in powers of η, $\eta^\dagger$ we obtain a sum of Feynman graphs involving $G_K(\mathbf{x}, \mathbf{y})$. According to (6.274) and (6.276), each of these $G_K(\mathbf{x}, \mathbf{y})$ can be written as

$$G_K(\mathbf{x}, \mathbf{y}) = \frac{1}{2M} \int_0^\infty ds \, e^{-s(m^2/2M)} \int \mathscr{D}x \, e^{-\int_{\mathbf{x}(0)=\mathbf{x}}^{\mathbf{x}(s)=\mathbf{y}} ds((M/2)\dot{\mathbf{x}}^2(s) + (1/2M)K(\mathbf{x}(s)))}. \tag{6.283}$$

In addition, the one-loop contribution $\mathrm{tr}\log G_K^{-1}$ has the representation

$$\frac{1}{2M} \int_0^\infty \frac{ds}{s} e^{-s(m^2/2M)} \int \mathscr{D}x \, e^{-\int_{\mathbf{x}(0)=\mathbf{x}}^{\mathbf{x}(s)=\mathbf{y}} ds((M/2)\dot{\mathbf{x}}^2(s) + (1/2M)K(\mathbf{x}(s)))}, \tag{6.284}$$

with a closed orbit $\mathbf{x}(s)$. In this way, the generating functional becomes the sum of path integrals over $N = 1, 2, 3, \ldots$ orbits

$$\prod_{i=1}^{N} \int \mathscr{D}x^{(i)} \, e^{-\sum_{i=1}^{N} \int ds^{(i)}((M/2)\dot{\mathbf{x}}^{(i)2}(s^{(i)}) + (1/2M)K(\mathbf{x}^{(i)}(s^{(i)})))}, \tag{6.285}$$

some orbits having endpoints, some being closed.

Let us now introduce the field interaction using formula (6.268). The differentiations with respect to $K(\mathbf{x})$ have the effect of changing each path integral (6.285) into

$$\prod_{i=1}^{N} \int \mathscr{D}x^{(i)}\, e^{-\sum_{i=1}^{N} \int ds^{(i)} (M/2) \dot{\mathbf{x}}^{(i)2}(s^{(i)}) - (1/(2TD^2M^2)) \int ds^{(i)}\, ds^{(j)}\, V(\mathbf{x}^{(i)}(s^{(i)}),\, \mathbf{x}^{(j)}(s^{(j)}))}. \quad (6.286)$$

This follows directly from the formula

$$\frac{\delta}{\delta K(\mathbf{x})} e^{-\sum_{i=1}^{N} \int ds^{(i)} (1/2M) K(\mathbf{x}^{(i)}(s))}$$

$$= -\frac{1}{2M} \sum_{i=1}^{N} \int ds^{(i)}\, \delta^{(3)}(\mathbf{x} - \mathbf{x}^{(i)}(s^{(i)}))\, e^{-\sum_{i=1}^{N} \int ds^{(i)} (1/2M) K(\mathbf{x}^{(i)}(s^{(i)}))}, \quad (6.287)$$

so that

$$\int d^3x\, d^3y \frac{\delta}{\delta K(\mathbf{x})} V(\mathbf{x},\, \mathbf{y}) \frac{\delta}{\delta K(\mathbf{y})} e^{-\sum_{i=1}^{N} \int ds^{(i)} (1/2M) K(\mathbf{x}^{(i)}(s^{(i)}))}$$

$$= \frac{1}{4M^2} \int d^3x\, d^3y\, V(\mathbf{x},\, \mathbf{y}) \int ds^{(i)} \int ds^{(j)}$$

$$\times \sum_{i,j=1}^{N} \delta^{(3)}(\mathbf{x} - \mathbf{x}^{(i)}(s^{(i)}))\, \delta^{(3)}(\mathbf{y} - \mathbf{y}^{(j)}(s^{(j)}))\, e^{-\sum_{i=1}^{N} \int ds (1/2M) K(\mathbf{x}(s))}$$

$$= \frac{1}{4M^2} \sum_{i,j=1}^{N} \int ds^{(i)} \int ds^{(j)}\, V(\mathbf{x}^{(i)}(s^{(i)}),\, \mathbf{x}^{(j)}(s^{(j)}))\, e^{-\sum_{i=1}^{N} \int ds^{(i)} (1/2M) K(\mathbf{x}^{(i)}(s^{(i)}))}. \quad (6.288)$$

Using now $M = D/a$ (recall (6.45)), we see that the partition function of the disorder field with a field interaction is indeed equivalent to the grand canonical ensemble of fluctuating random chains (6.286) in which the N-chain configuration has an additional Boltzmann factor

$$e^{-(1/2T) \sum_{i=1}^{N} \int (ds^{(i)}/a) \int (ds^{(j)}/a)\, V(\mathbf{x}^{(i)}(s^{(i)}), \mathbf{x}^{(j)}(s^{(j)}))}. \quad (6.289)$$

This is precisely the Boltzmann factor due to a two-body interaction between the elements of the chain.

Thus we have demonstrated that steric two-body interactions can be incorporated in the disorder field theory by means of a quartic field interaction. Usually, the steric interaction is so short-ranged that in the context of the long wavelength limit, which is implied when we are using

the continuous field theory, $V(\mathbf{x}, \mathbf{y})$ can be approximated by a δ-function:

$$V(\mathbf{x}, \mathbf{y}) \sim \frac{g}{12} D^{-2} a^4 \delta(\mathbf{x} - \mathbf{y}). \tag{6.290}$$

In this idealized case, the interaction (6.265) takes the $|\phi|^4$ form which was discussed in detail in Chapters 2 and 3,

$$E_{\text{int}} = \frac{g}{4!} \int dx |\varphi|^4. \tag{6.291}$$

The present techniques can be generalized to show that higher order field interactions correspond to higher many-body interactions between the chain elements. For example, the $|\phi|^6$ term in the field theory amounts to a three-body potential,

$$\begin{aligned} &\frac{D^3}{a^6} \int d^3x_1\, d^3x_2\, d^3x_3\, \varphi^\dagger(\mathbf{x}_1)\, \varphi(\mathbf{x}_1)\, \varphi^\dagger(\mathbf{x}_2)\, \varphi(\mathbf{x}_2)\, \varphi^\dagger(\mathbf{x}_3\, \varphi(\mathbf{x}_3)\, V_6(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\ &\quad \cong \sum_{i,j,k=1}^{N} \int \frac{ds^{(i)}}{a} \frac{ds^{(j)}}{a} \frac{ds^{(k)}}{a} V_6(\mathbf{x}^{(i)}(s^{(i)}), \mathbf{x}^{(j)}(s^{(j)}), \mathbf{x}^{(k)}(s^{(k)})) \end{aligned} \tag{6.292}$$

and so on.

6.17. INCORPORATING LONG-RANGE FORCES

The random chains we shall be interested in are line-like defects in various physical systems which can carry long-wavelength excitations like sound waves. Defect lines in such systems not only have a short-range steric interaction. Since they disturb the order of the system they couple also to the soft modes. As we shall see in detail later, this results in a long-range interaction of the same type as the magnetic interaction between electric current loops.

Recall that the interaction energy of an arbitrary current density $j_i(\mathbf{x})$ is given by the Biot-Savart law (in natural units with $c \equiv$ light velocity $\equiv 1$)

$$E_{\text{int}} = \frac{\mu}{2} \frac{1}{4\pi} \int dx^3\, d^3x'\, j_i(\mathbf{x})\, j_i(\mathbf{x}') \frac{1}{R}, \tag{6.293}$$

where $R \equiv |\mathbf{x} - \mathbf{x}'|$ is the distance between the current elements, μ is

the magnetic permeability of the embedding medium. For a particular ensemble of line-like currents $I_{(i)}$ running along the closed lines $L^{(i)}$ parametrized by $x_\ell^{(i)}(s^{(i)})$ the current density is

$$\mathbf{j}(\mathbf{x}) = \sum_{i=1}^{N} I^{(i)} \int_{L^{(i)}} ds^{(i)} \frac{d\mathbf{x}^{(i)}}{ds^{(i)}} \delta^{(3)}(\mathbf{x} - \mathbf{x}^{(i)}(s^{(i)})) \tag{6.294}$$

and the Biot-Savart law takes the form

$$E_{\text{int}} = \frac{1}{2}\frac{\mu}{4\pi} \sum_{i,j} I^{(i)} I^{(i)} \oint_{L^{(i)}} \oint_{L^{(i)}} d\mathbf{x}^{(i)} \cdot d\mathbf{x}^{(j)} \frac{1}{R}, \tag{6.295}$$

with $R = |\mathbf{x}^{(i)} - \mathbf{x}^{(j)}|$.

Energies of this $1/R$ type are found between line-like defects in many physical systems. The partition function of a grand canonical ensemble of lines with equal current strength has an additional Boltzmann factor

$$e^{-(1/2T)(\mu/4\pi)I^2 \Sigma_{i,j} \int d\mathbf{x}^{(i)} \cdot d\mathbf{x}^{(j)} 1/R}. \tag{6.296}$$

In the Section 6.10 we saw that the statistical mechanics of an ensemble of sterically interacting random chains can be advantageously described by a complex disorder field theory. If this type of field theory is to be applicable to defect lines it has to allow also for a simple inclusion of long-range interactions. This is indeed the case.

In order to see this we first notice that a Boltzmann factor (6.296) was encountered before in (3.28) in the general discussion of gauge fields. If we insert there the particular line distribution

$$\mathbf{j}(\mathbf{x}) = iI \sum_i \oint_{L^{(i)}} d\mathbf{x}^{(i)} \delta^{(3)}(\mathbf{x} - \mathbf{x}^{(i)}), \tag{6.297}$$

the partition function Z becomes equal to (6.286), up to the trivial factor $e^{-\text{tr}\log(-\partial^2)}$. The factor $i = \sqrt{-1}$ in front of the current strength I is necessary in order to obtain the correct sign in the exponent. Remembering the starting point (3.9) for formula (3.28) we can write (6.296) (up to a trivial factor) as a path integral over an auxiliary vector gauge field which is coupled locally to the line elements of (6.297) (see (3.9))

$$\begin{aligned} Z &= \int \mathscr{D}A\, \Phi[A]\, e^{-(1/T)\int d^3x [(1/2\mu)(\partial \times \mathbf{A})^2 - \mathbf{j}\cdot\mathbf{A}]} \\ &= \int \mathscr{D}A\, \Phi[A]\, e^{-(1/T)\int d^3x (1/2\mu)(\partial \times \mathbf{A})^2 - iI\Sigma_i \int_{L^{(i)}} d\mathbf{x}^{(i)} \cdot \mathbf{A}(\mathbf{x}^{(i)})]}. \end{aligned} \tag{6.298}$$

In this expression, the random line elements only occur in the exponential

$$e^{i(I/T)\Sigma_i \int d\mathbf{x}^{(i)} \cdot \mathbf{A}(\mathbf{x}^{(i)})}. \tag{6.299}$$

This is an important simplification which is crucial for the development of a gauge theory of defect lines.

Consider the partition function of a grand canonical ensemble of random chains (see (6.102)) without steric interactions

$$Z = \sum_N \frac{1}{N!} \prod_{i=1}^{N} \int_0^\infty \frac{ds^{(i)}}{s^{(i)}} e^{-(\varepsilon/T - \log 2D)s^{(i)}/a} \int \mathscr{D}^3 x^{(i)} \frac{\mathscr{D}^3 p^{(i)}}{(2\pi)^3} e^{\int_0^{s^{(i)}} ds_i^{(i)} (i\mathbf{p}^{(i)} \cdot \dot{\mathbf{x}}^{(i)} - \mathbf{p}^{(i)2}/2M)}. \tag{6.300}$$

If we rewrite the exponent in (6.299) as

$$e^{i(I/T)\Sigma_i \int ds^{(i)} \dot{\mathbf{x}}^{(i)} \cdot \mathbf{A}(\mathbf{x}^{(i)}(s^{(i)}))}, \tag{6.301}$$

we see that it can be incorporated into (6.277) by simply replacing the momentum variable $\mathbf{p}$ by $\mathbf{p} + q\mathbf{A}(\mathbf{x}(s))$ where $q = I/T$. In a given $\mathbf{A}(\mathbf{x})$ field, a single line has the partition function

$$Z_1 = \int \frac{ds}{s} e^{-(\varepsilon/T - \log 2D)(s/a)} \int_{\mathbf{x}(0) = \mathbf{x}(s)} \mathscr{D}^3 x \frac{\mathscr{D}^3 p}{(2\pi)^3} e^{\int ds [i(\mathbf{p} + q\mathbf{A}(\mathbf{x}(s))) \cdot \dot{\mathbf{x}} - (\mathbf{p}^2/2M)]}. \tag{6.302}$$

Changing the integration variable from $\mathbf{p}$ to $\mathbf{p} + q\mathbf{A}$ this becomes

$$Z_1 = \int \frac{ds}{s} e^{-(\varepsilon/T - \log 2D)(\varepsilon/a)} \int_{\mathbf{x}(0) = \mathbf{x}(s)} \mathscr{D}^3 x \frac{\mathscr{D}^3 p}{(2\pi)^3} e^{\int ds [i\mathbf{p} \cdot \dot{\mathbf{x}} - (\mathbf{p} - q\mathbf{A})^2/2M]}. \tag{6.303}$$

The functional integral is the spatial trace of the probability distribution (recall (6.87))

$$P(\mathbf{x}, \mathbf{y}, s) = \int \mathscr{D}^3 x \frac{\mathscr{D}^3 p}{(2\pi)^3} e^{\int ds [i\mathbf{p} \cdot \dot{\mathbf{x}} - (\mathbf{p} - q\mathbf{A})^2/2M]}. \tag{6.304}$$

By using the factorization property (6.84) of the path integral we can derive, just as we did in (6.91) and (6.92), the differential equation

$$-\frac{\partial}{\partial s} P(\mathbf{x}, \mathbf{y}, s) = \frac{1}{2M} \left(\frac{1}{i} \frac{\partial}{\partial \mathbf{x}} - q\mathbf{A}(\mathbf{x}) \right)^2 P(\mathbf{x}, \mathbf{y}, s). \tag{6.305}$$

In order to calculate $P(\mathbf{x}, \mathbf{y}, s)$ explicitly we have to solve the eigenvalue equations

$$\frac{1}{2M}\left(\frac{1}{i}\frac{\partial}{\partial \mathbf{x}} - q\mathbf{A}(\mathbf{x})\right)^2 u_n(\mathbf{x}) = E_n u_n(\mathbf{x}). \tag{6.306}$$

This is the same as the Schrödinger equation of a particle of charge q in an external magnetic vector potential. If $u_n(\mathbf{x})$ is a complete orthogonal set of solutions we can expand the probability as follows:

$$P_{\mathbf{A}}(\mathbf{x}, \mathbf{y}, s) = \sum_n e^{-E_n s} u_n(\mathbf{x}) u_n^\dagger(\mathbf{y}). \tag{6.307}$$

This can be inserted into (6.53) to obtain the partition function of a single random chain in the presence of the field $\mathbf{A}(\mathbf{x})$. In the continuum limit we have

$$Z_1 = \int_0^\infty \frac{ds}{s} \int d^3x\, P_{\mathbf{A}}(\mathbf{x}, \mathbf{y}, s)\, e^{-(\varepsilon/T - \log 2D)(s/a)}.$$

Using the orthonormality of $u_n(\mathbf{x})$ this becomes

$$Z_1 = -\sum_n \log\left(\frac{1}{a}\left(\frac{\varepsilon}{T} - \log 2D\right) + E_n\right) = -\sum_n \log\left(\frac{a^2}{2D}(m^2 + 2M E_n)\right), \tag{6.308}$$

where we have omitted an irrelevant integration constant.

The grand-canonical partition function is

$$Z = e^{Z_1} = \prod_n \frac{1}{\frac{a^2}{2D}(m^2 + 2M E_n)}. \tag{6.309}$$

Ignoring the irrelevant factor $\Pi_n a^2/2D$, this can be written as a product of auxiliary integrals

$$Z = \prod_n \left[\int \frac{\mathscr{D}\phi_n\, \mathscr{D}\phi_n^\dagger}{2\pi}\right] e^{-(1/2)\Sigma_n \phi_n^\dagger (2M E_n + m^2)\phi_n}, \tag{6.310}$$

which is the analogue of Eq. (6.63) derived for free random chains. Just as we did in that case, we can go to $\mathbf{x}$ space by using a complete set of spatial wave functions $u_n(\mathbf{x})$ and change the discrete variables φ_n into a complex field $\varphi(\mathbf{x})$,

$$\varphi(\mathbf{x}) = \sum_n \varphi_n u_n(\mathbf{x}). \tag{6.311}$$

Because of orthogonality, the measure $\Pi_n \int\int (d\varphi_n\, d\varphi_n^\dagger/2\pi)$ goes over into $\Pi_{\mathbf{x}} \int\int (d\varphi(\mathbf{x})\, d\varphi^\dagger(\mathbf{x})/2\pi a^3) \equiv \int \mathscr{D}\varphi(\mathbf{x})\, \mathscr{D}\varphi^\dagger(\mathbf{x})$. On the fields $\varphi(\mathbf{x})$, the eigenvalues E_n can be replaced by the differential operators (6.268) and we arrive at the following disorder field theory for the random chains in the external field

$$Z = \int \mathscr{D}\varphi(\mathbf{x})\, \mathscr{D}\varphi^\dagger(\mathbf{x})\, e^{-(1/2)\int d^3x \varphi^\dagger(\mathbf{x})(-(\partial - iq\mathbf{A})^2 + m^2)\varphi(\mathbf{x})}. \tag{6.312}$$

After a partial integration we arrive at the expression

$$Z = \int \mathscr{D}\varphi(\mathbf{x})\, \mathscr{D}\varphi^\dagger(\mathbf{x})\, e^{-(1/2)\int d^3x \{|(\partial - iq\mathbf{A})\varphi|^2 + m^2|\varphi|^2\}}. \tag{6.313}$$

This differs from the ensemble of the free random chain (6.60) only by the so-called "minimal substitution"

$$\partial \to \mathbf{D} \equiv \partial - iq\mathbf{A}, \tag{6.314}$$

familiar from quantum mechanics. It is the same replacement that was introduced previously in (3.50) for the purpose of constructing a gauge invariant coupling of a gauge potential to a complex fluctuating matter field.

Hence we can conclude: For the description of a grand canonical ensemble of random chains in an external vector potential $A_i(\mathbf{x})$, a disorder field theory can be used just as before but with the derivatives ∇ replaced by covariant derivatives $\nabla - iq\mathbf{A}$. With this result it is straightforward to study fluctuating defect lines in three dimensions with long-range Biot-Savart type interactions. All we have to do is to remember the equality of (6.296) and (6.298) and form the functional integral

$$Z = \int \mathscr{D}A\, \Phi[A] \int \mathscr{D}\varphi(\mathbf{x}) \mathscr{D}\varphi^\dagger(\mathbf{x})\, e^{-(1/2\mu T)\int d^3x (\partial \times \mathbf{A})^2 - \int d^3x \{(1/2)|(\partial - iq\mathbf{A})\varphi|^2 + (m^2/2)|\varphi|^2\}}. \tag{6.315}$$

This partition function can now be completed by the short-range steric interactions as illustrated in Section 6.10. Keeping only a repulsive contact interaction of the δ-function type we arrive at

$$Z = \int \mathscr{D}A\, \Phi[A] \int \mathscr{D}\varphi(\mathbf{x})\, \mathscr{D}\varphi^{\dagger}(\mathbf{x})$$

$$\times\, e^{-(1/2\mu T)\int d^3x (\partial \times \mathbf{A})^2 - \int d^3x \{(1/2)|(\partial - iq\mathbf{A})\varphi|^2 + (m^2/2)|\varphi|^2 + (g/4)|\varphi|^4\}}. \qquad (6.316)$$

This is precisely the field theory (3.53) discussed previously (except for a different field normalization $\varphi/\sqrt{T} \to \varphi$). The exponent involves the Ginzburg-Landau energy which had its first important appearance in quite a different context, namely, in the theory of superconductivity. Here it appears as a tool for describing a random set of defect lines with long-range Biot-Savart type and short-range steric interactions.

6.18. INTERACTING RANDOM CHAINS AND FEYNMAN DIAGRAMS

In Section 3.3 we described the perturbation expansion of the Ginzburg-Landau partition function in terms of Feynman diagrams. The result was that Z could be written as an exponential

$$Z[0] = e^{W[0]},$$

where $W[0]$ collects all connected loop diagrams involving oriented lines for the complex $\varphi(\mathbf{x})$ fields (3.60) and wiggly lines for the vector potential (3.58) (see Fig. 3.1). In Section 6.6 we saw that the one-loop- diagram in $W[0]$ of the free disorder field theory had a simple topological interpretation: It gave a direct picture of the free random chain. This was followed by the observation in Section 6.7 that lines in Feynman diagrams were pictures of open random chains. It is now obvious that the loop expansion in Fig. 3.1 for the full interacting disorder field theory can be interpreted as showing directly the different ways in which the segments of a single loop can interact with each other either by short-range interactions (via touching), or by long-range interactions, via the exchange of wiggly line fluctuations of the vector potential.

The arrows on the φ, $\varphi^{\dagger}$ lines, which in the perturbation theory of Section 3.3 were introduced to distinguish the φ and $\varphi^{\dagger}$ fields, have now the geometric meaning of distinguishing different orientations of the random loops.

APPENDIX 6A. GREEN FUNCTIONS AND FLUCTUATION DETERMINANTS FOR NON-CUBIC LATTICES

In general lattices an atom has q rather than $2D$ link vectors pointing to the q next neighbors. The number q is called the *coordination number* The n-step probability $P(\mathbf{x}, \mathbf{y}, n)$ satisfies the difference equation

$$P(\mathbf{x}, \mathbf{y}, n) = \frac{1}{q} \sum_{\pm \ell} P(\mathbf{x} + \boldsymbol{\ell}, \mathbf{y}, n - 1) \tag{6A.1}$$

or

$$P(\mathbf{x}, \mathbf{y}, n) - P(\mathbf{x}, \mathbf{y}, n - 1) = \frac{1}{q} \sum_{\pm \ell} (P(\mathbf{x} + \ell, \mathbf{y}, n - 1) - P(\mathbf{x}, \mathbf{y}, n - 1)). \tag{6A.2}$$

In contrast to the simple cubic lattice, the hopping operator is now given by the one-step probability

$$qP(\mathbf{x}, \mathbf{y}, 1) = \sum_{\pm \ell} (\nabla_\ell + 1) = q + \sum_{\ell} \bar{\nabla}_\ell \nabla_\ell, \tag{6A.3}$$

where $\boldsymbol{\ell}$ are the vectors pointing to the nearest neighbors. In b.c.c., f.c.c. and triangular lattices, the primitive translation vectors are

$$\boldsymbol{\ell}^{(1)} = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}),\ \boldsymbol{\ell}^{(2)} = \frac{a}{2}(-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}),\ \boldsymbol{\ell}^{(3)} = \frac{a}{2}(\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}), \text{ for b.c.c.,} \tag{6A.4}$$

$$\boldsymbol{\ell}^{(1)} = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}), \quad \boldsymbol{\ell}^{(2)} = \frac{a}{2}(\hat{\mathbf{y}} + \hat{\mathbf{z}}), \quad \boldsymbol{\ell}^{(3)} = \frac{a}{2}(\hat{\mathbf{z}} + \hat{\mathbf{x}}), \quad \text{for f.c.c.,} \tag{6A.5}$$

$$\boldsymbol{\ell}^{(1)} = a(1, 0), \quad \boldsymbol{\ell}^{(2)} = a\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \text{for triangular lattices.} \tag{6A.6}$$

The volume per cell is $v_c = \boldsymbol{\ell}^{(1)} \cdot (\boldsymbol{\ell}^{(2)} \times \boldsymbol{\ell}^{(3)}) = a^3/2,\ a^3/4,\ (\sqrt{3}/2)a^2$, respectively.

The b.c.c. lattice has 8 nearest neighbors corresponding to the eight permutations of the signs in front of the unit vectors in (A.4). This gives

$$qP(\mathbf{k}) = \sum_{\substack{\text{nearest}\\ \text{neigbours}}} e^{i\mathbf{k}\cdot\boldsymbol{\ell}} = (e^{ik_1a/2} + e^{-ik_1a/2})(e^{ik_2a/2} + e^{-ik_2a/2})(e^{ik_3a/2} + e^{-ik_3a/2})$$
$$= 8\cos\frac{k_1}{2}a\cos\frac{k_2}{2}a\cos\frac{k_3}{2}a. \tag{6A.7}$$

The f.c.c. lattice has 12 nearest neighbors and gives

$$qP(\mathbf{k}) = \sum_{\substack{\text{nearest}\\ \text{neigbours}}} e^{i\mathbf{k}\cdot\boldsymbol{\ell}} = (e^{ik_1a/2} + e^{-ik_1a/2})(e^{ik_2a/2} + e^{-ik_2a/2}) + (23) + (31)$$
$$= 4\left(\cos\frac{k_1}{2}a\cos\frac{k_2}{2}a + (23) + (31)\right). \tag{6A.8}$$

The triangular lattice has 6 nearest neighbors and

$$qP(\mathbf{k}) = \sum_{\substack{\text{nearest}\\ \text{neigbours}}} e^{i\mathbf{k}\cdot\boldsymbol{\ell}}$$
$$= 2\left(\cos k_1 + \cos\left(-\frac{k_1}{2} + \frac{\sqrt{3}}{2}k_2\right) + \cos\left(-\frac{k_1}{2} - \frac{\sqrt{3}}{2}k_2\right)\right). \tag{6A.9}$$

The massive lattice Green functions are given by

$$v_m(\mathbf{x}) = \frac{1}{N}\sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{m^2 + \mathbf{K}\cdot\bar{K}}, \tag{6A.10}$$

where $\mathbf{K}$, $\bar{\mathbf{K}}$ are the Fourier transforms of $q - qP(\mathbf{k})$ in the b.c.c., f.c.c. lattices. For the triangular lattice we define

$$\mathbf{K}\cdot\bar{\mathbf{K}} \equiv \tfrac{2}{3}(q - qP(\mathbf{k})) \tag{6A.11}$$

(and analogously for $-\bar{\nabla}\cdot\nabla$). The factor 2/3 ensures that the denominator in (A.10) has the small $\mathbf{k}$ behavior $m^2 + \mathbf{k}^2 + \ldots$ which is necessary to give m the proper meaning of a mass in the Green function.

The Green functions (A.10) determine the probability of a random chain of any length n to run from $\mathbf{x}$ to $\mathbf{y}$ (compare (6.46))

$$P(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{\infty} q^n e^{-(\epsilon/T)n} P(\mathbf{x}, \mathbf{y}, n)$$

via

$$P(\mathbf{x}, \mathbf{y}) = e^{\epsilon/T} v_m(\mathbf{x} - \mathbf{y}) = (4m^2 + q)\, v_m(\mathbf{x} - \mathbf{y}),$$

with

$$m^2 = \tfrac{1}{4}(e^{\epsilon/T} - q).$$

We now have to specify the momentum sum in (A.10). For this we consider the reciprocal basis vectors $\mathbf{g}^{(i)} = \boldsymbol{\ell}^{(i)} \times \boldsymbol{\ell}^{(k)}/[\boldsymbol{\ell}^{(i)} \cdot (\boldsymbol{\ell}^{(j)} \times \boldsymbol{\ell}^{(k)})]$

$$\mathbf{g}^{(1)} = \frac{1}{a}(\hat{\mathbf{x}} + \hat{\mathbf{y}}), \quad \mathbf{g}^{(2)} = \frac{1}{a}(\hat{\mathbf{y}} + \hat{\mathbf{z}}), \quad \mathbf{g}^{(3)} = \frac{1}{a}(\hat{\mathbf{z}} + \hat{\mathbf{x}}), \quad \text{for b.c.c.,} \tag{6A.12}$$

$$\mathbf{g}^{(1)} = \frac{1}{a}(\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}), \ \mathbf{g}^{(2)} = \frac{1}{a}(-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}), \ \mathbf{g}^{(3)} = \frac{1}{a}(\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}), \ \text{for f.c.c.,} \tag{6A.13}$$

$\mathbf{g}^{(1)} = (2/\sqrt{3}a)(\sqrt{3}/2, 1/2)$, $\mathbf{g}^{(2)} = (2/\sqrt{3}a)(0, 1)$, for a triangular lattice.

The momentum components along these vectors,

$$k^{(i)} = \mathbf{g}^{(i)} \cdot \mathbf{k}, \tag{6A.14}$$

are quantized such that

$$ak^{(i)} = \frac{2\pi}{L^{(i)}} n^{(i)}, \quad n^{(i)} = 0, 1, \ldots, L^{(i)} - 1, \tag{6A.15}$$

where $L^{(i)}$ is the number of lattice sites along the i-th primitive direction $\boldsymbol{\ell}^{(i)}$. The total number of sites is

$$N = \prod_{i=1}^{D} L^{(i)}. \tag{6A.16}$$

In the limit of an infinitely large lattice, the sums over $k^{(i)}$ can be replaced by integrals,

$$\sum_{\mathbf{k}} \to N \int \frac{d^D k^{(i)} a^D}{(2\pi)^D}. \tag{6A.17a}$$

These, in turn, can be expressed as integrals over the Cartesian momenta k_i

$$NJa^3 \int \frac{d^3k_i}{(2\pi)^D} \tag{6A.17b}$$

with Jacobian factor $J = 1/2,\ 1/4,\ \sqrt{3}/2$ for b.c.c., f.c.c., and triangular lattices, respectively. The factor Ja^3 is, of course, equal to the cell volume v_c. The disadvantage with the Cartesian integral is that the domain of integration is not just a trivial cube as in (A.17).

Consider now the lattice Green functions at the origin $v_m(\mathbf{0})$. Then there is no factor $\exp(i\mathbf{k}\cdot\mathbf{x})$ and it is possible to take advantage of the symmetries of the integrands and shift the bounderies as follows:

$$\int_{-\pi}^{\pi} \frac{d^{(3)}k^{(i)}a^3}{(2\pi)^3} = \frac{a^3}{2}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\int_{-\pi}^{\pi}\frac{dk_2}{2\pi}\int_{-2\pi}^{2\pi}\frac{dk_3}{2\pi} = a^3\int_{-2\pi}^{2\pi}\frac{d^3k_i}{(4\pi)^3}, \quad \text{b.c.c.,}$$

$$\int_{-\pi}^{\pi} \frac{d^{(3)}k^{(i)}a^3}{(2\pi)^3} = \frac{a^3}{4}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\int_{-2\pi}^{2\pi}\frac{dk_2}{2\pi}\int_{-2\pi}^{2\pi}\frac{dk_3}{2\pi} = a^3\int_{-2\pi}^{2\pi}\frac{d^3k_i}{(4\pi)^3}, \quad \text{f.c.c.} \tag{6A.18}$$

Thus we arrive at the formulas, for $a = 1$, .

$$v_m^{\text{bcc}}(\mathbf{0}) = \int_{-2\pi}^{2\pi}\frac{d^3k_i}{(4\pi)^3}\frac{1}{m^2 + 8 - 8\cos\frac{k_1}{2}\cos\frac{k_2}{2}\cos\frac{k_3}{2}}, \tag{6A.19}$$

$$v_m^{\text{fcc}}(\mathbf{0}) = \int_{-2\pi}^{2\pi}\frac{d^3k_i}{(4\pi)^3}\frac{1}{m^2 + 12 - 4\left(\cos\frac{k_1}{2}\cos\frac{k_2}{2} + (23) + (31)\right)}. \tag{6A.20}$$

For the triangular lattice, it is preferable to remain with the form

$$v_m^{\text{tr}} = \frac{3}{2}\int_{-\pi}^{\pi}\frac{d^2k^{(i)}}{(2\pi)^2}\frac{1}{\frac{3}{2}m^2 + 6 - 2[\cos k^{(1)} + \cos k^{(2)} + \cos(k^{(1)} + k^{(2)})]} \tag{6A.21}$$

Let us now perform the hopping expansion. For the b.c.c. lattice we find

$$v_m^{\text{bcc}}(\mathbf{0}) = \sum_{n=0,2,4,\ldots}^{\infty}\frac{H_n^{\text{bcc}}}{(m^2+8)^{n+1}}, \tag{6A.22}$$

where

$$H_n^{\mathrm{bcc}} = \int_{-2\pi}^{2\pi} \frac{d^{(3)}k^{(i)}}{(4\pi)^3} \left(8 \cos\frac{k_1}{2}\cos\frac{k_2}{2}\cos\frac{k_3}{2}\right)^n$$
$$= \left(\int_{-2\pi}^{2\pi} \frac{dk}{4\pi} 2\cos^n\frac{k}{2}\right)^3 = \left[\frac{n!}{\left(\frac{n}{2}!\right)^2}\right]^3, \quad n = \text{even}. \tag{6A.23}$$

For the f.c.c. lattice

$$v_m^{\mathrm{fcc}}(\mathbf{0}) = \sum_{n=0,1,\ldots}^{\infty} \frac{H_n^{\mathrm{f.c.c}}}{(m^2+12)^{n+1}}, \tag{6A.24}$$

where

$$H_n^{\mathrm{fcc}} = \int_{-2\pi}^{2\pi} \frac{d^{(3)}k^{(i)}}{(4\pi)^3} \left[4\left(\cos\frac{k_1}{2}\cos\frac{k_2}{2} + (23) + (31)\right)\right]^n$$
$$= \int_{-2\pi}^{2\pi} \frac{d^{(3)}k^{(i)}}{(4\pi)^3} \sum_{r+s+t=n} \frac{n!}{r!s!t!} \left(2\cos\frac{k_1}{2}\right)^{r+s} \left(2\cos\frac{k_2}{2}\right)^{s+t} \left(2\cos\frac{k_3}{2}\right)^{t+r}$$
$$= \sum_{\substack{r+s+t=n \\ r,s,t \text{ all even} \\ \text{or all odd}}} \frac{n!}{r!s!t!} \frac{(r+s)!(s+t)!(t+r)!}{\{[\frac{1}{2}(r+s)]![\frac{1}{2}(s+t)]![\frac{1}{2}(t+r)]!\}^2}. \tag{6A.25}$$

For the triangular lattice, we have similarly

$$v_m^{\mathrm{tr}}(\mathbf{0}) = \frac{3}{2} \sum_{n=0,1,2,\ldots}^{\infty} \frac{H_n^{\mathrm{tr}}}{(\frac{3}{2}m^2+6)^{n+1}}, \tag{6A.26}$$

where

$$H_n^{\mathrm{tr}} = \int_{-\pi}^{\pi} \frac{d^2k^{(i)}}{(2\pi)^2} [2(\cos k^{(1)} + \cos k^{(2)} + \cos(k^{(1)} + k^{(2)}))]^n$$
$$= \sum_{\substack{s,t \\ 2s+t=n}} \frac{n!}{s!t!} \sum_{\substack{q=0 \\ q+t=\text{even}}}^{s} 2^{s-q} \frac{(t+q)!}{\left(\frac{t+q}{2}\right)!^2} \frac{1}{q!(s-q)!}. \tag{6A.27}$$

TABLE 6.11. Values of coefficients of the hopping expansion for the lattice Green function $v_m(\mathbf{0}) = \sum_{n=0}^{\infty} \frac{H_n}{(m^2+q)^{n+1}}$, with $q = 6, 8, 12$ for triangular, b.c.c., f.c.c. lattices, respectively.

n	triangular	b.c.c.	f.c.c.
0	1	1	1
1	0		0
2	6	8	12
3	12		48
4	90	216	540
5	360		4320
6	2040	8000	42240
7	10080		403200
8	54810	343000	4038300
9	290640		40958400
10	1588356	16003008	423550512
11	8676360		4434978240
12	47977776	788889024	46982827584
13	266378112		502437551616
14	1488801600	40424237568	5417597053440
15	8355739392		58831951546368
16	47104393050	2131746903000	642874989479579
17	266482019232		7063600894137204
18	1512589408044	114933031928000	77991775777488013
19	8610448069080		864910651813115022
20	49144928795820	6306605327953216	9629507221597984651
21	281164160225521		107593162349093634344
22	1612061452900088	351047164190381568	1206070383916553040590
23	9261029179733800		13559567467471695731262
24	53299490722049776	19774031697705428416	152861211298075771440787
25	307262475209439815		1727559090926876367050428
26	1774040236685105835	1125058699232216000000	19569034453559283214435418
27	10257353534769383296		222143040275645868025031042
28	59385592508700544928	6456131305244229600000	2526717727938851698200014242
29	344239886217487370154		28792684069865467241793030332
30	1997744954148004550493	37323516777149988910080000	3286666300252419326580303930 56

The numerical values of the hopping coefficients H_n are listed in Table 6.11.

It is again possible to accelerate the convergence by using the same method as before in the simple cubic lattice. For this we calculate the small m behavior of $v_m(\mathbf{0})$. It is useful to take the k_i integration in (A.19) in the second form of Eq. (A.18) (with $a = 1$)

$$\frac{1}{2}\int \frac{d^3k_i}{(2\pi)^3}. \tag{6A.28}$$

Then only the $k \approx 0$ region contributes

$$v_m^{\text{bcc}}(\mathbf{0}) \xrightarrow[m\to 0]{} \frac{1}{2}\int \frac{d^3k_i}{(2\pi)^3}\frac{1}{m^2+\mathbf{k}^2} \to \text{const.} - \frac{1}{8\pi}m, \tag{6A.29}$$

which is, up to the factor $\frac{1}{2}$, the same as the result for the simple cubic case. In the same way, we take the k integration in (A.20) as

$$\frac{1}{4}\int \frac{d^3k_i}{(2\pi)^3} \tag{6A.30}$$

and find again

$$v_m^{\text{fcc}}(\mathbf{0}) \xrightarrow[m\to 0]{} \frac{1}{4}\int \frac{d^3k_i}{(2\pi)^3}\frac{1}{m^2+\mathbf{k}^2} \to \text{const.} - \frac{1}{8\pi}m. \tag{6A.31}$$

For the triangular lattice, we find

$$v_m^{\text{tr}}(\mathbf{0}) \xrightarrow[m\to 0]{} \frac{3}{2}\frac{\sqrt{3}}{2}\int \frac{d^2k}{(2\pi)^2}\frac{1}{\frac{3}{2}m^2+\frac{3}{2}\mathbf{k}^2} \to \text{const.} - \frac{\sqrt{3}}{2}\frac{1}{4\pi}\log m^2. \tag{6A.32}$$

We now construct the auxiliary hopping series with these small m behaviors using

$$\begin{aligned}
\sqrt{m^2+8\mp 8} &= \sqrt{m^2+8}\sum_{n=-1}^{\infty}\binom{\frac{1}{2}}{n+1}\frac{(\mp 8)^{n+1}}{(m^2+8)^{n+1}},\\
m = \sqrt{m^2+12-12} &= \sqrt{m^2+12}\sum_{n=-1}^{\infty}\binom{\frac{1}{2}}{n+1}\frac{(-12)^{n+1}}{(m^2+12)^{n+1}},
\end{aligned} \tag{6A.33}$$

$$\begin{aligned}
\log(\tfrac{3}{2}m^2) &= \log(\tfrac{3}{2}m^2+6-6)\\
&= \log(\tfrac{3}{2}m^2+6) - \sum_{n=0}^{\infty}\frac{1}{n+1}\frac{6^{n+1}}{(\frac{3}{2}m^2+6)^{n+1}},
\end{aligned} \tag{6A.34}$$

and obtain the rapidly convergent series

$$v_m^{\text{bcc}}(\mathbf{0}) = \frac{\sqrt{2}}{4\pi}\frac{\sqrt{m^2+16}-\sqrt{m^2}}{\sqrt{m^2+8}} + \sum_{n=0,2,\ldots}^{\infty}\frac{H_n-\bar{H}_n}{(m^2+8)^{n+1}}, \tag{6A.35}$$

with

$$\tilde{H}_n = \frac{\sqrt{2}}{\pi}\frac{(2n)!}{(n!)^2(n+1)}2^{n+1}. \tag{6A.36}$$

Similarly

$$v_m^{\mathrm{fcc}}(\mathbf{0}) = \frac{\sqrt{3}}{8\pi}\left(1 - \frac{\sqrt{m^2}}{\sqrt{m^2+12}}\right) + \sum_{n=0,1,2,\ldots} \frac{H_n - \tilde{H}_n}{(m^2+12)^{n+1}}, \tag{6A.37}$$

with

$$\tilde{H}_n = \frac{\sqrt{3}}{4\pi}\frac{(2n)!}{(n!)^2(n+1)}3^{n+1} \tag{6A.38}$$

and

$$v_m^{\mathrm{tr}}(\mathbf{0}) = \frac{3}{2}\left\{\frac{\sqrt{3}}{12\pi}\log\left(\frac{\frac{3}{2}m^2+6}{\frac{3}{2}m^2}\right) + \sum_{n=0,1,\ldots} \frac{H_n - \tilde{H}_n}{(\frac{3}{2}m+6)^{n+1}}\right\} \tag{6A.39}$$

for which

$$\tilde{H}_n = \frac{\sqrt{3}}{12\pi(n+1)}6^{n+1}. \tag{6A.40}$$

Table 6.12 shows the $\tilde{H}_n$ coefficients required for subtraction. The potentials themselves as a function of m^2 are listed in Table 6.13. For $m = 0$, we find

$$\begin{aligned} v_0^{\mathrm{bcc}}(\mathbf{0}) &= 0.174\,150, \\ v_0^{\mathrm{fcc}}(\mathbf{0}) &= 0.111\,055, \\ v_0^{\mathrm{tr}}(\mathbf{0}) &= \infty. \end{aligned} \tag{6A.41}$$

The accelerated sums may be truncated after $n = 2$ with less than 1/2% error.

It is possible to give the potentials $v_m^{\mathrm{bcc}}(\mathbf{0})$, $v_m^{\mathrm{fcc}}(\mathbf{0})$ also in terms of the elliptic integral

TABLE 6.12. Values of the asymptotic hopping coefficients $\bar{H}_n$ for triangular, b.c.c., f.c.c. lattices.

n	triangular	bcc	fcc
0	0.275664447710895	0.900316316157102	0.413496671566342
1	0.826993343132685		1.240490014699027
2	3.307973372530739	7.202530529256819	7.442940088194163
3	14.885880176388326		55.822050661456224
4	71.452224846663967	201.670854819190936	468.905225556231012
5	357.261124233319833		4220.147030006079108
6	1837.342924628502000	7605.872238895168255	39789.957711485717388
7	9646.050354299635501		387952.087686985744535
8	51445.601889598056005	329587.797018790047904	3879520.876869857445346
9	277806.250203829502425		39571112.944072501837484
10	1515306.819293615467772	15484633.954482721124821	410100625.056751791653438
11	8334187.506114885072748		4306056563.095893812361100
12	46158576.956943978864449	767084943.5912962522766	45710446592.86397174684567
13	257169214.474402167959072		489754784923.5429256216411
14	1440147601.056652140570804	39450082813.2665896535455	5289351677174.241325244621
15	8100830255.943668290710775		57521699489269.87441203525
16	45745864974.74071505342555	2086213202889.799021726	629355065000246.6688769705
17	259226568190.1973853027448		6922905715002677.582953104
18	1473498387607.437769089286	112728713419308.3218599	76516326323713618.45923622
19	8398940809362.395283808931		849331222193218555.2277699
20	47993947482070.83019319390	6196858417678544.499123	9463976475867264884.600628
21	274874426488223.8456519287		105824464230151654650.7353
22	1577540186801980.331567590	34545648665192944 4.05356	1187074424842563411652.522
23	9070856074111386.90651364		13354587279478838381090.88
24	52248130986881588.5815186	19483745847168773515.994	150639744512520932560913.8
25	301431524924316857.2010688		1703387880256960881359491.
26	1741604366229386286.050620	1109740874577201447673.2	1930506264291219104661065 2.
27	10076425261755734940.721444		2192503543016451280154658 59.
28	58373773930171154139.35181	6374176649049319291808 6.	2494917824811817853929227047.
29	338567888794992694008.2405		2844206320285466680692961 2710.
30	1965878063970925320047.8483	3687975237978980535774291.	3247900120584037574607297090 16.

TABLE 6.13. Lattice potentials at the origin $v_m(\mathbf{0})$, for simple cubic lattices with $D = 2$, 3, 4 and for b.c.c., f.c.c., triangular lattices.

m^2	2	3	4	b.c.c.	f.c.c.	triangular
0.0	∞	0.2527	0.1548	0.174150	0.112055	∞
0.5	0.3198	0.1943	0.1382	0.143117	0.096562	0.2994
1.0	0.2574	0.1710	0.1270	0.129724	0.089707	0.2436
1.5	0.2204	0.1544	0.1180	0.119729	0.084455	0.2102
2.0	0.1944	0.1414	0.1105	0.111630	0.080093	0.1866
2.5	0.1746	0.1308	0.1040	0.104804	0.076330	0.1685
3.0	0.1589	0.1218	0.0983	0.098913	0.073012	0.1540
3.5	0.1460	0.1141	0.0933	0.093744	0.070042	0.1420
4.0	0.1352	0.1074	0.0888	0.089154	0.067354	0.1319
4.5	0.1260	0.1015	0.0847	0.085038	0.064901	0.1233
5.0	0.1181	0.0963	0.0811	0.081319	0.062648	0.1158
5.5	0.1111	0.0916	0.0777	0.077936	0.060567	0.1091
6.0	0.1050	0.0874	0.0747	0.074843	0.058637	0.1033
6.5	0.0995	0.0835	0.0718	0.072002	0.056838	0.0980
7.0	0.0946	0.0800	0.0692	0.069380	0.055158	0.0933
7.5	0.0901	0.0768	0.0668	0.066951	0.053582	0.0890
8.0	0.0861	0.0739	0.0646	0.064695	0.052101	0.0851
8.5	0.0824	0.0712	0.0625	0.062592	0.050706	0.0816
9.0	0.0791	0.0686	0.0605	0.060627	0.049388	0.0783
9.5	0.0760	0.0663	0.0587	0.058786	0.048142	0.0753
10.0	0.0731	0.0641	0.0570	0.057057	0.046960	0.0725

$$K(k^2) = \int_0^{\pi/2} d\theta (1 - k^2 \sin^2\theta)^{-1/2}, \tag{6A.42}$$

which has the accurate expansion (6.135). For the bcc lattice one finds, after some calculation,

$$v_m^{\text{bcc}}(\mathbf{0}) = \int_{-2\pi}^{2\pi} \frac{d^3k}{(4\pi)^3} \frac{1}{m^2 + 8\left(1 - \cos\frac{k_1}{2}\cos\frac{k_3}{2}\cos\frac{k_3}{2}\right)}$$

$$= \frac{1}{\pi^2}\frac{8}{m^2+8}[K(k^2)]^2, \tag{6A.43}$$

where

$$k^2 \equiv \frac{1}{2} - \frac{1}{2}\left[1 - \left(\frac{8}{m^2+8}\right)^2\right]^{1/2}. \tag{6A.44}$$

For $m = 0$ this gives

$$v_0^{\mathrm{bcc}}(\mathbf{0}) = \frac{1}{2\pi^2}[K(\tfrac{1}{2})]^2 = \frac{1}{32\pi^3}[\Gamma(\tfrac{1}{4})]^4 = 0.174\,150\,491\,2. \quad \text{(6A.45)}$$

On an f.c.c. lattice we find

$$v_m^{\mathrm{fcc}}(\mathbf{0}) = \int_{-2\pi}^{2\pi} \frac{d^3k}{(4\pi)^3} \frac{1}{m^2 + 4\left[3 - \left(\cos\frac{k_1}{2}\cos\frac{k_2}{2} + (23) + (31)\right)\right]}$$

$$= \frac{1}{\pi^2}\frac{4}{m^2+16} K(k_+^2)\, K(k_-^2), \quad \text{(6A.46)}$$

where

$$k_\pm^2 \equiv \frac{1}{2}\left[1 \pm \frac{4\sqrt{t}\sqrt{t+1}}{(t+1)^3} - \frac{(t-1)\sqrt{t+1}\sqrt{t-3}}{(t+1)^2}\right]_{t=(m^2+12)/4}$$

$$= \frac{1}{2}\left[1 \pm \frac{16\sqrt{m^2+12}\sqrt{m^2+16}}{(m^2+16)^2} - \frac{(m^2+8)\sqrt{m^2+16}\sqrt{m^2}}{(m^2+16)^2}\right]. \quad \text{(6A.47)}$$

For $m = 0$

$$v_0^{\mathrm{fcc}} = \frac{1}{4\pi^2} K\left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right) K\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right). \quad \text{(6A.48)}$$

In the Tables one usually finds $K(\sin^2\alpha)$. The arguments $k_\pm^2$ correspond to $\alpha_+ = 75°$, $\alpha_- = 15°$ resulting in

$$v_0^{\mathrm{fcc}}(\mathbf{0}) = 0.112\,055\,098\,597. \quad \text{(6A.49)}$$

Consider now the logarithms of the fluctuation determinants

$$\frac{1}{N}\mathrm{tr}\log^{\mathrm{bcc}}(-\bar{\nabla}\cdot\nabla + m^2) = \int_{-2\pi}^{2\pi} \frac{d^3k_i}{(4\pi)^3} \log\left(m^2 + 8 - 8\cos\frac{k_1}{2}\cos\frac{k_2}{2}\cos\frac{k_3}{2}\right), \quad \text{(6A.50)}$$

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{fcc}}(-\bar{\nabla}\cdot\nabla+m^2)$$

$$=\int_{-2\pi}^{2\pi}\frac{d^3k_i}{(4\pi)^3}\log\left(m^2+12-4\left(\cos\frac{k_1}{2}\cos\frac{k_2}{2}+(23)+(31)\right)\right), \quad (6A.51)$$

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{tr}}(-\bar{\nabla}\cdot\nabla+m^2)$$

$$=\log\frac{2}{3}+\int_{-\pi}^{\pi}\frac{d^2k^{(i)}}{(2\pi)^2}\times\log(\tfrac{3}{2}m^2+6-2(\cos k^{(1)}+\cos k^{(2)}+\cos(k^{(1)}+k^{(2)}))). \quad (6A.52)$$

Their hopping expansions are

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{bcc}}(-\bar{\nabla}\cdot\nabla+m^2)=\log(m^2+8)-\sum_{n=2,4,6,\ldots}^{\infty}\frac{H_n^{\mathrm{bcc}}}{n(m^2+8)^n}, \quad (6A.53)$$

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{fcc}}(-\bar{\nabla}\cdot\nabla+m^2)=\log(m^2+12)-\sum_{n=1,2,3,\ldots}^{\infty}\frac{H_n^{\mathrm{bcc}}}{n(m^2+12)^n}, \quad (6A.54)$$

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{tr}}(-\bar{\nabla}\cdot\nabla+m^2)=\log(m^2+4)-\sum_{n=1,2,3,\ldots}^{\infty}\frac{H_n^{\mathrm{tr}}}{n(\frac{3}{2}m^2+6)^n}. \quad (6A.55)$$

Their convergence can be accelerated by using

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{bcc}}(-\bar{\nabla}\cdot\nabla+m^2)$$

$$=-\log 4+\frac{\sqrt{2}}{4\pi}\{-\sqrt{m^2(m^2+8)}+\sqrt{(m^2+16)(m^2+8)}-8$$

$$+8\log[(m+\sqrt{m^2+8})(\sqrt{m^2+16}+\sqrt{m^2+8})/4]$$

$$+\log(m^2+8)\left(1-\frac{2\sqrt{2}}{\pi}\right)-\sum_{n=2,4,\ldots}^{\infty}\frac{H_n-\tilde{H}_n}{n(m^2+8)^n} \quad (6A.56)$$

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{fcc}}(-\bar{\nabla}\cdot\nabla + m^2)$$

$$= -\log 4 + \frac{\sqrt{3}}{8\pi}\{m^2 - \sqrt{m^2}\sqrt{m^2+12}) + 6 + 12\log[(m + \sqrt{m^2+12})/2]$$

$$+ \log(m^2+12)\left(1 - \frac{3\sqrt{3}}{4\pi}\right) - \sum_{n=2,4,\ldots}^{\infty} \frac{H_n - \tilde{H}_n}{n(m^2+12)^n}, \tag{6A.57}$$

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{tr}}(-\bar{\nabla}\cdot\nabla + m^2) = \log(m^2+4)$$

$$+ \frac{\sqrt{3}}{12\pi}\left\{\frac{3}{2}m^2 + \log\left(\frac{\frac{3}{2}m^2+6}{\frac{3}{2}m^2}\right) - 6 - \sum_{n=1,2,\ldots}^{\infty} \frac{H_n - \tilde{H}_n}{n(\frac{3}{2}m^2+6)^n}\right\}. \tag{6A.58}$$

For $m = 0$, the exact values are

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{bcc}}(-\bar{\nabla}\cdot\nabla + m^2) = 0.603\,897,$$

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{fcc}}(-\bar{\nabla}\cdot\nabla + m^2) = 1.026\,626,$$

$$\frac{1}{N}\operatorname{tr}\log^{\mathrm{tr}}(-\bar{\nabla}\cdot\nabla + m^2) = 1.580, \tag{6A.59}$$

and the series up to $n = 2$ is everywhere more than 2% accurate.

The spatial dependence of the lattice potentials $v_m(\mathbf{x})$ for $m = 0$ and $m \neq 0$ is obtained by numerical integration. The asymptotic behaviors of the $v_m^{\mathrm{bcc}}(\mathbf{x})$ and $v_m^{\mathrm{fcc}}(\mathbf{x})$ potentials are the same as in the continuum case up to a factor J. The asymptotic limits are reached quite quickly. In Table 6.15 we have listed the values of the subtracted Coulomb potential $v_0^{\mathrm{tr}}(\mathbf{x})$ for the triangular lattice. Just as in the simple cubic case, it is quite easy to calculate the asymptotic behavior explicitly. The potential is defined by

$$v_0'^{\mathrm{tr}}(\mathbf{x}) \equiv v_0^{\mathrm{tr}}(\mathbf{x}) - v_0^{\mathrm{tr}}(\mathbf{0})$$

$$= \int_{-\pi}^{\pi} \frac{d^2k^{(i)}}{(2\pi)^2} \frac{\cos(k^{(1)}x^{(1)} + k^{(2)}x^{(2)}) - 1}{4 - \frac{4}{3}(\cos k^{(1)} + \cos k^{(2)} + \cos(k^{(1)} + k^{(2)}))}. \tag{6A.60}$$

TABLE 6.14. Logarithms of the functional determinants $(1/N)\,\mathrm{tr}\log(-\bar{\nabla}\cdot\nabla + m^2)$ for simple cubic lattices of $D = 2, 3, 4$ and b.c.c., f.c.c., triangular lattices.

m^2	2	3	4	b.c.c.	f.c.c.	triangular
0.0	1.1662	1.6733	1.9997	0.603 979	1.026 708	0.176 084
0.5	1.3655	1.7797	2.0731	0.680 827	1.077 626	0.106 812
1.0	1.5060	1.8703	2.1397	0.748 820	1.124 079	0.077 945
1.5	1.6233	1.9511	2.2011	0.811 077	1.167 563	0.060 357
2.0	1.7255	2.0247	2.2583	0.868 851	1.208 665	0.048 460
2.5	1.8166	2.0924	2.3119	0.922 913	1.247 747	0.039 915
3.0	1.8991	2.1553	2.3625	0.973 808	1.285 064	0.033 523
3.5	1.9747	2.2141	2.4104	1.021 946	1.320 814	0.028 595
4.0	2.0444	2.2694	2.4559	1.067 648	1.355 151	0.024 703
4.5	2.1093	2.3215	2.4992	1.111 178	1.388 206	0.021 571
5.0	2.1700	2.3708	2.5406	1.152 752	1.420 085	0.019 008
5.5	2.2270	2.4177	2.5803	1.192 552	1.450 882	0.016 883
6.0	2.2807	2.4624	2.6184	1.230 736	1.480 677	0.015 100
6.5	2.3316	2.5051	2.6550	1.267 437	1.509 541	0.013 588
7.0	2.3800	2.5459	2.6902	1.302 775	1.537 535	0.012 295
7.5	2.4260	2.5851	2.7242	1.336 850	1.564 716	0.011 179
8.0	2.4699	2.6227	2.7571	1.369 755	1.591 133	0.010 211
8.5	2.5119	2.6590	2.7888	1.401 570	1.616 831	0.009 363
9.0	2.5522	2.6939	2.8196	1.432 369	1.641 852	0.008 618
9.5	2.5909	2.7276	2.8493	1.462 218	1.666 231	0.007 959
10.0	2.6281	2.7602	2.8783	1.491 174	1.690 004	0.007 373

TABLE 6.15. The subtracted Coulomb potential $-v_0'^{\mathrm{tr}}(x^{(1)}, x^{(2)})$ [see Eq. (A.60)] on a triangular lattice $\mathbf{x} = x^{(1)}(1, 0) + x^{(2)}(-1/2, \sqrt{3}/2)$. Hexagonal symmetry implies $v_0'^{\mathrm{tr}}(n, n) = v_0'^{\mathrm{tr}}(n, 0) = v_0'^{\mathrm{tr}}(0, n)$. Notice that the rows are labelled by $-x^{(2)}$.

$-x^{(2)}$ \ $x^{(1)}$	0	1	2	3	4	5	6	7	8	9
0	0000	0.250	0.346	0.402	0.442	0.472	0.49	0.519	0.537	0.553
1		0.327	0.385	0.428	0.461	0.488	0.511	0.530	0.5474	0.563
2			0.422	0.454	0.481	0.504	0.524	0.542	0.5575	0.571
3				0.478	0.500	0.520	0.538	0.553	0.5677	0.582
4					0.519	0.535	0.551	0.565	0.5780	0.580
5						0.550	0.564	0.576	0.5882	0.599
6							0.576	0.587	0.5981	0.608
7								0.598	0.608	0.617
8									0.617	0.626
9										0.634

Consider the line $x^{(1)} = x^{(2)} = n$ (= integer). Just as in Eq. (6.189) we introduce the momenta

$$p = (k^{(1)} + k^{(2)})/2,$$
$$q = (k^{(1)} - k^{(2)})/2, \tag{6A.61}$$

and obtain

$$\begin{aligned} v_0'^{\mathrm{tr}}(n, n) &= \int_0^{2\pi} \frac{dp}{2\pi} \int_0^{\pi} \frac{dq}{\pi} \frac{\cos(2pn) - 1}{4 - \frac{4}{3}(2\cos p \cos q + \cos 2p)} \\ &= \frac{3}{8} \int_0^{2\pi} \frac{dp}{2\pi} \int_0^{\pi} \frac{dq}{\pi} \frac{\cos(2pn) - 1}{2 - \cos p \cos q - \cos^2 p} \\ &= \frac{3}{8} \int_0^{2\pi} \frac{dp}{2\pi} \frac{\cos(2pn) - 1}{\sqrt{(2 - \cos^2 p)^2 - \cos^2 p}} \\ &= \frac{\sqrt{3}}{8} \int_0^{2\pi} \frac{dp}{2\pi} \frac{\cos(2pn) - 1}{|\sin p|} \frac{\sqrt{3}}{\sqrt{4 - \cos^2 p}}. \end{aligned} \tag{6A.62}$$

It is now convenient to express this in terms of the previously calculated expression for the diagonal values of the potential on a simple cubic lattice (Eq. (6.192)) and write

$$v_0'^{\mathrm{tr}}(n, n) = \frac{\sqrt{3}}{2} \left\{ v_0'(n, n) + \frac{1}{4} \int_0^{2\pi} \frac{dp}{2\pi} \frac{\cos(2pn) - 1}{|\sin p|} \left[\frac{\sqrt{3}}{\sqrt{3 + \sin^2 p}} - 1 \right] \right\}. \tag{6A.63}$$

The asymptotic behavior of $v_0'(n, n)$ was given in Eq. (6.196):

$$v_0'(n, n) \to -\frac{1}{2\pi} \log(n \cdot 4e^{\gamma}). \tag{6A.64}$$

In the limit $n \to \infty$, the additional term contributes only via the integral[g]

$$-\int_0^{2\pi} \frac{dp}{2\pi} \frac{1}{|\sin p|} \left[\frac{1}{\sqrt{1 + \frac{1}{3}\sin^2 p}} - 1 \right]. \tag{6A.65}$$

[g]The $\cos(2pn)$ contribution vanishes due to fast oscillations.

In order to calculate this we introduce the dummy parameter μ (to be taken $\to 0$ at the end) and consider

$$-\int_0^{2\pi}\frac{dp}{2\pi}\left\{\frac{1}{(1+\frac{1}{3}\sin^2 p)^{(\mu+1)/2}}\sin^{\mu-1}p-\sin^{\mu-1}p\right\}. \tag{6A.66}$$

These are well-known integrals and give

$$-\frac{1}{2\pi}\left[\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}\left(\frac{3}{4}\right)^{\mu/2}-2^{\mu-2}\frac{\Gamma\left(\frac{\mu}{2}\right)^2}{\Gamma(\mu)}\right]. \tag{6A.67}$$

Expanding the above Γ functions, we find

$$\begin{aligned}\Gamma\left(\frac{\mu}{2}\right)&=\frac{2}{\mu}\left(1-\frac{\mu}{2}\gamma+\ldots\right),\\ \Gamma\left(\frac{\mu+1}{2}\right)&=\Gamma\left(\frac{1}{2}\right)\left(1-\frac{\mu}{2}(\gamma+\log 4)+\ldots\right),\\ \Gamma(\mu)&=\frac{1}{\mu}(1-\mu\gamma+\ldots).\end{aligned} \tag{6A.68}$$

The $\mu\to 0$ limit of (A.67) is therefore

$$-\frac{1}{2\pi}\log\frac{\sqrt{3}}{2}. \tag{6A.69}$$

Hence we find for the triangular lattice

$$v_0'^{\mathrm{tr}}(n,n)=-\frac{\sqrt{3}}{2}\frac{1}{2\pi}\log(n\,2\sqrt{3}\,e^{\gamma}). \tag{6A.70}$$

The proper length of the vector $(x^{(1)}, x^{(2)}) = (n, n)$ is given by

$$|\mathbf{x}|=a\left|n(1,0)+n\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)\right|=an. \tag{6A.71}$$

Hence as a function of $|\mathbf{x}|$, $v_0'^{\mathrm{tr}}(\mathbf{x})$ behaves as follows:

$$v_0'^{\mathrm{tr}}(\mathbf{x}) \to -\frac{\sqrt{3}}{2}\frac{1}{2\pi}\log\left(\frac{|\mathbf{x}|}{a}2\sqrt{3}e^{\gamma}\right), \tag{6A.72}$$

where $2\sqrt{3}e^{\gamma} \approx 6.16982191$. Notice that due to hexagonal symmetry, $v'(n, n)$ is the same as $v'(n, 0) = v'(0, n)$.

The factor $\sqrt{3}/2$ is a consequence of our normalization convention in which $v_0(\mathbf{x})$ satisfies

$$-\bar{\nabla}\cdot\nabla v_0(\mathbf{x}) = \delta_{\mathbf{x},\mathbf{0}}. \tag{6A.73a}$$

Some authors define $v_0(\mathbf{x})$ with the normalization

$$-\bar{\nabla}\cdot\nabla v_0(\mathbf{x}) = \delta_{\mathbf{x},\mathbf{0}}2/\sqrt{3}. \tag{6A.73b}$$

This has the advantage that in the continuum limit, where

$$-\bar{\nabla}\cdot\nabla \to -a^2\partial^2,$$

$$\delta_{\mathbf{x},\mathbf{0}} \to \delta^2(\mathbf{x})\sqrt{3}a^2/2, \tag{6A.74}$$

the potential $v_0(\mathbf{x})$ of (6A.74) goes directly over into the usual Coulomb potential in the continuum, which satisfies

$$-\partial^2 v(\mathbf{x}) = \delta(\mathbf{x}). \tag{6A.75}$$

It has, therefore, the same large distance behaviour, i.e.

$$v_0(\mathbf{x}) \to -(2\pi)^{-1}\ln(|\mathbf{x}|2\sqrt{3}e^{\gamma}/a). \tag{6A.76}$$

Similarly, the Coulomb potentials on a b.c.c. and f.c.c. lattice have the usual long-distance behaviour $1/4\pi|\mathbf{x}|$ only after multiplication by factor $J^{-1} = 2$ and 4, respectively [compare (6A.17) and (6A.18)].

The exact evaluation of the Green function $v_4''(n, n)$ for arbitrary n proceeds by expanding $\cos(2np)$ of Eq. (6A.62) in powers of $\sin p$, up to $\sin^{2n}p$,

$$\cos(2pn) = 1 - \frac{(2n)^2}{2!}\sin^2 p + (2n)^2\frac{((2n)^2 - 2^2)}{4!}\sin^4 p$$

$$\frac{(2n)^2((2n)^2 - 2^2)((2n)^2 - 4^2)}{6!}\sin^6 p + \ldots \tag{6A.77}$$

So that we are faced with a sequence of integrals

$$\int_0^{\pi/2} dp \sin^\alpha p \frac{1}{\sqrt{1+(1/3)\sin^2 p}} \tag{6A.78}$$

for $\alpha = 1, 3, \ldots, 2n-1$. These are evaluated using the formula

$$\int_0^{\pi/2} dp \sin^\alpha p \frac{1}{(1-z\sin^2 p)^\rho} = \frac{1}{2} B((1+\alpha)/2,\ 1/2) F((1+\alpha)/2,\ \rho,\ 1+\alpha/2,\ z)$$

$$= \frac{1}{2} B((1+\alpha)/2,\ 1/2)(1-z)^{-\rho}\Gamma(1/2,\ \rho,\ 1+\alpha/2,\ z/(z-1)) \tag{6A.79}$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ are beta and hypergeometric functions, respectively. The lowest integral (6A.80) involves $\alpha = 1$, $\rho = 1/2$ and thus

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, w\right) = w^{-1} \operatorname{arsin} w. \tag{6A.80}$$

The higher cases, $\alpha = 3, 5, \ldots$, etc. can be found recursively, using the well-known relation for the Legendre functions

$$(t^2-1)\frac{d}{dt} P_\nu^\mu(t) = (\nu+\mu)(\nu-\pi+1)(t^2-1)^{1/2} P_\nu^{\mu-1}(t) - \mu t P_\nu^\mu(t). \tag{6A.81}$$

This translates into a recursion relation for F,

$$F(-\nu,\ \nu+1,\ 2-\mu,\ (1-t)/2) = [(1-\mu)(\nu+\mu)(\nu-\mu+1)]$$

$$\times \left\{ (1+t)(1-t)]^{-\mu/2} \frac{d}{dt} [(1+t)(1-t)]^{\mu/2} - \frac{\mu t}{1-t^2} \right\}$$

$$F\left(-\nu,\ \nu+1,\ 1-\mu,\ \frac{1-t}{2}\right) \tag{6A.82}$$

and permits calculating the desired functions

$$h(\lambda,\ w) = F\left(\frac{1}{2}, \frac{1}{2}, \lambda, w\right),$$

TABLE 6.16. Exact values of subtracted lattice Green functions $v'_2(\mathbf{x})$ on a triangular lattice, as in Table 6.15 [Kleinert (1988)].

$x^{(1)}$	$-x^{(2)}$	$v'_2(\mathbf{x})$
0	1	$-1/4$
0	2	$3\sqrt{3}/\pi - 2$
0	3	$36\sqrt{3}/\pi - 81/4$
0	4	$420\sqrt{3}/\pi - 232$
0	5	$5100\sqrt{3}/\pi - 11249/4$
0	6	$318807\sqrt{3}/5\pi - 35154$
0	7	$4063416\sqrt{3}/5\pi - 1792225/4$
1	1	$-3\sqrt{3}/2\pi + 1/2$
1	2	$-15\sqrt{3}/2\pi + 15/4$
1	3	$-96\sqrt{3}/\pi + 105/2$
1	4	$-1212\sqrt{3}/\pi + 2671/4$
1	5	$-155787\sqrt{3}/10\pi + 17177/2$
1	6	$-2020287\sqrt{3}/10\pi + 445535/4$
1	7	$-92434836\sqrt{3}/35\pi + 2912113/2$
2	2	$21\sqrt{3}/\pi - 12$
2	3	$333\sqrt{3}/2\pi - 369/4$
2	4	$13581\sqrt{3}/5\pi - 1498$
2	5	$381909\sqrt{3}/10\pi - 84225/4$
2	6	$18541821\sqrt{3}/35\pi - 292076$
2	7	$508856151\sqrt{3}/70\pi - 16031265/4$
3	3	$-5967\sqrt{3}/10\pi + 657/2$
3	4	$-23364\sqrt{3}/5\pi + 10303/4$
3	5	$-2836992\sqrt{3}/35\pi + 89377/2$
3	6	$-84341997\sqrt{3}/70\pi + 265715/4$
3	7	$-245959593\sqrt{3}/14\pi + 19372089/2$
4	4	$608388/35\sqrt{3}/\pi - 9584$
4	5	$4825704\sqrt{3}/35\pi - 304065/4$
4	6	$86914383\sqrt{3}/35\pi - 1369098$
4	7	$1339015704\sqrt{3}/35\pi - 84370065/4$
5	5	$-36772521\sqrt{3}/70\pi + 579249/2$
5	6	$-293797737\sqrt{3}/70\pi + 9255951/4$
5	7	$-4258672416\sqrt{3}/55\pi + 85379241/2$
6	6	$6248287053\sqrt{3}/385\pi - 8947692$
6	7	$14334701607\sqrt{3}/110\pi - 287386737/4$
7	7	$-5095277250087\sqrt{3}/10010\pi + 561273441/2$

iteratively, using

$$f(\lambda, w) = \left(\lambda - \frac{3}{2}\right)^{-2} [(\lambda-2)(\lambda-1)f(\lambda-1, w) + (\lambda-1)(1-w)\partial_w f(\lambda-1, w)], \tag{6A.83}$$

and starting from (6A.81).

These functions become rapidly very involved. We calculate $v'_2(n, 0)$ up to some $n_{\max}$. Then we set $v'_2(n, -1) = v'_2(n+1, 1)$ and calculate $v'_2(n, -m)$ by solving the Laplace equation $-\bar{\nabla}\cdot\nabla v'_2(n, -m) = 0$ for $v'_2(n, -m-1)$, letting successively, $m = 0, 1, 2, \ldots, (n_{\max} - 1)$ and, for each m, $n = 1, 2, \ldots, n_{\max} - m$. In this way we fill up the entire triangle $n + |m| \leq n_{\max}$ $(n > 0, m < 0)$. The resulting potentials are listed in Table 6.16.

NOTES AND REFERENCES

The path-integral representation for study fluctuating lines was used extensively by S. Edwards and K.F. Freed in the context of polymers. Some of the original references are S.F. Edwards, *Proc. Roy. Soc.* **85** (1965) 613, Natl. Bur. Stand. Misc. Publ. **273** (1967) 255, H.P. Gillis and K.F. Freed, *J. Chem. Phys.* **63** (1975) 852.

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