PART IV

DIFFERENTIAL GEOMETRY OF DEFECTS AND GRAVITY WITH TORSION

Qua propter locus est intactus,
inane vacansque. Quod si non esset,
nulla ratione moveri res possent.

(Therefore, space is untouchable,
free and empty. For if it were not so,
matter could not move.)

Lucretius, *De Rerum Natura*, Rome, 57 B.C.

CHAPTER ONE

INTRODUCTION

The defects in different physical systems have the common property that, in the continuum limit, certain closed contour integrals over field variables do not vanish due to singularities. If the field variables are spatial distortions, as in the case of crystals, the defects in the continuum may be efficiently described by means of differential geometry. A crystal filled with dislocations and disclinations turns out to have the same geometric properties as an affine space with torsion and curvature, respectively. Now, according to Einstein and later researchers, such a space forms the basis for a coordinate independent description of gravity. Mass points generate curvature, spinning matter gives rise to curvature and torsion. We may therefore expect many features of gravitational matter to coincide with those of crystalline defects.

In the previous discussion of line-like defects we saw that defects can always be described in terms of defect-gauge fields. This was a consequence of their closed-loop nature and the associated vanishing divergence of the defect density. The gauge transformations amount to movements of the physically irrelevant Volterra surfaces. The standard geometric description of gravitational matter, on the other hand, is given in terms of a *metric tensor* and a *connection* field which govern the distances and laws of parallelism in space.

The relation between the two descriptions is quite simple. Since Sciama

(1962) and Kibble (1961) it has been known that geometric quantities may be interpreted as gauge quantities associated with local transformations of the space group. They showed that the theory of gravitational matter is invariant under local translations and local rotations.[a] We shall see that, structurally, this is precisely the gauge invariance found previously for defect systems when subjected to deformations of the Volterra surfaces.

Differential geometry has one disadvantage, however. It cannot cope with the discreteness of the lattice. The gauge fields of defects previously studied were discrete. In the geometric discussion to follow these will always be continuous. Up to now there exists no analogue of differential geometry which would be applicable to discrete defect systems. This is why we must restrict ourselves to a continuum approximation of the defected crystal. While the continuum approximation is physically not quite correct, it has, at least, the advantage of being mathematically manageable and consistent. We had seen that the description of disclinations in terms of integer gauge fields ran into certain difficulties. The non-Abelian nature of the group of rotations was not properly respected and we were forced to adopt a certain "tangential approximation." In the gauge formulation of differential geometry which works entirely with infinitesimal defects, such mathematical difficulties will certainly be absent.

To be specific it will be necessary to review the ordinary differential geometry in such a way as to make the connection with defect theory most transparent. Einstein's original theory ignored the spin of gravitational matter and the geometry he employed was free of torsion. The role of spin was recognized only much later. Quantitatively, its effects are very small and, up to the present, the different possible equations of motion involving spin and torsion have not been tested experimentally.

After generalizing Einstein's theory to spaces with curvature *and* torsion we shall use our experience and develop, in the same type of space, a general differential geometric theory of stresses and defects.

[a] It goes without saying that "space" in gravity includes the time variable so that the local translations and local rotations comprise local time displacements and local Lorentz transformations, respectively.

CHAPTER TWO

METRIC-AFFINE SPACES

2.1. GRAVITY AND GEOMETRY

Einstein's theory is deeply rooted in the philosophy of Plato, who postulated the relevance of simple geometric laws in nature. It starts with the observation that in the macrocosmos, that is, over length scales exceeding by far the distances of the planets, all but gravitational forces are irrelevant.[a] These forces, on the other hand, are quite simple. They are determined by one single parameter per point-like object, its mass, and are independent of any other internal structure. Moreover, the gravitational acceleration of arbitrary mass points is universal. It is independent of the size of the mass of the particle due to the marvelous equality of inertial and gravitational mass. It is this universality which forms the basis of Einstein's geometric description of the motion of mass points in gravitational fields.

In the absence of gravitational fields, relativistic physics is described in inertial frames in a Lorentz invariant fashion using coordinate four vectors $x^a = (ct, x^1, x^2, x^3)$, $a = 0, 1, 2, 3$, to specify space-time points (where c is the light velocity in vacuum $c = 2.9979250(10) \times 10^{10}$ cm/s) and vectors and tensors, v^a, $t^{a_1, \ldots, a_n}$, to specify physical observables. Such quantities make it easy to formulate any equation of motion in a

[a]Recently it has become apparent, however, that magnetic fields are more important than previously thought.

Lorentz covariant form, i.e., in a form which does not depend on the particular coordinate system. All one has to do is to define products of vector and tensor quantities with the help of the Minkowski metric

$$\eta_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{2.1}$$

and subsequent contraction of indices. The scalar product of two vectors, for example, is given by $v \cdot w \equiv v^a w^b \eta_{ab}$.

Free massive particles run on straight world lines in four-dimensional space-time. Starting at a space-time point x_1, these arrive at another space-time point x_2 following straight world lines which extremize the action

$$\mathscr{A} = -m \int_{x_1}^{x_2} ds, \tag{2.2}$$

where ds is the Lorentz invariant quantity

$$ds = (\eta_{ab} dx^a dx^b)^{1/2}, \tag{2.3}$$

and will be referred to as the *invariant length parameter*. Its quotient with the light velocity, $ds/c \equiv d\tau$, gives the elapsed time measured by a clock attached to the particle. Unlike material particles, light rays behave like "timeless particles": for them, no time elapses, since they follow paths given by $ds = 0$. In the following we will not mention time explicitly and use space for space-time.

Einstein realized that the universality of the gravitational acceleration, imparted upon an arbitrary mass point by a gravitational force, makes it possible to eliminate these forces completely, at least locally, i.e., in a small neighborhood of the point. This is achieved by going to an inertial coordinate system which itself follows this acceleration. This is called a "freely falling frame." This local frame is then again of the Minkowski type. In it, free particles trace out straight lines.

The situation can also be looked upon from the opposite or *active* view, in contrast to the previous *passive view*: A coordinate system x^μ in which gravitational forces are observed can be thought of, locally, as being an accelerated piece of an inertial frame x^a with the acceleration simulating the gravitational forces. It is a direct consequence of this observation that the motion of a free particle in a gravitational field can be described locally by the same laws as those valid in Minkowski space. In order to achieve this, the Minkowski metric η_{ab} has to be replaced by another one, called $g_{\mu\nu}(x)$, which arises from η_{ab} after a *local coordinate transformation* from the freely falling frame x^a to the actual frame x^μ, where the gravitational forces are present. Let this local transformation be given by

$$x^a \to x^\mu(x^a). \tag{2.4}$$

From the above construction it is clear that it can be defined at most in some *small* neighborhood of x^a. Under such a transformation the Lorentz invariant distance (2.3) goes over into

$$ds = [g_{\mu\nu}(x)\,dx^\mu dx^\nu]^{1/2}, \tag{2.5a}$$

where the new metric $g_{\mu\nu}(x)$ is given by

$$g_{\mu\nu}(x) = \eta_{ab}\frac{\partial x^a}{\partial x^\mu}\frac{\partial x^b}{\partial x^\nu}. \tag{2.5b}$$

Light rays still satisfy $ds = 0$ and the world line of a free massive particle minimizes the action (2.2), but now with ds from (2.5a). It is straightforward to find the differential equations for this movement as described in the coordinates x^μ. If we parametrize the path directly in terms of the variable s, we can calculate

$$\begin{aligned}\delta\int_{x_1}^{x_2} ds &= \delta\int_{s_1}^{s_2} ds\left(g_{\mu\nu}(x(s))\frac{dx^\mu(s)}{ds}\frac{dx^\nu(s)}{ds}\right)^{1/2}\\ &= \frac{1}{2}\int_{s_1}^{s_2} ds\left[(\partial_\lambda g_{\mu\nu})\,\delta x^\lambda\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + 2g_{\lambda\nu}\frac{d\delta x^\lambda}{ds}\frac{dx^\nu}{ds}\right].\end{aligned} \tag{2.6}$$

Partially integrating the last term gives,

$$g_{\mu\nu}\,\delta x^\mu\frac{dx^\nu}{ds}\Big|_{s1}^{s_2} - \int_{s_1}^{s_2} ds\frac{d}{ds}\left(g_{\lambda\nu}\frac{dx^\nu}{ds}\right)\delta x^\lambda.$$

Since δx^μ must be set equal to zero at the end points we find, from the extremal principle,

$$\delta\int_{x_1}^{x_2} ds = \frac{1}{2}\int_{s_1}^{s_2} ds\left[(\partial_\lambda g_{\mu\nu} - 2\partial_\mu g_{\lambda\nu})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2g_{\lambda\nu}\frac{d^2x^\nu}{ds^2}\right]\delta x^\lambda = 0 \tag{2.7}$$

and the differential equation for particle motion becomes

$$g_{\lambda\nu}\frac{d^2x^\nu}{ds^2}+\left(\partial_\mu g_{\lambda\nu}-\frac{1}{2}\partial_\lambda g_{\mu\nu}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=0. \tag{2.8}$$

It is convenient to introduce a quantity called the *Christoffel symbol of the first kind*

$$\{\mu\nu,\lambda\}=\tfrac{1}{2}(\partial_\mu g_{\nu\lambda}+\partial_\nu g_\mu-\partial_\lambda g_{\mu\nu}), \tag{2.9}$$

which describes the so-called *connection* of the space in the parametrization x^μ. Then

$$g_{\lambda\mu}\frac{d^2x^\mu}{ds^2}+\{\mu\nu,\lambda\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=0. \tag{2.10}$$

It is also useful to introduce the *Christoffel symbol of the second kind*

$$\left\{\begin{matrix}\varkappa\\ \mu\nu\end{matrix}\right\}=g^{\varkappa\lambda}\{\mu\nu,\lambda\}, \tag{2.11}$$

in terms of which Eq. (2.8) takes the form

$$\frac{d^2x^\lambda}{ds^2}+\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{dx}=0. \tag{2.12}$$

Observe that the form of this equation does not at all imply that there really *is* a gravitational field. After all, the same equation would hold if the coordinate transformation (2.4) did not involve any acceleration. The gravitational field is hidden in the Christoffel symbol in a somewhat subtle way. In order to see precisely how, we have to take a closer look at the geometry inherent in the metric $g_{\mu\nu}(x)$.

2.2. MINKOWSKI GEOMETRY FORMULATED IN GENERAL COORDINATES

2.2.1. *Local basis tetrads*

If we want to understand the geometry of a space with gravitational forces it is important to learn to distinguish between inessential complications which are merely due to the formulation in terms of general coordinates, and the proper manifestations of these forces. For this purpose it is useful to look first at a space *without* these forces, i.e., ordinary Minkowski space x^a with a metric $ds = (\eta_{ab} dx^a dx^b)^{1/2}$ but described in terms of general curvilinear coordinates $x^\mu(x^a)$. The basis vectors in Minkowski space will be denoted by $\mathbf{e}_a$ so that $\mathbf{x} = \mathbf{e}_a x^a$ are the vectors pointing to specific points in this space. The basis vectors $\mathbf{e}_a$ define what is called an *inertial frame* of reference. They can be taken to be orthonormal with respect to the Minkowski metric η_{ab}, i.e.,

$$\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab}. \tag{2.13}$$

Consider now an arbitrary new set of coordinates x^μ for the same points in space whose values are given by a mapping

$$x^a \to x^\mu = x^\mu(x^a). \tag{2.14}$$

In order that x^μ provide a reparametrization of Minkowski space we shall assume that the functions $x^\mu(x^a)$ possess an inverse $x^a = x^a(x^\mu)$ and are sufficiently smooth so that $x^\mu(x^a)$ and $x^a(x^\mu)$ are twice differentiable. Then by Schwarz' lemma, the second derivatives commute with each other. In other words, the general coordinate transformation (2.14) and its inverse $x^a(x^\mu)$ are supposed to satisfy the integrability conditions

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) x^a(x^\lambda) = 0, \qquad (\partial_a \partial_b - \partial_b \partial_a) x^\mu(x^c) = 0. \tag{2.15}$$

The derivatives $\partial x^a / \partial x^\mu$ define a network of new coordinate lines whose tangent vectors are given by

$$\mathbf{e}_\mu(x) \equiv \mathbf{e}_a e^a{}_\mu(x) = \mathbf{e}_a \frac{\partial x^a}{\partial x^\mu}. \tag{2.16}$$

These are called *local basis vectors*. Their components $e^a{}_\mu(x)$ are called *local basis tetrads*. The difference vector between two neighboring points $\mathbf{x}$ and $\mathbf{x}'$ has, in the inertial reference frame, the description $d\mathbf{x} = \mathbf{e}_a(x'^a - x^a) = \mathbf{e}_a dx^a$. On going to coordinates x'^μ, x^μ, this becomes

$$d\mathbf{x} = \mathbf{e}_a \frac{\partial x^a}{\partial x^\mu} dx^\mu = \mathbf{e}_a e^a{}_\mu dx^\mu = \mathbf{e}_a e^a{}_\mu (x'^\mu - x^\mu) = \mathbf{e}_\mu (x'^\mu - x^\mu).$$

Thus, the same vector is obtained by taking the differences of the x^μ

coordinates and contracting them with basis vectors $\mathbf{e}_\mu$ of the local reference frame. A direct consequence of this is that the length of an infinitesimal vector, $ds = \sqrt{d\mathbf{x}^2}$, is given by

$$ds = \sqrt{d\mathbf{x}^2} = \sqrt{(\mathbf{e}_\mu dx^\mu)^2} = \sqrt{\mathbf{e}_\mu \cdot \mathbf{e}_\nu dx^\mu dx^\nu}.$$

Therefore, the metric in curvilinear coordinates is given by the scalar product of the local basis vectors: $g_{\mu\nu}(x) = \mathbf{e}_\mu(x) \cdot \mathbf{e}_\nu(x)$. In fact, inserting (2.16) we find

$$g_{\mu\nu}(x) = \mathbf{e}_\mu \cdot \mathbf{e}_\nu = \mathbf{e}_a \cdot \mathbf{a}_b \frac{\partial x^a}{\partial x^\mu} \frac{\partial x^b}{\partial x^\nu} = \eta_{ab} \frac{\partial x^a}{\partial x^\mu} \frac{\partial x^b}{\partial x^\nu},$$

an expression which was given in Eq. (2.5b).

Since the general coordinate transformation (2.14) was assumed to have an inverse, we can also calculate the derivatives $\partial x^\mu / \partial x^a$. These are reciprocal to the derivatives $\partial x^a / \partial x^\mu$, i.e.,

$$\frac{\partial x^\mu}{\partial x^a} \frac{\partial x^a}{\partial x^\nu} = \delta^\mu{}_\nu, \qquad \frac{\partial x^a}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^b} = \delta^a{}_b.$$

It is useful to denote the derivatives $\partial x^\mu / \partial x^a$ by $e_a{}^\mu$ and to call the vectors $\mathbf{e}^\mu = \mathbf{e}_a \eta^{ab} e_b{}^\mu$ *reciprocal basis vectors*. Their components satisfy

$$g_{\mu\nu}(x) = e^a{}_\mu(x) e_{a\nu}(x) \tag{2.17}$$

and

$$e_a{}^\mu(x) e^a{}_\nu(x) = \delta^\mu{}_\nu, \qquad e^a{}_\mu(x) e_b{}^\mu(x) = \delta^a{}_b. \tag{2.18}$$

Further, we shall freely raise and lower the latin index using the metric $\eta_{ab} = \eta^{ab}$ and define

$$e^{a\mu} \equiv \eta^{ab} e_b{}^\mu, \qquad e_{a\mu} \equiv \eta_{ab} e^b{}_\mu.$$

2.2.2. *Vectors and tensors*

In formulating the laws of physics in Minkowski space it is important to analyze physical quantities according to their transformation properties under Lorentz transformations. These are defined by linear coordinate changes from one inertial frame of reference to another,

$$x^a \to x'^a \equiv (\Lambda x)^a = \Lambda^a{}_b x^b,$$

where $\Lambda^a{}_b$ comprises all 4×4 matrices which preserve the metric η_{ab}, i.e., the length elements $ds = \sqrt{\eta_{ab}\, dx^a\, dx^b} = \sqrt{\eta_{ab}\, dx'^a\, dx'^b}$ are the same in both coordinate frames x^a and x'^a. This implies that the $\Lambda^a{}_b$'s coincide with all matrices satisfying

$$\eta_{ab} \Lambda^a{}_{a'} \Lambda^b{}_{b'} = (\Lambda^T \eta \Lambda)_{a'b'} = \eta_{a'b'}$$

or

$$(\eta\Lambda)^T = \eta\Lambda^{-1}.$$

Infinitesimally, one can parametrize $\Lambda^a{}_b$, $(\Lambda^{-1})^a{}_b$ as follows,

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b, \qquad (\Lambda^{-1})^a{}_b = \delta^a{}_b - \omega^a{}_b.$$

The relation $(\eta\Lambda)^T = \eta\Lambda^{-1}$ implies that

$$\omega_{ab} \equiv \eta_{aa'} \omega^{a'}{}_b$$

is an antisymmetric matrix, i.e., $\omega_{ab} = -\omega_{ba}$. It has six independent elements. The three components $\omega_i = (½)\varepsilon_{ijk}\,\omega_{jk}$ parametrize infinitesimal rotations, $|\boldsymbol{\omega}|$ being the rotation angle and $\boldsymbol{\omega}/|\boldsymbol{\omega}|$ the axis. The three components ω_{0i} are associated with the infinitesimal relative velocity of the two coordinate frames.

Since the physical events are the same before and after a Lorentz transformation, the basis vectors $\mathbf{e}_a$ change according to the law

$$\mathbf{e}_a \to \mathbf{e}'_a \equiv (\mathbf{e}\Lambda^{-1})_a = \mathbf{e}_b (\Lambda^{-1})^b{}_a = (\mathbf{e}\eta\Lambda^T\eta)_a = (\eta\Lambda\eta)_a{}^b \mathbf{e}_b \equiv \Lambda_a{}^b \mathbf{e}_b.$$

Indeed this gives

$$\mathbf{x} \equiv \mathbf{e}_a x^a \equiv \mathbf{e}x \to \mathbf{x}' = \mathbf{e}\eta\Lambda^T\eta\Lambda x = \mathbf{e}x = \mathbf{x},$$

so that the vectors of physical events remain unchanged. It is customary to call lower indices transforming with $\Lambda_a{}^b$, covariant and the upper indices transforming with $\Lambda^a{}_b$, contravariant.

Consider now a physical observable at a point P whose position corresponds to a *fixed vector* $\mathbf{x}$ in space and which is itself a vector quantity

$$\mathbf{v}(\mathbf{x}) = \mathbf{e}_a v^a(x).$$

Under arbitrary Lorentz transformations of the coordinates x^a, the basis vectors $\mathbf{e}_a$ change. The observable vector $\mathbf{v}(\mathbf{x})$, however, must remain the unchanged at the same point in space, i.e.,

$$\mathbf{v}'(\mathbf{x}) = \mathbf{v}(\mathbf{x}).$$

Writing this as

$$\mathbf{v}'(\mathbf{x}) = \mathbf{e}_a v'^a(x') = \mathbf{v}(\mathbf{x}) = \mathbf{e}_a v^a(x),$$

we see that the components of the vector in the two frames have to be related in the same way as the coordinates x'^a and x^a, i.e.,

$$v'^a(x') = \Lambda^a{}_b v'^b(x),$$

or, written differently,

$$v'^a(x) = \Lambda^a{}_b v^b(\Lambda^{-1}x). \tag{2.19}$$

For infinitesimal transformations,

$$\Lambda^a{}_b = (\delta^a{}_b + \omega^a{}_b)x^b$$

with

$$(\Lambda^{-1}x)^a = x^a - \omega^a{}_b x^b,$$

we see that $v^a(x)$ has the transformation law

$$v'^a(x) = v^a(x) + \omega^a{}_b v^b(x) - \omega^{b'}{}_b x^b \partial_{b'} v^a(x).$$

It is conventional to denote the infinitesimal local change of a function $f(x)$ when evaluated at the *same numerical values of the coordinates* x [which correspond to two *different points* P in space, namely $\mathbf{x} = \mathbf{e}_a x^a$ and $\mathbf{x}' = \mathbf{e}_a(\Lambda^{-1}x)^a$] by $\delta f(x)$,

$$\delta f(x) \equiv f'(x) - f(x). \tag{2.20}$$

This is sometimes referred to as the *substantial change* of $f(x)$. Then the infinitesimal transformation law of a vector reads

$$\delta v^a(x) = v'^a(x) - v^a(x) = \omega^a{}_b v^b(x) - \omega^b{}_{b'} x^{b'} \partial_b v^a(x). \tag{2.21}$$

To be able to construct Lorentz invariant quantities it is necessary to contract contravariant indices with covariant ones, e.g., $v_a(x) = \eta_{ab} v^b(x)$. Its transformation property is found to be

$$\begin{aligned}\delta v_a(x) &= \omega_a{}^b v_b(x) - \omega^{b'}{}_b x^b \partial_{b'} v_a(x)\\ &= \omega_a{}^b v_b(x) + \omega_b{}^{b'} x^b \partial_{b'} v_a(x),\end{aligned} \tag{2.22}$$

where we have introduced the matrix elements

$$\omega_b{}^{b'} = \eta_{ab} \eta^{a'b'} \omega^a{}_{a'} = \eta^{a'b'} \omega_{ba'}. \tag{2.23}$$

The derivatives of covariant and contravariant vector fields with respect to the coordinates of x^a are tensor fields of higher rank. For infinitesimal transformations the derivatives change via the *sum* of contraction with ω_{ab}, one applied to each index. This follows directly from (2.22) using the commutation rule $[\partial_a, x_b] = \eta_{ab}$:

$$\begin{aligned}\delta \partial_b v_a &= \partial_b \delta v_a = \partial_b (\omega_a{}^{a'} v_{a'} + \omega_c{}^{c'} x^c \partial_{c'} v_a)\\ &= \omega_a{}^{a'} \partial_b v_{a'} + \omega_b{}^{b'} \partial_{b'} v_a + \omega_{c'} x^c \partial_{c'} \partial_b v_a.\end{aligned} \tag{2.24}$$

Notice that, since the arguments in f and f' in (2.20) are the same, the operation "substantial change" commutes with the derivative. The simple rule (2.24) can easily be extended to arbitrary higher derivatives thereby obtaining the transformation properties of tensor fields of higher rank.

Consider now the same physical objects but now described in terms of curvilinear coordinates $x^\mu(x^a)$. Then the components of $\mathbf{v}$ are measured not with respect to the basis $\mathbf{e}_a$ but with respect to the *local* basis $\mathbf{e}_\mu(x) = \mathbf{e}_a e^a{}_\mu(x)$ so that it is natural to specify $\mathbf{v}$ in terms of its local components $v^\mu(x) = v^a(x) e_a{}^\mu(x)$. On such fields one may perform Lorentz transformations as well as any general coordinate transformations (2.14), $x^\mu \to x'^\mu(x^\mu)$, which will be referred to as *Einstein transformations*. Under these transformations, the components $e_a{}^\mu(x)$, being derivatives of the coordinate transformation functions $x^\mu(x^a)$, undergo the following changes:

$$e_a{}^\mu \to e'_a{}^\mu(x') \equiv \frac{\partial x'^\mu}{\partial x^a} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial x^a} = \alpha^\mu{}_\nu e_a{}^\nu(x),$$

$$e^a{}_\mu \to e'^a{}_\mu(x') \equiv \frac{\partial x^a}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^a}{\partial x^\nu} = \alpha_\mu{}^\nu e^a{}_\nu(x). \tag{2.25}$$

The matrices

$$\alpha^\mu{}_\nu \equiv \frac{\partial x'^\mu}{\partial x^\nu} \equiv (\alpha)^\mu{}_\nu, \qquad \alpha_\mu{}^\nu \equiv \frac{\partial x^\nu}{\partial x'^\mu} \tag{2.26}$$

are reciprocal to each other,

$$\alpha^\nu{}_\lambda \alpha_\nu{}^\mu = \delta_\lambda{}^\mu, \qquad \alpha_\nu{}^\mu \alpha^\lambda{}_\mu = \delta_\nu{}^\lambda, \tag{2.27}$$

i.e.,

$$(\alpha^{-1})^\nu{}_\lambda = \alpha_\lambda{}^\nu \tag{2.28}$$

is the inverse of the matrix $\alpha_\nu{}^\mu$. For infinitesimal changes we put

$$x'^\mu \equiv x^\mu - \xi^\mu(x) \tag{2.29}$$

and see that Einstein transformations can be interpreted as *local translations*. The infinitesimal transformation matrices are

$$\alpha^\lambda{}_\nu \approx \delta^\lambda{}_\nu - \partial_\nu \xi^\lambda(x), \qquad \alpha_\mu{}^\nu \approx \delta_\mu{}^\nu + \partial_\mu \xi^\nu(x), \tag{2.30}$$

and the substantial changes of $e_a{}^\mu(x)$ are given by

$$\begin{aligned}\delta e_a{}^\mu &\equiv e'_a{}^\mu(x) - e_a{}^\mu(x) = e'_a{}^\mu(x') - e_a{}^\mu(x') \\ &= e_a{}^\mu(x) - e_a{}^\mu(x') + e_a{}'^\mu(x') - e_a{}^\mu(x) = \xi^\lambda \partial_\lambda e_a{}^\mu(x) - \partial_\lambda \xi^\mu e_a{}^\lambda(x) \\ \delta e^a{}_\mu &= \xi^\lambda \partial_\lambda e^a{}_\mu(x) + \partial_\mu \xi^\lambda e^a{}_\lambda(x).\end{aligned} \tag{2.31}$$

When we write $\partial_\mu \xi^\lambda e^a{}_\lambda$ we mean $(\partial_\mu \xi^\lambda) e^a{}_\lambda$. Otherwise we write $\partial_\mu(\xi^\lambda e^a{}_\lambda)$.

Analogous transformation laws can be derived for the components of the vector fields v^{μ}, v_{μ}. They follow from the fact that the components $v^{a}(x^{b})$, $v_{a}(x^{b})$ are trivially independent of the change of general coordinates from x^{μ} to x'^{μ} since the inertial basis and coordinates are unchanged. Thus we have the obvious relation

$$v'^{a}(x^{b}) = v^{a}(x^{b}).$$

When reparametrizing the point x^{b} in the two different coordinates x'^{μ} and x^{μ}, this relation takes the form

$$v'^{a}(x') = v^{a}(x), \tag{2.32}$$

where we have omitted the Greek superscripts of x', x. Thus the substantial changes, at the same values of the general coordinates x^{μ}, are

$$\delta v^{a}(x) = v'^{a}(x) - v^{a}(x) = \xi^{\lambda}\partial_{\lambda}v^{a}(x). \tag{2.33}$$

Using this and (2.31), we derive from (2.32)

$$v'^{\mu}(x') = \alpha^{\mu}{}_{\nu}v^{\nu}(x), \qquad v'_{\mu}(x') = \alpha_{\mu}{}^{\nu}v_{\nu}(x), \tag{2.34}$$

with the substantial changes

$$\begin{aligned} \delta v^{\mu}(x) &= v'^{\mu}(x) - v^{\mu}(x) = \xi^{\lambda}\partial_{\lambda}v^{\mu} - \partial_{\lambda}\xi^{\mu}v^{\lambda}, \\ \delta v_{\mu}(x) &= v'_{\mu}(x) - v_{\mu}(x) = \xi^{\lambda}\partial_{\lambda}v_{\mu} + \partial_{\mu}\xi^{\lambda}v_{\lambda}. \end{aligned} \tag{2.35}$$

Any four-component field with these transformation properties is called a contra- and covariant *Einstein vector* or *world vector*, respectively.

This definition can be extended trivially to any *Einstein-* or *world tensors*. All one has to do is to apply separately the transformation matrices (2.26) to each index. In particular, the metric $g_{\mu\nu}(x)$ transforms as

$$g'^{\lambda\varkappa}(x') = \alpha^{\lambda}{}_{\mu}\alpha^{\varkappa}{}_{\nu}g^{\mu\nu}(x), \qquad g'_{\lambda\varkappa}(x') = \alpha_{\lambda}{}^{\mu}\alpha_{\varkappa}{}^{\nu}g_{\mu\nu}(x), \tag{2.36}$$

or, in infinitesimal form,

$$\begin{aligned} \delta g^{\mu\nu} &= \xi^{\lambda}\partial_{\lambda}g^{\mu\nu} - \partial_{\lambda}\xi^{\mu}g^{\lambda\nu} - \partial_{\lambda}\xi^{\nu}g^{\mu\lambda}, \\ \delta g_{\mu\nu} &= \xi^{\lambda}\partial_{\lambda}g_{\mu\nu} + \partial_{\mu}\xi^{\lambda}g_{\lambda\nu} + \partial_{\nu}\xi^{\lambda}g_{\mu\lambda}. \end{aligned} \tag{2.37}$$

It is now obvious from (2.27) that one may multiply world tensors with each other by a simple contraction of upper and lower indices and always obtain new world tensors. In particular, one obtains an *Einstein-* or *world invariant* if such a contraction is complete, i.e., if no index is left uncontracted.

2.2.3. *Connections and covariant derivatives*

The multiplication rules for world tensors are completely analogous to those for Lorentz tensors. There is, however, one important difference between the two tensor fields. Unlike the Lorentz case, derivatives are no longer tensors. In curvilinear coordinates, certain modification of the derivatives are required in order to make them proper tensors.[b] It is quite easy to find this modification and to construct covariant objects analogous to the derivative tensors in the Lorentz frames. All we have to do is rewrite these derivative tensors in terms of the general curvilinear components. Take, for example, the tensor $\partial_b v_a(x)$. Going over to curvilinear components x^μ we can write this as

$$\partial_b v_a = \partial_b (e_a{}^\mu v_\mu).$$

But if we take the derivative ∂_b past the basis tetrad $e_a{}^\mu$ we find

$$\partial_b v_a = e_a{}^\mu \partial_b v_\mu + \partial_b e_a{}^\mu v_\mu . \tag{2.38}$$

Using the relation

$$\partial_b = e_b{}^\lambda \partial_\lambda , \tag{2.39}$$

we see that

$$\partial_b v_a = e_a{}^\mu e_b{}^\nu \partial_\nu v_\mu + (e_b{}^\nu \partial_\nu e_a{}^\lambda) v_\lambda .$$

The right-hand side can be rewritten in the covariant form

$$\partial_b v_a \equiv e_a{}^\mu e_b{}^\nu D_\nu v_\mu , \tag{2.40}$$

[b]Notice that $\sqrt{g} = \sqrt{\det g_{\mu\nu}}$ transforms like $\delta\sqrt{g} = \xi^\lambda \partial_\lambda \sqrt{g} + \partial_\lambda \xi^\lambda \sqrt{g} = \partial_\lambda (\xi^\lambda \sqrt{g})$. Hence $\int d^4x \sqrt{g}$ is invariant. A quantity transforming this way is called a *density* [see Eq. (3.5) for the consequences].

where the symbol D_ν stands for the *covariant* derivative

$$D_\nu v_\mu = \partial_\nu v_\mu - e_c{}^\lambda \partial_\nu e_\mu{}^c v_\lambda \equiv \partial_\nu v_\mu - \Gamma_{\nu\mu}{}^\lambda v_\lambda \,. \tag{2.41}$$

The explicit form on the right-hand side follows from the simple relation

$$\partial_\nu e_a{}^\lambda = -e_a{}^\mu (e_c{}^\lambda \partial_\nu e^c{}_\mu), \qquad \partial_\nu e^a{}_\lambda = -e^a{}_\mu (e^c{}_\lambda \partial_\nu e_c{}^\mu), \tag{2.42}$$

which, in turn, follows from differentiation of the reciprocity relation $e_a{}^\lambda e^b{}_\lambda = \delta_a{}^b$.

Similarly, we can find the Einstein version of the derivative of a contravariant vector field $\partial_b v^a(x)$, which can be rewritten as

$$\partial_b v^a = \partial_b (e^a{}_\mu v^\mu) = e^a{}_\mu e_b{}^\nu \partial_\nu v^\mu + (e_b{}^\nu \partial_\nu e^a{}_\lambda) v^\lambda \tag{2.43}$$

and cast in the form

$$e^a{}_\mu e_b{}^\nu D_\nu v^\mu, \tag{2.44}$$

with a covariant derivative

$$\begin{aligned} D_\nu v^\mu &= \partial_\nu v^\mu - e^c{}_\lambda \partial_\nu e_c{}^\mu v^\lambda = \partial_\nu v^\mu + e_c{}^\mu \partial_\nu e^c{}_\lambda v^\lambda \\ &\equiv \partial_\nu v^\mu + \Gamma_{\nu\lambda}{}^\mu v^\lambda. \end{aligned} \tag{2.45}$$

The expression

$$\Gamma_{\mu\nu}{}^\lambda \equiv e_a{}^\lambda \partial_\mu e^a{}_\nu \equiv -e^a{}_\nu \partial_\mu e_a{}^\lambda, \tag{2.46}$$

is called the *affine connection*. In general, a space with metric $g_{\mu\nu}$ and an affine connection (both single-valued) defining covariant derivatives, is called a *metric-affine space* and the geometry, carried by $g_{\mu\nu}$, $\Gamma_{\mu\nu}{}^\lambda$, an *affine geometry*. Observe that, by definition, the covariant derivatives of $e^\nu{}_a$, $e^a{}_\nu$ vanish:

$$\begin{aligned} D_\mu e_\nu{}^a &= \partial_\mu e^a{}_\nu - \Gamma_{\mu\nu}{}^\lambda e^a{}_\lambda = 0, \\ D_\mu e_a{}^\nu &= \partial_\mu e_a{}^\nu + \Gamma_{\mu\lambda}{}^\nu e_a{}^\lambda = 0. \end{aligned} \tag{2.47}$$

Since $g_{\mu\nu} = e^{a}{}_{\mu} e_{a\nu}$, that same property holds for the metric tensor[b]

$$
\begin{aligned}
D_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - \Gamma_{\lambda\mu}{}^{\sigma} g_{\sigma\nu} - \Gamma_{\lambda\nu}{}^{\sigma} g_{\mu\sigma} = 0, \\
D_\lambda g^{\mu\nu} &= \partial_\lambda g^{\mu\nu} + \Gamma_{\lambda\sigma}{}^{\mu} g^{\sigma\nu} + \Gamma_{\lambda\sigma}{}^{\nu} g^{\mu\sigma} = 0.
\end{aligned}
\tag{2.48}
$$

It is worth noting that the metric satisfies once more similar relations with the connections replaced by Christoffel symbols. In fact, from the definition (2.8) we can verify directly that

$$
\begin{aligned}
&\partial_\lambda g_{\mu\nu} - \left\{ \begin{matrix} \sigma \\ \lambda\mu \end{matrix} \right\} g_{\sigma\nu} - \left\{ \begin{matrix} \sigma \\ \lambda\nu \end{matrix} \right\} g_{\mu\sigma} = 0, \\
&\partial_\lambda g^{\mu\nu} + \left\{ \begin{matrix} \mu \\ \lambda\sigma \end{matrix} \right\} g^{\sigma\nu} + \left\{ \begin{matrix} \nu \\ \lambda\sigma \end{matrix} \right\} g^{\mu\sigma} = 0.
\end{aligned}
\tag{2.49}
$$

Since the expressions on left-hand sides of Eqs. (2.40), (2.43) are tensors with respect to Lorentz transformations, the right-hand quantities, $D_\nu v_\mu$, $D_\nu v^\mu$, must also be tensors with respect to general coordinate transformations, i.e., world tensors. In fact, one can easily verify that they transform covariantly,

$$
D'_{\mu'} v_{\nu'}(x') = \alpha_{\mu'}{}^{\mu} \alpha_{\nu'}{}^{\nu} D_\mu v_\nu(x).
$$

The term, which disturbs the covariant behavior of ordinary derivatives

$$
\partial'_{\mu'} v_{\nu'}(x') = \alpha_{\mu'}{}^{\mu} \partial_\mu (\alpha_{\nu'}{}^{\nu} v_\nu(x)) = \alpha_{\mu'}{}^{\mu} \alpha_{\nu'}{}^{\nu} \partial_\mu v_\nu(x) + \alpha_{\mu'}{}^{\mu} \partial_\mu \alpha_{\nu'}{}^{\nu} v_\nu(x)
$$

is compensated by the non-tensorial behavior of $\Gamma_{\mu\nu}{}^{\lambda}$:

[c]Right now, when the space is still flat, this is a rather trivial statement. The identities (2.47), (2.48) will, however, remain valid also if the space acquires curvature or torsion. In general relativity there have been theories based on spaces in which this is not satisfied. The object $Q_{\lambda\mu\nu} = -D_\lambda g_{\mu\nu}$ then becomes a dynamical field to be determined from field equations. See Th. de Donder, *La gravitation de Weyl-Eddington-Einstein* (Gauthier-Villars, Paris, 1924); H. Weyl, *Phys. Z.* **22** (1921) 473, *Ann. Phys.* **59** (1919) 101, **65** (1921) 541; A.S. Eddington, *Proc. Roy. Soc.* **99** (1921) 104 and *The Mathematical Theory of Relativity* (Springer, Berlin, 1925); F.W. Hehl, J.D. McCrea, E.W. Mielke, in *Exakte Wissenschaften und ihre philosophische Grundlegung*, ed: by W. Deppert, K. Hübner, A. Oberschelp, V. Weidemann (Verlag Peter Lang, Frankfurt, 1988). In these spaces, the connection is defined by $\Gamma_{\mu\nu}{}^{\lambda} \equiv e_a{}^{\lambda}(\partial_\mu - D_\mu) e^a{}_\nu$ and can be decomposed as follows:

$$
\Gamma_{\mu\nu}{}^{\lambda} \equiv e_a{}^{\lambda}(\partial_\mu - D_\mu) e^a{}_\nu = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} - (S_\mu{}^{\lambda}{}_\nu - S^{\lambda}{}_{\nu\mu} + S_{\nu\mu}{}^{\lambda}) + \frac{1}{2}(Q_\mu{}^{\lambda}{}_\nu - Q^{\lambda}{}_{\nu\mu} + Q_{\nu\mu}{}^{\lambda})
$$

where $S_{\mu\nu}{}^{\lambda} \equiv (1/2)(\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\nu\mu}{}^{\lambda})$ as is defined in (2.53).

$$\Gamma'_{\mu'\nu'}{}^{\lambda'}(x') = e'_a{}^{\lambda'} \partial'_{\mu'} e'^a{}_{\nu'} = \alpha^{\lambda'}{}_\lambda \alpha_{\mu'}{}^\mu e_a{}^\lambda \partial_\mu (\alpha_{\nu'}{}^\nu e^a{}_\nu)$$
$$= \alpha_{\mu'}{}^\mu(\alpha_{\nu'}{}^\nu \alpha^{\lambda'}{}_\lambda \Gamma_{\mu\nu}{}^\lambda(x) + \alpha^{\lambda'}{}_\nu \partial_\mu \alpha_{\nu'}{}^\nu), \tag{2.50a}$$

or

$$\Gamma'_{\mu'\nu'}{}^{\lambda'}(x') = -e'^a{}_{\nu'} \partial_{\mu'} e_a{}^{\lambda'} = -\alpha_{\nu'}{}^\nu \alpha_{\mu'}{}^\mu e^a{}_\nu \partial_\mu (\alpha^{\lambda'}{}_\lambda e_a{}^\lambda)$$
$$= \alpha_{\mu'}{}^\mu(\alpha_{\nu'}{}^\nu \alpha^{\lambda'}{}_\lambda \Gamma_{\mu\nu}{}^\lambda(x) - \alpha_{\nu'}{}^\nu \partial_\mu \alpha^{\lambda'}{}_\nu). \tag{2.50b}$$

Infinitesimally, $\alpha_\mu{}^\nu \approx \delta_\mu{}^\nu + \partial_\mu \xi^\nu$, $\alpha^\mu{}_\nu \approx \delta_\nu{}^\mu - \partial_\nu \xi^\mu$ and we see that the covariant derivatives $D_\mu v_\nu$, $D_\mu v^\nu$ have the correct substantial variations of world tensors:

$$\delta D_\mu v_\nu = \xi^\lambda \partial_\lambda D_\mu v_\nu + \partial_\mu \xi^\lambda D_\lambda v_\nu + \partial_\nu \xi^\lambda D_\mu v_\lambda,$$
$$\delta D_\mu v^\nu = \xi^\lambda \partial_\lambda D_\mu v^\nu + \partial_\mu \xi^\lambda D_\lambda v_\nu \; - \; \xi^\nu D_\mu v^\lambda. \tag{2.51}$$

Again the last non-covariant piece in

$$\delta \partial_\mu v_\nu = \partial_\mu \delta v_\mu = \partial_\mu (\xi^\lambda \partial_\lambda v_\nu + \partial_\nu \xi^\lambda v_\lambda)$$
$$= \xi^\lambda \partial_\lambda \partial_\mu v_\nu + \partial_\mu \xi^\lambda \partial_\lambda v_\nu + \partial_\nu \xi^\lambda \partial_\mu v_\lambda + \partial_\mu \partial_\nu \xi^\lambda v_\lambda$$

is cancelled by the last, non-tensorial, piece in $\delta(\Gamma_{\mu\nu}{}^\kappa v_\kappa)$

$$\delta \Gamma_{\mu\nu}{}^\kappa = \xi^\lambda \partial_\lambda \Gamma_{\mu\nu}{}^\kappa + \partial_\mu \xi^\lambda \Gamma_{\lambda\nu}{}^\kappa + \partial_\nu \xi^\lambda \Gamma_{\mu\lambda}{}^\kappa + \partial_\mu \partial_\nu \xi^\kappa. \tag{2.50c}$$

It can easily be checked that the same cancellation occurs if the covariant derivative of an arbitrary tensor field is defined by

$$D_\mu v_{\nu_1 \ldots \nu_n}{}^{\nu'_1 \ldots \nu'_m} = \partial_\mu v_{\nu_1 \ldots \nu_n}{}^{\nu'_1 \ldots \nu'_m} - \sum_{i=1}^n \Gamma_{\mu\nu_i}{}^{\lambda_i} v_{\nu_1, \ldots \lambda_i \ldots \nu_n}{}^{\nu'_1 \ldots \nu'_m},$$
$$+ \sum_{j=1}^m \Gamma_{\mu\lambda'_j}{}^{\nu'_j} v_{\nu_1 \ldots \nu_n}{}^{\nu'_1 \ldots \lambda'_j \ldots \nu'_m}. \tag{2.52}$$

2.3. TORSION TENSOR

Since the coordinate transformations $x'^\lambda = x^\lambda - \xi^\lambda(x)$ were assumed to be integrable, the derivatives of the infinitesimal, local translation field $\xi^\lambda(x)$ commute, i.e., $(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \xi^\lambda(x) = 0$. As a consequence the anti-symmetric part of the connection

$$S_{\mu\nu}{}^{\lambda} = \tfrac{1}{2}(\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\nu\mu}{}^{\lambda}) \tag{2.53}$$

transforms like a tensor [just use (2.50c)]. It is called the *torsion tensor*.

Minkowski space has no torsion. For, if we describe it in terms of the coordinates x^a, the basis tetrads would be $e^a{}_\mu = \delta^a{}_\mu$ and the connection vanishes. If we now perform a general coordinate transformation to curvilinear coordinates $x^\mu(x^a)$, the connection would not be zero in general. The torsion, however is a tensor and therefore remains zero in all coordinate systems.

It is useful to realize that with the help of the torsion tensor, the connection can be decomposed into a Christoffel part, given by Eq. (2.11), which depends only on the metric $g_{\mu\nu}(x)$, and a second part, called *contortion tensor*, which is a combination of torsion tensors. To see this, let us define the modified connection $\Gamma_{\mu\nu\lambda} \equiv \Gamma_{\mu\nu}{}^{\varkappa} g_{\varkappa\lambda} = e_{a\lambda}\,\partial_\mu e^a{}_\nu$ and regroup the right-hand side as follows,

$$\begin{aligned}\Gamma_{\mu\nu\lambda} &= e_{a\lambda}\,\partial_\mu e^a{}_\nu \\ &= \tfrac{1}{2}\{e_{a\lambda}\,\partial_\mu e^a{}_\nu + \partial_\mu e_{a\lambda}\, e^a{}_\nu + e_{a\mu}\,\partial_\nu e^a{}_\lambda + \partial_\nu e_{a\mu}\, e^a{}_\lambda - e_{a\mu}\,\partial_\lambda e^a{}_\nu - \partial_\lambda e_{a\mu}\, e^a{}_\nu\} \\ &\quad + \tfrac{1}{2}\{e_{a\lambda}\partial_\mu e^a{}_\nu - e_{a\lambda}\partial_\nu e^a{}_\mu - e_{a\mu}\partial_\nu e^a{}_\lambda + e_{a\mu}\partial_\lambda e^a{}_\nu + e_{a\nu}\partial_\lambda e^a{}_\mu - e_{a\nu}\partial_\mu e^a{}_\lambda\}.\end{aligned} \tag{2.54}$$

The first part can be written as $(1/2)(\partial_\mu(e_{a\lambda}e^a{}_\nu) + \partial_\nu(e_{a\mu}e^a{}_\lambda) - \partial_\lambda(e_{a\mu}e^a{}_\nu)) = (1/2)(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$ so that it coincides with the Christoffel symbol (2.9). The second part is the contortion tensor. It is a combination of torsion tensors $S_{\mu\nu\lambda} \equiv S_{\mu\nu}{}^{\varkappa} g_{\varkappa\lambda}$. Using (2.46) we see that

$$\Gamma_{\mu\nu\lambda} = \{\mu\nu, \lambda\} + S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu} = \{\mu\nu, \lambda\} + K_{\mu\nu\lambda}. \tag{2.55}$$

Note that the contortion tensor $K_{\mu\nu\lambda}$ is antisymmetic in the last two indices. The three S terms are easy to remember: The first starts out with the same indices as $K_{\mu\nu\lambda}$. The second and third are shifted cyclically with alternating signs. Spaces with $S = K = 0$ are called *symmetric* or *Riemannian spaces*. Spaces with curvature and non-zero torsion are also referred to as *Riemann-Cartan* spaces.

2.4. CURVATURE TENSOR AS A COVARIANT CURL OF THE CONNECTION

In the last section we noted that even though the connection $\Gamma_{\mu\nu}{}^{\lambda}$ is not a tensor, its antisymmetric part, the torsion $S_{\mu\nu}{}^{\lambda}$, is. The question arises whether it is possible to form a covariant object which contains information on the symmetric Christoffel part of the connection. Such a tensor does indeed exist.

Looking back at the transformation property (2.50c) we see that the proper tensor character is destroyed by the last term which is additive in the derivative of an arbitrary function $\partial_\mu \partial_\nu \xi^\varkappa(x)$. Such additive transformation laws are familiar from gauge fields. Recall that the gauge fields of magnetism transform additively with a first derivative

$$\delta A_i(x) = \partial_i \Lambda, \tag{2.56a}$$

where $\Lambda(x)$ is an arbitrary gauge function. Now, in magnetism there was a simple way of constructing a gauge invariant quantity, namely, the antisymmetric combination of derivatives

$$F_{ij} = \partial_i A_j - \partial_j A_i, \tag{2.56b}$$

which gives a measurable magnetic field strength via $B_i = (1/2)\varepsilon_{ijk} F_{jk}$. This suggests that a similar construction might exist also for the connection.

Such construction is straightforward if we observe that $\Gamma_{\mu\nu}{}^\lambda$ has precisely the transformation properties of a non-Abelian gauge field discussed before in Part I [Eq. (3.125)]. To establish contact with the notation employed there we must consider $\Gamma_{\mu\nu}{}^\lambda$ as the matrix elements of the four 4×4 matrices Γ_μ,

$$\Gamma_{\mu\nu}{}^\lambda = (\Gamma_\mu)_\nu{}^\lambda. \tag{2.57}$$

Then we may rewrite (2.50a) as the matrix equation

$$\Gamma'_{\mu'}(x') = \alpha_{\mu'}{}^\mu (\alpha \Gamma_\mu(x) \alpha^{-1} + \partial_\mu \alpha \; \alpha^{-1}). \tag{2.58}$$

This shows that Γ_μ transforms in exactly the same way as the non-abelian gauge field A_μ in (I.3.128).

This is not surprising if we recall that the original purpose for introducing these fields was to form the covariant derivatives (I.3.127) and (2.41). Thus the connection may be viewed as a non-Abelian gauge field of the group of general coordinate transformations $\alpha^\mu{}_\nu$; Einstein vectors and tensors are the associated gauge covariant quantities. From the discussion in Part I, Section 3.5, it is then clear how to form from $\Gamma_{\mu\nu}{}^\alpha$ a tensor invariant under general coordinate transformations. All we have to do is take the covariant curl [see (I.3.132)]

$$F_{\mu\nu} \equiv \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu - [\Gamma_\mu, \Gamma_\nu]. \tag{2.59}$$

The matrix elements of $F_{\mu\nu}$ are commonly denoted by

$$R_{\mu\nu\lambda}{}^{\sigma} = \partial_\mu \Gamma_{\nu\lambda}{}^{\sigma} - \partial_\nu \Gamma_{\mu\lambda}{}^{\sigma} - \Gamma_{\mu\lambda}{}^{\delta}\Gamma_{\nu\delta}{}^{\sigma} + \Gamma_{\nu\lambda}{}^{\sigma}\Gamma_{\mu\delta}{}^{\sigma}. \tag{2.60}$$

The covariance properties of $F_{\mu\nu}$ follow in the same way as (I.3.133). In the present case there is another simple way of deriving them by realizing that in terms of the basis tetrad $e_a{}^{\mu}$, the covariant curl has the simple representation

$$R_{\mu\nu\lambda}{}^{\sigma} = e_a{}^{\sigma}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\lambda = -e^a{}_\lambda (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e_a{}^{\sigma}. \tag{2.61}$$

The first part follows directly from inserting $\Gamma_{\mu\nu}{}^{\lambda} = e_a{}^{\lambda}\partial_\mu e^a{}_\nu$ into (2.59) and evaluating the derivatives,

$$\begin{aligned}
&\partial_\mu \Gamma_{\nu\lambda}{}^{\kappa} - (\Gamma_\mu \Gamma_\nu)_\lambda{}^{\kappa} - (\mu \leftrightarrow \nu) \\
&= (\partial_\mu e_a{}^{\kappa} \partial_\nu e^a{}_\lambda + e_a{}^{\kappa}\partial_\mu \partial_\nu e^a{}_\lambda + e_b{}^{\rho}\partial_\mu e^b{}_\lambda e^a{}_\rho \partial_\nu e_a{}^{\kappa}) - (\mu \leftrightarrow \nu) \\
&= e_a{}^{\kappa}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\lambda,
\end{aligned}$$

where, in the third term, we have used the alternative representation $\Gamma_{\nu\rho}{}^{\kappa} = -e^a{}_\rho \partial_\nu e_a{}^{\kappa}$. The second part of (2.61) follows from the opposite choice of the two representations $\Gamma_{\mu\nu}{}^{\lambda} = -e^a{}_\nu \partial_\mu e_a{}^{\lambda}$, $\Gamma_{\nu\rho}{}^{\kappa} = e_a{}^{\kappa}\partial_\nu e^a{}_\rho$.

From the tetrad expression for $R_{\mu\nu\lambda}{}^{\kappa}$ the tensor transformation law is easily found [using (2.25)]

$$\begin{aligned}
R_{\mu\nu\lambda}{}^{\kappa}(x) \to R_{\mu'\nu'\lambda'}{}'^{\kappa'}(x') &= e_a{}^{\kappa'}(x')(\partial_{\mu'}{}' \partial_{\nu'}{}' - \partial_{\nu'}{}' \partial_{\mu'}{}') e'^a{}_{\lambda'}(x') \\
&= \alpha^{\kappa'}{}_\kappa \alpha_{\mu'}{}^{\mu} \alpha_{\nu'}{}^{\nu} e_a{}^{\kappa}(x)(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)(\alpha_{\lambda'}{}^{\lambda} e^a{}_\lambda(x)) \\
&= \alpha_{\mu'}{}^{\mu}\alpha_{\nu'}{}^{\nu}\alpha_{\lambda'}{}^{\lambda}\alpha^{\kappa'}{}_\kappa R_{\mu\nu\lambda}{}^{\kappa}(x) \\
&\quad + \alpha_{\mu'}{}^{\mu}\alpha_{\nu'}{}^{\nu}\alpha^{\kappa'}{}_\lambda [(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\alpha_{\lambda'}{}^{\lambda}].
\end{aligned} \tag{2.62}$$

Since general coordinate transformations are assumed to be smooth, the derivatives in front of $\alpha_{\lambda'}{}^{\lambda}$ commute and $R_{\mu\nu\lambda}{}^{\kappa}$ is a proper tensor. It is called the *curvature tensor*.

By construction, this curvature tensor is antisymmetric in the first index pair. What is not so easy to see is that it is also antisymmetric with respect to the second index pair, namely,

$$R_{\mu\nu\lambda\varkappa} = -R_{\mu\nu\varkappa\lambda} \tag{2.63}$$

where $R_{\mu\nu\lambda\varkappa} \equiv R_{\mu\nu\lambda}{}^{\sigma} g_{\varkappa\sigma}$. Indeed, if we calculate this difference using the definition (2.61) we find

$$\begin{aligned} R_{\mu\nu\lambda\varkappa} + R_{\mu\nu\varkappa\lambda} &= e_{a\varkappa}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\lambda + e_{a\lambda}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\varkappa \\ &= \partial_\mu \partial_\nu (e_{a\varkappa} e^a{}_\lambda) - \partial_\nu \partial_\mu (e_{a\varkappa} e^a{}_\lambda) \\ &= (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) g_{\lambda\varkappa} \end{aligned} \tag{2.64}$$

By assumption, the coordinate transformations $x^a(x^\mu)$ are smooth functions satisfying the integrability condition (2.15). As a consequence, the metric $g_{\lambda\varkappa}(x) = (\partial x^a/\partial x^\lambda)(\partial x_a/\partial x^\varkappa)$ is a smooth function. In the following we shall always assume that it is at least doubly differentiable. Since the metric is an observable quantity it must be single-valued. This implies the integrability condition

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) g_{\lambda\varkappa} = 0. \tag{2.65}$$

It is this property which makes the curvature tensor antisymmetric in the last two indices.[c] The curvature tensor gives a covariant characterization of the connection which includes information on the Christoffel symbol.

Since $R_{\mu\nu\lambda}{}^{\varkappa}$ is a tensor, it can be contracted using the metric tensor to form covariant quantities of lower rank. There are two possibilities:

$$R_{\mu\nu} \equiv R_{\varkappa\mu\nu}{}^{\varkappa}, \tag{2.66a}$$

called the *Ricci tensor* and

$$R = R_{\mu\nu} g^{\mu\nu}, \tag{2.66b}$$

called the *scalar curvature*. A combination of both

[d]If the space were more general and had $D_\mu g_{\lambda\varkappa} = -Q_{\mu\lambda\varkappa} \neq 0$ (see footnote c), there would be a symmetric part

$$R_{\mu\nu\lambda\varkappa} + R_{\mu\nu\varkappa\lambda} = [D_\mu Q_{\nu\lambda\varkappa} - (\nu\mu)] + 2S_{\mu\nu}{}^{\rho} Q_{\rho\lambda\varkappa}.$$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \tag{2.67}$$

was introduced by Einstein and is therefore called the *Einstein curvature tensor*.

It turns out that there is a related tensor which deals more exclusively with the Christoffel part of the connection than the curvature tensor (2.60). Since the contortion $K_{\mu\nu}{}^{\lambda}$ is a tensor the Christoffel part $\left\{ {\lambda \atop \mu\nu} \right\}$ of $\Gamma_{\mu\nu}{}^{\lambda}$ has the same transformation properties (2.50c) as $\Gamma_{\mu\nu}{}^{\lambda}$, and we can form the *Riemann curvature tensor*

$$\overset{\{\}}{R}_{\mu\nu\lambda}{}^{\varkappa} = \partial_\mu \left\{ {\varkappa \atop \nu\lambda} \right\} - \partial_\nu \left\{ {\varkappa \atop \mu\lambda} \right\} - \left(\left\{ {\rho \atop \mu\lambda} \right\} \left\{ {\varkappa \atop \nu\rho} \right\} - \left\{ {\rho \atop \nu\lambda} \right\} \left\{ {\varkappa \atop \mu\rho} \right\} \right). \tag{2.68}$$

Unlike $R_{\mu\nu\lambda}{}^{\varkappa}$, this curvature tensor can be expressed completely in terms of derivatives of the metric [recall (2.9), (2.11)]. The difference between the two tensors is a function of the contortion,

$$R_{\mu\nu\lambda}{}^{\varkappa} - \overset{\{\}}{R}_{\mu\nu\lambda}{}^{\varkappa} = \overset{\{\}}{D}_\mu K_{\nu\lambda}{}^{\varkappa} - \overset{\{\}}{D}_\nu K_{\mu\lambda}{}^{\varkappa} - (K_{\mu\lambda}{}^{\rho} K_{\nu\rho}{}^{\varkappa} - K_{\nu\lambda}{}^{\rho} K_{\mu\rho}{}^{\varkappa}), \tag{2.69}$$

where $\overset{\{\}}{D}_\mu$ denotes a covariant derivative which is formed with only the Christoffel part of the connection. Notice that the Riemannian curvature tensor has the same antisymmetry in the first and second index pairs as $R_{\mu\nu\lambda}{}^{\varkappa}$. For the first pairs this is trivial, by the construction (2.68); for the second, this follows from (2.69) and the antisymmetry of the contortion $K_{\nu\lambda\varkappa}$ in the second index pair [see (2.55)]. In addition, it is symmetric under the exchange of the first and the second index pair

$$\overset{\{\}}{R}_{\mu\nu\lambda\varkappa} = \overset{\{\}}{R}_{\lambda\varkappa\mu\nu}. \tag{2.70}$$

To see this one writes the first two terms in (2.68) explicitly as derivatives of the metric tensor

$$\overset{\{\}}{R}_{\mu\nu\lambda\varkappa} = \left[g_{\varkappa\varkappa'} \partial_\mu \frac{g^{\varkappa'\sigma}}{2} (\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) \right] - [\mu \leftrightarrow \nu]$$
$$- g_{\varkappa\varkappa'} \left(\left\{ {\rho \atop \mu\lambda} \right\} \left\{ {\varkappa' \atop \nu\rho} \right\} - \left\{ {\rho \atop \nu\lambda} \right\} \left\{ {\varkappa' \atop \mu\rho} \right\} \right)$$

and uses (2.49) to express $\partial_\mu g^{\varkappa'\sigma}$ in terms of Christoffel symbols, so that we can write

$$\begin{aligned}
g_{\varkappa\varkappa'}\partial_\mu g^{\varkappa'\sigma} &= -(\partial_\mu g_{\varkappa\varkappa'})g^{\varkappa'\sigma} \\
&= -\left(\begin{Bmatrix}\tau \\ \mu\varkappa\end{Bmatrix} g_{\tau\varkappa'} + \begin{Bmatrix}\tau \\ \mu\varkappa'\end{Bmatrix} g_{\varkappa\tau}\right) g^{\varkappa'\sigma} \\
&= -\begin{Bmatrix}\sigma \\ \mu\varkappa\end{Bmatrix} - \{\mu\varkappa', \varkappa\}\, g^{\varkappa'\sigma}
\end{aligned}$$

and hence

$$\begin{aligned}
\overset{\{\}}{R}_{\mu\nu\lambda\varkappa} &= \frac{1}{2}[(\partial_\mu\partial_\lambda g_{\nu\varkappa} - \partial_\mu\partial_\varkappa g_{\nu\lambda}) - (\mu \leftrightarrow \nu)] \\
&\quad - \frac{1}{2}\left[\left(\begin{Bmatrix}\sigma \\ \mu\varkappa\end{Bmatrix} + \{\mu\varkappa', \varkappa\} g^{\varkappa'\sigma}\right)(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) - (\mu \leftrightarrow \nu)\right] \\
&\quad - \left(\begin{Bmatrix}\rho \\ \mu\lambda\end{Bmatrix}\{\nu\rho, \varkappa\} - \begin{Bmatrix}\rho \\ \nu\lambda\end{Bmatrix}\{\mu\rho, \varkappa\}\right).
\end{aligned}$$

A further use of relation (2.49) brings the second line to

$$\begin{aligned}
-\Bigg\{\frac{1}{2}\left(\begin{Bmatrix}\sigma \\ \mu\varkappa\end{Bmatrix} + \{\mu\varkappa', \varkappa\} g^{\varkappa'\sigma}\right)\Bigg[\left(\begin{Bmatrix}\rho \\ \nu\lambda\end{Bmatrix} g_{\rho\sigma} + \begin{Bmatrix}\rho \\ \nu\sigma\end{Bmatrix} g_{\lambda\rho}\right) + (\lambda \leftrightarrow \nu) \\
- \begin{Bmatrix}\rho \\ \sigma\nu\end{Bmatrix} g_{\rho\lambda} - \begin{Bmatrix}\rho \\ \sigma\lambda\end{Bmatrix} g_{\nu\rho}\Bigg]\Bigg\} + (\mu \leftrightarrow \nu)
\end{aligned}$$

and we find that almost all terms cancel, by the symmetry of $\begin{Bmatrix}\lambda \\ \mu\nu\end{Bmatrix}$ in $\mu\nu$. Only

$$-\left(\begin{Bmatrix}\sigma \\ \mu\varkappa\end{Bmatrix}\{\nu\lambda, \sigma\} + \{\mu\varkappa', \varkappa\}\begin{Bmatrix}\varkappa' \\ \nu\lambda\end{Bmatrix}\right) + (\mu \leftrightarrow \nu)$$

remains. The second term in this expression cancels the third line in $\overset{\{\}}{R}_{\mu\nu\lambda\varkappa}$ which therefore becomes

$$\overset{\{\}}{R}_{\mu\nu\lambda\varkappa} = \frac{1}{2}[(\partial_\mu\partial_\lambda g_{\nu\varkappa} - \partial_\mu\partial_\varkappa g_{\nu\lambda}) - (\mu \leftrightarrow \nu)] - \left(\begin{Bmatrix}\sigma \\ \mu\varkappa\end{Bmatrix}\begin{Bmatrix}\sigma' \\ \nu\lambda\end{Bmatrix} - (\mu\nu)\right) g_{\sigma\sigma'}. \tag{2.71}$$

This expression shows manifestly the symmetry $\mu\nu \leftrightarrow \lambda\varkappa$ [using $(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) g_{\lambda\varkappa} = 0$, as well as antisymmetry under $\lambda \leftrightarrow \varkappa$ and $\mu \leftrightarrow \nu$].

Another property of Minkowski space now emerges. Just as this space had a vanishing torsion tensor for any curvilinear parametrization, it also has a vanishing curvature tensor. This follows from the obvious fact that $R_{\mu\nu\lambda}{}^{\varkappa} = e_a{}^{\varkappa}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\lambda \equiv 0$ for the special choice of the basis tetrad $e_a{}^{\varkappa} = \delta_a{}^{\varkappa}$. Together with the tensor transformation law (2.62) we find $R_{\mu\nu}{}^{\varkappa} \equiv 0$ in all transformed coordinates.

2.5. TORSION AND CURVATURE FROM DEFECTS

In the last two sections we saw that a Minkowski space had neither torsion nor curvature. The absence of torsion followed from its tensor property, which was a consequence of the commutativity of derivatives in front of the infinitesimal translation field,

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\, \xi^{\varkappa}(x) = 0. \tag{2.72a}$$

The absence of curvature, on the other hand, was a consequence of the integrability condition (2.15) of the transformation matrices,

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\, \alpha^{\varkappa}{}_\lambda(x) = 0.$$

This implies that

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\, \partial_\lambda \xi^{\varkappa}(x) = 0, \tag{2.72b}$$

i.e., that derivatives commute in front of *derivatives* of the infinitesimal translation field. This suggests a simple way of constructing general affine spaces with torsion or curvature or both from a Minkowski space by performing *singular* coordinate transformations which do not satisfy (2.72a) and (2.72b).

Let us study the properties of a space at which we can arrive starting from basis tetrads $e_a{}^\mu = \delta_a{}^\mu$, $e^a{}_\mu = \delta^a{}_\mu$ via such *infinitesimal singular* coordinate transformations $\xi^{\varkappa}(x)$. According to (2.30), the new basis tetrads are

$$e_a{}^\mu = \delta_a{}^\mu - \partial_a \xi^\mu, \qquad e^a{}_\mu = \delta^a{}_\mu + \partial_\mu \xi^a, \tag{2.73}$$

and the metric is [compare (2.37)]

$$g_{\mu\nu} = e^{a}{}_{\mu} e_{a\nu} = \eta_{\mu\nu} + (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu). \tag{2.74a}$$

The connection associated with the tetrads (2.73) is [see (2.50c)]

$$\Gamma_{\mu\nu}{}^{\lambda} = \partial_\mu \partial_\nu \xi^{\lambda} \tag{2.74b}$$

and the curvature tensor becomes, accordingly [from (2.61)],

$$R_{\mu\nu\lambda}{}^{\varkappa} = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \partial_\lambda \xi^{\varkappa}. \tag{2.74c}$$

Since ξ^{λ} are infinitesimal, we can lower the index in both equations [with an error which is only of the order of $(\xi)^2$ and thus negligible] so that

$$\Gamma_{\mu\nu\lambda} = \partial_\mu \partial_\nu \xi_\lambda, \qquad S_{\mu\nu\lambda} = \tfrac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \xi_\lambda, \qquad R_{\mu\nu\lambda\varkappa} = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \partial_\lambda \xi_\varkappa . \tag{2.75}$$

For singular $\xi_\mu(x)$, the metric and the connection are, in general, also singular. This could cause difficulties in performing consistent length measurements and parallel displacements. To avoid such difficulties, Einstein postulated that the metric $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}{}^{\lambda}$ should be so smooth as to permit two differentiations which commute as stated for $g_{\mu\nu}$ in (2.65). By (2.74a) and (2.74b), this implies that we must consider only such singular coordinate transformations which satisfy the conditions

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)(\partial_\lambda \xi_\varkappa + \partial_\varkappa \xi_\lambda) = 0, \qquad (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \partial_\sigma \partial_\lambda \xi_\varkappa = 0. \tag{2.76}$$

Observe that the curvature tensor is now trivially antisymmetric in the last two indices — an immediate consequence of the integrability condition (2.65) of the metric.

For completeness, let us also write down the decomposition (2.55) of the connection into the Christoffel symbol and the contortion tensor [by inserting (2.73) into (2.54) and (2.53), and (2.74a) into (2.9)]

$$\Gamma_{\mu\nu\varkappa} = \{\mu\nu, \varkappa\} + K_{\mu\nu\varkappa}, \tag{2.77}$$

with

$$\{\mu\nu, \varkappa\} = \tfrac{1}{2}\partial_\mu (\partial_\nu \xi_\varkappa + \partial_\varkappa \xi_\nu) + \tfrac{1}{2}\partial_\nu (\partial_\mu \xi_\varkappa + \partial_\varkappa \xi_\mu) - \tfrac{1}{2}\partial_\varkappa (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu), \tag{2.78}$$

$$\begin{aligned} K_{\mu\nu\lambda} &= \tfrac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \xi_\lambda - \tfrac{1}{2}(\partial_\nu \partial_\lambda - \partial_\lambda \partial_\nu) \xi_\mu + \tfrac{1}{2}(\partial_\lambda \partial_\mu - \partial_\mu \partial_\lambda) \xi_\nu \\ &= \tfrac{1}{2}\partial_\mu (\partial_\nu \xi_\lambda - \partial_\lambda \xi_\nu) + \tfrac{1}{2}\partial_\lambda (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu) - \tfrac{1}{2}\partial_\nu (\partial_\lambda \xi_\mu + \partial_\mu \xi_\lambda). \end{aligned} \tag{2.79}$$

From the Christoffel symbol we find the Riemann curvature tensor,

$$\overset{\{\}}{R}_{\mu\nu\lambda\varkappa} = \tfrac{1}{2}\partial_\mu[\partial_\nu(\partial_\lambda \xi_\varkappa + \partial_\varkappa \xi_\lambda) + \partial_\lambda(\partial_\nu \xi_\varkappa + \partial_\varkappa \xi_\nu) - \partial_\varkappa(\partial_\nu \xi_\lambda + \partial_\lambda \xi_\nu)]$$
$$-\tfrac{1}{2}\partial_\nu[\partial_\mu(\partial_\lambda \xi_\varkappa + \partial_\varkappa \xi_\lambda) + \partial_\lambda(\partial_\nu \xi_\varkappa + \partial_\varkappa \xi_\nu) - \partial_\varkappa(\partial_\mu \xi_\lambda + \partial_\lambda \xi_\mu)].$$

Due to the integrability condition (2.76), the first terms in each line cancel and we obtain

$$\overset{\{\}}{R}_{\mu\nu\lambda\varkappa} = \tfrac{1}{2}[(\partial_\mu \partial_\lambda(\partial_\nu \xi_\varkappa + \partial_\varkappa \xi_\nu) - (\mu \leftrightarrow \nu) - (\lambda \leftrightarrow \varkappa)] \qquad (2.80)$$

In order to understand the geometric properties of a space generated by the infinitesimal singular transformations,

$$x^a \to x^\mu = (x^a - \xi^a(x^b))\,\delta_a{}^\mu, \qquad (2.81a)$$

we recall that such transformations have been encountered before in the context of crystalline defects. There, we considered infinitesimal displacements of atoms $u_i(\mathbf{x})$ containing defects, i.e.,

$$x_i \to x_i' = x_i + u_i(\mathbf{x}), \qquad (2.81b)$$

where x_i' are the shifted positions, as seen from an outside observer and $u_i(\mathbf{x})$ is the total (i.e., elastic plus plastic) distortion field, $u_i(x) = u_i^{el}(\mathbf{x}) + u_i^p(\mathbf{x})$ [with derivatives commuting neither in front of $u_i(\mathbf{x})$ nor in front of $\partial_i u_j(\mathbf{x})$]. Thus, if we change the sign of the displacement field, $u_i(\mathbf{x}) \to -u_i(\mathbf{x})$, the transformation law (2.81b) is of the same form as (2.81a) and the non-commutativity of derivatives in front of singular coordinate changes $\xi^a(x^b)$ is completely analogous to that in front of total displacements $u_i(\mathbf{x})$. Moreover in Chapter 2, Part III, we analyzed the defect structure of $u_i(\mathbf{x})$ in terms of dislocations and disclinations. A similar analysis can be given here for the coordinate changes $\xi^a(x^b)$.

To be specific, let us restrict our considerations to the three-dimensional Euclidean subspace of Minkowski space. Then we have to identify the physical coordinates of material points x^a for $a = 1, 2, 3$ with the previous spatial coordinates x_i for[c] $i = 1, 2, 3$ and $\partial_a = \partial/\partial x^a$ $(a = i)$ with the previous derivatives ∂_i. The infinitesimal translations in (2.80), $\xi^{a=i}(\mathbf{x})$, are equal to the displacements total $u_i(\mathbf{x})$ (with the reversed sign convention), so that the basis tetrads are

$$e_a{}^i = \delta_a{}^i - \partial_a u_i, \qquad e^a{}_i = \delta^a{}_i + \partial_i u_a, \tag{2.82}$$

and the metric becomes, in linear approximation,

$$g_{ij} = e_{ai} e^a{}_j = \delta_{ij} + \partial_i u_j + \partial_j u_i. \tag{2.83}$$

Apart from the trivial unit matrix it coincides with twice the total strain tensor $u_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$. Invariant distances $ds = \sqrt{g_{ij} dx^i dx^j}$ are measured by counting atomic steps within the distorted crystal.

The connection is simply

$$\Gamma_{ijk} = \partial_i \partial_j u_k \tag{2.84}$$

with torsion and curvature tensors

$$S_{ijk} = \tfrac{1}{2}(\partial_i \partial_j - \partial_j \partial_i) u_k, \qquad R_{ijk\ell} = (\partial_i \partial_j - \partial_j \partial_i) \partial_k u_\ell. \tag{2.85}$$

We now recall that the expression for the curvature tensor appeared before in Eq. (III.2.51). There it was introduced purely as a matter of convenience. In fact we did not yet know its fundamental differential geometric meaning.

The integrability conditions (2.76) imply

$$(\partial_i \partial_j - \partial_j \partial_i)(\partial_k u_\ell + \partial_\ell u_k) = 0, \quad (\partial_i \partial_j - \partial_j \partial_i) \partial_m (\partial_k u_\ell + \partial_\ell u_k) = 0,$$

$$(\partial_i \partial_j - \partial_j \partial_i) \partial_m (\partial_k u_\ell - \partial_\ell u_k) = 0. \tag{2.85$'$}$$

They state that the strain tensor, its derivative, and the derivative of the local rotation field are all twice-differentiable single-valued functions everywhere. We argued in (III.2.47)–(III.2.50) that this was true in a crystal. We may take advantage of the first condition and write the curvature tensor alternatively as

$$R_{ijk\ell} = (\partial_i \partial_j - \partial_j \partial_i) \tfrac{1}{2} (\partial_k u_\ell - \partial_\ell u_k). \tag{2.86}$$

[e]When working in Minkowski space our convention is to consider vector components with *upper* indices as physical components. In purely three-dimensional calculations one usually employs the metric $\eta_{ab} = \delta_{ab}$ so that $x^{a=i}$ and $x_{a=i}$ are the same.

The antisymmetry in ij and $k\ell$ suggests, in three dimensions, the introduction of a tensor of second rank,[f]

$$G_{ji} \equiv \tfrac{1}{4} e_{ik\ell} e_{jmn} R^{k\ell mn}, \tag{2.87}$$

where

$$e_{ijk} = \sqrt{g}\, \varepsilon_{ijk} = g_{ii'} g_{jj'} g_{kk'} e^{i'j'k'} = g_{ii'} g_{jj'} g_{kk'} \left(\frac{1}{\sqrt{g}} \varepsilon^{i'j'k'} \right)$$

is the covariant version of the ε-tensor in general metric-affine spaces.

The tensor G_{ji} happens to coincide with the Einstein tensor as defined in (2.67). Indeed, if we use the identity

$$\begin{aligned} e_{ik\ell} e_{jmn} = {} & g_{ij} g_{km} g_{\ell n} + g_{im} g_{kn} g_{\ell j} + g_{in} g_{kj} g_{\ell m} \\ & - g_{ij} g_{\ell m} g_{kn} - g_{im} g_{\ell n} g_{kj} - g_{in} g_{\ell j} g_{km} \end{aligned}$$

and insert it into (2.87), we find, using (2.66a)

$$G_{ji} = R_{ji} - \tfrac{1}{2} g_{ji} R_k{}^k .$$

To linear approximation, G_{ij} becomes [using (2.85)]

$$G_{ji} = \varepsilon_{ik\ell} \partial_k \partial_\ell \left(\tfrac{1}{2} \varepsilon_{jmn} \partial_m u_n \right).$$

The second factor is the local rotation $\omega_j = (1/2)\, \varepsilon_{jmn} \partial_m u_n$ introduced in Eq. (III.2.3), and we see that the Einstein tensor can be written as

$$G_{ji} = \varepsilon_{ik\ell} \partial_k \partial_\ell \omega_j . \tag{2.88}$$

Let us also form the Einstein tensor $\overset{\{\}}{G}_{ij}$ associated with the Riemannian curvature tensor $\overset{\{\}}{R}_{ijk\ell}$. Using (2.80), we find

$$\overset{\{\}}{G}_{ji} = \varepsilon_{ik\ell} \varepsilon_{jmn} \partial_k \partial_m \tfrac{1}{2} (\partial_\ell u_n + \partial_n u_\ell). \tag{2.89}$$

[f] Notice that in 4 dimensions, there is an equation similar to (2.87),

$$G^{\nu\mu} = \tfrac{1}{4} e_\alpha{}^{\mu\beta\gamma} e^{\nu\alpha\delta\tau} R_{\beta\gamma\delta\tau} .$$

In the discussion of crystal defects we introduced the following measures for the non-commutativity of derivatives: the dislocation density (III.2.42a)

$$\alpha_{ij} = \varepsilon_{ik\ell}\,\partial_k\,\partial_\ell\,u_j, \tag{2.90a}$$

the disclination density (III.2.42b)

$$\Theta_{ij} = \varepsilon_{ik\ell}\,\partial_k\,\partial_\ell\,\omega_j \tag{2.90b}$$

and the defect density (III.2.76)[g]

$$\eta_{ij} = \varepsilon_{ik\ell}\,\varepsilon_{jmn}\,\partial_k\,\partial_m\,u_{\ell n}\,. \tag{2.91}$$

Comparison with (2.85) shows that α_{ij} is directly related to the torsion tensor $S_{k\ell}{}^i = (1/2)(\Gamma_{k\ell}{}^i - \Gamma_{\ell k}{}^i)$:

$$\alpha_{ij} \equiv \varepsilon_{ik\ell}\,\Gamma_{k\ell j} \equiv \varepsilon_{ik\ell}\,S_{k\ell j}\,. \tag{2.92a}$$

Hence torsion is a measure of the translational defects contained in singular coordinate transformations.

We may also use the decomposition (2.55) and write, by the symmetry of the Christoffel symbol $\{k\ell, j\}$ in $k\ell$,

$$\alpha_{ij} = \varepsilon_{ik\ell}\,K_{k\ell j}\,,$$

where $K_{k\ell j}$ is the contortion tensor. Since this is antisymmetric in ℓj, it is useful to introduce the second rank tensor,

$$K_{\ell n} = \tfrac{1}{2} K_{k\ell j}\,\varepsilon_{\ell jn}\,.$$

Inserting this into (2.92b) we see that

$$\alpha_{ij} = -K_{ji} + \delta_{ij}\,K_{\ell\ell}\,,$$

that is, K_{ij} coincides with Nye's contortion tensor which was introduced previously in Part III, Eq. (2.79a). This can be seen once more using the explicit decomposition of K_{ijk} as given in (2.79), which reads in terms of the displacement field $u_i(\mathbf{x})$

[g] Recall that these total displacements $u_i(x)$ were defined with the opposite sign convention as those in (2.82)–(2.89) so that all the following identification carry a factor -1. For simplicity this factor will be suppressed.

$$K_{ijk} = \partial_i(\partial_j u_k - \partial_k u_j)/2 - (\partial_j(\partial_k u_i + \partial_i u_k)/2 - (j \leftrightarrow k))$$
$$= \partial_i \omega_{jk} - (\partial_j u_{ki} - (j \leftrightarrow k)). \tag{2.92b}$$

Contracting this with $(1/2)\varepsilon_{\ell jk}$ yields

$$K_{i\ell} = \partial_i \omega_\ell - \varepsilon_{\ell jk} \partial_j u_{ki},$$

which is precisely the defining equation (III.2.78).

Consider now the disclination density Θ_{ij}. Comparing (2.90) with (2.88) we see that it coincides exactly with the Einstein tensor G_{ji} formed from the full curvature tensor

$$\Theta_{ij} \equiv G_{ji}. \tag{2.93}$$

The defect density (2.91), finally, coincides with the Einstein tensor formed from the Riemannian curvature tensor, (2.89),

$$\eta_{ij} = \overset{\{\}}{G}_{ij}.$$

Hence we can conclude: *A space with torsion and curvature can be generated from a Minkowski space via singular coordinate transformations and is completely equivalent to a crystal which has undergone plastic deformation being filled with dislocations and disclinations.*

In Minkowski space, the trajectories of free particles are straight lines. In defected space, free particles follow a complicated path, which is no longer straight since defects may lie in its way. According to Einstein's theory, the motion of mass points in a gravitational field is governed by the principle of shortest path as defined by the defected metric $g_{\mu\nu}$. This metric contains all gravitational effects. They may be viewed as a consequence of disclinations present in the "world crystal." The natural length scale of gravitation is the Planck length

$$\ell_P = \left(\frac{c^3}{8\pi G\hbar}\right)^{-1/2},$$

where c is the light velocity ($\approx 3 \times 10^{10}$ cm/s), $\hbar$ is Planck's constant ($\approx 1.05459 \times 10^{-24}$ erg/s) and G is Newton's gravitational constant ($\approx 6.670 \times 10^{-8}$ cm^3/(g $\cdot$ s^2)). The Planck length is an extremely small quantity ($\approx 8.09 \times 10^{-33}$ cm) which at present is beyond any experimental resolution. This may be imagined as the lattice constant of the world crystal. In the presence of torsion, particles with spin, move along the most straight possible path (called auto parallel) which may no longer be the shortest [see F.W. Hehl et al. (1976)].

2.6. DIFFERENTIAL GEOMETRIC PROPERTIES OF METRIC-AFFINE SPACES WITH CURVATURE AND TORSION

At this juncture we have studied only such affine spaces which were obtained from a Minkowski space by introducing an infinitesimal amount of defects. In reality, defects can pile up and the space must be described by the full nonlinear formulation of affine spaces. At the linear level we have learned how dislocations and disclinations manifest themselves in certain non-vanishing contour integrals around Burgers circuits. In this section we discuss these geometric aspects, emphasizing on their non-linear properties.

The metric-affine space will be characterized by the same type of integrability conditions as the space with infinitesimal defects. Explicitly, the metric and the connection are *single-valued*, *twice-differentiable functions* which satisfy the *integrability conditions*

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) g_{\lambda\varkappa} = 0, \tag{2.94}$$

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \Gamma_{\sigma\lambda}{}^{\varkappa} = 0. \tag{2.95}$$

Remember that the first condition ensures the antisymmetry of the curvature tensor in the last two indices [see (2.64)]. By antisymmetrizing the second condition in $\sigma\lambda$ it becomes the integrability condition for the torsion

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) S_{\sigma\lambda}{}^{\varkappa} = 0. \tag{2.95a$'$}$$

Moreover, using the decomposition (2.55), the Christoffel symbol is seen to be integrable as well:

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \left\{ \begin{matrix} \varkappa \\ \sigma\lambda \end{matrix} \right\} = 0. \tag{2.95b$'$}$$

Since $\partial_\mu g_{\lambda\varkappa}$ can be expressed in terms of products of Christoffel symbols and metric tensors [see (2.49)], and since products of integrable functions are integrable[h], this implies that the derivatives of $g_{\lambda\varkappa}$ also satisfy

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \partial_\sigma g_{\lambda\varkappa} = 0. \tag{2.95a''}$$

Conversely, with the Christoffel symbol consisting of products of $g_{\lambda\varkappa}$ and $\partial_\mu g_{\lambda\varkappa}$, this condition implies (2.95b) and thus is completely equivalent to it [in the presence of (2.94)].

In order to understand the geometric properties of such a metric-affine space let us first introduce the concept of *local parallelism*. Consider a vector field $\mathbf{v}(x) = \mathbf{e}_a v^a(x)$ which is parallel in the inertial frame in the naive sense that all vectors point in the same direction. This simply means $\partial_b \mathbf{v}(x) = \mathbf{e}_a \partial_b v^a = 0$. When we change the coordinates to x^μ, we find

$$\partial_b v^a = \partial_b (e^a{}_\mu v^\mu) = e_b{}^\nu \partial_\nu (e^a{}_\mu v^\mu) = e_b{}^\nu e^a{}_\mu D_\nu v^\mu = 0. \tag{2.96}$$

Thus parallel vector fields have their local components v^μ change in such a way that their covariant derivatives vanish:

$$D_\nu v^\mu = \partial_\nu v^\mu + \Gamma_{\nu\lambda}{}^\mu v^\lambda = 0. \tag{2.97}$$

Similarly,

$$D_\nu v_\mu = \partial_\nu v_\mu - \Gamma_{\nu\mu}{}^\lambda v_\lambda = 0. \tag{2.98}$$

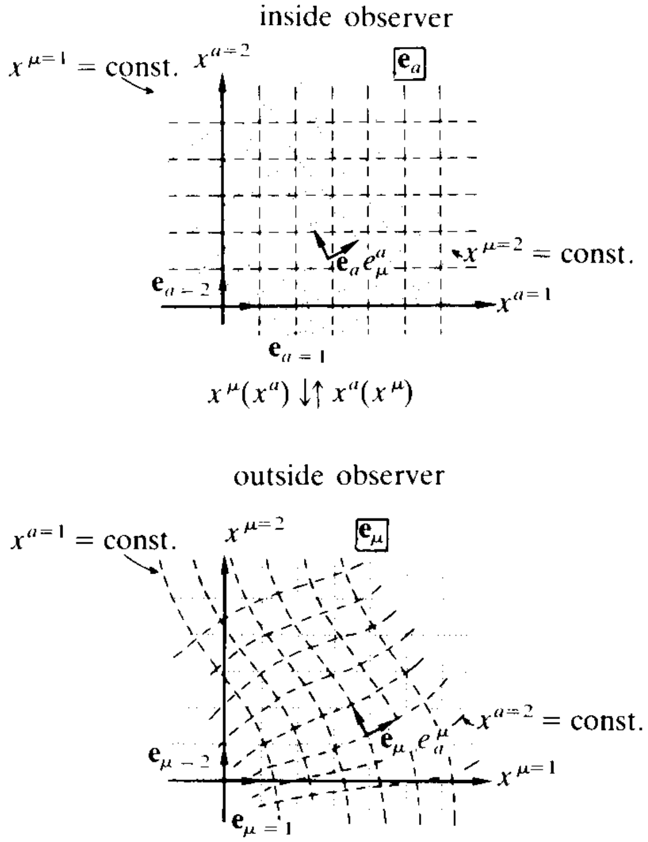
Notice that the basis tetrads $e_a{}^\nu$, $e^a{}_\nu$ are parallel vector fields, by construction [see (2.47)].

From the standpoint of a world crystal without defects, parallelism has a simple meaning. Consider Fig. 2.1b. Let the distorted coordinate system $x^a = \text{const.}$ (the dashed curves) be the crystal planes of an elastically distorted crystal as seen from the local frame x^μ, which are identified with the coordinates of an outside observer. An observer within the distorted crystal orients himself by the planes $x^a = \text{const.}$ He measures distances and directions by counting atoms along the crystal directions $\mathbf{e}_a$. The above definition of parallelism amounts to vectors being defined as parallel if they are so from his point of view, i.e., if they

[h]This follows from the chain rule of differentiation

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)(fg) = [(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) f] g + f(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) g.$$

FIG. 2.1a,b. Illustration of crystal planes (x^a = const.) before and after an elastic distortion, once seen from within the crystal (a) and once from outside (b).

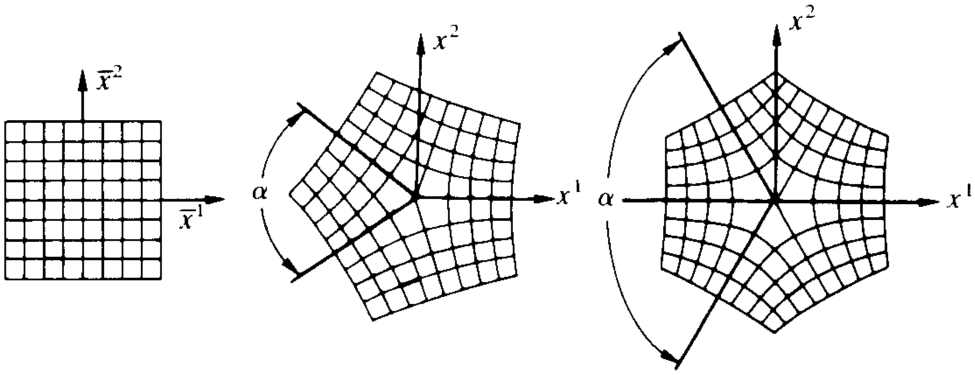


have been parallel in the ideal reference crystal before distortion took place. Thus the normal vectors to the dashed coordinate planes x^a = const. are parallel to each other. Indeed, they form the vector fields $e_a{}^\mu(x)$, which always satisfy $D_\nu e_a{}^\mu = 0$ [see (2.47)].

If the mapping $x^\mu(x^a)$ contains defects, it is impossible, in general, to find a global definition of parallelism. Consider, for example, a wedge disclination which is shown in Fig. 2.2., say the $-90°$ one. The crystal has been cut from the left, and new crystalline material has been inserted in the Volterra construction process. The crystalline coordinate planes define parallel planes. With the right-hand piece stemming from the original crystal, there exists a completely consistent definition of local parallelism. For example, the almost horizontal lines are all parallel. The lines cutting these vertically are also parallel by definition. On the left-hand side, the vertical lines continue smoothly into the inserted new crystalline material from above and below. Where they meet they turn out to be orthogonal. This shows that there is no global parallelism. Still, the coordinate planes define local parallelism in any small region inside the original as well as the inserted material, except on the disclination line.

Let us study this situation more generally. Given an arbitrary

FIG. 2.2. Lattice planes in a crystal in which two types of disclinations of $-90°$ and $-180°$ have been formed by Volterra process.



connection $\Gamma_{\mu\nu}{}^{\lambda}$ we first inquire under what condition it is possible to find a parallel vector field in the whole space. For this we consider the vector field $v^{\mu}(x)$ at a point x_0 where it has the value $v^{\mu}(x_0)$. Let us now move to the neighboring position $x_0 + dx$. There the field has components

$$v^{\mu}(x_0 + dx) = v^{\mu}(x_0) + \partial_{\nu} v^{\mu}(x_0) dx^{\nu}.$$

If $v^{\mu}(x)$ is a parallel vector field with $D_{\nu} v^{\mu} = 0$, then the derivative satisfies

$$\partial_{\nu} v^{\mu} = -\Gamma_{\nu\varkappa}{}^{\mu} v^{\varkappa}. \qquad (2.99)$$

This differential equation is integrable over a finite region of space if and only if the condition of Schwarz' Lemma, i.e.,

$$(\partial_{\lambda} \partial_{\nu} - \partial_{\nu} \partial_{\lambda}) v^{\mu} = 0, \qquad (2.100)$$

is fulfilled. If we calculate

$$(\partial_{\lambda} \partial_{\nu} - \partial_{\nu} \partial_{\lambda}) v^{\mu} = -\partial_{\lambda} (\Gamma_{\nu\varkappa}{}^{\mu} v^{\varkappa}) + \partial_{\nu} (\Gamma_{\lambda\varkappa}{}^{\mu} v^{\varkappa}), \qquad (2.101)$$

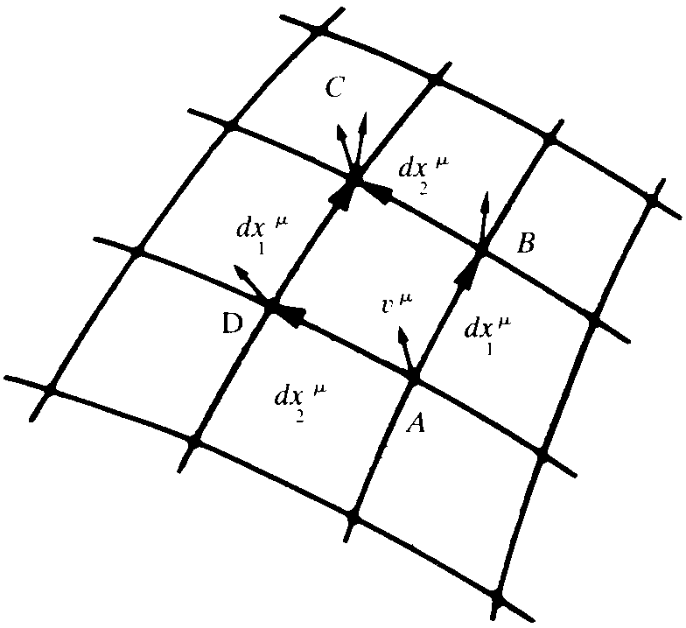
we find

$$-(\partial_{\lambda} \Gamma_{\nu\varkappa}{}^{\mu} - \partial_{\nu} \Gamma_{\lambda\varkappa}{}^{\mu}) v^{\varkappa} - \Gamma_{\nu\varkappa}{}^{\mu} \partial_{\lambda} v^{\varkappa} + \Gamma_{\lambda\varkappa}{}^{\mu} \partial_{\nu} v^{\varkappa} \qquad (2.102)$$

and thus, using once more (2.99),

$$(\partial_{\lambda} \partial_{\nu} - \partial_{\nu} \partial_{\lambda}) v^{\mu} = -R_{\lambda\nu\varkappa}{}^{\mu} v^{\varkappa}. \qquad (2.103)$$

FIG. 2.3. Illustration of parallel transport of a vector around a closed circuit ABCD.



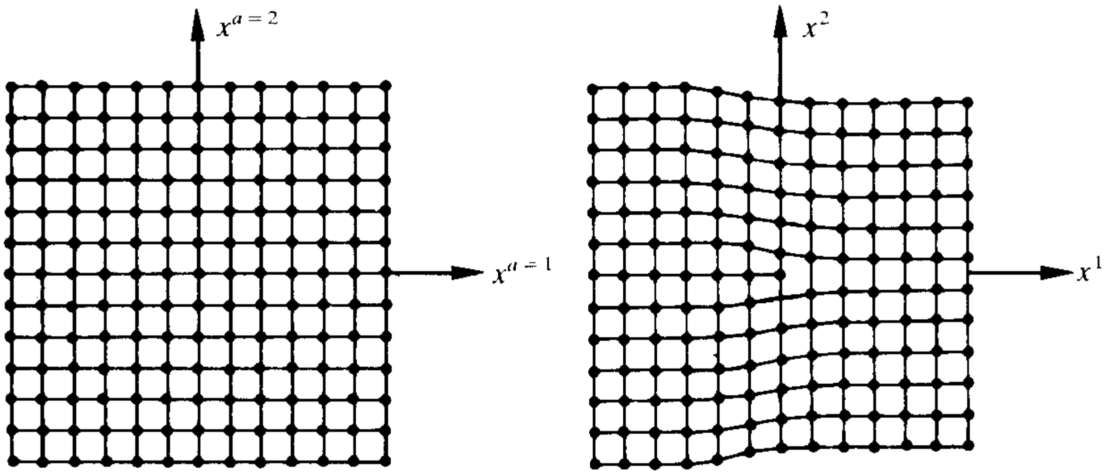
Thus the parallel field $v^\mu(x)$ exists in the whole space if and only if the curvature tensor vanishes everywhere.

If $R_{\lambda\nu\varkappa}{}^\mu$ is nonzero, the concept of parallel vectors cannot be carried over from Minkowski space to the affine space over any finite distance. In other words in curved spaces there exists no *teleparalellism*. Such spaces are called *curved.*

We illustrated earlier that this was the case in the presence of disclinations. Disclinations represent curvature, i.e., a crystal containing disclinations is curved in the differential geometric sense. This is in accordance with the previous observation that the disclination density Θ_{ij} coincides with the Einstein tensor G_{ij}.

In the illustration we also saw that even in the presence of a disclination it is still meaningful to define a vector field as *locally parallel.* The condition for this is that the covariant derivatives vanish at that point x_0:$D_\nu v^\mu(x_0) = 0$. If this condition is satisfied, the neighboring vector $v^\mu(x)$, close to x_0, differs from $v^\mu(x_0)$ by terms of order $(x - x_0)^2$ at most [rather than $(x - x_0)$ for non-parallel vectors]. In order to see this let us draw an infinitesimal quadrangle $ABCD$ in the coordinate frame x^μ spanned by $AB = \underset{1}{dx}{}^\mu = DC$ and $BC = \underset{2}{dx}{}^\mu = AD$ (see Fig. 2.3). We now calculate the change of direction of a vector as it makes a complete circuit around the quadrangle while being kept parallel. When passing from A at x^μ to B at $x^\mu + \underset{1}{dx}{}^\mu$ the vector components change from $v^\mu = v_A{}^\mu(x)$ to

FIG. 2.4. Atomic positions in a crystal with and without a dislocation of the edge type.



ideal dislocated

$$v_B{}^{\mu} = v^{\mu}(x + \underset{1}{dx}) = v_A{}^{\mu} + \partial_{\nu} v^{\mu} \underset{1}{dx}{}^{\nu} = v_A{}^{\mu} - \overset{A}{\Gamma}_{\nu\lambda}{}^{\mu} v^{\lambda} \underset{1}{dx}{}^{\nu} \qquad (2.104)$$

On continuing to C at $x_0{}^{\mu} + \underset{1}{dx}{}^{\mu} + \underset{2}{dx}{}^{\mu}$ we have

$$\begin{aligned} v_C{}^{\mu} &= v_B{}^{\mu} - \overset{B}{\Gamma}_{\tau\varkappa}{}^{\mu} v_B{}^{\varkappa} \underset{2}{dx}{}^{\tau} \\ &= v_A{}^{\mu} - \overset{A}{\Gamma}_{\nu\lambda}{}^{\mu} v^{\lambda} \underset{1}{dx}{}^{\nu} - \overset{B}{\Gamma}_{\tau\varkappa}{}^{\mu} v_A{}^{\varkappa} \underset{2}{dx}{}^{\tau} + \overset{B}{\Gamma}_{\tau\varkappa}{}^{\mu} \overset{B}{\Gamma}_{\nu\lambda}{}^{\varkappa} v_A{}^{\lambda} \underset{1}{dx}{}^{\nu} \underset{2}{dx}{}^{\tau} \\ &= v_A{}^{\mu} - \overset{A}{\Gamma}_{\nu\lambda}{}^{\mu} v^{\lambda} (\underset{1}{dx}{}^{\nu} + \underset{2}{dx}{}^{\nu}) - \partial_{\nu} \overset{A}{\Gamma}_{\tau\varkappa}{}^{\mu} v_A{}^{\varkappa} \underset{1}{dx}{}^{\nu} \underset{2}{dx}{}^{\tau} \\ &\quad + \overset{A}{\Gamma}_{\tau\varkappa}{}^{\mu} \overset{A}{\Gamma}_{\nu\lambda}{}^{\varkappa} v_A{}^{\lambda} \underset{1}{dx}{}^{\nu} \underset{2}{dx}{}^{\tau} + O(dx^3). \end{aligned} \qquad (2.105)$$

We can now repeat the same procedure, but along the line ADC and find the same result with $\underset{1}{dx} \leftrightarrow \underset{2}{dx}$ interchanged. The difference between the two results is

$$v^{\mu}{}_{ABC} - v^{\mu}{}_{ADC} = -\tfrac{1}{2} R_{\nu\tau\varkappa}{}^{\mu} v_A{}^{\varkappa} ds^{\nu\tau} + O(dx^3), \qquad (2.106)$$

where $ds^{\nu\tau} = (\underset{1}{dx}{}^{\mu} \underset{2}{dx}{}^{\tau} - \underset{1}{dx}{}^{\tau} \underset{2}{dx}{}^{\nu})$ is the infinitesimal surface element of the quadrangle. Thus the vectors $v^{\mu}{}_{ABC}$ and $v^{\mu}{}_{ADC}$ differ indeed by terms of second order in dx. The second order difference is governed by the curvature tensor. For zero-curvature, the difference is of the third order.

There exists a similar geometric illustration for the torsion property $S_{\mu\nu}{}^{\lambda} \neq 0$. Consider a crystal with an edge dislocation (see Fig. 2.4). Let us focus attention upon a closed circuit with the form of a parallelogram in

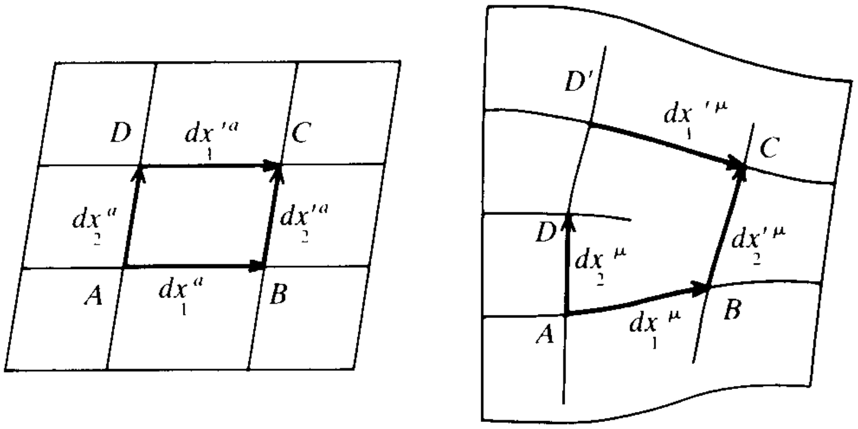


FIG. 2.5. Illustration of non-closure of a parallelogram after inserting an edge dislocation.

an ideal reference crystal (i.e., in the frame x^a). Suppose its image in x^μ space encloses the dislocation line (see Fig. 2.5). We now recall that, in the Volterra process of constructing the dislocation, the reference crystal was cut open, and a layer of atoms was inserted. In this process, the original parallelogram is opened so that the dislocated crystal has a gap between the open ends. The gap vector is precisely the Burgers vector. To be specific, let the parallelogram in the ideal reference crystal be spanned by the vectors $AB = \underset{1}{dx}{}^a = DC$, $AD = \underset{2}{dx}{}^a = BC$. In the defected space x^μ these become $AB = \underset{1}{dx}{}^\mu$, $AD = \underset{2}{dx}{}^\mu$, $D'C = \underset{1}{dx}{}'^\mu$, $BC = \underset{2}{dx}{}'^\mu$. Since $\underset{1}{dx}{}'^\mu$, $\underset{2}{dx}{}'^\mu$ are parallel in the ideal reference crystal, they are parallel vectors, i.e., the vectors $v^\mu(x) = \underset{2}{dx}{}^\mu$, $v^\mu(x^\mu + \underset{1}{dx}{}^\mu) = \underset{2}{dx}{}'^\mu$ satisfy (2.99) when going from A to B:

$$\underset{2}{dx}{}'^\mu = \underset{2}{dx}{}^\mu - \Gamma_{\nu\lambda}{}^\mu \underset{1}{dx}{}^\nu \underset{2}{dx}{}^\lambda. \tag{2.107}$$

Similarly the vectors $\underset{1}{dx}{}^\mu$ and $\underset{1}{dx}{}'^\mu$ are parallel and therefore related by

$$\underset{1}{dx}{}'^\mu = \underset{1}{dx}{}^\mu - \Gamma_{\nu\lambda}{}^\mu \underset{2}{dx}{}^\nu \underset{1}{dx}{}^\lambda. \tag{2.108}$$

From this follows the Burgers vector

$$b^\mu = (\underset{2}{dx}{}' + \underset{1}{dx})^\mu - (\underset{1}{dx}{}' + \underset{2}{dx})^\mu = -S_{\nu\lambda}{}^\mu ds^{\nu\lambda}. \tag{2.109}$$

In Minkowski space, the torsion vanishes and the image is again a closed parallelogram. Einstein's original theory of gravitation assumed the absence of torsion, $S_{\mu\nu}{}^\lambda = 0$.

2.7. CIRCUIT INTEGRALS IN METRIC-AFFINE SPACES WITH CURVATURE AND TORSION

In order to establish contact with the circuit definitions of disclinations and dislocations in crystals, let us rephrase the differential results (2.106) and (2.109) in terms of contour integrals. Given a vector field $v^\mu(x)$ which is locally *parallel*, i.e., which has $D_\nu v^\mu(x) = 0$. Consider the change sustained by $v^\mu(x)$ as it is transported around a closed contour:

$$\Delta v^\mu = \oint_{C(x^\mu)} dx^\nu \partial_\nu v^\mu(x). \tag{2.110}$$

By decomposing C into a large set of infinitesimal surface elements we can apply (2.106) and find

$$\Delta v^\mu = \oint_{C(x^\mu)} dx^\nu \partial_\nu v^\mu = -\frac{1}{2}\int_{S(x^\mu)} ds^{\tau\nu} R_{\tau\nu\varkappa}{}^\mu(x) v^\varkappa(x). \tag{2.111}$$

Notice that the tetrad fields $e_a{}^\mu$ are locally parallel by definition, so that they satisfy

$$\Delta e_a{}^\mu = -\oint_{C(x^\mu)} dx^\nu \partial_\nu e_a{}^\mu = -\frac{1}{2}\int_{S(x^\mu)} ds^{\tau\nu} R_{\tau\nu\varkappa}{}^\mu(x) e_a{}^\varkappa(x). \tag{2.112}$$

Actually, this relation follows directly from Stokes' theorem:

$$\Delta e_a{}^\mu = \oint_{C(x^\mu)} dx^\nu \partial_\nu e_a{}^\mu = \int_{S(x^\mu)} ds^{\tau\nu} \partial_\tau \partial_\nu e_a{}^\mu = -\frac{1}{2}\int_{S(x^\mu)} ds^{\tau\nu} R_{\tau\nu\varkappa}{}^\mu e_a{}^\varkappa. \tag{2.113}$$

For an infinitesimal circuit, we can remove the tetrad from the integral and find

$$\Delta e_a{}^\mu \approx \left\{-\frac{1}{2}\int_{S(x^\mu)} ds^{\tau\nu} R_{\tau\nu\varkappa}{}^\mu\right\} e_a{}^\varkappa \equiv \omega^\mu{}_\varkappa e^\varkappa{}_a. \tag{2.114}$$

The matrix $\omega^\mu{}_\varkappa$ has the property that $\omega_{\mu\varkappa} = g_{\mu\lambda}\omega^\lambda{}_\varkappa$ is antisymmetric, due to the antisymmetry of $R_{\tau\nu\varkappa\mu}$ in $\varkappa\mu$. Hence $\omega^\mu{}_\varkappa$ can be interpreted as the parameters of an infinitesimal local Lorentz transformation. In three dimensions, it reduces to a local rotation, in agreement with what we observed previously: curvature is a signal for disclinations and these are rotational defects.

Let us now give an integral characterization of torsion. For this we consider an arbitrary closed contour $C(x^a)$ in the inertial frame (which generalizes the parallelogram used in the previous discussion). In the defected space this contour has an image $C'(x^a)$ which does not necessarily close. In order to find how much is missing, we form the integral

$$\oint_{C(x^a)} dx^\mu = \oint_{C(x^a)} dx^a \frac{\partial x^\mu}{\partial x^a} = \oint_{C(x^a)} dx^a e_a{}^\mu(x^a).$$

By Stokes' theorem, this becomes

$$\frac{1}{2}\int_{S(x^a)} ds^{ab}(\partial_a e_b{}^\mu - \partial_b e_a{}^\mu) = \frac{1}{2}\int_{S(x^a)} ds^{ab}(e_a{}^\nu \partial_\nu e_b{}^\mu - (a \leftrightarrow b)) = -\int_{S(x^a)} ds^{ab} S_{ab}{}^\mu. \tag{2.115}$$

The quantity

$$S_{ab}{}^\mu \equiv -\tfrac{1}{2} e_a{}^\nu(\partial_\nu e_b{}^\mu - (a \leftrightarrow b)) \tag{2.116}$$

is called the *anholonomity* of the mapping $x^a \to x^\mu$. It is related to the torsion $S_{\lambda\varkappa}{}^\mu$ by conversion of the lower indices from the local to the inertial form,

$$\begin{aligned} S_{ab}{}^\mu = e_a{}^\lambda e_b{}^\varkappa S_{\lambda\varkappa}{}^\mu &= -\tfrac{1}{2}(e_a{}^\lambda e_b{}^\varkappa[e^c{}_\varkappa \partial_\lambda e_c{}^\mu - (a \leftrightarrow b)]) \\ &\equiv -\tfrac{1}{2}[e_a{}^\lambda \partial_\lambda e_b{}^\mu - (a \leftrightarrow b)]. \end{aligned} \tag{2.117}$$

If the tetrad vectors are known as functions of the external coordinates x^a, we may also use $e_a{}^\lambda \partial_\lambda = \partial/\partial x^a = \partial_a$ and write the anholonomity in the form

$$S_{ab}{}^\mu \equiv -\tfrac{1}{2}(\partial_a e_b{}^\mu - (a \leftrightarrow b)). \tag{2.118}$$

Sometimes one also converts the upper Einstein index μ into a Lorentz index c and works with

$$S_{ab}{}^c = e^c{}_\mu S_{ab}{}^\mu = -\tfrac{1}{2}(e^c{}_\mu \partial_a e_b{}^\mu - (a \leftrightarrow b)). \tag{2.119}$$

If there is no torsion, the integral (2.115) vanishes. Otherwise the image of the closed contour $C(x^a)$ has a gap and thus defines the Burgers vector

$$b^\mu = \oint_{C'(x^\mu)} dx^\mu = -\oint_{C(x^a)} ds^{ab} S_{ab}{}^\mu. \tag{2.120}$$

Note that oftentimes the circuit integrals measuring curvature and torsion are executed in the opposite way by forming closed circuits $C(x^\mu)$ around the defect in the space x^μ and studying the properties of the image circuit $C'(x^a)$ in the ideal reference crystal (this is the so-called Cartan transport). In the case of torsion, one measures how much the image $C'(x^a)$ fails to close. This gives the Burgers vector

$$b^a = \oint_{C'(x^a)} dx^a = \oint_{C(x^\mu)} dx^\mu \frac{\partial x^a}{\partial x^\mu} = \oint_{C(x^\mu)} dx^\mu e^a{}_\mu , \tag{2.121}$$

which, by Stokes' theorem, can be rewritten as

$$b^a = \int_{S(x^\mu)} dx^\mu \partial_\nu e^a{}_\mu = \int_{S(x^\mu)} ds^{\nu\mu} S_{\nu\mu}{}^\lambda e^a{}_\lambda(x). \tag{2.122}$$

The tensor $S_{\nu\mu}{}^a = S_{\nu\mu}{}^\lambda e^a{}_\lambda = (1/2)(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu)$ is obviously a converse form of the anholonomity (2.116), with Einstein indices exchanged for Lorentz indices.

There is an analogous circuit integral characterizing the curvature from the standpoint of the x^a coordinates. For this, one introduces a partial Lorentz tensor related to $R_{\mu\nu\rho}{}^\kappa$:

$$R_{ab\lambda}{}^\kappa \equiv e_c{}^\kappa(\partial_a \partial_b - \partial_b \partial_a) e^c{}_\lambda . \tag{2.123}$$

Then the circuit integral reads

$$\Delta e^a{}_\mu = -\frac{1}{2}\int_{S(x^\mu)} ds^{cd} R_{cd\mu}{}^\lambda e^a{}_\lambda . \tag{2.124}$$

If one wishes to calculate $\overline{R}_{ab\lambda}{}^\kappa$ from the usual curvature tensor $R_{\mu\nu\lambda}{}^\kappa$ one must note that under the anholonomic mapping $x^a \to x^\mu$, R is not a tensor. In fact, a simple manipulation shows that:

$$\begin{aligned}R_{\mu\nu\lambda}{}^{\varkappa} &= e_d{}^{\varkappa}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)e^d{}_\lambda = e_d{}^{\varkappa}(e^a{}_\mu\partial_a e^b{}_\nu\partial_b - (\mu\leftrightarrow\nu))e^d{}_\lambda \\ &= e^a{}_\mu e^b{}_\nu e_d{}^{\varkappa}(\partial_a\partial_b - \partial_b\partial_a)e^d{}_\lambda + [e^a{}_\mu e_d{}^{\varkappa}(\partial_a e^b{}_\nu)(\partial_b e^d{}_\lambda) - (\mu\leftrightarrow\nu)] \\ &= e^a{}_\mu e^b{}_\nu R_{ab\lambda}{}^{\varkappa} + e^a{}_\mu e_d{}^{\varkappa} e_a{}^\sigma\Gamma_{\sigma\nu}{}^b e_b{}^\tau\Gamma_{\tau\lambda}{}^d - (\mu\leftrightarrow\nu) \\ &= e^a{}_\mu e^b{}_\nu R_{ab\lambda}{}^{\varkappa} + \Gamma_{\mu\nu}{}^\sigma\Gamma_{\sigma\lambda}{}^{\varkappa} - (\mu\leftrightarrow\nu) \\ &= e^a{}_\mu e^b{}_\nu R_{ab\lambda}{}^{\varkappa} + 2S_{\mu\nu}{}^\sigma\Gamma_{\sigma\lambda}{}^{\varkappa}. \end{aligned} \tag{2.125}$$

Let us define

$$R_{abc}{}^d = e_c{}^\lambda e^d{}_\kappa R_{ab\lambda}{}^\kappa = e_c{}^\lambda(\partial_a\partial_b - \partial_b\partial_a)e^d{}_\lambda\,. \tag{2.126}$$

The torsion $S_{ab}{}^c$ was expressed in terms of derivatives $\partial/\partial x^a = \partial_a$ in (2.119) as follows

$$\begin{aligned}S_{ab}{}^c &= e_a{}^\mu e_b{}^\nu e^c{}_\lambda S_{\mu\nu}{}^\lambda = -\tfrac{1}{2}e^c{}_\mu\partial_a e_b{}^\mu \\ &= \tfrac{1}{2}e_b{}^\mu\partial_a e^c{}_\mu\,. \end{aligned} \tag{2.127}$$

For the connection we define, similarly,

$$\begin{aligned}\Gamma_{ab}{}^c &= e_a{}^\mu e_b{}^\nu e^c{}_\lambda\Gamma_{\mu\nu}{}^\lambda = -e_a{}^\mu e_b{}^\nu e^c{}_\lambda e^d{}_\nu\partial_\mu e_d{}^\lambda \\ &= -e_a{}^\mu e^c{}_\lambda\partial_\mu e_b{}^\lambda = -e^c{}_\lambda\partial_a e_b{}^\lambda \equiv e^c{}_\lambda\Gamma_{ab}{}^\lambda \\ &= e_b{}^\lambda\partial_a e^c{}_\lambda\,. \end{aligned} \tag{2.128}$$

Explicitly,

$$\Gamma_{ab}{}^\lambda \equiv e_c{}^\lambda\Gamma_{ab}{}^c \equiv e_a{}^\mu e_b{}^\nu\Gamma_{\mu\nu}{}^\lambda.$$

Then the $R_{abc}{}^d$ of (2.126) can be written as

$$R_{abc}{}^d = \partial_a\Gamma_{bc}{}^d - \partial_b\Gamma_{ac}{}^d + [\Gamma_a, \Gamma_b]_c{}^d. \tag{2.129}$$

It should be pointed out that as a consequence of nonzero curvature the covariant derivatives no longer commute. If we form

$$D_\nu D_\mu v_\lambda - D_\mu D_\nu v_\lambda\,, \tag{2.130}$$

we find after some algebra

$$\begin{aligned}D_\nu D_\mu v_\lambda - D_\mu D_\nu v_\lambda &= -R_{\nu\mu\lambda}{}^{\varkappa}v_{\varkappa} - 2S_{\nu\mu}{}^\rho D_\rho v_\lambda, \\ D_\nu D_\mu v^{\varkappa} - D_\mu D_\nu v^{\varkappa} &= R_{\nu\mu\lambda}{}^{\varkappa}v^\lambda - 2S_{\nu\mu}{}^\rho D_\rho v^{\varkappa}. \end{aligned} \tag{2.131}$$

2.8. SOME EXAMPLES OF COORDINATE SYSTEMS WITH DEFECTS

It may be useful to give a few explicit examples of defected mappings $x^\mu(x^a)$. We shall do so by appealing to actual physical situations. For simplicity, we consider two dimensions. Imagine an ideal crystal with atoms placed at $x^a = (n^1, n^2, n^3) \cdot b$ with infinitesimal lattice constant b. The simplest defect was shown in Fig. 2.4, namely, the edge dislocation. The mapping transforms the lattice points to new distorted positions of which $x^\mu(x^a)$ are the cartesian coordinates. There exists no one-to-one mapping between the two figures since the excessive atoms in the middle horizontal layer $x^1 < 0$, $x^2 = 0$ have no correspondence in x^a space. In the continuum limit of an infinitesimally small Burgers vector, the mapping can be described by *multivalued* function

$$x^{\bar{1}} = x^1, \qquad x^{\bar{2}} = x^2 - \frac{b}{2\pi} \tan^{-1} \frac{x^2}{x^1},$$

where the function $\tan^{-1}$ may be defined to be equal to $\pm\pi$ for $x^1 < 0$, $x^2 = \pm\varepsilon$. We have used the notation $x^{\bar{a}} \equiv x^a$ in order to distinguish $x^{a=1,2}$ from $x^{\mu=1,2}$. In differential forms we have

$$dx^{\bar{1}} = dx^1, \qquad dx^{\bar{2}} = dx^2 + \frac{b}{2\pi} \frac{1}{(x^1)^2 + (x^2)^2} (x^2 dx^1 - x^1 dx^2), \tag{2.132}$$

with the basis components[g] $e^a{}_\mu = \partial x^a / \partial x^\mu$

$$e^a{}_\mu = \begin{pmatrix} 1 & 0 \\ \dfrac{b}{2\pi} \dfrac{x^2}{(x^1)^2 + (x^2)^2} & 1 - \dfrac{b}{2\pi} \dfrac{x^1}{(x^1)^2 + (x^2)^2} \end{pmatrix}. \tag{2.133}$$

The mapping (2.133) is not in stress equilibrium. It represents only a plastic deformation in a specific defect gauge. To reach equilibrium an elastic deformation has to be superimposed[i] so as to minimize the elastic energy. Incidentally the mapping $x^{\bar{1}} = x^1$, $x^{\bar{2}} = x^2$, $x^{\bar{3}} = x^3 + \tan^{-1}(x^2/x^1)$ gives the plastic distortion of a screw dislocation which happens to be in stress equilibrium.

Let us now integrate dx^μ over a Burgers circuit which consists of a closed circuit $C(x^\mu)$ in x^μ space around the origin,

[i] Notice that $-(e^2{}_\mu - \delta^2{}_\mu)$ is the analogue of the superflow around a vortex line of strength b [see Part II, Eq. (1.58)].

$$b^a = \int_{C(x^\mu)} dx^a = \int_{C(x^\mu)} dx^\mu \frac{\partial x^a}{\partial x^\mu} = \int_{C(x^\mu)} dx^\mu e^a{}_\mu .$$

Inserting (2.132) we see that

$$b^{\bar{1}} = \oint_{C(x^\mu)} dx^{\bar{1}} = 0, \qquad b^{\bar{2}} = \oint_{C(x^\mu)} dx^{\bar{2}} = -b. \qquad (2.134)$$

It is easy to calculate the torsion tensor $S_{\mu\nu}{}^\lambda$ of (2.132). Because of its antisymmetry, only $S_{12}{}^1$ and $S_{12}{}^2$ are independent:

$$S_{12}{}^{\bar{2}} = \partial_1 e^{\bar{2}}{}_2 - \partial_2 e^{\bar{2}}{}_1 = \partial_1 \frac{\partial x^{\bar{2}}}{\partial x^2} - \partial_2 \frac{\partial x^{\bar{2}}}{\partial x^1}$$

$$S_{12}{}^{\bar{1}} = \partial_1 e^{\bar{1}}{}_2 - \partial_2 e^{\bar{1}}{}_1 = \partial_1 \frac{\partial x^{\bar{1}}}{\partial x^2} - \partial_2 \frac{\partial x^{\bar{1}}}{\partial x^1}. \qquad (2.135)$$

Smearing out the singularity in the denominator of (2.132) by adding a small ε^2, as in Part II, Eq. (1.64), we find

$$S_{12}{}^{\bar{2}} = -b\delta^{(2)}(x^\mu), \qquad S_{12}{}^{\bar{1}} = 0. \qquad (2.136)$$

We may write this result, with the Burgers vector $b^a = (0, -b)$, in the form

$$S_{\mu\nu}{}^a = \varepsilon_{\mu\nu} b^a \, \delta^{(2)}(x^\lambda). \qquad (2.137)$$

Let us now calculate the curvature tensor for this defect,

$$R_{\mu\nu\lambda\kappa} = e_{a\kappa}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\lambda . \qquad (2.138)$$

Since $e^a{}_\mu$ in (2.133) is single-valued, derivatives in front of it commute. Hence, $R_{\mu\nu\lambda\kappa}$ vanishes identically,

$$R_{\mu\nu\lambda\kappa} \equiv 0. \qquad (2.139)$$

Thus a pure dislocation gives rise to torsion but not to curvature.

In contrast to this, consider now a wedge disclination (see Fig. 2.2). A cut has been made along the left half $x^{\bar{2}}$ axis, the lips separated by an angle α, and new material has been filled in, fitting into the crystalline structure and balancing the forces. The lattice unit vectors $\mathbf{e}_\mu = \mathbf{e}_a e^a{}_\mu$ in x^μ space are obviously rotated with respect to $\mathbf{e}_a$ in x^a space even at a large distance away from the origin. At the wedge, i.e., for $x^{\bar{2}} < 0$, $x^{\bar{1}} \approx 0$, they are rotated by $\pm\alpha/2$ for $x^{\bar{1}} = \pm\varepsilon$. For $x^{\bar{2}} > 0$, they remain unchanged. We can represent this operation mathematically by using $\varphi \equiv \tan^{-1}(x^2/x^1)$ defined to have the jump from 180° to −180° for $x^2 < 0$, $x^1 = \pm 0$ and taking

$$e^a{}_\lambda = \begin{pmatrix} \cos(\varphi/n) & -\sin(\varphi/n) \\ \sin(\varphi/n) & \cos(\varphi/n) \end{pmatrix}^a{}_\lambda = e_{a\lambda},$$

$$e_a{}^\lambda = \begin{pmatrix} \cos(\varphi/n) & -\sin(\varphi/n) \\ \sin(\varphi/n) & \cos(\varphi/n) \end{pmatrix}_a{}^\lambda = e^{a\lambda} \tag{2.140}$$

as the desired mapping, with $n \equiv 2\pi/\alpha$. In order for real crystalline material to fit in between the lips, α must be a multiple of $\pi/4$. The cases $\alpha = -\pi/2$, $-\pi$ are displayed in Fig. 2.2. At the level of differential geometry, however, α is assumed to be infinitesimal, i.e., $n \to \infty$. The need for this unphysical limit represents a basic weakness of the differential geometric approach to defects.

It can now be easily verified that the mapping (2.140) gives rise to curvature. Differentiating once we find

$$\partial_\nu e_{a\lambda} = \frac{1}{n}\begin{pmatrix} -\sin(\varphi/n) & -\cos(\varphi/n) \\ \cos(\varphi/n) & -\sin(\varphi/n) \end{pmatrix}_{a\lambda} \partial_\nu \varphi(x) \tag{2.141}$$

and

$$R_{\mu\nu\lambda}{}^\varkappa = e^{a\varkappa}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e_{a\lambda} = \frac{1}{n}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^\varkappa{}_\lambda (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\varphi(x). \tag{2.142}$$

Indeed, $\partial_\mu \partial_\nu \varphi - (\mu\nu)$ is nonzero, since

$$(\partial_1, \partial_2)\varphi = \left(-\frac{x^2}{(x^1)^2 + (x^2)^2 + \varepsilon^2}, \frac{x^1}{(x^1)^2 + (x^2)^2 + \varepsilon^2}\right), \tag{2.143}$$

and

$$\begin{aligned} \partial_1 \partial_2 \varphi - \partial_2 \partial_1 \varphi &= 2\frac{1}{(x^1)^2 + (x^2)^2 + \varepsilon^2} - \frac{2[(x^1)^2 + (x^2)^2]}{[(x^1)^2 + (x^2)^2 + \varepsilon^2]^2} \\ &= \frac{2\varepsilon^2}{[(x^1)^2 + (x^2)^2 + \varepsilon^2]^2} = 2\pi\delta^{(2)}(x^\lambda). \end{aligned} \tag{2.144}$$

Only one independent component of the curvature tensor emerges,

$$R_{12\,12} = -\frac{1}{n} 2\pi\delta^{(2)}(x^\lambda) = -\alpha\delta^{(2)}(x^\lambda).$$

As before, the small parameter ε smears out the singularity and helps to display the δ function at the origin. Of course, the same result could have been deduced without an ε by applying Stokes' theorem:

$$\oint_{C(x^\mu)} dx^\lambda \partial_\lambda \varphi = \int_{S(x^\mu)} ds^{\lambda\kappa} \partial_\lambda \partial_\kappa \varphi = \Delta\varphi = -\frac{2\pi}{n} = -\alpha. \qquad (2.145)$$

2.9. IDENTITIES FOR CURVATURE AND TORSION TENSORS

Because of their physical importance, it is useful to derive a few important properties of curvature and torsion tensors. As noted before, the curvature tensor is antisymmetric in $\mu\nu$, by construction, and in $\lambda\kappa$, due to the integrability condition (2.7) of the metric tensor. In addition, it satisfies a *fundamental identity*[j] which follows directly from the representation (2.59), by adding terms in which $\mu\nu\lambda$ are interchanged cyclically:

$$R_{\underbrace{\nu\mu\lambda}}{}^{\kappa} = 2D_{\underbrace{\nu}}S_{\underbrace{\mu\lambda}}{}^{\kappa} - 4S_{\underbrace{\nu\mu}}{}^{\rho}S_{\underbrace{\lambda}\rho}{}^{\kappa}, \qquad (2.146)$$

where the symbol $\sqcup\!\sqcup$ means the sum of cyclic permutations. In symmetric spaces (i.e., spaces with no torsion), this implies an additional symmetry property,

$$R_{\mu\nu\lambda\kappa} + R_{\nu\lambda\mu\kappa} + R_{\lambda\mu\nu\kappa} = 0. \qquad (2.147)$$

Using the antisymmetry in $\mu\nu$ and $\lambda\kappa$ we are led once more to the property (2.70),

$$R_{\mu\nu\lambda\kappa} = R_{\lambda\kappa\mu\nu}, \qquad (2.148)$$

as it should, since in symmetric spaces $\overset{\{\}}{R}_{\mu\nu\lambda\kappa} = R_{\mu\nu\lambda\kappa}$.

Another important identity is the Bianchi identity.[j] It follows from the global existence of the affine connection which implies, as we had postulated previously in (2.76), that the connection also satisfies the integrability condition

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Gamma_{\lambda\kappa}{}^{\rho} = 0. \qquad (2.149)$$

[j]Schouten has called the antisymmetry in $\mu\nu$ the *first* identity and (2.146) the *second*. The Bianchi identity was first found by Voss in 1880, by Ricci in 1889, and by Bianchi in 1902.

To derive it, consider the mixed object

$$\mathbf{R}_{\sigma\nu\mu} \equiv (\partial_\sigma \partial_\nu - \partial_\nu \partial_\sigma)\, \mathbf{e}_\mu \,, \tag{2.150}$$

which determines the curvature tensor $R_{\sigma\nu\mu}{}^{\lambda}$ via its scalar product with $\mathbf{e}^{\lambda}$. Applying the covariant derivative we have

$$D_\tau \mathbf{R}_{\sigma\nu\mu} = \partial_\tau \mathbf{R}_{\sigma\nu\mu} - \Gamma_{\tau\sigma}{}^{\kappa}\mathbf{R}_{\kappa\nu\mu} - \Gamma_{\tau\nu}{}^{\kappa}\mathbf{R}_{\sigma\kappa\mu} - \Gamma_{\tau\mu}{}^{\kappa}\mathbf{R}_{\sigma\nu\kappa} \tag{2.151}$$

or

$$D_\tau \mathbf{R}_{\sigma\nu\mu} = \partial_\tau \mathbf{R}_{\sigma\nu\mu} - \Gamma_{\tau\mu}{}^{\kappa}\mathbf{R}_{\sigma\nu\kappa} + 2S_{\tau\sigma}{}^{\kappa}\mathbf{R}_{\nu\kappa\mu} \,. \tag{2.152}$$

Now we use

$$\partial_\sigma \partial_\nu \mathbf{e}_\mu = \partial_\sigma (\Gamma_{\nu\mu}{}^{\kappa}\mathbf{e}_\kappa) = \Gamma_{\nu\mu}{}^{\kappa}\partial_\sigma \mathbf{e}_\kappa + \partial_\sigma \Gamma_{\nu\mu}{}^{\kappa}\mathbf{e}_\kappa \tag{2.153}$$

to derive

$$\partial_\tau \partial_\sigma \partial_\nu e_\mu = \partial_\tau \Gamma_{\nu\mu}{}^{\kappa}\partial_\sigma \mathbf{e}_\kappa + (\tau\sigma) + \partial_\tau \partial_\sigma \Gamma_{\nu\mu}{}^{\kappa}\mathbf{e}_\kappa + \Gamma_{\nu\mu}{}^{\kappa}\partial_\tau \partial_\sigma \mathbf{e}_\kappa . \tag{2.154}$$

Antisymmetrizing in $\sigma\tau$ gives

$$\partial_\tau \partial_\sigma \partial_\nu \mathbf{e}_\mu - \partial_\sigma \partial_\tau \partial_\nu \mathbf{e}_\mu = \Gamma_{\nu\mu}{}^{\kappa}\mathbf{R}_{\tau\sigma\kappa} + [(\partial_\tau \partial_\sigma - \partial_\sigma \partial_\tau)\Gamma_{\nu\mu}{}^{\kappa}]\,\mathbf{e}_\kappa \,. \tag{2.155}$$

Invoking the integrability condition for the connection, (2.149), and performing cyclic sums over $\tau\sigma\nu$ while using the antisymmetry of $\mathbf{R}_{\sigma\nu\mu}$ in $\sigma\nu$ leads to

$$\partial_\tau \mathbf{R}_{\sigma\nu\mu} - \Gamma_{\nu\mu}{}^{\kappa}\mathbf{R}_{\tau\sigma\kappa} = 0. \tag{2.156}$$

Inserting this into (2.152) and multiplying by $\mathbf{e}^{\kappa}$ we obtain an expression involving the covariant derivative of the curvature tensor,

$$D_\tau R_{\sigma\nu\mu}{}^{\kappa} - 2S_{\tau\sigma}{}^{\lambda}R_{\nu\lambda\mu}{}^{\kappa} = 0. \tag{2.157}$$

This is the *Bianchi identity*, which guarantees the integrability of the connection.

Within the defect interpretation of torsion and curvature, we are now prepared to demonstrate that these two identities have a simple physical interpretation. They are just the nonlinear versions of the conservation laws for dislocation and disclination densities. Recall that these were given in Part III, Eqs. (2.45), (2.46) as follows,

$$\partial_i \alpha_{ij} = -\varepsilon_{jk\ell} \Theta_{k\ell} , \tag{2.158}$$

$$\partial_i \Theta_{ij} = 0. \tag{2.159}$$

They state that disclination lines never end while dislocation lines can end at most at a disclination line.

Consider first Eq. (2.157). Linearizing, we deduce

$$\partial_\tau R_{\sigma\nu\mu}{}^\lambda + \partial_\sigma R_{\nu\tau\mu}{}^\lambda + \partial_\nu R_{\tau\mu\sigma}{}^\lambda = 0. \tag{2.160}$$

Contracting ν with μ and τ with λ, we obtain,

$$\partial_\tau R_{\sigma\nu}{}^{\nu\tau} + \partial_\sigma R_{\nu\lambda}{}^{\nu\lambda} + \partial_\nu R_\tau{}^\nu{}_\sigma{}^\tau = 0 = 2\partial_\tau R_\sigma{}^\tau + \partial_\sigma R = 2\partial_\tau G_\sigma{}^\tau = 0. \tag{2.161}$$

Since in three dimensions, the Einstein tensor $G_{\mu\nu}$ is the same as the disclination density $\Theta_{\mu\nu}$ [see (2.93)] we see that (2.161) indeed coincides with the conservation law (2.159).

Equation (2.146) on the other hand has the linearized form

$$2(\partial_\nu S_{\mu\lambda}{}^\varkappa + \partial_\mu S_{\lambda\nu}{}^\varkappa + \partial_\lambda S_{\nu\mu}{}^\varkappa) = (R_{\nu\mu\lambda}{}^\varkappa + R_{\mu\lambda\nu}{}^\varkappa + R_{\lambda\nu\mu}{}^\varkappa). \tag{2.162}$$

Contracting ν and $\varkappa$ we find

$$2(\partial_\nu S_{\mu\lambda}{}^\nu + \partial_\mu S_{\lambda\nu}{}^\nu - \partial_\lambda S_{\mu\nu}{}^\nu) = R_{\nu\mu\lambda}{}^\nu + R_{\mu\lambda\varkappa}{}^\varkappa + R_{\lambda\nu\mu}{}^\nu = R_{\mu\lambda} - \mathrm{R}_{\lambda\mu} \tag{2.163}$$

where we have used the antisymmetry of $R_{\nu\mu\lambda\varkappa}$ in the last two indices (which was a consequence of the integrability condition for the metric tensor). The right-hand side is the same as $G_{\mu\lambda} - G_{\lambda\mu}$.

In three dimensions we may contract this equation with the ε tensor and obtain

$$\varepsilon_{jk\ell}(\partial_i S_{k\ell i} + \partial_k S_{\ell nn} - \partial_\ell S_{knn}) = \varepsilon_{jk\ell} G_{k\ell} . \tag{2.164}$$

Recalling the identification [see (2.92a), (2.93)]

$$G_{\ell k} = \Theta_{k\ell} , \qquad S_{k\ell j} = \tfrac{1}{2} \varepsilon_{k\ell i} \alpha_{ij} ,$$

we find

$$\partial_i \alpha_{ij} = -\varepsilon_{jk\ell} \Theta_{k\ell}, \tag{2.165}$$

which is just (2.158).

This is no miracle, since in Eqs. (III.2.45)–(III.2.46) we derived the conservation laws from the integrability conditions (III.2.47), (III.2.50) which are the same as Eqs. (2.76) in the present discussion. In general curved space, the corresponding integrability conditions are those of the metric and the connection. In the context of Einstein's theory of the gravitational field, to be discussed below, Eqs. (2.146), (2.157) appear once more with the physical meaning of being the conservation laws for the angular momentum and energy-momentum density, respectively.

2.10. CURVATURE FROM EMBEDDING

Instead of mappings from the space x^a to x^μ with rotational defects, there is another way of obtaining curvature. This is by embedding the space x^μ into a higher dimensional "Minkowski" space x^A, $A = 1, \ldots, N$ with metric η_{AB} consisting only of diagonal elements ± 1. The mapping $x^A(x^\mu)$ is smooth but cannot be inverted to $x^\mu(x^A)$. Thus there are N basis vectors $\mathbf{e}_A$ in the embedding space and

$$\mathbf{e}_\lambda(x^\mu) = \mathbf{e}_A e^A{}_\lambda(x^\mu) = \mathbf{e}_A \frac{\partial x^A}{\partial x^\lambda} \tag{2.166}$$

form four local tangent vectors in the 4-dimensional submanifold $x^A(x^\mu)$. They induce a metric

$$g_{\lambda\varkappa}(x^\mu) = \mathbf{e}_\lambda(x^\mu)\, \mathbf{e}_\varkappa(x^\mu), \tag{2.167}$$

which we employ to define the contravariant components of $e^A{}_\lambda$,

$$e^{A\lambda}(x^\mu) = g^{\lambda\lambda'}(x^\mu) e^A{}_{\lambda'}(x^\mu). \tag{2.168}$$

But these are no longer reciprocal to $e^A{}_\lambda(x^\mu)$, i.e.,

$$e^{A\lambda} e_{B\lambda} \neq \delta^A{}_B, \tag{2.169}$$

since there are not enough of them to span the N-dimensional space. They do fulfill, however, the completeness relation in the four-dimensional subspace:

$$e^A{}_\lambda e_A{}^\varkappa = \delta_\lambda{}^\varkappa. \tag{2.170}$$

Let us take an example: the surface of a sphere of radius a in three dimensions with the mapping

$$x^A = (x^1, x^2, x^3) = a(\sin\Theta\cos\varphi, \sin\Theta\sin\varphi, \cos\Theta) \tag{2.171}$$

has the tangent vectors

$$\begin{aligned} e^A{}_1(x^\mu) &= a(\cos\Theta\cos\varphi, \cos\Theta\sin\varphi, -\sin\Theta) = e_{A1}, \\ e^A{}_2(x^\mu) &= a(-\sin\Theta\sin\varphi, \sin\Theta\cos\varphi, 0) = e_{A2}, \end{aligned} \tag{2.172}$$

where we have set $x^{\mu=1} = \Theta$, $x^{\mu=2} = \varphi$. The metric is

$$g_{\mu\nu} = a^2\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\Theta \end{pmatrix}, \qquad g^{\mu\nu} = \frac{1}{a^2}\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\Theta} \end{pmatrix}, \tag{2.173}$$

so that

$$a^2 e_A{}^1 = e_{A1} = e^A{}_1, \qquad e_A{}^2 = \frac{1}{a}\left(-\frac{\sin\varphi}{\sin\Theta}, \frac{\cos\varphi}{\sin\Theta}, 0\right). \tag{2.174}$$

It follows that

$$\begin{aligned} \Gamma_{221} &= e_{A1}\partial_2 e^A{}_2 = e_{A1}a(-\sin\Theta\cos\varphi, -\sin\Theta\sin\varphi, 0) \\ &= -a^2\sin\Theta\cos\Theta = -\Gamma_{212} = -\Gamma_{122}. \end{aligned} \tag{2.175}$$

All other components vanish. The space is obviously symmetric; it carries no torsion. For raised components we find

$$\Gamma_{22}{}^1 = -\sin\Theta\cos\Theta, \qquad \Gamma_{21}{}^2 = \cot\Theta. \tag{2.176}$$

The curvature tensor is simply

$$\begin{aligned} R_{122}{}^1 &= \partial_1\Gamma_{22}{}^1 - \partial_2\Gamma_{12}{}^1 - \Gamma_{12}{}^1\Gamma_{21}{}^1 - \Gamma_{12}{}^2\Gamma_{22}{}^1 + \Gamma_{22}{}^1\Gamma_{11}{}^1 + \Gamma_{22}{}^2\Gamma_{12}{}^1 \\ &= -\cos^2\Theta + \sin^2\Theta + \cot\Theta\sin\Theta\cos\Theta = \sin^2\Theta, \end{aligned} \tag{2.177}$$

and

$$R_{12}{}^{21} = \frac{1}{a^2}. \tag{2.178}$$

All other components can be obtained by using the antisymmetry of $R_{\mu\nu\lambda\varkappa}$ in $\mu \to \nu$, $\lambda \to \varkappa$, and the symmetry under $\mu\nu \leftrightarrow \lambda\varkappa$, Eq. (2.64). Thus we can form the Ricci tensor

$$R_{\mu\nu}{}^{\lambda\mu} = R_{\nu}{}^{\lambda} = \frac{1}{a^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.179}$$

and the curvature scalar

$$R = R_{\nu}{}^{\nu} = \frac{2}{a^2}. \tag{2.180}$$

Observe that for noninvertible vectors $e^{A}{}_{\lambda}$, the curvature has to be calculated from (2.59). Formula (2.61) can no longer be used since in the derivation of this formula one must have

$$\partial_\mu e^a{}_\nu = \Gamma_{\mu\nu}{}^{\lambda} e^a{}_\lambda, \tag{2.181}$$

which no longer follows from the correct relation

$$\Gamma_{\mu\nu}{}^{\lambda} = e_A{}^{\lambda} \partial_\mu e^A{}_\nu, \tag{2.182}$$

due to the noninvertibility $e^A{}_\lambda e_B{}^\lambda \neq \delta^A{}_B$.

2.11. GEODESIC COORDINATES IN CURVED SPACE

To a local observer, curved space looks flat in his immediate neighborhood. After all, this was why man believed for a long time that the earth was a flat disc. In four-dimensional space-time, the equivalent statement is that, in a freely-falling elevator cabin, an observer would not experience any gravitational force, as long as the cabin is small enough to make higher nonlinear effects negligible. The cabin constitutes an inertial frame of reference for the motion of a mass point. Its coordinates in a curved geometry can be determined from the requirement that the Christoffel symbol, $\{\mu'\lambda', \varkappa'\}$, vanishes [recall (2.9)], which, in turn, amounts to

$$\partial_{\lambda'} g_{\mu'\lambda'}(x') = 0, \qquad \partial_{\lambda'} g^{\mu'\lambda'}(x') = -g^{\mu'\sigma'} g^{\lambda'\tau'} \partial_{\lambda'} g_{\sigma'\tau'}(x') = 0. \tag{2.183}$$

Given an arbitrary set of coordinates x, the derivatives of the metric are related by

$$\partial_{\lambda'} g^{\mu'\nu'}(x') = \alpha_{\lambda'}{}^{\lambda}\partial_\lambda (\alpha^{\mu'}{}_{\mu}\alpha^{\nu'}{}_{\nu} g^{\mu\nu}(x))$$
$$= \alpha_{\lambda'}{}^{\lambda}(\partial_\lambda \alpha^{\mu'}{}_{\mu}\alpha^{\nu'}{}_{\nu} g^{\mu\nu}(x) + \alpha^{\mu'}{}_{\mu}\partial_\lambda \alpha^{\nu'}{}_{\nu} g^{\mu\nu}(x) + \alpha^{\mu'}{}_{\mu}\alpha^{\nu'}{}_{\nu}\partial_\lambda g^{\mu\nu}). \tag{2.184}$$

These are $n^2(n+1)/2$ partial differential equations for the n coordinates $x'^{\mu'}(x)$ which cannot, in general, have a solution over a finite region. This fact is also obvious from (2.71) since, if $\partial_{\lambda'} g_{\mu'\lambda'}$ were to vanish over a finite region, the space would necessarily have $\overset{\{\}}{R}_{\mu\nu\lambda}{}^{\varkappa} = 0$. So we can, at best, achieve

$$\partial_{\lambda'} g_{\mu'\nu'}(x_0') = 0 \tag{2.185}$$

at some point $x_0' = x_0$.

In this situation, a mass point would move force free at x_0. Any deviation from x_0 would lead to gravitational forces which are small, i.e., of order $O(x - x_0)$. Let us try and solve (2.185) by an expansion

$$x'^{\mu'} = x_0{}^{\mu} + a^{\mu}{}_{\lambda}(x - x_0)^{\lambda} + \frac{1}{2!}a^{\mu}{}_{\lambda\varkappa}(x - x_0)^{\lambda}(x - x_0)^{\varkappa}$$
$$+ \frac{1}{3!}a^{\mu}{}_{\lambda\varkappa\delta}(x - x_0)^{\lambda}(x - x_0)^{\varkappa}(x - x_0)^{\delta} + \dots \tag{2.186}$$

By definition, the matrix $\alpha^{\mu'}{}_{\mu} \equiv \partial x'^{\mu'}/\partial x^{\mu}$ becomes

$$\alpha^{\mu'}{}_{\mu} = a^{\mu'}{}_{\mu} + a^{\mu'}{}_{\mu\lambda}(x - x_0)^{\lambda} + \frac{1}{2!}a^{\mu'}{}_{\mu\lambda\varkappa}(x - x_0)^{\lambda}(x - x_0)^{\varkappa} + \dots,$$
$$\partial_\lambda \alpha^{\mu'}{}_{\mu} = a^{\mu'}{}_{\mu\lambda} + a^{\mu'}{}_{\mu\lambda\varkappa}(x - x_0)^{\varkappa} + \dots. \tag{2.187}$$

Inserting this into (2.184) we find

$$\partial_\lambda g^{\mu'\nu'}|_{x_0} = a^{\mu'}{}_{\mu}a^{\nu'}{}_{\nu}\partial_\lambda g^{\mu\nu}|_{x_0} + (a^{\mu'}{}_{\sigma\lambda}a^{\nu'}{}_{\tau}2g^{\sigma\tau}|_{x_0} + (\mu' \leftrightarrow \nu'))$$
$$= 0 + O(x - x_0), \tag{2.188}$$

which is solved by

$$a^{\mu'}{}_{\mu} = \delta^{\mu'}{}_{\mu}|_{x_0}, \qquad a^{\mu}{}_{\lambda\varkappa} = \frac{1}{2}\left\{\begin{matrix}\mu \\ \lambda\varkappa\end{matrix}\right\}\Bigg|_{x_0}, \tag{2.189}$$

as a direct consequence of (2.49). Hence the coordinates which are locally geodesic at x_0 are given by

$$x'^{\mu} = x_0{}^{\mu} + \frac{1}{2}\left\{\begin{matrix}\mu \\ \lambda\varkappa\end{matrix}\right\}(x - x_0)^{\lambda}(x - x_0)^{\varkappa} + O[(x - x_0)^3]. \tag{2.190}$$

Notice that while the Christoffel symbols vanish in the geodesic frame at x_0, their derivatives do not, due to the nonzero curvature at x_0.

As far as the crystalline defects are concerned, the possibility of constructing geodesic coordinates is related to the fact that, in the regions between defects, the crystal can always be distorted to form a regular array of atoms. In the continuum limit, these regions shrink to zero but so do the Burgers vectors of the defects. Therefore even if an arbitrarily small neighborhood did contain some defects, these themselves would be infinitesimal so that the regularity of the crystal would be disturbed only to higher orders in $(x - x_0)$.

CHAPTER THREE

FIELD EQUATIONS FOR GRAVITATION

3.1. INVARIANT ACTION

In the last chapter, we analyzed the space in which a particle in a gravitational field follows the same equations of motion (when expressed in general curvilinear coordinates) as a particle in Minkowski space. The only difference lay in certain properties of the metric. We may now ask how such a metric associated with a gravitational massive object is to be calculated. For this, the ten components of the metric tensor $g_{\mu\nu}(x)$ have to be considered as dynamical variables and we will need an action principle for it.

Since the equation of motion for $g_{\mu\nu}(x)$ must also be independent of the general coordinates employed, this action must be an invariant under Einstein transformations,

$$x^{\mu} \rightarrow x'^{\mu'}(x^{\mu}), \qquad dx^{\mu} \rightarrow dx'^{\mu'} = \alpha^{\mu'}{}_{\mu}(x) dx^{\mu}. \tag{3.1}$$

An action involves an integral over the full space,

$$\mathscr{A} = \int dx\, L(x). \tag{3.2}$$

where dx is a short notation for the volume element d^4x in four-dimensional space-time, and which transforms as

$$dx \to dx' = dx \det \alpha . \tag{3.3}$$

The simplest object $L(x)$ which leaves $\mathcal{A}$ invariant can be formed from the determinant of the metric

$$g = \det(g_{\mu\nu}) . \tag{3.4}$$

Since $g'_{\mu'\nu'}(x') = g_{\mu\nu}(\alpha^{-1})^{\mu}{}_{\mu'}(\alpha^{-1})^{\nu}{}_{\nu'}$ [see Eq. (2.36)] we have

$$g \to g' = g \det \alpha^{-2} . \tag{3.5}$$

Therefore, an action proportional to the 4-volume of space,

$$\mathcal{A} = \Lambda \int dx \sqrt{-g} , \tag{3.6}$$

is an invariant. It is referred to as the "cosmological term." However, this action is not capable of giving equations of motion for the gravitational field since it contains no derivatives of $g_{\mu\nu}$ so that the $g_{\mu\nu}$ field cannot propagate. We must find some scalar Lagrangian L containing $g_{\mu\nu}$ and $\partial g_{\mu\nu}$. Then

$$\mathcal{A} = \int dx \sqrt{-g}\, L(g, \partial g) \tag{3.7}$$

will be a possible gravitational action.

Now, the only scalar which occurred in the previous geometric analysis and which involved $\partial_\lambda g_{\mu\nu}$ was R, the scalar curvature. Therefore, Einstein postulated, with Hilbert, the following gravitational field action,

$$\mathcal{A}_f = -\frac{1}{2\kappa} \int dx \sqrt{-g}\, R \tag{3.8}$$

Here κ is a constant related to Newton's gravitational coupling $G = 6.670 \times 10^{-8}\ \mathrm{cm^3 g^{-1} s^{-2}}$ via

$$\frac{1}{\kappa} = \frac{c^3}{8\pi G} = \frac{\hbar}{\ell_P^2} , \tag{3.9}$$

where G is defined such that the force between two mass points is given by

$$F = -G\, mm'/r^2 . \tag{3.10}$$

With (3.8), a system consisting of a set of mass points with world lines $x_i^{\mu}(s_i)$ and their gravitational fields has the total action [recall Eq. (2.2)]

$$\mathcal{A} = \mathcal{A}_f - \sum_i m_i \int ds_i \equiv \mathcal{A}_f + \mathcal{A}_m. \tag{3.11}$$

For the following formulas it will be convenient to set $\kappa = 1$ since it can always be reintroduced as a relative factor between the field and matter parts in all field equations to be derived.

Variation with respect to $\delta x_i^{\mu}(s_i)$ at fixed $g_{\mu\nu}$ gives the equations of motion discussed in the beginning. In addition, the action (3.11) permits us to find the gravitational field generated by the presence of these mass points[a]. The equations of motion for $g_{\mu\nu}(x)$ are

$$\frac{\delta \mathcal{A}}{\delta g_{\mu\nu}} = 0, \tag{3.12}$$

with ten independent components. These are not the only equations. The curvature tensor $R_{\mu\nu\lambda}{}^{\kappa}$ also contains the torsion tensor $S_{\mu\nu}{}^{\lambda}$, via the contortion tensor $K_{\mu\nu}{}^{\lambda}$. These are 24 more independent degrees of freedom of the geometry to be determined by the second set of equations of motion

$$\frac{\delta \mathcal{A}}{\delta K_{\mu\nu}{}^{\lambda}} = 0. \tag{3.13}$$

We had seen in Eq. (2.10) that point particles do not couple to torsion; conversely, we do not expect these particles to generate a space with nonvanishing torsion. This is why Einstein considered only symmetric (Riemannian) spaces from the outset. For spinning matter, however, the situation is different and torsion is necessary for a complete dynamical theory. The action (3.8) in a space without torsion is called a *Hilbert-Einstein* action, in a space with torsion an *Einstein-Cartan* action.

3.2. ENERGY-MOMENTUM TENSOR AND SPIN DENSITY

It is useful to study the derivatives of the different pieces of the action with respect to $g^{\mu\nu}$ and $K_{\mu\nu}{}^{\lambda}$, separately. In view of the physical interpretations to be given later we shall introduce the tensors $T^{\mu\nu}$, $\Sigma^{\nu\lambda,\mu}$ via

[a]It is worth noting that, strictly speaking, there cannot be any point particles in general relativity. They have to be much larger than their Schwarzschild radius, i.e., $\gg \kappa m_i c$.

$$\left.\frac{\delta \mathscr{A}_m}{\delta g_{\mu\nu}}\right|_{S_{\mu\nu}{}^{\lambda}} \equiv -\frac{1}{2}\sqrt{-g}\,\overset{m}{T}{}^{\mu\nu}, \tag{3.14}$$

$$\left.\frac{\delta \mathscr{A}_f}{\delta g_{\mu\nu}}\right|_{S_{\mu\nu}{}^{\lambda}} \equiv -\frac{1}{2}\sqrt{-g}\,\overset{f}{T}{}^{\mu\nu}, \tag{3.15}$$

as the matter and field *symmetric energy-momentum tensors* and via

$$\left.\frac{\delta \mathscr{A}_m}{\delta K_{\mu\nu}{}^{\lambda}}\right|_{g_{\mu\nu}} \equiv -\frac{1}{2}\sqrt{-g}\,\overset{m}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\mu}, \tag{3.16}$$

$$\left.\frac{\delta \mathscr{A}_f}{\delta K_{\mu\nu}{}^{\lambda}}\right|_{g_{\mu\nu}} \equiv -\frac{1}{2}\sqrt{-g}\,\overset{f}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\mu}, \tag{3.17}$$

as the *spin current density* of matter and field, respectively. The field equations (3.12), (3.13) can then be simply stated as

$$\overset{f}{T}_{\mu\nu} + \overset{m}{T}_{\mu\nu} = 0, \tag{3.18}$$

$$\overset{f}{\Sigma}_{\mu\varkappa}{}^{\cdot\nu} + \overset{m}{\Sigma}_{\mu\varkappa}{}^{\cdot\nu} = 0, \tag{3.19}$$

i.e. the total energy momentum and spin densities vanish.

Consider first the action of matter. In order to exhibit the dependence on the metric tensor, we parametrize each world line by an arbitrary variable s_i' (not equal to the invariant length ds_i) and write

$$ds_i = \sqrt{g_{\mu\nu}(x_i)\,dx_i^{\mu}dx_i^{\nu}} = \sqrt{g_{\mu\nu}(x_i(s_i'))\frac{dx_i^{\mu}}{ds_i'}\frac{dx_i^{\nu}}{ds_i'}}\,ds_i'$$

so that the matter action becomes

$$\mathscr{A}_m = -\sum_i m_i \int ds_i = -\sum_i m_i \int \sqrt{g_{\mu\nu}(x_i(s_i'))\frac{dx_i^{\mu}}{ds_i'}\frac{dx_i^{\nu}}{ds_i'}}\,ds_i'.$$

Variation with respect to $g_{\mu\nu}$ and $K_{\mu\nu}{}^{\lambda}$ gives

$$\begin{aligned}\frac{\delta \mathscr{A}_m}{\delta g_{\mu\nu}} &= -\frac{1}{2}\sum_i m_i \int ds_i' \frac{1}{\dfrac{ds_i}{ds_i'}}\frac{dx^{\mu}}{ds_i'}\frac{dx^{\nu}}{ds_i'}\delta^{(4)}(x - x_i(s_i')) \\ &= -\frac{1}{2}\sum_i m_i \int ds_i \frac{dx^{\mu}}{ds_i}\frac{dx^{\nu}}{ds_i}\delta^{(4)}(x - x_i(s_i)),\end{aligned} \tag{3.20}$$

$$\frac{\delta \mathscr{A}_m}{\delta K_{\mu\nu}{}^{\lambda}} = 0. \tag{3.21}$$

We may therefore make the identification

$$\overset{m}{T}{}^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \sum_i m_i \int ds_i \frac{dx_i{}^{\mu}}{ds_i} \frac{dx_i{}^{\nu}}{ds_i} \delta^{(4)}(x - x_i(s_i)), \tag{3.22}$$

$$\overset{m}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\mu} = 0. \tag{3.23}$$

Let us calculate these quantities for the gravitational field. We first perform the variation of $\sqrt{-g}$ with respect to $\delta g_{\mu\nu}$. For this we write

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g \tag{3.24}$$

and observe that, by varying $g_{\mu\nu}(x)$, the variation of the determinant g involves the cofactors, which in fact are equal to g times the inverse, $g^{\mu\nu}$, i.e.,

$$\delta g = g\, g^{\mu\nu}\, \delta g_{\mu\nu}. \tag{3.25}$$

Moreover, due to $g^{\mu\nu}g_{\nu\lambda} = \delta^{\mu}{}_{\lambda}$, we have

$$g^{\lambda\mu}\,\delta g_{\mu\nu} = -g_{\nu\varkappa}\,\delta g^{\lambda\varkappa} \tag{3.26}$$

so that $\delta g^{\lambda\varkappa} = -g^{\lambda\mu}g^{\varkappa\nu}\,\delta g_{\mu\nu}$ and

$$\delta g = g g^{\mu\nu}\,\delta g_{\mu\nu} = -g g_{\mu\nu}\,\delta g^{\mu\nu},$$

$$\delta\sqrt{-g} = \tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu} = -\tfrac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}. \tag{3.27}$$

Notice that (3.26) implies a change of sign [compared with (3.14), (3.15)] if we calculate the energy-momentum tensor from the variation with respect to $\delta g^{\mu\nu}$

$$\left.\frac{\delta \mathscr{A}_m}{\delta g^{\mu\nu}}\right|_{S_{\mu\nu}{}^{\lambda}} = \frac{1}{2}\sqrt{-g}\,\overset{m}{T}_{\mu\nu}, \tag{3.14$'$}$$

$$\left.\frac{\delta \mathscr{A}_f}{\delta g^{\mu\nu}}\right|_{S_{\mu\nu}{}^{\lambda}} = \frac{1}{2}\sqrt{-g}\,\overset{f}{T}_{\mu\nu}. \tag{3.15$'$}$$

Therefore, we can write the variation of $\mathcal{A}_f = -(1/2)\int dx\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}$ as

$$\begin{aligned}\delta\mathcal{A}_f &= -\frac{1}{2}\int dx\sqrt{-g}\left\{-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}R + \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right\}\\ &= -\frac{1}{2}\int dx\sqrt{-g}\left[\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + g^{\mu\nu}\delta R_{\mu\nu}\right].\end{aligned} \tag{3.28}$$

The factor accompanying $\delta g^{\mu\nu}$ is the Einstein tensor,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R. \tag{3.29}$$

In general this tensor is not necessarily symmetric (only in symmetric spaces would this be true). The variation with respect to $\delta g^{\mu\nu}$, however, picks out only the symmetrized part of it [in contrast, see (6.13)].

Consider now the variation of the Ricci tensor,

$$\begin{aligned}\delta R_{\mu\nu} = {}& \partial_\varkappa \delta\Gamma_{\mu\nu}{}^\varkappa - \partial_\mu \delta\Gamma_{\varkappa\nu}{}^\varkappa - \delta\Gamma_{\varkappa\nu}{}^\tau\Gamma_{\mu\tau}{}^\varkappa - \Gamma_{\varkappa\nu}{}^\tau\delta\Gamma_{\mu\tau}{}^\varkappa\\ &+ \delta\Gamma_{\mu\nu}{}^\tau\Gamma_{\varkappa\tau}{}^\varkappa + \Gamma_{\mu\nu}{}^\tau\delta\Gamma_{\varkappa\tau}{}^\varkappa.\end{aligned} \tag{3.30}$$

In treating this relation further it is useful to realize that unlike $\Gamma_{\mu\nu}{}^\varkappa$, $\delta\Gamma_{\mu\nu}{}^\varkappa$ is a tensor [observe that $\delta\Gamma_{\mu\nu\varkappa}$ is not!]. This follows directly from the transformation law (2.50c): The last, non-holonomic piece, $\partial_\mu\partial_\nu\xi^\varkappa$ in $\Gamma_{\mu\nu}{}^\varkappa$ cancels out in $\delta\Gamma_{\mu\nu}{}^\varkappa$ since it is the same for Γ and $\Gamma + \delta\Gamma$. Therefore we may rewrite (3.30) in terms of covariant derivatives,

$$\delta R_{\mu\nu} = D_\varkappa\delta\Gamma_{\mu\nu}{}^\varkappa - D_\mu\delta\Gamma_{\varkappa\nu}{}^\varkappa + 2S_{\varkappa\mu}{}^\tau\delta\Gamma_{\tau\nu}{}^\varkappa. \tag{3.31}$$

This gives

$$\begin{aligned}\delta R_{\mu\nu} = {}& \partial_\varkappa\delta\Gamma_{\mu\nu}{}^\varkappa - \partial_\mu\delta\Gamma_{\varkappa\nu}{}^\varkappa - \Gamma_{\varkappa\mu}{}^\tau\delta\Gamma_{\tau\nu}{}^\varkappa - \Gamma_{\varkappa\nu}{}^\tau\delta\Gamma_{\mu\tau}{}^\varkappa + \Gamma_{\varkappa\tau}{}^\varkappa\delta\Gamma_{\mu\nu}{}^\tau\\ &+ \Gamma_{\mu\varkappa}{}^\tau\delta\Gamma_{\tau\nu}{}^\varkappa + \Gamma_{\mu\nu}{}^\tau\delta\Gamma_{\varkappa\tau}{}^\varkappa - \Gamma_{\mu\tau}{}^\varkappa\delta\Gamma_{\varkappa\nu}{}^\tau + 2S_{\varkappa\mu}{}^\tau\delta\Gamma_{\tau\nu}{}^\varkappa,\end{aligned}$$

which is obviously the same as (3.30). In symmetric spaces this relation was first used by Palatini.

We must now express this variation in terms of $\delta g^{\mu\nu}$ and $\delta S_{\mu\nu}{}^\lambda$. It is useful to perform all operations inside the action integral,

$$-\frac{1}{2}\int dx\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}. \tag{3.32a}$$

Because of the tensor nature of $\delta\Gamma_{\mu\nu}{}^{\varkappa}$ we can take $g^{\mu\nu}$ through the covariant derivative and write this as

$$-\frac{1}{2}\int dx\sqrt{-g}\,(D_{\varkappa}\,\delta\Gamma_{\mu}{}^{\mu\varkappa}-D_{\mu}\,\delta\Gamma_{\varkappa}{}^{\mu\varkappa}+2S_{\varkappa}{}^{\mu\tau}\,\delta\Gamma_{\tau\mu}{}^{\varkappa})$$

$$=-\frac{1}{2}\int dx\sqrt{-g}\,(\partial_{\varkappa}\,\delta\Gamma_{\mu}{}^{\mu\varkappa}-\partial_{\mu}\,\delta\Gamma_{\varkappa}{}^{\mu\varkappa}+\Gamma_{\varkappa\lambda}{}^{\varkappa}\,\delta\Gamma_{\mu}{}^{\mu\lambda}$$
$$-\Gamma_{\mu\lambda}{}^{\mu}\delta\Gamma_{\varkappa}{}^{\lambda\varkappa}+2S_{\varkappa}{}^{\mu\tau}\,\delta\Gamma_{\tau\mu}{}^{\varkappa}). \tag{3.32b}$$

Further, we use

$$\partial_{\varkappa}\sqrt{-g}=\sqrt{-g}\,\Gamma_{\varkappa\lambda}{}^{\lambda} \tag{3.33}$$

to rewrite $\sqrt{-g}\,\partial_{\varkappa}\,\delta\Gamma_{\mu\nu}{}^{\varkappa}$ as

$$\partial_{\varkappa}(\sqrt{-g}\,\delta\Gamma_{\mu\nu}{}^{\varkappa})-\sqrt{-g}\,\Gamma_{\varkappa\lambda}{}^{\lambda}\delta\Gamma_{\mu\nu}{}^{\varkappa} \tag{3.34}$$

and split (3.32b) into a pure surface term

$$-\frac{1}{2}\int dx\,\{\partial_{\varkappa}\sqrt{-g}\,\delta\Gamma_{\mu}{}^{\mu\varkappa}-\partial_{\mu}\sqrt{-g}\,\delta\Gamma_{\varkappa}{}^{\mu\varkappa}\}, \tag{3.35}$$

plus terms originating from the connection,

$$-\frac{1}{2}\int dx\sqrt{-g}\,(-\Gamma_{\varkappa\lambda}{}^{\lambda}\delta\Gamma^{\nu}{}_{\nu}{}^{\varkappa}+\Gamma^{\nu}{}_{\lambda}{}^{\lambda}\delta\Gamma_{\varkappa\nu}{}^{\varkappa}+\Gamma_{\varkappa\lambda}{}^{\varkappa}\delta\Gamma_{\nu}{}^{\nu\lambda}$$
$$-\Gamma_{\mu\lambda}{}^{\mu}\delta\Gamma_{\varkappa}{}^{\lambda\varkappa}+2S_{\varkappa}{}^{\nu\tau}\,\delta\Gamma_{\tau\nu}{}^{\varkappa})$$
$$=-\frac{1}{2}\int dx\sqrt{-g}\,(-2S_{\varkappa}\,\delta\Gamma^{\nu}{}_{\nu}{}^{\varkappa}+2S^{\nu}\,\delta\Gamma_{\varkappa\nu}{}^{\varkappa}+2S_{\varkappa}{}^{\nu\tau}\,\delta\Gamma_{\tau\nu}{}^{\varkappa}), \tag{3.36}$$

where we have abbreviated

$$S_{\varkappa}\equiv S_{\varkappa\lambda}{}^{\lambda},\qquad S^{\varkappa}\equiv S^{\varkappa}{}_{\lambda}{}^{\lambda}, \tag{3.37}$$

It is useful to state the result as follows

$$-\frac{1}{2}\int dx\sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu}=-\frac{1}{2}\int dx\sqrt{-g}\,S_{\kappa}{}^{\mu,\tau}\,\delta\Gamma_{\tau\mu}{}^{\kappa}, \tag{3.38}$$

valid up to irrelevant surface terms, where

$$\tfrac{1}{2} S_{\mu\varkappa}{}^{,\tau} \equiv S_{\mu\varkappa}{}^{\tau} + \delta_{\mu}{}^{\tau} S_{\varkappa} - \delta_{\varkappa}{}^{\tau} S_{\mu} . \tag{3.39}$$

This combination of torsion tensors is referred to as the *Palatini tensor*.

We are now going to express $\delta\Gamma_{\tau\mu}{}^{\varkappa}$ in terms of $\delta g_{\mu\nu}$ and $\delta S_{\mu\nu}{}^{\lambda}$. To do this we note that the varied metric $g_{\mu\rho} + \delta g_{\mu\rho}$ certainly satisfies the identity (2.48),

$$\overset{\Gamma+\delta\Gamma}{D_\tau}(g_{\mu\rho} + \delta g_{\mu\rho}) = 0, \tag{3.40}$$

where $\overset{\Gamma+\delta\Gamma}{D}$ is the covariant derivative formed with the varied connection. For the variations $\delta g_{\mu\rho}$ this implies,

$$\overset{\Gamma}{D_\mu} \delta g_{\tau\rho} = \delta\Gamma_{\mu\tau\rho} + \delta\Gamma_{\mu\rho\tau} , \tag{3.41}$$

where we have introduced

$$\delta\Gamma_{\mu\tau\rho} \equiv g_{\rho\lambda}\, \delta\Gamma_{\mu\tau}{}^{\lambda}. \tag{3.42}$$

This gives

$$\begin{aligned}\tfrac{1}{2}(\overset{\Gamma}{D_\tau} \delta g_{\mu\rho} + \overset{\Gamma}{D_\mu} \delta g_{\rho\tau} - \overset{\Gamma}{D_\rho} \delta g_{\tau\mu}) &= \delta\Gamma_{\tau\mu\rho} - \delta S_{\tau\mu\rho} + \delta S_{\mu\rho\tau} - \delta S_{\rho\tau\mu} \\ &= \delta\Gamma_{\tau\mu\rho} - \delta K_{\tau\mu\rho} ,\end{aligned} \tag{3.43}$$

where $\delta S_{\tau\mu\rho} \equiv g_{\rho\lambda}\, \delta S_{\tau\mu}{}^{\lambda} \equiv g_{\rho\lambda} \cdot (1/2)\delta(\Gamma_{\tau\mu}{}^{\lambda} - \Gamma_{\mu\tau}{}^{\lambda})$ and $\delta K_{\tau\mu\rho} \equiv \delta S_{\tau\mu\rho} - \delta S_{\mu\rho\tau} + \delta S_{\rho\tau\mu}$ is the result of a variation of $S_{\mu\nu}{}^{\lambda}$ at fixed $g_{\mu\nu}$. Notice that even though $\Gamma_{\mu\nu}{}^{\lambda} = \{{}^{\lambda}_{\mu\nu}\} + K_{\mu\nu}{}^{\lambda}$, the left-hand side cannot be identified with $g_{\rho\varkappa}\delta\{{}^{\varkappa}_{\tau\mu}\}$, since $\delta K_{\mu\nu}{}^{\lambda}$ contains contributions from $\delta S_{\mu\nu}{}^{\lambda}$ at fixed $g_{\mu\nu}$ and from $g_{\mu\nu}$ at fixed $S_{\mu\nu}{}^{\lambda}$.

Using (3.43) we can rewrite (3.38) as

$$\begin{aligned}&-\frac{1}{2}\int dx \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} \\ &= \frac{1}{2}\int dx \sqrt{-g}\, S^{\mu\rho,\tau}\left[\delta K_{\tau\mu\rho} + \frac{1}{2}(D_\tau \delta g_{\mu\rho} + D_\mu \delta g_{\rho\tau} - D_\rho \delta g_{\tau\mu})\right].\end{aligned}$$

The first term at the right-hand side shows that the Palatini tensor $S^{\rho\mu,\tau}$

plays the role of the spin current of the gravitational field [recall the definition (3.17)] up to a factor $-1/\kappa$,

$$\overset{f}{\Sigma}_{\mu\rho,\tau} = -\frac{1}{\kappa} S_{\mu\rho,\tau}. \tag{3.44}$$

The second term is partially integrated, with the result

$$\text{surface term} - \frac{1}{4}\int dx \sqrt{-g}\{D_\tau{}^* S^{\mu\rho,\tau}\delta g_{\mu\rho} + D_\mu{}^* S^{\mu\rho,\tau}\delta g_{\rho\tau} - D_\rho{}^* S^{\mu\rho,\tau}\delta g_{\tau\mu}\}.$$

Here $D_\mu{}^*$ is defined as

$$D_\mu{}^* \equiv D_\mu + 2S_\mu. \tag{3.45}$$

This modified covariant derivative is a convenient tool for partial integration in integrals containing the metric factor $\sqrt{-g}$. Take any tensors $U^{\mu\ldots\nu_i\ldots}$, $V_{\ldots\nu_i\ldots}$ and consider, for example,

$$-\int dx \sqrt{-g}\, U^{\mu\ldots\nu_i\ldots} D_\mu V_{\ldots\nu_i\ldots}. \tag{3.46}$$

Partial integration gives

$$\text{surface term} + \sum_i \int dx \left[\partial_\mu \sqrt{-g}\, U^{\mu\ldots\nu_i\ldots} V_{\ldots\nu_i\ldots} + \sqrt{-g}\, U^{\mu\ldots\nu_i\ldots} \Gamma_{\mu\nu_i}{}^{\lambda_i} V_{\ldots\lambda_i\ldots}\right], \tag{3.47}$$

where $\Sigma_i \Gamma_{\mu\nu_i}{}^{\lambda_i}$ indicates the sum of connections, each acting on one of the indices of $V_{\ldots\lambda_i\ldots}$ and contracted with ν_i in $U^{\mu\ldots\nu_i\ldots}$. But

$$\partial_\mu \sqrt{-g} = \sqrt{-g}\,\Gamma_{\mu\varkappa}{}^{\varkappa} = \sqrt{-g}(2S_\mu + \Gamma_{\varkappa\mu}{}^{\varkappa})$$

so that (3.47) becomes

$$\text{surface term} + \sum_i \int dx \sqrt{-g}\left[\left(\partial_\mu U^{\mu\ldots\lambda_i\ldots} + \sum_i \Gamma_{\mu\nu_i}{}^{\lambda_i} U^{\mu\ldots\nu_i\ldots} + \Gamma_{\varkappa\mu}{}^{\varkappa} U^{\mu\ldots\lambda_i\ldots}\right) V_{\ldots\lambda_i\ldots} + 2S_\mu \sum_i U^{\mu\ldots\lambda_i\ldots} V_{\ldots\lambda_i\ldots}\right].$$

Now, the terms in parentheses are just the covariant derivative of $U^{\mu\ldots\nu_i\ldots}$ so that we arrive at

$$+\int dx\sqrt{-g}\,D_\mu{}^* U^{\mu\ldots\nu_i\ldots} V_{\ldots\nu_i\ldots}\,. \tag{3.48}$$

Relabeling the indices in (3.43), we arrive at the following derivative of the action[b] with respect to $\delta g_{\mu\nu}$

$$-\frac{1}{2}\int dx\sqrt{-g}\left\{G^{\mu\nu}-\frac{1}{2}D_\lambda{}^*(S^{\mu\nu,\lambda}-S^{\nu\lambda,\mu}+S^{\lambda\mu,\nu})\right\}.$$

Thus the complete energy-momentum tensor of the field reads

$$\overset{f}{T}{}^{\mu\nu}=-\frac{1}{\kappa}\left[G^{\mu\nu}-\frac{1}{2}D_\lambda{}^*(S^{\mu\nu,\lambda}-S^{\nu\lambda,\mu}+S^{\lambda\mu,\nu})\right]. \tag{3.49}$$

Actually, the variation $\delta g^{\mu\nu}$ yields only the symmetrized part of $\overset{f}{T}_{\mu\nu}$. This specification is, however, unnecessary. We shall demonstrate later that total angular momentum conservation [cf. Eq. (5.13)] makes $\overset{f}{T}{}^{\mu\nu}$ symmetric as it stands (even though $G^{\mu\nu}$ is not).

We finally arrive at the following field equations:

$$-\kappa\overset{f}{\Sigma}_{\mu\varkappa}{}^{\cdot\tau}=S_{\mu\varkappa}{}^{\cdot\tau}=\kappa\overset{m}{\Sigma}_{\mu\varkappa}{}^{\cdot\tau}, \tag{3.50}$$

$$-\kappa\overset{f}{T}{}^{\mu\nu}=G^{\mu\nu}-\tfrac{1}{2}D_\lambda{}^*(S^{\mu\nu,\lambda}-S^{\nu\lambda,\mu}+S^{\lambda\mu,\nu})=\kappa\overset{m}{T}{}^{\mu\nu}. \tag{3.51}$$

For a set of point particles we insert (3.22), (3.23) and have

$$S_{\mu\varkappa}{}^{\cdot\tau}=0, \tag{3.50$'$}$$

$$G^{\mu\nu}=\kappa\overset{m}{T}{}^{\mu\nu}. \tag{3.51$'$}$$

[b]Recall that $\delta g^{\mu\nu}G_{\mu\nu}=-\delta g_{\mu\nu}G^{\mu\nu}$, by (3.26).

3.3. SYMMETRIC ENERGY-MOMENTUM TENSOR OF THE GRAVITATIONAL FIELD AND DEFECT DENSITY

The field energy-momentum tensor obtained in (3.49) has a direct defect interpretation. For simplicity, let us go to three dimensions. Then the linearized version of (3.49) reads

$$-\kappa \overset{f}{T}_{ij} = G_{ij} - \tfrac{1}{2}\partial_k (S_{ij,k} - S_{jk,i} - S_{ki,j}), \tag{3.52}$$

with the spin density (3.44), (3.39),

$$-\tfrac{1}{2}\kappa \overset{f}{\Sigma}_{ij,k} = \tfrac{1}{2} S_{ij,k} = S_{ijk} + \delta_{ik} S_j - \delta_{jk} S_i . \tag{3.53}$$

We insert the dislocation density according to (2.85), (2.92a)

$$S_{ijk} = \tfrac{1}{2}(\partial_i \partial_j - \partial_j \partial_i) u_k = \tfrac{1}{2}\varepsilon_{ij\ell}\,\alpha_{\ell k} . \tag{3.54}$$

Then the Palatini tensor reads

$$S_{ij,k} = \varepsilon_{ij\ell}\,\alpha_{\ell k} + \delta_{ik}\,\varepsilon_{jp\ell}\,\alpha_{\ell p} - \delta_{jk}\,\varepsilon_{ip\ell}\,\alpha_{\ell p} . \tag{3.55}$$

Since both sides are antisymmetric in (ij), we can contract them with ε_{ijn},

$$\begin{aligned} \varepsilon_{ijn} S_{ij,k} &= 2\alpha_{nk} + \varepsilon_{kjn}\,\varepsilon_{jp\ell}\,\alpha_{\ell p} - \varepsilon_{ikn}\,\varepsilon_{ip\ell}\,\alpha_{\ell p} \\ &= 2\alpha_{nk} - 2(\delta_{kp}\delta_{n\ell} - \delta_{k\ell}\,\delta_{np})\,\alpha_{\ell p} = 2\alpha_{kn} , \end{aligned}$$

and $S_{ij,k}$ becomes simply

$$S_{ij,k} = \varepsilon_{ij\ell}\,\alpha_{k\ell} . \tag{3.56}$$

and the spin density coincides with the dislocation density up to a factor $-\kappa$

$$-\kappa \overset{f}{\Sigma}_{ij,k} = \varepsilon_{ij\ell}\,\alpha_{k\ell} . \tag{3.57}$$

Thus the spin density is related to the dislocation density whose indices appear in the transposed order with respect to (3.54). The transposition insures that the spin density has vanishing divergence, since

$$\partial_k S_{ij,k} = \varepsilon_{ij\ell}\,\partial_k \alpha_{k\ell} = 0. \tag{3.58}$$

In terms of the derivatives of the displacement field $u_i(x)$, the Palatini tensor reads

$$S_{ij,k} = \varepsilon_{ij\ell}\,\varepsilon_{kmn}\,\partial_m\,\partial_n\,u_\ell\,, \tag{3.59}$$

a form which shows explicitly the conservation law (3.58).

Let us now form the three combinations of $S_{ij,k}$ required in (3.52),

$$\tfrac{1}{2}(S_{ij,k} - S_{jk,i} + S_{ki,j}) = \tfrac{1}{2}(\varepsilon_{ij\ell}\,\alpha_{k\ell} - \varepsilon_{jk\ell}\,\alpha_{i\ell} + \varepsilon_{ki\ell}\,\alpha_{j\ell}). \tag{3.60}$$

The identity

$$\varepsilon_{ij\ell}\,\delta_{km} + \varepsilon_{jk\ell}\,\delta_{im} + \varepsilon_{ki\ell}\,\delta_{jm} = \varepsilon_{ijk}\,\delta_{\ell m} \tag{3.61}$$

may be contracted with $\alpha_{m\ell}$ giving

$$\varepsilon_{ij\ell}\,\alpha_{k\ell} + \varepsilon_{jk\ell}\,\alpha_{i\ell} + \varepsilon_{ki\ell}\,\alpha_{j\ell} = \varepsilon_{ijk}\,\alpha_{\ell\ell}\,, \tag{3.62}$$

so that

$$\tfrac{1}{2}(S_{ij,k} - S_{jk,i} + S_{ki,j}) = -\varepsilon_{jk\ell}\,\alpha_{i\ell} + \tfrac{1}{2}\,\varepsilon_{ijk}\,\alpha_{\ell\ell}\,. \tag{3.63}$$

The right-hand side is recognized as

$$\varepsilon_{jk\ell}\,K_{\ell i} \tag{3.64}$$

where $K_{\ell j} = -\alpha_{j\ell} + (1/2)\,\delta_{\ell j}K_{kk}$ is Nye's contortion tensor which was defined in Part III, Eq. (2.79a). With this notation, Eq. (3.52) becomes

$$-\kappa\,\overset{f}{T}_{ij} = G_{ij} - \varepsilon_{jk\ell}\,\partial_k\,K_{\ell i}\,.$$

Now we recall that the Einstein curvature tensor G_{ij} of a metric $g_{ij} = \delta_{ij} + \partial_i u_j + \partial_j u_i$ with the total (elastic plus plastic) displacement $u_i(\mathbf{x})$ coincides with the disclination density Θ_{ji} [see (2.93)]. But then, comparison with Eq. (III.2.80a) shows that the symmetric energy-momentum tensor times $-\kappa$ is nothing but the total defect density η_{ij}:

$$-\kappa\,T_{ij} = \eta_{ij}\,. \tag{3.65}$$

CHAPTER FOUR

SPINNING PARTICLES

4.1. LOCAL LORENTZ INVARIANCE AND NON-HOLONOMIC COORDINATES

Until now we have been discussing the gravitational field interacting with massive point particles without any intrinsic spin. Let us now see how spin can be incorporated into this geometric framework.

Spin was originally defined in Lorentz invariant theories as follows: A particle moving with velocity $\mathbf{v}$ is brought to rest by a Lorentz transformation. Then its quantum mechanical description requires several components which, under rotation, transform according to an irreducible representation of the rotation group. For spin 1/2, this property is automatically accounted for by describing the spinning particles in terms of a Dirac field $\psi_\alpha(x)$ which extremizes the action

$$\mathcal{A}_m = \frac{1}{2}\int d^4x\,\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\,\psi(x) + \text{h.c.} \tag{4.1}$$

In order to allow for the presence of a gravitational field, this action has to be generalized to arbitrary curved space-time. Naively we would expect an expression like

$$\mathcal{A}_m = \frac{1}{2} \int dx \sqrt{-g}\, \bar{\psi}(x)(i\, \gamma^\mu(x)\, D_\mu - m)\, \psi(x) + \text{h.c.}, \tag{4.2}$$

where the matrices γ^μ satisfy the Dirac algebra

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x), \tag{4.3}$$

which can be solved by x-dependent matrices $\gamma^\mu(x)$. What is unknown is the covariant derivative D_μ of a spinor which depends on the transformation properties of $\psi(x)$ under Lorentz transformations.

The most convenient theoretical framework for solving this problem is based on the introduction of non-holonomic coordinates. They are related to x^μ by some differential coordinate transformation,

$$dx^\alpha = dx^\mu h^\alpha{}_\mu(x), \tag{4.4}$$

which satisfies the following conditions.

1. It has an inverse

$$dx^\mu = dx^\alpha h_\alpha{}^\mu(x) \tag{4.5}$$

i.e., the matrices $h^\alpha{}_\mu$, $h_\alpha{}^\mu$ satisfy

$$h_\alpha{}^\mu h^\beta{}_\mu = \delta_\alpha{}^\beta, \qquad h^\alpha{}_\mu h_\alpha{}^\nu = \delta_\mu{}^\nu. \tag{4.6}$$

2. The transformation matrices $h^\alpha{}_\mu(x)$, $h_\alpha{}^\mu(x)$ obey the integrability condition

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) h^\alpha{}_\lambda = 0. \tag{4.7}$$

This condition has the consequence that, if we define a curvature tensor $\overset{h}{R}_{\mu\nu\lambda}{}^\varkappa$ in terms of $h_\alpha{}^\mu$ in the same way as $R_{\mu\nu\lambda}{}^\varkappa$ was defined in terms of $e_a{}^\mu$ [compare (2.61)], this vanishes identically,

$$\overset{h}{R}_{\mu\nu\lambda}{}^\varkappa = h_\alpha{}^\varkappa (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) h^\alpha{}_\lambda \equiv 0. \tag{4.8}$$

3. The matrix $h_\alpha{}^\mu(x)$ is chosen so as to bring the metric $g_{\alpha\beta}$ to the flat form at every point in space.

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\alpha\beta}. \tag{4.9}$$

This implies that the metric $g_{\mu\nu}(x)$ is the square of the matrices $h^{\alpha}{}_{\mu}(x)$ in the same way as it is of the basis tetrads $e^{a}{}_{\mu}(x)$ [recall (2.17)]:

$$g_{\mu\nu}(x) = h^{\alpha}{}_{\mu}(x) h^{\beta}{}_{\nu}(x) \eta_{\alpha\beta} \equiv h^{\alpha}{}_{\mu}(x) h_{\beta\nu}(x). \tag{4.10}$$

Still, they are completely different objects with different integrability properties, While $h^{\alpha}{}_{\lambda}$ satisfies (4.7), $e^{a}{}_{\lambda}$ does not, with the commutator of the derivatives determining the curvature tensor [see (2.61)].

The 16 component transformation matrices $h_{\alpha}{}^{\mu}(x)$ are called *vierbein fields* and $h^{\alpha}{}_{\mu}(x)$ the *reciprocal vierbein fields*. As in the case of the basis tetrads $e^{a}{}_{\mu}$, $e_{a}{}^{\mu}$ we shall freely raise and lower the indices α, β, γ, ... using the metric $\eta_{\alpha\beta} = \eta^{\alpha\beta}$ and define

$$h^{\alpha\mu} \equiv \eta^{\alpha\beta} h_{\beta}{}^{\mu}, \qquad h_{\alpha\mu} \equiv \eta_{\alpha\beta} h^{\beta}{}_{\mu}.$$

From the defect point of view, the matrices $h_{\alpha}{}^{\mu}$ create an intermediate coordinate system dx^{α} which, by the integrability condition (4.7), has the same disclination content as the coordinates x^{μ} but is completely free of dislocations. We shall see later in Chapter 7 that $h_{\alpha\lambda}$ is the geometric analogue of the plastic distortion $\beta^{p}_{\lambda\alpha}$ discussed in Part III, Eq. (2.62). The metric in the new coordinate system x^{α} is locally Minkowski-like, at each point in space. Still, the coordinates x^{α} do not form a Minkowski space since they differ from the inertial coordinates x^{a} by the presence of disclinations, i.e., there are wedge-like pieces missing with respect to an ideal reference crystal.

Observe that in order to specify space-time points we have to parametrize them in terms of the original variables x^{μ}. Only *derivatives* can be executed in x^{α} space and vector directions be fixed with respect to the intermediate local axes,

$$\begin{aligned} \mathbf{e}_{\alpha}(x) &\equiv \mathbf{e}_{\mu}(x) \frac{\partial x^{\mu}}{\partial x^{\alpha}} = \mathbf{e}_{\mu}(x) h_{\alpha}{}^{\mu}(x) \\ &\equiv \mathbf{e}_{a} e^{a}{}_{\mu}(x) h_{\alpha}{}^{\mu}(x) \equiv \mathbf{e}_{a} e^{a}{}_{\alpha}(x), \end{aligned} \tag{4.11}$$

from which one can go back to the local basis via the reciprocal vierbein fields

$$\mathbf{e}_\mu(x) = \mathbf{e}_a(x)\frac{\partial x^\alpha}{\partial x^\mu} = \mathbf{e}_\alpha(x)\, h^\alpha{}_\mu(x). \tag{4.12}$$

Thus, an arbitrary vector may be transformed as follows,

$$\begin{aligned}\mathbf{v}(x) &\equiv \mathbf{e}_a v^a(x) = \mathbf{e}_a e^a{}_\mu v^\mu = \mathbf{e}_a e^a{}_\alpha (h^\alpha{}_\mu v^\mu) = \mathbf{e}_a e^{a\alpha} h_\alpha{}^\mu v_\mu \\ &\equiv \mathbf{e}_a e^a{}_\alpha v^\alpha \equiv \mathbf{e}_a e^{a\alpha} v_\alpha ,\end{aligned} \tag{4.13}$$

where we have introduced the components

$$v^\alpha(x) \equiv v^\mu(x)\, h^\alpha{}_\mu(x), \qquad v_\alpha(x) \equiv v_\mu(x) h_\alpha{}^\mu(x). \tag{4.14}$$

The orthogonality relations (4.6) imply the inverse relations

$$v_\mu(x) = v_\alpha(x) h^\alpha{}_\mu(x), \qquad v^\mu(x) = v^\alpha(x) h_\alpha{}^\mu(x). \tag{4.15}$$

In the intermediate basis $\mathbf{e}_\alpha(x)$ the covariant derivatives of the vector fields v_β, v^β is found to be

$$D_\alpha v_\beta = \partial_\alpha v_\beta - \bar{\Gamma}_{\alpha\beta}{}^\gamma v_\gamma, \qquad D_\alpha v^\beta = \partial_\alpha v^\beta + \bar{\Gamma}_{\alpha\gamma}{}^\beta v^\gamma \tag{4.16}$$

where $\bar{\Gamma}_{\alpha\beta}{}^\gamma$ is calculated with $e^a{}_\beta$ rather than with $e^a{}_\mu$ [compare (2.46)]:

$$\bar{\Gamma}_{\alpha\beta}{}^\gamma = e_a{}^\gamma \partial_\alpha e^a{}_\beta = -e^a{}_\beta \partial_\alpha e_a{}^\gamma. \tag{4.17}$$

It is called the *spin connection*. Written out in terms of $e^a{}_\mu$ and $h_a{}^\mu$ this becomes

$$\begin{aligned}\bar{\Gamma}_{\alpha\beta}{}^\gamma &= e_a{}^\lambda h^\gamma{}_\lambda h_\alpha{}^\mu \partial_\mu (e^a{}_\nu h_\beta{}^\nu) \\ &= h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu \Gamma_{\mu\nu}{}^\lambda + h^\gamma{}_\lambda h_\alpha{}^\mu \delta^\lambda{}_\nu \partial_\mu h_\beta{}^\nu \\ &= h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu (\Gamma_{\mu\nu}{}^\lambda + h^\delta{}_\nu \partial_\mu h_\delta{}^\lambda) \\ &= h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu (\Gamma_{\mu\nu}{}^\lambda - \overset{h}{\Gamma}_{\mu\nu}{}^\lambda),\end{aligned} \tag{4.18}$$

where $\overset{h}{\Gamma}_{\mu\nu}{}^\lambda$ is defined in terms of h in the same way as $\Gamma_{\mu\nu}{}^\lambda$ is defined in terms of e [see (2.46)]. Alternatively we may also write

$$\bar{\Gamma}_{\alpha\beta}{}^\gamma = h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu \Gamma_{\mu\nu}{}^\lambda - h_\alpha{}^\mu h_\beta{}^\nu \partial_\mu h^\gamma{}_\nu$$

$$\bar{\Gamma}_{\alpha\beta}{}^{\gamma} = h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} (\Gamma_{\mu\nu}{}^{\lambda} - h_{\delta}{}^{\lambda} \partial_{\mu} h^{\delta}{}_{\nu})$$
$$= h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} (\Gamma_{\mu\nu}{}^{\lambda} - \overset{h}{\Gamma}_{\mu\nu}{}^{\lambda}). \tag{4.19}$$

The second line implies that $h_{\alpha}{}^{\mu}$ satisfies identities like $e_{a}{}^{\mu}$ in (2.47):

$$D_{\alpha} h_{\beta}{}^{\mu} = 0, \qquad D_{\alpha} h^{\beta}{}_{\mu} = 0. \tag{4.20}$$

If we now decompose the two connections on the right-hand side into Christoffel parts and contortion tensors [recall (2.54), (2.55)] we realize that, by the identity

$$g_{\mu\nu}(x) = e^{a}{}_{\mu}(x) e^{b}{}_{\nu}(x) \eta_{ab} \equiv h^{\alpha}{}_{\mu}(x) h^{\beta}{}_{\nu}(x) \eta_{\alpha\beta}, \tag{4.21}$$

the two Christoffel parts in $\Gamma_{\mu\nu}{}^{\lambda}$ and $\overset{h}{\Gamma}_{\mu\nu}{}^{\lambda}$ are the same so that $\bar{\Gamma}_{\alpha\beta}{}^{\gamma}$ becomes simply

$$\bar{\Gamma}_{\alpha\beta}{}^{\gamma} = h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \left(\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} + K_{\mu\nu}{}^{\lambda} - \overset{h}{\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}} - \overset{h}{K}_{\mu\nu}{}^{\lambda} \right)$$
$$= h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} (K_{\mu\nu}{}^{\lambda} - \overset{h}{K}_{\mu\nu}{}^{\lambda}), \tag{4.22}$$

where $K_{\mu\nu}{}^{\lambda}$ is the contortion as given in (2.54), (2.55) and $\overset{h}{K}_{\mu\nu}{}^{\lambda}$ the analogous expression in terms of $h_{\alpha}{}^{\mu}$ instead of $e_{a}{}^{\mu}$. Explicitly

$$K_{\mu\nu}{}^{\lambda} = S_{\mu\nu}{}^{\lambda} - S_{\nu}{}^{\lambda}{}_{\mu} + S^{\lambda}{}_{\mu\nu}, \tag{4.23}$$

$$\overset{h}{K}_{\mu\nu}{}^{\lambda} = \overset{h}{S}_{\mu\nu}{}^{\lambda} - \overset{h}{S}_{\nu}{}^{\lambda}{}_{\mu} + \overset{h}{S}{}^{\lambda}{}_{\mu\nu}. \tag{4.24}$$

Notice that due to (4.21) the spin connection is antisymmetric in the last two indices.

It will be helpful to use $h^{\alpha}{}_{\mu}$, $h_{\alpha}{}^{\mu}$ freely for changing indices α into μ, for instance,

$$K_{\alpha\beta}{}^{\gamma} \equiv h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} K_{\mu\nu}{}^{\lambda}, \tag{4.25}$$

$$\overset{h}{K}_{\alpha\beta}{}^{\gamma} = h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \overset{h}{K}_{\mu\nu}{}^{\lambda}. \tag{4.26}$$

Observe that by introducing the vierbein fields $h_\mu{}^\alpha$, $h_\alpha{}^\mu$ the description of gravitational effects in terms of the 10 metric components $g_{\mu\nu}$ and the 24 torsion components $K_{\mu\nu}{}^\lambda$ has been replaced by 16 components $h_\mu{}^\alpha$ and the 24 $K_{\mu\nu}{}^\lambda$. We mentioned earlier that the vierbein fields satisfy the same relation

$$h_{\alpha\mu} h^\alpha{}_\nu = g_{\mu\nu} \tag{4.27}$$

as $e_{a\mu}(x)$. Thus they can be considered as another "square root" of the metric $g_{\mu\nu}$ different from $e_{a\mu}$. Obviously, such a "square root" is defined only up to an arbitrary local Lorentz transformation which accounts for the six additional degrees of freedom of the $h_\alpha{}^\mu(x)$ with respect to the $g_{\mu\nu}(x)$ description. In fact, by introducing Lorentz transformations with rotational defects $dx^\alpha = dx^a \Lambda_a{}^\alpha(x)$, where $\Lambda_a{}^\alpha(x)$ are non-integrable functions of x we could make $h_\alpha{}^\mu(x)$ coincide with the underlying fully defected coordinates dx^a. We shall not do so, however, since the intended introduction of spin into the gravitational field requires local Lorentz transformations without defects.

Local Lorentz transformations connecting dx^a and the dislocated dx^α are given precisely by the matrices $e^a{}_\alpha(x)$ introduced previously [see (4.11)] as can be seen directly from Eq. (4.9) which implies

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\alpha\beta} = h_\alpha{}^\mu h_{\beta\mu} = e^a{}_\alpha e_a{}^\mu e^b{}_\beta e_{b\mu} \tag{4.28}$$

$$= e^a{}_\alpha e^b{}_\beta \eta_{ab} = e^a{}_\alpha e^b{}_\beta \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{ab} . \tag{4.29}$$

Thus the matrix

$$\Lambda^a{}_\alpha(x) \equiv e^a{}_\alpha(x) \tag{4.30}$$

satisfies indeed the defining equation of Lorentz transformations,

$$\eta = \Lambda(x)\,\eta\,\Lambda^T(x). \tag{4.31}$$

Notice that, due to (4.17) $e^a{}_\alpha(x)$, $e_a{}^\alpha(x)$ satisfy identities like (2.47), (4.20),

$$D_\alpha e^a{}_\beta = 0, \qquad D_\alpha e_a{}^\beta = 0. \tag{4.32}$$

Any theory which is formulated in a generally invariant way can be recast in terms of non-holonomic coordinates dx^α. Since the metric is $\eta^{\alpha\beta}$, Minkowskian invariant actions have the same form as those in a flat space except that the derivatives are replaced by covariant ones,

$$\partial_\alpha v_\beta \rightarrow D_\alpha v_\beta = \partial_\alpha v_\beta - \bar{\Gamma}_{\alpha\beta}{}^\gamma v_\gamma.$$

For example,

$$\mathscr{A} = \int d^4x^\alpha D_\alpha v_\beta(x^\mu) D^\alpha v^\beta(x^\mu) \tag{4.33}$$

is the non-holonomic form of a generally covariant action. As we said in the beginning, the specification of space-time points must be made with the x^μ coordinates. For this reason the action is preferably written as

$$\mathscr{A} = \int d^4x^\mu \sqrt{-g}\, D_\alpha v_\beta(x^\mu) D^\alpha v^\beta(x^\mu). \tag{4.34}$$

Under a general coordinate transformation *à la* Einstein, $dx^\mu \rightarrow dx'^{\mu'} = dx^\mu \alpha_\mu{}^{\mu'}$, the indices α are inert. For instance, $h_\alpha{}^\mu$ itself transforms as

$$h_\alpha{}^\mu(x) \underset{E}{\rightarrow} h_\alpha{}^{\mu'}(x') = h_\alpha{}^\mu(x)\, \alpha_\mu{}^{\mu'}. \tag{4.35}$$

Vectors and tensors with indices α, β, ... experience only changes of their arguments $x \rightarrow x - \xi$ so that their infinitesimal substantial changes are

$$\delta_E v_\alpha(x) = \xi^\lambda \partial_\lambda v_\alpha(x) \tag{4.36}$$

$$\delta_E D_\alpha v_\beta(x) = \xi^\lambda \partial_\lambda D_\alpha v_\beta(x). \tag{4.37}$$

The freedom in choosing $h_\alpha{}^\mu(x)$ up to a local Lorentz transformation, when taking the "square root" of $g_{\mu\nu}(x)$ in (4.27), implies that the theory should be invariant under

$$\delta_L dx^\alpha = \omega^\alpha{}_\beta(x) dx^\beta, \tag{4.38}$$

$$\delta_L h_\alpha{}^\mu(x) = \omega_\alpha{}^\beta(x) h_\beta{}^\mu(x). \tag{4.39}$$

Here $\omega_\alpha{}^{\alpha'}(x)$ are the local versions of the infinitesimal angles introduced in (2.21) and (2.22).

Indeed the action (4.34) is automatically invariant if every index α is transformed accordingly.

$$\delta_L v_\alpha(x) = \omega_\alpha{}^{\alpha'}(x) v_{\alpha'}(x), \tag{4.40}$$

$$\delta_L D_\alpha v_\beta(x) = \omega_\alpha{}^{\alpha'}(x) D_{\alpha'} v_\beta(x) + \omega_\beta{}^{\beta'}(x) D_\alpha v_{\beta'}(x). \tag{4.40$'$}$$

The variables x^μ are unchanged since (4.38) refers only to the differentials dx^α and leaves dx^μ unchanged.

It is useful to verify explicitly how the covariant derivatives guarantee local Lorentz invariance. Consider

$$\delta_L v_\alpha = \omega_\alpha{}^{\alpha'}(x) v_{\alpha'}(x), \qquad \delta_L \partial_\alpha = \omega_\alpha{}^{\alpha'}(x) \partial_{\alpha'}. \tag{4.41}$$

Then the derivative $\partial_\alpha v_\beta$ transforms as

$$\begin{aligned} \delta_L \partial_\alpha v_\beta &= (\delta_L \partial_\alpha) v_\beta + \partial_\alpha (\delta_L v_\beta) \\ &= \omega_\alpha{}^{\alpha'} \partial_{\alpha'} v_\beta + \partial_\alpha (\omega_\beta{}^{\beta'} v_{\beta'}) \\ &= \omega_\alpha{}^{\alpha'} \partial_{\alpha'} v_\beta + \omega_\beta{}^{\beta'} \partial_\alpha v_{\beta'} + (\partial_\alpha \omega_\beta{}^{\beta'}) v_{\beta'}. \end{aligned} \tag{4.42}$$

The spin connection behaves as follows: Due to the factors $h_\lambda{}^\gamma h_\alpha{}^\mu h_\beta{}^\nu$ in (4.19), the first piece of $\bar\Gamma_{\alpha\beta}{}^\gamma$, call it $\bar\Gamma'_{\alpha\beta}{}^\gamma$, transforms like a local Lorentz tensor:

$$\delta_L \bar\Gamma'_{\alpha\beta}{}^\gamma = \omega_\alpha{}^{\alpha'} \bar\Gamma'_{\alpha'\beta}{}^\gamma + \omega_\beta{}^{\beta'} \bar\Gamma'_{\alpha\beta'}{}^\gamma + \omega^\gamma{}_{\gamma'} \bar\Gamma'_{\alpha\beta}{}^{\gamma'}. \tag{4.43}$$

But from the second piece $\overset{h}{\Gamma}_{\mu\nu}{}^\lambda$ there is a non-tensorial derivative contribution,

$$\begin{aligned} \delta_L \overset{h}{\Gamma}_{\mu\nu}{}^\lambda &= (\delta h_\delta{}^\lambda) \partial_\mu h^\delta{}_\nu + h_\delta{}^\lambda \partial_\mu (\delta h^\delta{}_\nu) \\ &= \omega_\delta{}^{\delta'} h_{\delta'}{}^\lambda \partial_\mu h^\delta{}_\nu + h_\delta{}^\lambda \partial_\mu (\omega^\delta{}_{\delta'} h^{\delta'}{}_\nu) \\ &= \omega_\delta{}^{\delta'} h_{\delta'}{}^\lambda \partial_\mu h^\delta{}_\nu + \omega^\delta{}_{\delta'} h_\delta{}^\lambda \partial_\mu h^{\delta'}{}_\nu + \partial_\mu \omega^\delta{}_{\delta'} (h_\delta{}^\lambda h^{\delta'}{}_\nu) \\ &= \partial_\mu \omega^\delta{}_{\delta'} h_\delta{}^\lambda h^{\delta'}{}_\nu = -\partial_\mu \omega_{\delta'}{}^\delta h_\delta{}^\lambda h^{\delta'}{}_\nu \end{aligned} \tag{4.44}$$

the cancellation in the third line being due to the antisymmetry of $\omega_{\delta}{}^{\delta'} = -\omega^{\delta'}{}_{\delta}$. Thus we arrive at

$$\delta_L \overset{h}{\Gamma}_{\mu\nu}{}^{\lambda} = \partial_\mu \omega^{\delta}{}_{\delta'} h_\delta{}^{\lambda} h^{\delta'}{}_{\nu}, \qquad \delta_L \overline{\Gamma}_{\alpha\beta}{}^{\gamma} = \delta_{L_0} \overline{\Gamma}_{\alpha\beta}{}^{\gamma} + \partial_\alpha \omega_\beta{}^{\gamma}, \quad (4.45)$$

where δ_{L_0} denotes the proper Lorentz tensor transformation law satisfied by $\overline{\Gamma}_{\alpha\beta}{}^{\gamma}$ in (4.43). The last term is precisely what is required to cancel the last non-tensorial piece of (4.42), when transforming $D_\alpha v_\beta$, so that we indeed obtain the covariant transformation law (4.40′).

Armed with these transformation laws it is now straightforward to introduce spinor fields into a gravitational theory. In a freely falling elevator, which is a local inertial frame, a spinor field $\psi(x)$ transforms like

$$\delta_L \psi(x) = -\frac{i}{2} \omega^{\alpha\beta}(x) \Sigma_{\alpha\beta} \psi(x), \quad (4.46)$$

when locally changing from one such frame of reference to another Lorentz transformed one. Here $\Sigma_{\alpha\beta}$ are the spin representation matrices of the local Lorentz group. They are antisymmetric in α, β and satisfy the commutation relations

$$[\Sigma_{\alpha\beta}, \Sigma_{\alpha\gamma}] = -i\,\eta_{\alpha\alpha} \Sigma_{\beta\gamma}. \quad (4.47)$$

For spin-1 they are given explicitly by

$$(\Sigma_{\alpha\beta})_{\alpha'\beta'} = i(\eta_{\alpha\alpha'} \eta_{\beta\beta'} - (\alpha \leftrightarrow \beta)),$$

$$(\Sigma_{\alpha\beta})_{\alpha'}{}^{\beta'} = i(\eta_{\alpha\alpha'} \delta_\beta{}^{\beta'} - (\alpha \leftrightarrow \beta)), \quad (4.48)$$

and (4.46) coincides with (4.40):

$$\delta_L v_\alpha = -\frac{i}{2} \omega^{\gamma\delta} i(\eta_{\gamma\alpha} \delta_\delta{}^{\beta} - (\alpha \leftrightarrow \beta)) v_\beta = \omega_\alpha{}^{\beta} v_\beta. \quad (4.49)$$

For spin-$\frac{1}{2}$, $\Sigma_{\alpha\beta}$ is expressed in terms of Dirac matrices as

$$\Sigma_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta]. \tag{4.50}$$

The derivative of ψ changes as

$$\begin{aligned}\delta_L \partial_\alpha \psi &= \omega_\alpha{}^{\alpha'} \partial_{\alpha'} \psi + \partial_\alpha \delta_L \psi \\ &= \omega_\alpha{}^{\alpha'} \partial_{\alpha'} \psi - \frac{i}{2} \partial_\alpha (\omega^{\beta\gamma} \Sigma_{\beta\gamma}) \psi \\ &= \omega_\alpha{}^{\alpha'} \partial_{\alpha'} \psi - \frac{i}{2} \omega^{\beta\gamma} \Sigma_{\beta\gamma} \partial_\alpha \psi - \frac{i}{2} (\partial_\alpha \omega^{\beta\gamma}) \Sigma_{\beta\gamma} \psi. \end{aligned} \tag{4.51}$$

The first two terms describe the normal Lorentz behavior of $\partial_\alpha \psi$. The last term is due to the dependence of the angles $\omega^{\beta\gamma}(x)$ on x. It can be removed by using the connection $\Gamma_{\alpha\beta}{}^\gamma$ and defining the covariant derivative by

$$D_\alpha \psi(x) \equiv \partial_\alpha \psi(x) + \frac{i}{2} \bar{\Gamma}_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma \psi(x). \tag{4.52}$$

For, if we form

$$\delta_L \frac{i}{2} \bar{\Gamma}_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma \psi, \tag{4.53}$$

we obtain two terms. There is a term with the regular Lorentz transformation property

$$\delta_{L,\omega} \frac{i}{2} \bar{\Gamma}_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma \psi = -\frac{i}{2} \omega^{\sigma\tau} \Sigma_{\sigma\tau} \left(\frac{i}{2} \bar{\Gamma}_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma \psi \right), \tag{4.54}$$

as follows from

$$\frac{i}{2} \delta_L \bar{\Gamma}_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma \psi + \frac{i}{2} \bar{\Gamma}_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma \delta_L \psi \tag{4.55}$$

after applying the commutation rule (4.47). A second term arises from $\partial_\alpha \omega_\beta{}^\gamma$, which is

$$\frac{i}{2} \partial_\alpha \omega_\beta{}^\gamma \Sigma^\beta{}_\gamma \psi \tag{4.56}$$

and cancels against the last term in (4.51). Thus $D_\alpha \psi$ behaves like

$$\delta_L D_\alpha \psi = \omega_\alpha{}^{\alpha'}(x) D_{\alpha'} \psi - \frac{i}{2} \omega^{\beta\gamma}(x) \Sigma_{\beta\gamma} D_\alpha \psi \tag{4.57}$$

and represents, therefore, a proper covariant derivative which generalizes the standard Lorentz transformation behavior to the case of local transformations $\omega_\alpha{}^\beta(x)$.

We can now immediately construct the spin-$\frac{1}{2}$ action for a Dirac particle in a gravitational field (dx is again short notation for d^4x)

$$\begin{aligned}\mathcal{A}_m[h, K, \psi] &= \frac{1}{2} \int dx \sqrt{-g}\, \bar{\psi}(\gamma^\alpha D_\alpha - m)\psi(x) + \text{h.c.} \\ &\equiv \frac{1}{2} \int dx \sqrt{-g}\, \bar{\psi} \gamma^\alpha \left(\partial_\alpha + \frac{i}{2} \bar{\Gamma}_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma \right) \psi(x) + \text{h.c.}\end{aligned} \tag{4.58}$$

If we wish, we may change the derivatives from ∂_α to ∂_μ by using $\partial_\alpha = h_\alpha{}^\mu \partial_\mu$ and $\gamma^\alpha = h^\alpha{}_\mu(x)\gamma^\mu(x)$ so that $\Sigma_{\alpha\beta}(x) = (i/4)[\gamma_\alpha(x), \gamma_\beta(x)]$ and, expressing $\bar{\Gamma}_{\alpha\beta}{}^\gamma$ by (4.19), the action reads

$$\begin{aligned}\mathcal{A}_m[h, K, \psi] &= \frac{1}{2} \int dx \sqrt{-g}\, \bar{\psi}(x) \\ &\quad \times \left[\gamma^\mu(x) \left(\partial_\mu + \frac{i}{2} (K_{\mu\nu}{}^\lambda - \overset{h}{K}_{\mu\nu}{}^\lambda) \Sigma^\nu{}_\lambda \right) - m \right] \psi(x) + \text{h.c.}\end{aligned} \tag{4.59}$$

This is of the form (4.2). Due to the x dependence of γ^μ and $\Sigma^{\mu\nu}{}_\lambda$, this form is, however, not very convenient. Much more useful is the initial expression (4.58)

$$\mathcal{A}_m[h, K, \psi] = \frac{1}{2} \int dx \sqrt{-g}\, \bar{\psi}(x) \{\gamma^\alpha D_\alpha - m\} \psi(x) + \text{h.c.}, \tag{4.60}$$

with the covariant derivative written in the form

$$D_\alpha = h_\alpha{}^\mu(x)\partial_\mu + \frac{i}{2} \bar{\Gamma}_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma = h_\alpha{}^\mu D_\mu \equiv h_\alpha{}^\mu \left(\partial_\mu + \frac{i}{2} \bar{\Gamma}_{\mu\beta}{}^\gamma \Sigma^\beta{}_\gamma \right), \tag{4.61}$$

involving the spin connection

$$\bar{\Gamma}_{\mu\beta}{}^\gamma \equiv h^\alpha{}_\mu \bar{\Gamma}_{\alpha\beta}{}^\gamma. \tag{4.61'}$$

This can easily be generalized to any higher spin if desired.

4.2. FIELD EQUATIONS WITH GRAVITATIONAL SPINNING MATTER

Consider the action of a spin-1/2 field interacting with a gravitation field:

$$\mathcal{A}[h, K, \psi] = -\frac{1}{2\kappa}\int dx\,\sqrt{-g}\,R + \frac{1}{2}\int dx\,\sqrt{-g}\,\bar{\psi}(\gamma^\alpha D_\alpha - m)\,\psi(x) + \text{h.c.}$$
$$\equiv \mathcal{A}_f[h, K] + \mathcal{A}_m[h, K, \psi]. \tag{4.62}$$

It is a functional of the vierbein field $h_\alpha{}^\mu$, the torsion $K_{\mu\nu}{}^\lambda$, and the Dirac field $\psi(x)$. Varying $\mathcal{A}$ with respect to $\bar{\psi}$ we obtain the equation of motion

$$\frac{\delta \mathcal{A}_m}{\delta \bar{\psi}} = \sqrt{-g}\,(\gamma^\alpha D_\alpha - m)\,\psi(x) = 0 \tag{4.63}$$

of a Dirac particle in a general affine space.

To obtain the gravitational field equations we again define the spin-current density, just as we did in (3.17), by differentiating with respect to $K_{\mu\nu}{}^\lambda$, at fixed $h_\alpha{}^\mu$, and find for the gravitational field

$$\frac{\delta \mathcal{A}_f}{\delta K_{\mu\nu}{}^\lambda} = -\frac{1}{2}\sqrt{-g}\,\overset{f}{\Sigma}{}^\nu{}_\lambda{}^{\cdot\mu}, \tag{4.64}$$

as given in (3.53).

From the matter action (4.59) we obtain,

$$\sqrt{-g}\,\overset{m}{\Sigma}{}^\nu{}_\lambda{}^{\cdot\mu} \equiv 2\frac{\delta \mathcal{A}_m}{\delta K_{\mu\nu}{}^\lambda} = \sqrt{-g}\left\{-\frac{i}{2}\bar{\psi}(x)\,\gamma^\mu(x)\,\Sigma^\nu{}_\lambda(x)\,\psi(x) + \text{h.c.}\right\}$$
$$= h_\gamma{}^\lambda h_\alpha{}^\mu h_\beta{}^\nu\sqrt{-g}\left\{-\frac{i}{2}\bar{\psi}(x)\,\gamma^\alpha\Sigma^\beta{}_\gamma\,\psi(x) + \text{h.c.}\right\}$$
$$= h_\gamma{}^\lambda h_\alpha{}^\mu h_\beta{}^\nu\sqrt{-g}\,\overset{m}{\Sigma}{}^\beta{}_\gamma{}^{\cdot\alpha}. \tag{4.65}$$

The expression $\overset{m}{\Sigma}{}^\beta{}_\gamma{}^{\cdot\alpha}$ is recognized as the canonical spin current of a Dirac particle in Minkowski space and $\overset{m}{\Sigma}{}^\nu{}_\lambda{}^{\cdot\mu}$ is its generally covariant analogue. Thus, for the spin-1/2 field, the definition (3.16) of the spin current density is consistent with the canonical definition,

$$\overset{m}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\mu} \equiv -i \sum_i \pi_i{}^{\mu} \Sigma^{\nu}{}_{\lambda}\, \varphi_i = -i \sum_i \frac{\partial L}{\partial D_{\mu}\varphi_i} \Sigma^{\nu}{}_{\lambda}\, \varphi_i . \tag{4.66}$$

where the sum over i covers all independent matter fields of the system. This is also true, in general, by the fact that the general Einstein invariant matter action has the functional form [compare (4.59)]

$$\mathcal{A}_m = \mathcal{A}_m[h, K, \varphi_i] = \int dx\, \sqrt{-g}\, L(h_{\alpha}{}^{\mu}, \varphi_i, D_{\mu}\varphi_i), \tag{4.67}$$

so that indeed, for fixed $h_{\alpha}{}^{\mu}$,

$$2\frac{\delta \mathcal{A}_m}{\delta K_{\mu\nu}{}^{\lambda}}\bigg|_{h_{\alpha}{}^{\mu}} = 2\sqrt{-g} \sum_i \frac{\partial L}{\partial D_{\mu}\varphi_i} \frac{i}{2} \Sigma^{\nu}{}_{\lambda}\, \varphi_i$$

$$\equiv i\sqrt{-g} \sum_i \pi_i{}^{\mu} \Sigma^{\nu}{}_{\lambda}\, \varphi_i = -\overset{m}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\mu}. \tag{4.68}$$

The field equations associated with $\delta K_{\mu\nu}{}^{\lambda}$ are therefore

$$-\kappa \overset{f}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\mu} = \kappa \overset{m}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\mu}, \tag{4.69}$$

thus extending Eq. (3.50) to systems with spinning matter.

Let us now turn to the equations for $h_{\alpha}{}^{\mu}$. It will be useful to define the total symmetric energy-momentum tensor as

$$\sqrt{-g}\, T_{\mu}{}^{\alpha}(x) \equiv \frac{\delta \mathcal{A}}{\delta h_{\alpha}{}^{\mu}(x)}\bigg|_{S_{\mu\nu}{}^{\lambda}}, \quad \sqrt{-g}\, T_{\alpha}{}^{\mu} = -\frac{\delta \mathcal{A}}{\delta h^{\alpha}{}_{\mu}}\bigg|_{S_{\mu\nu}{}^{\lambda}}, \tag{4.70}$$

with the derivative formed at fixed $S_{\mu\nu}{}^{\lambda}$. For the pure gravitational action which depends only on $g^{\mu\nu} = h^{\alpha\mu} h_{\alpha}{}^{\nu}$ and $K_{\mu\nu}{}^{\lambda}$, this definition leads trivially to the same symmetric energy-momentum tensor as that introduced earlier in (3.15) except that one index has the α form. This follows from the chain rule of differentiation together with (3.15)

$$\sqrt{-g}\, \overset{f}{T}_{\mu}{}^{\alpha} \equiv \frac{\delta \mathcal{A}_f}{\delta h_{\alpha}{}^{\mu}} = \frac{\delta \mathcal{A}_f}{\delta g^{\lambda\kappa}} \frac{\partial g^{\lambda\kappa}}{\partial h_{\alpha}{}^{\mu}} = \sqrt{-g}\, \overset{f}{T}_{\mu\kappa} h^{\alpha\kappa}. \tag{4.71}$$

For matter, the actual calculation of the symmetric energy-momentum tensor is most conveniently performed in two steps. Take, for instance, the Dirac field. As a first step we differentiate $\sqrt{-g}$ and $\gamma^\alpha h_\alpha{}^\mu \partial_\mu$ with respect to $h_\alpha{}^\mu$ while keeping, for the moment, $D_\mu = \text{const}$. The result is the so-called canonical energy-momentum tensor,

$$\sqrt{-g}\,\overset{m}{\Theta}_\mu{}^\alpha \equiv \sqrt{-g}\,\frac{1}{2}\left(\bar{\psi}\gamma^\alpha i D_\mu \psi - h^\alpha{}_\mu L\right) + \text{h.c.} \tag{4.72}$$

A moment's thought teaches us that this is a general feature: The derivative of (4.67) with respect to $h_\alpha{}^\mu$ which couples D_μ to the spin indices α gives

$$\frac{\delta \mathscr{A}_m}{\delta h_\alpha{}^\mu} \to \sqrt{-g}\sum_i \frac{\partial L}{\partial D_\nu \varphi_i} D_\mu \varphi_i h^{\alpha\nu}, \tag{4.73}$$

while the derivative of the $\sqrt{-g}$ part adds

$$\frac{\delta \mathscr{A}_m}{\delta h_\alpha{}^\mu} \to -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu} L h^{\alpha\nu}. \tag{4.74}$$

Therefore one always obtains

$$\overset{m}{\Theta}_\mu{}^\alpha = \left(\sum_i \frac{\partial L}{\partial D_\nu \varphi_i} D_\mu \varphi_i - g_{\mu\nu} L\right) h^{\alpha\nu}, \tag{4.75}$$

which is indeed the canonical energy-momentum tensor for an arbitrary Lagrangian containing covariant derivatives. In the particular case of a pure gravitational field we can compare this first step of differentiation at fixed D_μ with the variation (3.28) and find the symmetric part of the equation

$$\overset{f}{\Theta}_\mu{}^\alpha = -\frac{1}{\kappa} G_{\mu\nu} h^\alpha{}_\nu. \tag{4.76}$$

We will see below that this holds, in fact, without symmetrization. Thus the canonical energy-momentum tensor of the gravitational field is equal to minus $1/\kappa$ times the Einstein tensor.

We now turn to the second step, the calculation of the remaining derivative with respect to $h_\alpha{}^\mu$. This is somewhat tedious. Let us write the additional contribution to $\overset{m}{\Theta}_\kappa{}^\delta$ as

$$\sqrt{-g}\,\Delta\overset{m}{\Theta}_{\varkappa}{}^{\delta} = \int dx \frac{\delta\mathscr{A}_m}{\delta K_{\mu\beta}{}^{\gamma}} \frac{\delta\bar{\Gamma}_{\mu\beta}{}^{\gamma}}{\delta h_{\delta}{}^{\varkappa}}\bigg|_{S_{\mu\nu}{}^{\lambda}} = -\frac{1}{2}\int dx\,\sqrt{-g}\,\overset{m}{\Sigma}{}^{\beta}{}_{\gamma}{}^{\cdot\,\mu} \frac{\delta\bar{\Gamma}_{\mu\beta}{}^{\gamma}}{\delta h_{\delta}{}^{\varkappa}}\bigg|_{S_{\mu\nu}{}^{\lambda}}, \tag{4.77}$$

and use for the spin connection the explicit form

$$\bar{\Gamma}_{\mu\beta}{}^{\gamma} = h^{\gamma}{}_{\lambda} h_{\beta}{}^{\nu}(\Gamma_{\mu\nu}{}^{\lambda} - \overset{h}{\Gamma}_{\mu\nu}{}^{\lambda}) = -h_{\beta}{}^{\nu}\overset{\Gamma}{D}_{\mu} h^{\gamma}{}_{\nu} = h^{\gamma}{}_{\nu}\overset{\Gamma}{D}_{\mu} h_{\beta}{}^{\nu} \tag{4.78}$$

where $\overset{\Gamma}{D}_{\mu}$ denotes the part of the covariant derivative containing only the ordinary connection $\Gamma_{\mu\nu}{}^{\lambda}$. If we vary $\delta h_{\mu\beta}{}^{\gamma}$ and hold $\Gamma_{\mu\nu}{}^{\lambda}$ fixed we have

$$\delta\bar{\Gamma}_{\mu\beta}{}^{\gamma}|_{\Gamma_{\mu\nu}{}^{\lambda}} = \delta h^{\gamma}{}_{\nu}\overset{\Gamma}{D}_{\mu} h_{\beta}{}^{\nu} + h^{\gamma}{}_{\nu}\overset{\Gamma}{D}_{\mu}\,\delta h_{\beta}{}^{\nu} \tag{4.79}$$

Since $D_{\mu} h^{\gamma}{}_{\nu} = 0$ [recall (4.20)] we see that $\overset{\Gamma}{D}_{\mu} h_{\beta}{}^{\nu} = \bar{\Gamma}_{\mu\beta}{}^{\lambda} h_{\lambda}{}^{\nu}$ and we may write

$$\delta\bar{\Gamma}_{\mu\beta}{}^{\gamma}|_{\Gamma_{\mu\nu}{}^{\lambda}} = h^{\gamma}{}_{\nu} D_{\mu}\,\delta h_{\beta}{}^{\nu} \tag{4.80}$$

Inserting this into (4.77), a partial integration gives the first contribution

$$\Delta_1\overset{m}{\Theta}_{\varkappa}{}^{\delta} = -(1/2) D_{\mu}\overset{m}{\Sigma}_{\varkappa}{}^{\delta,\,\mu}. \tag{4.81a}$$

We now include the contribution from $\delta\Gamma_{\mu\nu}{}^{\lambda}$. Using the decomposition (3.43) with $\delta S_{\mu\nu\lambda} = 0$, i.e., $\delta K_{\mu\nu\lambda} = 0$, we find

$$\Delta_2\overset{m}{\Theta}_{\varkappa}{}^{\delta} = \frac{1}{4}\left[D_{\mu}\left(\overset{m}{\Sigma}{}^{\nu\sigma,\,\mu} - \overset{m}{\Sigma}{}^{\sigma\mu,\,\nu} + \overset{m}{\Sigma}{}^{\mu\nu,\,\sigma}\right)\right]\frac{\partial g_{\nu\sigma}}{\partial h_{\delta}{}^{\varkappa}}. \tag{4.81b}$$

With

$$\frac{\partial g_{\nu\sigma}}{\partial h_{\delta}{}^{\varkappa}} = -g_{\nu\varkappa} h^{\delta}{}_{\sigma} + (\nu \leftrightarrow \sigma),$$

this gives, altogether,

$$\Delta\overset{m}{\Theta}_{\varkappa}{}^{\delta}(x) = (-1/2)\,D^{*}{}_{\mu}\left(\overset{m}{\Sigma}_{\varkappa}{}^{\delta,\,\mu} - \overset{m}{\Sigma}{}^{\delta\mu}{}_{,\,\varkappa} + \overset{m}{\Sigma}{}^{\mu}{}_{\varkappa,\,\delta}\right) \tag{4.82}$$

This is precisely the same type of correction $\Delta\Theta_{\varkappa}{}^{\delta} = \Delta\Theta_{\varkappa}{}^{\nu} h^{\delta}{}_{\nu}$ that had been added to the canonical energy-momentum tensor $\Theta_{\varkappa\delta}$ of the gravitational field in (3.49), in order to produce the symmetric one $T_{\varkappa\delta}$. Here, it is obtained for arbitrary spinning matter fields:

$$\overset{m}{T}_{\varkappa\nu} = \overset{m}{\Theta}_{\varkappa\nu} + \Delta\overset{m}{\Theta}_{\varkappa\nu} = \overset{m}{\Theta}_{\varkappa\nu} - \frac{1}{2} D^{*}{}_{\mu} \left(\overset{m}{\Sigma}_{\varkappa\nu,}{}^{\mu} - \overset{m}{\Sigma}_{\nu}{}^{\mu}{}_{,\varkappa} + \overset{m}{\Sigma}{}^{\mu}{}_{\varkappa,\nu} \right). \quad (4.83)$$

For spin-$\frac{1}{2}$ this is the expression first found by Belinfante in 1939. We have lowered the index ν on both sides which is permissible due to the covariant form of the equation.

In terms of $\overset{m}{T}_{\mu\nu}$, the field equations which follows from variations of the action with respect to $\delta h_{\alpha}{}^{\mu}$ have once more the simple form (3.51′):

$$-\kappa \overset{h}{T}{}^{\mu\nu} = \kappa \overset{m}{T}{}^{\mu\nu}, \quad (4.84)$$

now derived in the presence of spinning matter.

CHAPTER FIVE

COVARIANT CONSERVATION LAWS

According Noether's theorem, the invariance of the action under general coordinate transformations and local Lorentz transformations must be associated with certains conservation laws. For the following considerations, it will be convenient to consider $h_\alpha{}^\mu(\mathbf{x})$ and $\bar{\Gamma}_{\mu\beta}{}^\alpha$ as independent variables and rename $\bar{\Gamma}_{\mu\beta}{}^\alpha$ as $A_{\mu\beta}{}^\alpha$. Then, from the derivation in (4.71) and (4.72), it follows that varying the action in $h_\alpha{}^\mu$ at *fixed* $A_{\mu\beta}{}^\gamma$ gives the canonical energy-momentum tensor

$$\frac{\delta\mathcal{A}[h_\alpha{}^\mu,\ A_{\mu\beta}{}^\gamma]}{\delta h_\alpha{}^\mu} = \sqrt{-g}\,\Theta_\mu{}^\alpha, \tag{5.1}$$

while the variation with respect to $A_{\mu\beta}{}^\gamma$ produces the spin current density[a]

$$\begin{aligned}\frac{\delta\mathcal{A}[h_\alpha{}^\mu,\ A_{\mu\beta}{}^\gamma]}{\delta A_{\mu\beta}{}^\gamma} &= -\frac{1}{2}\sqrt{-g}\,\Sigma^\beta{}_\gamma{}^{\cdot\alpha} h_\alpha{}^\mu \\ &\equiv -\frac{1}{2}\sqrt{-g}\,\Sigma^\beta{}_\gamma{}^{\cdot\mu}.\end{aligned} \tag{5.2}$$

[a] Recall that the field $A_{\mu\beta}{}^\gamma$ has the pure contortion form, $A_{\mu\beta}{}^\gamma = h^\gamma{}_\lambda h_\beta{}^\nu (K_{\mu\nu}{}^\lambda - \overset{h}{K}_{\mu\nu}{}^\lambda)$ and thus is antisymmetric in β, γ, as is the case with $\bar{\Gamma}_{\alpha\beta}{}^\gamma$.

These quantities will now be shown to satisfy *covariant conservation laws.*

5.1. SPIN DENSITY

Consider first local Lorentz transformations. Under these the vierbein fields behave vectorially in the index α,

$$\delta_L h_\alpha{}^\mu(x) = \omega_\alpha{}^{\alpha'}(x) h_{\alpha'}{}^\mu(x). \tag{5.3}$$

Similarly, the field $A_{\mu\beta}{}^\gamma$ is a tensor in β, γ plus a derivative term [see (4.49)]

$$\delta_L A_{\mu\beta}{}^\gamma = \omega_\beta{}^{\beta'}(x) A_{\mu\beta'}{}^\gamma + \omega^\gamma{}_{\gamma'}(x) A_{\mu\beta}{}^{\gamma'} + \partial_\mu \omega_\beta{}^\gamma(x). \tag{5.4}$$

The change of the action has to vanish. This gives

$$\begin{aligned}\delta_L \mathscr{A} = &\int dx \left\{ \frac{\delta \mathscr{A}}{\delta h_\alpha{}^\mu(x)} \omega_\alpha{}^{\alpha'}(x) h_{\alpha'}{}^\mu(x) \right.\\ &\left. + \frac{\delta \mathscr{A}}{\delta A_{\mu\beta}{}^\gamma(x)} (\omega_\beta{}^{\beta'}(x) A_{\mu\beta'}{}^\gamma(x) + \omega^\gamma{}_{\gamma'}(x) A_{\mu\beta}{}^{\gamma'}(x) + \partial_\mu \omega_\beta{}^\gamma(x)) \right\}\\ = &\int dx \sqrt{-g} \left\{ \Theta_\mu{}^\alpha \omega_\alpha{}^{\alpha'} h_{\alpha'}{}^\mu \right.\\ &\left. - \frac{1}{2} \Sigma^\beta{}_\gamma{}^{\cdot\,\mu} (\omega_\beta{}^{\beta'} A_{\mu\beta'}{}^\gamma + \omega^\gamma{}_{\gamma'} A_{\mu\beta}{}^{\gamma'} + \partial_\mu \omega_\beta{}^\gamma) \right\}.\end{aligned} \tag{5.5}$$

Partially integrating the last term gives

$$\begin{aligned}\int dx &\left\{ \sqrt{-g}\, \Theta_\mu{}^\alpha \omega_\alpha{}^{\alpha'} h_{\alpha'}{}^\mu + \frac{1}{2} \partial_\mu (\sqrt{-g}\, \Sigma^\beta{}_\gamma{}^{\cdot\,\mu}) \omega_\beta{}^\gamma \right.\\ &\left. - \frac{1}{2} \sqrt{-g}\, \Sigma^\beta{}_\gamma{}^{\cdot\,\mu} (\omega_\beta{}^{\beta'} A_{\mu\beta'}{}^\gamma + \omega^\gamma{}_{\gamma'} A_{\mu\beta}{}^{\gamma'}) \right\}.\end{aligned} \tag{5.6}$$

Since $\omega_\beta{}^\gamma(x')$ is an arbitrary antisymmetric function of x' it can be chosen to be zero everywhere except at some place x and we find

$$\frac{1}{2}\sqrt{-g}\,(\Theta_\mu{}^\beta h_\gamma{}^\mu - \Theta_{\mu\gamma} h^{\beta\mu}) + \frac{1}{2}\partial_\mu \sqrt{-g}\,\Sigma^\beta{}_\gamma{}^{,\mu}$$

$$-\frac{1}{2}\sqrt{-g}\,(\Sigma^\beta{}_{\delta'}{}^\mu A_{\mu\gamma}{}^\delta + \Sigma^\delta{}_{\beta'}{}^\mu A_{\mu\delta}{}^\beta). \tag{5.7}$$

Defining

$$\Theta_\gamma{}^\beta \equiv \Theta_\mu{}^\beta h_\gamma{}^\mu \tag{5.8}$$

and raising the index γ with the Minkowski metric $\eta^{\gamma\gamma'}$, this reads

$$\frac{1}{2}(\Theta^{\gamma\beta} - (\beta\gamma)) + \frac{1}{2}\Gamma_{\mu\sigma}{}^\sigma \Sigma^{\beta\gamma,\mu} + \frac{1}{2}\overset{L}{D}_\mu \Sigma^{\beta\gamma\mu} = 0. \tag{5.9}$$

where $\overset{L}{D}_\mu$ is the covariant derivative for the local Lorentz index γ, i.e., for a vector

$$\overset{L}{D}_\mu v_\alpha = \partial_\mu v_\alpha - A_{\mu\alpha}{}^\beta v_\beta = h^\beta{}_\mu D_\beta v_\alpha, \tag{5.10}$$

$$\overset{L}{D}_\mu v^\alpha = \partial_\mu v^\alpha - A_\mu{}^\alpha{}_\beta v^\beta = \partial_\mu v^\alpha + A_{\mu\beta}{}^\alpha v^\beta = h^\beta{}_\mu D_\beta v^\alpha. \tag{5.11}$$

The derivative $\overset{L}{D}_\mu \Sigma^{\beta\gamma,\mu}$ can be made completely covariant also in the Einstein index μ, by going to

$$\overset{L}{D}_\mu \Sigma^{\beta\gamma,\nu} \equiv \overset{L}{D}_\mu \Sigma^{\beta\gamma,\nu} - \Gamma_{\mu\lambda}{}^\nu \Sigma^{\beta\gamma,\lambda}. \tag{5.12}$$

But the last term cancels part of to the middle one in (5.9) and we have [recall (3.45)]

$$\frac{1}{2}(\Theta^{\gamma\beta} - \Theta^{\beta\gamma}) + \frac{1}{2}D_\mu{}^* \Sigma^{\beta\gamma,\mu} = 0. \tag{5.13}$$

Being a covariant relation, this can be multiplied by $h_\beta{}^\lambda h_\gamma{}^\varkappa$ and the vierbeins can be moved under the derivative, yielding

$$\Theta^{[\varkappa,\lambda]} + \frac{1}{2}h_\beta{}^\lambda h_\gamma{}^\varkappa D_\mu{}^* \Sigma^{\beta\gamma,\mu} = \Theta^{[\varkappa,\lambda]} + \frac{1}{2}D_\mu{}^* \Sigma^{\lambda\varkappa,\mu} = 0. \tag{5.14}$$

For a vector this type of operation is demonstrated as follows:

$$
\begin{aligned}
h_\alpha{}^\nu D_\mu v^\alpha &= h_\alpha{}^\nu(\partial_\mu v^\alpha + A_{\mu\beta}{}^\alpha v^\beta) \\
&= \partial_\mu(h_\alpha{}^\nu v^\alpha) + h_\alpha{}^\nu A_{\mu\beta}{}^\alpha v^\beta - (\partial_\mu h_\alpha{}^\nu)v^\alpha \\
&= \partial_\mu v^\nu + h_\alpha{}^\nu h^\alpha{}_\lambda h_\beta{}^\nu(\Gamma_{\mu\nu}{}^\lambda - \overset{h}{\Gamma}_{\mu\nu}{}^\lambda)v^\beta + \overset{h}{\Gamma}_{\mu\lambda}{}^\nu h_\alpha{}^\lambda v^\alpha \\
&= \partial_\mu v^\nu + \Gamma_{\mu\lambda}{}^\nu v^\lambda \equiv D_\mu v^\nu,
\end{aligned}
\tag{5.15}
$$

and the extension to tensors is obvious.

5.2. ENERGY-MOMENTUM DENSITY

Let us now deduce the consequence of local Einstein invariance. In this case the space-time coordinates must be transformed as well and the action is invariant in the following sense (again dx stands for d^4x)

$$
\mathscr{A} = \int dx\,\sqrt{-g(x)}\,L(h(x),\,A(x)) = \int dx'\,\sqrt{-g'(x')}\,L(h'(x'),\,A'(x'))
\tag{5.16}
$$

If we change the variables x' to x in the second integral we see that the difference

$$
\int dx\left\{\sqrt{-g'(x)}\,L(h'(x),\,A'(x)) - \sqrt{-g(x)}\,L(h(x),\,A(x))\right\}
\tag{5.17}
$$

must be concentrated in the neighborhood of the surface of the integration volume. This is because the original integrations $\int d^4x'$, $\int d^4x$ covered the *same* volume so that, after the change of variables $x' \to x$, the first integral runs through a slightly different region. Infinitesimally this amounts to the statement that

$$
\delta_E\mathscr{A} = \int dx\,\delta_E[\sqrt{-g(x)}\,L(h(x),\,A(x))]
\tag{5.18}
$$

is a pure surface term. Recall that δ_E is the substantial change at *fixed* argument x [see (2.20)].

Now, under Einstein transformations the metric transforms as

$$
\delta_E\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta_E g^{\mu\nu} = \tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta_E g_{\mu\nu},
\tag{5.19}
$$

which, upon inserting (2.37), yields

$$\tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu}[\xi^\lambda\partial_\lambda g_{\mu\nu} + (\partial_\mu \xi^\lambda)g_{\lambda\nu} + (\partial_\nu\xi^\lambda)g_{\mu\lambda}]. \tag{5.20}$$

Therefore

$$\delta_E\sqrt{-g} = \xi^\lambda\partial_\lambda\sqrt{-g} + \sqrt{-g}\,\partial_\lambda\xi^\lambda = \partial_\lambda(\xi^\lambda\sqrt{-g}) \tag{5.21}$$

and

$$\delta_E\int dx\sqrt{-g} = \int dx\sqrt{g}\,D_\lambda\xi^\lambda = \int dx\,\partial_\lambda(\xi^\lambda\sqrt{-g}). \tag{5.22}$$

This shows that the trivial action $\int dx\sqrt{-g}$ indeed changes by a pure surface term. There is complete invariance if we require $\xi^\lambda(x)$ to vanish at the surface.

The same result holds for a general action if L is a scalar Lagrangian satisfying

$$L'(x') = L(x) \tag{5.23}$$

and therefore

$$\begin{aligned}\delta_E L(x) &\equiv L'(x) - L(x) = L'(x') - L(x')\\ &= L(x) - L(x') = \xi^\lambda\partial_\lambda L(x).\end{aligned} \tag{5.24}$$

The variation of $\mathscr{A}$ is

$$\begin{aligned}\delta_E\mathscr{A} = \delta_E\int dx(\sqrt{-g}\,L(x) &= \int dx\left\{[\delta_E\sqrt{-g}]L(x) + \sqrt{-g}\,\delta_E L(x)\right\}\\ = \int dx&\left\{\partial_\lambda[\xi^\lambda\sqrt{-g}]L(x) + \sqrt{-g}\,\xi^\lambda\partial_\lambda L(x)\right\}\\ = \int dx\,&\partial_\lambda(\xi^\lambda\sqrt{-g}\,L(x)).\end{aligned} \tag{5.25}$$

We can now derive the covariant conservation law associated with Einstein invariance by using the substantial variations $\delta_E h_\alpha{}^\mu$ and $\delta_E A_{\mu\beta}{}^\gamma$ and calculating $\delta_E\mathscr{A}$ once more as follows:

$$\delta_E \mathscr{A} = \int dx \left(\frac{\delta \mathscr{A}}{\delta h_\alpha{}^\mu} \delta_E h_\alpha{}^\mu + \frac{\delta \mathscr{A}}{\delta A_{\mu\beta}{}^\gamma} \delta_E A_{\mu\beta}{}^\gamma \right)$$
$$= \int dx \left(\sqrt{-g}\, \Theta_\mu{}^\alpha \delta_E h_\alpha{}^\mu - \frac{1}{2} \sqrt{-g}\, \Sigma^\beta{}_\gamma{}^{\cdot\mu} \delta_E A_{\mu\beta}{}^\gamma \right). \qquad (5.26)$$

The substantial variations of the vierbein fields $h_\alpha{}^\mu$ and $A_{\mu\beta}{}^\gamma$ are those of a vector in the index μ:

$$\delta_E h_\alpha{}^\mu = \xi^\lambda \partial_\lambda h_\alpha{}^\mu - \partial_\varkappa \xi^\mu h_\alpha{}^\varkappa, \qquad \delta_E A_{\mu\beta}{}^\gamma = \xi^\lambda \partial_\lambda A_{\mu\beta}{}^\gamma + \partial_\mu \xi^\lambda A_{\lambda\beta}{}^\gamma. \qquad (5.27)$$

Plugging these into (5.26), we have

$$\delta_E \mathscr{A} = \int dx \Big\{ \sqrt{-g}\, \Theta_\mu{}^\alpha (\xi^\lambda \partial_\lambda h_\alpha{}^\mu - \partial_\varkappa \xi^\mu h_\alpha{}^\varkappa)$$
$$- \frac{1}{2} \sqrt{-g}\, \Sigma^\beta{}_\gamma{}^{\cdot\mu} (\xi^\lambda \partial_\lambda A_{\mu\beta}{}^\gamma + \partial_\mu \xi^\lambda A_{\lambda\beta}{}^\gamma) \Big\}. \qquad (5.28)$$

After partial integrations and letting ξ^λ be zero everywhere, except for a δ-function singularity at some place x, gives

$$\partial_\varkappa (\sqrt{-g}\, \Theta_\lambda{}^\alpha h_\alpha{}^\varkappa) + \sqrt{-g}\, \Theta_\mu{}^\alpha \partial_\lambda h_\alpha{}^\mu$$
$$+ \frac{1}{2} \partial_\mu (\sqrt{-g}\, \Sigma^\beta{}_\gamma{}^{\cdot\mu} A_{\lambda\beta}{}^\gamma) - \frac{1}{2} \sqrt{-g}\, \Sigma^\beta{}_\gamma{}^{\cdot\mu} \partial_\lambda A_{\mu\beta}{}^\gamma = 0. \qquad (5.29)$$

The second line can be rewritten as

$$\frac{1}{2} \partial_\mu (-\sqrt{-g}\, \Sigma^\beta{}_\gamma{}^{\cdot\mu}) A_{\lambda\beta}{}^\gamma + \frac{1}{2} \sqrt{-g}\, \Sigma^\beta{}_\gamma{}^{\cdot\mu} (\partial_\mu A_{\lambda\beta}{}^\gamma - \partial_\lambda A_{\mu\beta}{}^\gamma). \qquad (5.30)$$

If we introduce the covariant curl of the A field,

$$F_{\mu\lambda\beta}{}^\gamma \equiv \partial_\mu A_{\lambda\beta}{}^\gamma - \partial_\lambda A_{\mu\beta}{}^\gamma - (A_{\mu\beta}{}^\delta A_{\lambda\delta}{}^\gamma - (\mu \leftrightarrow \nu)), \qquad (5.31)$$

then (5.30) becomes

$$\frac{1}{2}\partial_\mu(\sqrt{-g}\,\Sigma^{\beta}{}_{\gamma}{}^{\cdot\mu})A_{\lambda\beta}{}^{\gamma}+\frac{1}{2}\sqrt{-g}\,\Sigma^{\beta}{}_{\gamma}{}^{\cdot\mu}(A_{\mu\beta}{}^{\delta}A_{\lambda\delta}{}^{\gamma}-(\mu\leftrightarrow\lambda))$$
$$+\frac{1}{2}\sqrt{-g}\,\Sigma^{\beta}{}_{\gamma}{}^{\cdot\mu}F_{\mu\lambda\beta}{}^{\gamma}. \tag{5.32}$$

But the first three terms can be collected into a covariant derivative, i.e.,

$$\frac{1}{2}\sqrt{-g}\,D_\mu{}^{*}\Sigma\ {}^{\beta}{}_{\gamma}{}^{\cdot\mu}A_{\lambda\beta}{}^{\gamma}, \tag{5.33}$$

so that the second line in (5.29) becomes, with the conservation law (5.13)

$$-\sqrt{-g}\,\Theta_{\gamma}{}^{\beta}A_{\lambda\beta}{}^{\gamma}+\frac{1}{2}\sqrt{-g}\,\Sigma^{\beta}{}_{\gamma}{}^{\cdot\mu}F_{\mu\lambda\beta}{}^{\gamma}. \tag{5.34}$$

In the first line of (5.31) we write

$$\Theta_\mu{}^{\alpha}\partial_\lambda h_\alpha{}^{\mu}=\Theta_\mu{}^{\alpha}D^{L}{}_{\lambda}h_\alpha{}^{\mu}+\Theta_\mu{}^{\alpha}A_{\lambda\alpha}{}^{\beta}h_\beta{}^{\mu} \tag{5.35}$$

and (5.29) takes the form

$$\partial_\varkappa(\sqrt{-g}\,\Theta_\lambda{}^{\varkappa})+\sqrt{-g}\,\Theta_\mu{}^{\alpha}D^{L}{}_{\lambda}h_\alpha{}^{\mu}-\frac{1}{2}\sqrt{-g}\,\Sigma^{\beta}{}_{\gamma}{}^{\cdot\mu}F_{\lambda\mu\beta}{}^{\gamma}=0. \tag{5.36}$$

This equation is covariant under local Lorentz transformations but not yet manifestly so under Einstein transformations. In order to verify the latter we observe that the derivative D^{L} of h can be rewritten as

$$D^{L}{}_{\lambda}h_\alpha{}^{\mu}=\partial_\lambda h_\alpha{}^{\mu}-A_{\lambda\alpha}{}^{\beta}h_\beta{}^{\mu}$$
$$=-\overset{h}{\Gamma}_{\lambda\varkappa}{}^{\mu}h_\alpha{}^{\varkappa}-(\Gamma_{\lambda\sigma}{}^{\mu}-\overset{h}{\Gamma}_{\lambda\sigma}{}^{\mu})h_\alpha{}^{\sigma}=-\Gamma_{\lambda\sigma}{}^{\mu}h_\alpha{}^{\sigma}, \tag{5.37}$$

in accordance with the identity $D_\lambda h_\alpha{}^{\mu}=0$. Then the second term is

$$-\sqrt{-g}\,\Gamma_{\lambda\sigma}{}^{\mu}\Theta_\mu{}^{\sigma}, \tag{5.38}$$

we now rewrite the first term as

$$\sqrt{-g}(D^{*}{}_{\varkappa}\Theta_\lambda{}^{\varkappa}+\Gamma_{\varkappa\lambda}{}^{\tau}\Theta_\tau{}^{\varkappa}). \tag{5.39}$$

Then the completely covariant conservation law for the energy-momentum tensor is

$$D^*_{\varkappa}\Theta_{\lambda}{}^{\varkappa} + 2S_{\varkappa\lambda}{}^{\tau}\Theta_{\tau}{}^{\varkappa} - \frac{1}{2}\Sigma^{\beta}{}_{\gamma}{}^{\cdot\,\mu}\mathrm{F}_{\lambda\mu\beta}{}^{\gamma} = 0. \tag{5.40}$$

5.3. COVARIANT DERIVATION OF CONSERVATION LAWS

It should be noted that the conservation laws of energy, momentum and angular momentum can be derived somewhat more efficiently, if some initial effort is spent in preparing the Einstein and local Lorentz transformations (5.27), (5.3), (5.4) of $h_{\alpha}{}^{\mu}$ and $A_{\mu\alpha}{}^{\beta}$ in covariant form. Take $\delta_E h_{\alpha}{}^{\mu}$. It can be rewritten as

$$\delta_E h_{\alpha}{}^{\mu} = \xi^{\lambda}\partial_{\lambda} h_{\alpha}{}^{\mu} - D_{\lambda}\xi^{\mu} h_{\alpha}{}^{\lambda} + \Gamma_{\lambda\varkappa}{}^{\mu} h_{\alpha}{}^{\lambda}\xi^{\varkappa} \tag{5.41}$$

where as

$$\partial_{\lambda} h_{\alpha}{}^{\mu} = -\overset{h}{\Gamma}_{\lambda\nu}{}^{\mu} h_{\alpha}{}^{\nu} = A_{\lambda\alpha}{}^{\beta} h_{\beta}{}^{\mu} - \Gamma_{\lambda\nu}{}^{\mu} h_{\alpha}{}^{\nu}, \tag{5.42}$$

so that we arrive at the covariant form

$$\delta_E h_{\alpha}{}^{\mu} = -D_{\alpha}\xi^{\mu} + (A_{\lambda\alpha}{}^{\mu} - 2S_{\lambda\alpha}{}^{\mu})\xi^{\lambda} \tag{5.43}$$

The reciprocal field $h^{\alpha}{}_{\mu}$ transforms via

$$\delta_E h^{\alpha}{}_{\mu} = D_{\mu}\xi^{\alpha} - (A_{\beta\mu}{}^{\alpha} - 2S_{\beta\mu}{}^{\alpha})\xi^{\beta}. \tag{5.43'}$$

Similarly, we find

$$\begin{aligned}\delta_E A_{\mu\alpha}{}^{\beta} &= \xi^{\lambda}\partial_{\lambda} A_{\mu\alpha}{}^{\beta} + D_{\mu}\xi^{\lambda} A_{\lambda\alpha}{}^{\beta} - \Gamma_{\mu\varkappa}{}^{\lambda} A_{\lambda\alpha}{}^{\beta}\xi^{\varkappa} \\ &= D_{\mu}(\xi^{\lambda} A_{\lambda\alpha}{}^{\beta}) - \xi^{\lambda}(D_{\mu} A_{\lambda\alpha}{}^{\beta} - \partial_{\lambda} A_{\mu\alpha}{}^{\beta}) - \Gamma_{\mu\varkappa}{}^{\lambda} A_{\lambda\alpha}{}^{\beta}\xi^{\varkappa} \\ &= D_{\mu}(\xi^{\lambda} A_{\lambda\alpha}{}^{\beta}) - \xi^{\lambda} F_{\mu\lambda\alpha}{}^{\beta}.\end{aligned} \tag{5.44}$$

Under local Lorentz transformations, the vierbein field has already its simplest possible form,

$$\delta_L h_{\alpha}{}^{\mu} = \omega_{\alpha}{}^{\beta} h_{\beta}{}^{\mu}, \tag{5.45}$$

while $A_{\mu\alpha}{}^{\beta}$ acquires the typical additive term of a gauge field

$$\delta_L A_{\mu\alpha}{}^{\beta} = D_\mu \omega_\alpha{}^{\beta}. \tag{5.46}$$

Using these covariant transformation rules, the variations of the action (5.5), (5.28) become

$$\delta_L \mathscr{A} = \int dx \sqrt{-g}\, \{\Theta_\beta{}^{\alpha}\omega_\alpha{}^{\beta} h_\beta{}^{\mu} - (1/2)\, \Sigma^{\alpha}{}_{\beta}{}^{\cdot\,\mu} D_\mu \omega_\alpha{}^{\beta}\}, \tag{5.47}$$

$$\delta_E \mathscr{A} = \int dx \sqrt{-g}\, \{\Theta_\mu{}^{\alpha}(-D_\lambda \xi^{\mu} h_\alpha{}^{\lambda} + (A_{\lambda\alpha}{}^{\mu} - 2S_{\lambda\alpha}{}^{\mu})\xi^{\lambda}) - (1/2)\, \Sigma^{\alpha}{}_{\beta}{}^{\cdot\,\mu}[D_\mu(\xi^{\lambda} A_{\lambda\alpha}{}^{\beta}) - \xi^{\lambda} F_{\mu\lambda\alpha}{}^{\beta}]\}. \tag{5.48}$$

A partial integration of (5.47) [using (3.46), (3.48)] then gives directly the divergence of the spin current (5.13). A partial integration of (5.48) leads to

$$D^*{}_\lambda \Theta_\mu{}^{\lambda} + (A_{\mu\alpha}{}^{\beta} - 2S_{\mu\alpha}{}^{\beta})\Theta_\beta{}^{\alpha} + (1/2) D^*{}_\nu \Sigma^{\alpha}{}_{\beta}{}^{\cdot\,\nu} A_{\mu\alpha}{}^{\beta} + (1/2)\, \Sigma^{\alpha}{}_{\beta}{}^{\cdot\,\nu} F_{\nu\mu\alpha}{}^{\beta} = 0 \tag{5.49}$$

which, after inserting (5.13), reduces correctly to the covariant, conservation law for the canonical energy-momentum tensor (5.40).

5.4. MATTER WITH INTEGER SPIN

If matter fields only carried integer spin it would not be necessary to introduce the $h^{\alpha}{}_{\mu}$, $A_{\mu\alpha}{}^{\beta}$ fields. Then there would only be invariance under Einstein transformations from symmetry considerations. The law of angular momentum conservation requires the use of equation of motion. The action may be written in terms of $g_{\mu\nu}$ and $K_{\mu\nu}{}^{\lambda}$ with the aid of $\Gamma_{\mu\nu}{}^{\lambda} = \{{}^{\lambda}_{\mu\nu}\} + K_{\mu\nu}{}^{\lambda}$ and Einstein invariance amounts to

$$\delta_E \mathscr{A} = \int dx \left(\left.\frac{\delta \mathscr{A}}{\delta g_{\mu\nu}}\right|_{S_{\mu\nu}{}^{\lambda}} \delta_E g_{\mu\nu} + \left.\frac{\delta \mathscr{A}}{\delta K_{\mu\nu}{}^{\lambda}}\right|_{g_{\mu\nu}} \delta_E K_{\mu\nu}{}^{\lambda} \right)$$

$$= -\frac{1}{2}\int dx \sqrt{-g}\, \{T^{\mu\nu}(\xi^{\lambda}\partial_\lambda g_{\mu\nu} + \partial_\mu \xi^{\lambda} g_{\lambda\nu} + \partial_\nu \xi^{\lambda} g_{\mu\lambda}) + \Sigma^{\nu}{}_{\varkappa}{}^{\cdot\,\mu}(\xi^{\lambda}\partial_\lambda K_{\mu\nu}{}^{\varkappa} + \partial_\mu \xi^{\lambda} K_{\lambda\nu}{}^{\varkappa} + \partial_\nu \xi^{\lambda} K_{\mu\lambda}{}^{\varkappa} - \partial_\lambda \xi^{\varkappa} K_{\mu\nu}{}^{\lambda})\}, \tag{5.50}$$

where we have used the definition (3.14)–(3.17) and inserted the transformation laws under general coordinate transformations. We have omitted the matter label m since the equations in this section apply just as well to the gravitational action $\mathscr{A}_f$. The calculations are simplified if we define the symmetrized canonical energy-momentum tensor by

$$\left.\frac{\delta \mathscr{A}}{\delta g_{\mu\nu}}\right|_{\Gamma_{\mu\nu}{}^{\lambda}=\text{const.}} \equiv -\frac{1}{2}\sqrt{-g}\,(\Theta_{\mu\nu}+\Theta_{\nu\mu}). \tag{5.51}$$

It is easy to see that this definition agrees with (4.72) (by differentiating with respect to $h_\alpha{}^\mu$ at fixed $A_{\mu\alpha}{}^\beta$ and changing the index α to ν) if there are no spin-$\frac{1}{2}$ fields. It may also be verified by forming

$$\left.\frac{\delta \mathscr{A}}{\delta g_{\mu\nu}(x)}\right|_{S_{\mu\nu}{}^{\lambda}=\text{const.}} = \left.\frac{\delta \mathscr{A}}{\delta g_{\mu\nu}(x)}\right|_{\Gamma_{\mu\nu}{}^{\lambda}=\text{const.}} + \int dy \left.\frac{\delta \mathscr{A}}{\delta \Gamma_{\sigma\tau}{}^{\lambda}(x)}\right|_{g_{\mu\nu}=\text{const.}} \left.\frac{\delta \Gamma_{\sigma\tau}{}^{\lambda}(y)}{\delta g_{\mu\nu}(x)}\right|_{S_{\mu\nu}{}^{\lambda}=\text{const.}}, \tag{5.52}$$

so that one obtains the standard Belinfante relation (4.83) between $T_{\mu\nu}$ and $\Theta_{\mu\nu}$. For pure gravity, (5.51) is in accord with (4.76) which states that $\Theta_{\mu\nu}$ is the Einstein tensor [recall (4.76)] up to a factor $-\kappa$

$$-\kappa\Theta_{\mu\nu} = G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R,$$

as can be seen from (3.28) and the Belinfante relation (4.83) again coincides with (3.49).

Thus we can evaluate the consequences of Einstein invariance by using Θ and Σ and considering, instead of (5.50), the variation

$$\begin{aligned}0 = \delta_E\mathscr{A} &= \int dx \left\{ \left.\frac{\delta \mathscr{A}}{\delta g_{\mu\nu}}\right|_{\Gamma_{\mu\nu}{}^{\lambda}} \delta_E g_{\mu\nu} + \left.\frac{\delta \mathscr{A}}{\delta \Gamma_{\mu\nu}{}^{\lambda}}\right|_{g_{\mu\nu}} \delta_E \Gamma_{\mu\nu}{}^{\lambda} \right\} \\ &= -\frac{1}{2}\int dx \sqrt{-g}\,\{\Theta^{\mu\nu}(\xi^\lambda\partial_\lambda g_{\mu\nu} + \partial_\mu\xi^\lambda g_{\lambda\nu} + \partial_\nu\xi^\lambda g_{\mu\lambda}) \\ &\quad - \Sigma^{\nu}{}_{\varkappa}{}^{\cdot\mu}(\xi^\lambda\partial_\lambda\Gamma_{\mu\nu}{}^{\varkappa} + \partial_\mu\xi^\lambda\Gamma_{\lambda\nu}{}^{\varkappa} + \partial_\nu\xi^\lambda\Gamma_{\mu\lambda}{}^{\varkappa} - \partial_\lambda\xi^{\varkappa}\Gamma_{\mu\nu}{}^{\lambda} + \partial_\mu\partial_\nu\xi^{\varkappa})\}.\end{aligned}$$

It is again useful to bring the variations $\delta_E g_{\mu\nu}$, $\delta_E\Gamma_{\mu\nu}{}^{\lambda}$ into covariant form. We rewrite the Einstein variation of the metric as

$$\delta_E g_{\mu\nu} = \overset{\{\}}{D}_\mu \xi_\nu + \overset{\{\}}{D}_\nu \xi_\mu = D_\mu \xi_\nu + D_\nu \xi_\mu + [K_{\mu\nu}{}^\lambda + (\mu \leftrightarrow \nu)]\,\xi_\lambda$$
$$= D_\mu \xi_\nu + D_\nu \xi_\mu + 2[S_{\lambda\mu\nu} + (\mu \leftrightarrow \nu)]\,\xi^\lambda, \qquad (5.53)$$

and the variation of the connection

$$\delta_E \Gamma_{\mu\nu}{}^\varkappa = D_\mu D_\nu \xi^\varkappa - 2D_\mu(S_{\nu\lambda}{}^\varkappa \xi^\lambda) + R_{\lambda\mu\nu}{}^\varkappa \xi^\lambda. \qquad (5.54)$$

Inserting this into (5.52) gives

$$\delta_E \mathscr{A} = \int d^4x \sqrt{-g}\,\{(\Theta^{\nu\mu} + \Theta^{\mu\nu})(D_\nu \xi_\mu + 2S_{\lambda\mu\nu})\,\xi^\lambda$$
$$+ \Sigma^\nu{}_\varkappa{}^{\cdot\mu}[D_\mu D_\nu \xi^\varkappa - 2D_\mu(S_{\nu\lambda}{}^\varkappa \xi^\lambda) + R_{\lambda\mu\nu}{}^\varkappa \xi^\lambda]\}.$$

By partially integrating the Σ term and using the spin divergence law (5.13), we obtain immediately

$$\delta_E \mathscr{A} = 2\int d^4x \sqrt{-g}\,\{-D_\mu \Theta_\lambda{}^\mu - 2S_{\mu\lambda}{}^\nu \Theta_\nu{}^\mu + (1/2)\,\Sigma^\nu{}_\varkappa{}^{\cdot\mu} R_{\lambda\mu\nu}{}^\varkappa\}\,\xi^\lambda, \quad (5.55)$$

leading directly to the covariant conservation law

$$D_\mu{}^* \Theta_\lambda{}^\mu + 2S_{\nu\lambda}{}^\varkappa \Theta_\varkappa{}^\nu - \frac{1}{2}\Sigma^\nu{}_\varkappa{}^{\cdot\mu} R_{\lambda\mu\nu}{}^\varkappa = 0. \qquad (5.56)$$

This is not in manifest agreement with (5.40) since the last term is $\Sigma^\nu{}_\varkappa{}^{\cdot\mu} R_{\lambda\mu\nu}{}^\varkappa$, while we had $\Sigma^\beta{}_\gamma{}^{\cdot\mu} F_{\lambda\mu\beta}{}^\gamma$ in (5.40).

We shall now demonstrate that

$$R_{\lambda\mu\nu}{}^\varkappa = F_{\lambda\mu\beta}{}^\gamma h^\beta{}_\nu h_\gamma{}^\varkappa \qquad (5.57)$$

and hence that both formulas coincide. This is a direct consequence of the fact that the vierbein fields h_α are integrable, i.e., $(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) h_\alpha{}^\lambda = 0$. Consider

$$F_{\mu\nu\beta}{}^\gamma = \partial_\mu A_{\nu\beta}{}^\gamma - \partial_\nu A_{\mu\beta}{}^\gamma - A_{\mu\beta}{}^\delta A_{\nu\delta}{}^\gamma + A_{\nu\beta}{}^\delta A_{\mu\delta}{}^\gamma$$
$$= \partial_\mu[(\Gamma_{\nu\lambda}{}^\varkappa - \overset{h}{\Gamma}_{\nu\lambda}{}^\varkappa) h_\beta{}^\lambda h^\alpha{}_\varkappa] - (\mu \leftrightarrow \nu)$$
$$- (\Gamma - \overset{h}{\Gamma})_{\mu\lambda}{}^\tau (\Gamma - \overset{h}{\Gamma})_{\nu\tau}{}^\varkappa h_\beta{}^\lambda h^\gamma{}_\varkappa - (\mu \leftrightarrow \nu)$$
$$= [\partial_\mu \Gamma_{\nu\lambda}{}^\varkappa - (\Gamma_\mu \Gamma_\nu)_\lambda{}^\varkappa - (\mu \leftrightarrow \nu)]\, h_\beta{}^\lambda h^\gamma{}_\varkappa$$

$$+ \{\Gamma_{\nu\lambda}{}^{\varkappa}\partial_\mu(h_\beta{}^\lambda h^\gamma{}_\varkappa) - \partial_\mu(\overset{h}{\Gamma}_{\nu\lambda}{}^{\varkappa}h_\beta{}^\lambda h^\gamma{}_\varkappa) - (\mu \leftrightarrow \nu)$$
$$+ (\overset{h}{\Gamma}_\mu \overset{h}{\Gamma}_\nu + \overset{h}{\Gamma}_\mu \Gamma_\nu - \overset{h}{\Gamma}_\mu \overset{h}{\Gamma}_\nu)_\lambda{}^{\varkappa}h_\beta{}^\lambda h^\gamma{}_\varkappa - (\mu \leftrightarrow \nu)\}. \tag{5.58}$$

Once we have demonstrated the vanishing of the terms in curly brackets, (5.57) is verified. The first term inside these brackets is

$$\Gamma_{\nu\lambda}{}^{\varkappa}\partial_\mu h_\beta{}^\lambda h^\gamma{}_\varkappa + \Gamma_{\nu\lambda}{}^{\varkappa}h_\beta{}^\lambda \partial_\mu h^\gamma{}_\varkappa - (\mu \leftrightarrow \nu)$$
$$= -\Gamma_{\nu\lambda}{}^{\gamma}\overset{h}{\Gamma}_{\mu\beta}{}^{\lambda} + \Gamma_{\nu\beta}{}^{\varkappa}\overset{h}{\Gamma}_{\mu\varkappa}{}^{\gamma} + (\mu \leftrightarrow \nu), \tag{5.59}$$

and the second gives [using (2.46) in terms of $h^\gamma{}_\lambda$]

$$-\partial_\mu(h_\beta{}^\lambda \partial_\nu h^\gamma{}_\lambda) - (\mu \leftrightarrow \nu) = \overset{h}{\Gamma}_{\mu\beta}{}^{\lambda}\overset{h}{\Gamma}_{\nu\lambda}{}^{\gamma} - (\mu \leftrightarrow \nu) - h_\beta{}^\lambda(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)h^\gamma{}_\lambda \tag{5.60}$$

and, indeed, by recalling (4.8)

$$F_{\mu\nu\beta}{}^{\gamma} = (R_{\mu\nu\lambda}{}^{\varkappa} - \overset{h}{R}_{\mu\nu\lambda}{}^{\varkappa})h_\beta{}^\lambda h^\gamma{}_\varkappa = R_{\mu\nu\lambda}{}^{\gamma}h_\beta{}^\lambda h^\gamma{}_\varkappa. \tag{5.61}$$

This identity between the curls of A and Γ is related to a fundamental algebraic property of covariant derivatives. Consider a vector field v_λ and apply $D_\mu D_\nu - D_\nu D_\mu$ to it. We find

$$[D_\mu, D_\nu]v_\lambda = \partial_\mu(\partial_\nu v_\lambda - \Gamma_{\nu\lambda}{}^{\varkappa}v_\varkappa) - \Gamma_\mu{}^\tau D_\tau v_\lambda$$
$$- \Gamma_{\mu\lambda}{}^{\tau}(\partial_\nu v_\tau - \Gamma_{\nu\tau}{}^{\varkappa}v_\varkappa) - (\mu \leftrightarrow \nu). \tag{5.62}$$

Since $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)v_\lambda = 0$ we obtain the so-called Ricci identity

$$[D_\mu, D_\nu]v_\lambda = -R_{\mu\nu\lambda}{}^{\varkappa}v_\varkappa - 2S_{\mu\nu}{}^{\tau}D_\tau v_\lambda. \tag{5.63}$$

For a general tensor, R acts additively on each index. Now, a similar relation may be calculated for the vector $\mathbf{v}$ in the dislocated basis, v_β:

$$[D_\mu, D_\nu]v_\beta = \partial_\mu(\partial_\nu v_\beta - A_{\nu\beta}{}^{\gamma}v_\gamma) - \Gamma_{\mu\nu}{}^{\tau}D_\tau v_\beta$$
$$- A_{\mu\beta}{}^{\gamma}(\partial_\nu v_\gamma - A_{\nu\gamma}{}^{\delta}v_\delta) - (\mu \leftrightarrow \nu)$$
$$= -F_{\mu\nu\beta}{}^{\gamma}v_\gamma - 2S_{\mu\nu}{}^{\tau}D_\tau v_\beta. \tag{5.64}$$

For a field of arbitrary spin this generalizes to

$$[D_\mu, D_\nu]\psi = \frac{i}{2}F_{\mu\nu\beta}{}^{\gamma}\Sigma^{\beta}{}_{\gamma}\psi - 2S_{\mu\nu}{}^{\tau}D_\tau\psi. \tag{5.65}$$

From the complete covariance of (5.62) and (5.64) we may multiply (5.62) by $h_\beta{}^\lambda$ and move this factor through the covariant derivatives (which, in this process, change their connection since they are applied to different objects before and after). But then the R term in (5.63) and the F term in (5.65) must be simply related by (5.57).

When expressing the energy-momentum tensor and the spin-current density in terms of the Einstein and Palatini tensor, the two covariant conservation laws (5.13) and (5.56) of a pure gravitational field take the form [recall (3.44), (4.76)]

$$\frac{1}{2} D^*{}_\mu S^{\lambda\varkappa,\mu} = G^{[\lambda,\varkappa]}, \tag{5.66}$$

$$D_\mu{}^* G_\lambda{}^\mu + 2S_{\nu\lambda}{}^\varkappa G_\varkappa{}^\nu - \frac{1}{2} S^\nu{}_\varkappa{}^{\cdot\mu} R_{\lambda\mu\nu}{}^\varkappa = 0, \tag{5.67}$$

with $(1/2)S^\nu{}_\varkappa{}^{\cdot\mu} = S^\nu{}_\varkappa{}^\mu + g^{\nu\mu}S_\varkappa - \delta_\varkappa{}^\mu S^\nu$, $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R$.

5.5. RELATION BETWEEN CONSERVATION LAWS AND FUNDAMENTAL IDENTITIES

It is a fact that for the gravitational field, by itself, both covariant laws are automatically satisfied irrespective of the presence of matter due to the fundamental identities (2.146) and (2.157). To see this we apply (3.45) to (3.39) and obtain

$$\begin{aligned} \frac{1}{2} D^*{}_\lambda S_{\nu\mu}{}^{\cdot\lambda} &= D_\lambda{}^* (S_{\nu\mu}{}^\lambda + \delta_\nu{}^\lambda S_\mu - \delta_\mu{}^\lambda S_\nu) \\ &= D_\lambda S_{\nu\mu}{}^\lambda + D_\nu S_\mu - D_\mu S_\nu \\ &= D_\lambda S_{\nu\mu}{}^\lambda + D_\nu S_\mu - D_\mu S_\nu + 2S_\lambda S_{\nu\mu}{}^\lambda. \end{aligned} \tag{5.68}$$

Now we take (2.146) and contract ν and $\varkappa$. This gives

$$\begin{aligned} R_{\mu\lambda} - R_{\lambda\mu} &= 2(D_\varkappa S_{\mu\lambda}{}^\varkappa + D_\mu S_{\lambda\varkappa}{}^\varkappa + D_\lambda S_{\varkappa\mu}{}^\varkappa) \\ &\quad - 4(S_{\varkappa\mu}{}^\rho S_{\lambda\rho}{}^\varkappa + S_{\mu\lambda}{}^\rho S_{\varkappa\rho}{}^\varkappa + S_{\lambda\varkappa}{}^\rho S_{\mu\rho}{}^\varkappa), \end{aligned} \tag{5.69}$$

from which follows

$$G_{\mu\lambda} - G_{\lambda\mu} = 2(D_\varkappa S_{\mu\lambda}{}^\varkappa + D_\mu S_\lambda - D_\lambda S_\mu) + 4S_\rho S_{\mu\lambda}{}^\rho. \tag{5.70}$$

Equation (5.68) now takes the form

$$D_\lambda {}^*S_{\nu\mu}{}^{\cdot\lambda} = G_{\nu\mu} - G_{\mu\nu} \tag{5.71}$$

in agreement with (5.66). Similarly, using (2.157) and permuting the indices we have

$$D_\tau R_{\sigma\nu\mu}{}^\tau + D_\sigma R_{\nu\tau\mu}{}^\tau + D_\nu R_{\tau\sigma\mu}{}^\tau = 2S_{\tau\sigma}{}^\lambda R_{\nu\lambda\mu}{}^\tau + 2S_{\sigma\nu}{}^\lambda R_{\tau\lambda\mu}{}^\tau + 2S_{\nu\tau}{}^\lambda R_{\sigma\lambda\mu}{}^\tau . \tag{5.72}$$

Contracting ν and μ, this becomes

$$\begin{aligned} 2D_\tau R_\sigma{}^\tau - D_\sigma R = 2D_\tau G_\sigma{}^\tau &= -2S_{\tau\sigma}{}^\lambda R_\lambda{}^\tau + 2S_\sigma{}^{\mu\lambda} R_{\lambda\mu} + 2S^\mu{}_\tau{}^\lambda R_{\sigma\lambda\mu}{}^\tau \\ &= -4S_{\tau\sigma}{}^\lambda R_\lambda{}^\tau + 2S^\mu{}_\tau{}^\lambda R_{\sigma\lambda\mu}{}^\tau \end{aligned} \tag{5.73}$$

or

$$D_\mu {}^*G_\sigma{}^\mu - 2S_\mu (R_\lambda{}^\mu - \tfrac{1}{2}\delta_\lambda{}^\mu R) + 2S_{\tau\sigma}{}^\lambda (G_\lambda{}^\tau + \tfrac{1}{2}\delta_\lambda{}^\tau R) - S^\mu{}_\tau{}^\lambda R_{\sigma\lambda\mu}{}^\tau = 0, \tag{5.74}$$

in agreement with (5.67).

Within the defect interpretation, we had observed before that the fundamental identities were a nonlinear generalization of the conservation laws of defect densities. From what we have just learned, these conservation laws can be obtained as the conservation laws of energy-momentum and angular momentum from an Einstein action.

The two laws follow from the invariance of the Einstein action under general coordinate (Einstein) transformations, which may be considered as local translations, and under local Lorentz transformations, respectively.

These transformations correspond to elastic deformations (translational and rotational) of the "world crystal" and the invariance of the action expresses the fact that elastic deformations do not change the defect structure.

It is important to realize that due to the relation between the conservation laws and the fundamental identities, they remain valid in the presence of any matter distribution. Then, by the field equations (4.69), (4.84), the spin density and energy-momentum tensor of the matter fields have to satisfy the same divergence laws once more by themselves. Indeed, it can easily be seen that this is a direct consequence of the Einstein invariance of the matter action in a *fixed arbitrary* affine space, i.e., a space whose geometry is not determined by the matter fields under consideration.

CHAPTER SIX

GRAVITATION OF SPINNING MATTER AS A GAUGE THEORY

6.1. LOCAL LORENTZ TRANSFORMATIONS

The alert reader will have noticed that the theory of gravity of spinning matter, when formulated in terms of fields $h_\alpha{}^\mu$, $A_{\mu\alpha}{}^\mu$ is really a gauge theory of local Lorentz transformations. Gauge properties had shown up before in (2.50c) when we observed that the connection $\Gamma_{\mu\nu}{}^\lambda$ transformed like a non-Abelian gauge field under general coordinate transformations. But, at that early stage, we could not have properly spoken about a gauge theory since the connection $\Gamma_{\mu\nu}{}^\lambda$ was not an independent field. In the present formulation the situation has changed. Now it is easy to convince ourselves that $A_{\mu\alpha}{}^\beta$ is an independent field and, according to Eq. (5.4), $A_{\mu\alpha}{}^\beta$ is a gauge field with respect to local Lorentz transformation.

Let us recall that under infinitesimal Lorentz transformations a vector behaves like

$$\delta v_\alpha = \omega_\alpha{}^\beta v_\beta, \qquad \delta v^\alpha = \omega^\alpha{}_\beta v^\beta. \tag{6.1}$$

From the antisymmetry of the matrix ω this can also be written as

$$\delta v^\alpha = -v^\beta \omega_\beta{}^\alpha. \tag{6.2}$$

For a tensor this amounts to

$$\delta t_\alpha{}^\beta = \omega_\alpha{}^{\alpha'} t_{\alpha'}{}^\beta + \omega^\beta{}_{\beta'} t_\alpha{}^{\beta'} = (\omega t)_\alpha{}^\beta - (t\omega)_\alpha{}^\beta. \tag{6.3}$$

In order to compare this with the general discussion of Section 3.4 of Part I we may identify the representation matrices of the Lorentz group as

$$\ell_{\beta\delta}^{\alpha\gamma} = -i(\delta_\alpha{}^\gamma \delta_\beta{}^\delta - \delta_\mu{}^\gamma \delta_\nu{}^\lambda). \tag{6.4}$$

Indeed, in terms of these matrices the infinitesimal Lorentz transformation,

$$dx'^\alpha = \Lambda^\alpha{}_\beta dx^\beta = dx^\alpha + \omega^\alpha{}_\beta dx^\beta, \tag{6.5}$$

can be written in the general form

$$\delta dx^\alpha = i(\omega^{\gamma\delta}/2)(\ell_{\gamma\delta})^{\alpha\beta} dx_\beta \tag{6.6}$$

just as in (I.3.122), with $\ell_{\gamma\delta}$ playing the role of the matrices ℓ_a and $\omega^{\gamma\delta}/2$ that of the rotation angles α^a. Thus $(\ell_{\gamma\delta}\omega^{\gamma\delta}/2)^{\alpha\beta}$ coincides with the infinitesimal matrix $(\ell_a \alpha^a)$ in (I.3.73). Now, by (6.4) this matrix is nothing but $-i\omega^{\alpha\beta}$.

This shows that the transformation law (I.3.108b) coincides precisely with (5.4) for the special case of the local Lorentz group,

$$\delta_L A_{\mu\beta}{}^\gamma = \omega_\beta{}^{\beta'} A_{\mu\beta'}{}^\gamma + \omega^\gamma{}_{\gamma'} A_{\mu\beta}{}^{\gamma'} + \partial_\mu \omega_\beta{}^\gamma. \tag{6.7}$$

Observe that the space-time variables x^μ are not transformed so that the Lorentz group plays the same role as an internal symmetry group. There is, however, a certain similarity with external gauge symmetries discussed in Section 3.5, Part I. This is because $h_\alpha{}^\mu$ can couple Lorentz and Einstein indices, just as in (I.3.135), thus giving rise to more invariants. For instance, there is no need of forming $(F_{\mu\nu\alpha}{}^\beta)^2$ in order to get an invariant action. There also exists an invariant expression linear in the field strength,

$$\mathcal{A}_f = -\frac{1}{2\kappa}\int dx \sqrt{-g}\, h^{\alpha\mu} h_\beta{}^\nu F_{\nu\mu\alpha}{}^\beta. \tag{6.8}$$

In fact, from (5.57) this is just the Einstein-Cartan action (3.8).

For completeness, let us see once more how the spin current and energy-momentum tensor follow from this action with independent fields $h_\alpha{}^\mu$, $A_{\mu\alpha}{}^\beta$. First we calculate the spin current of the field. By definition,

$$\frac{1}{2}\sqrt{-g}\,\overset{f}{\Sigma}_{\alpha\beta}{}^{\cdot\mu}(x) = -\frac{\delta\mathcal{A}_f}{\delta A_\mu{}^{\alpha\beta}(x)}$$

$$= \frac{1}{2\kappa}\frac{\delta}{\delta A_\mu{}^{\alpha\beta}(x)}\int \mathcal{A}_f\sqrt{-g}\,h^{\alpha'\mu'}\,h_{\beta'}{}^{\nu'}(\partial_{\nu'}A_{\mu'\alpha'}{}^{\beta'}$$

$$-\,\partial_{\mu'}A_{\nu'\alpha'}{}^{\beta'} - A_{\nu'\alpha'}{}^{\gamma}A_{\mu'\gamma}{}^{\beta'} + A_{\mu'\alpha'}{}^{\gamma}A_{\nu'\gamma}{}^{\beta'})$$

$$= -\frac{1}{2\kappa}\{\partial_\nu\sqrt{-g}\,(h_\alpha{}^\mu h_\beta{}^\nu - (\alpha\leftrightarrow\beta)) + \sqrt{-g}\,(A_{\nu\alpha'\alpha}h^{\alpha'\mu}h_\beta{}^\nu$$

$$+\,A_{\nu\beta'\beta}h_\alpha{}^\mu h^{\beta'\nu}) - (\alpha\leftrightarrow\beta)\}. \tag{6.9}$$

We may write this in terms of the partially covariant derivatives (5.10), (5.11) as

$$-\kappa\,\overset{f}{\Sigma}_{\alpha\beta}{}^{\cdot\mu} = \overset{L}{D}_\nu(h_\alpha{}^\mu h_\beta{}^\nu - (\alpha\leftrightarrow\beta)) + \Gamma_{\nu\sigma}{}^{\sigma}(h_\alpha{}^\mu h_\beta{}^\nu - (\alpha\leftrightarrow\beta)).$$

Applying the chain rule of differentiation this becomes

$$-\kappa\,\overset{f}{\Sigma}_{\alpha\beta}{}^{\cdot\mu} = (\overset{L}{D}_\beta h_\alpha{}^\mu - h_\alpha{}^\mu\overset{L}{D}_\nu h_\beta{}^\nu + h_\alpha{}^\mu\Gamma_{\beta\sigma}{}^{\sigma}) - (\alpha\leftrightarrow\beta). \tag{6.10}$$

We now observe that, due to the identity $D_\mu h_\alpha{}^\nu \equiv 0$, the connection can be rewritten as

$$\Gamma_{\mu\nu}{}^\lambda = h^{\alpha\lambda}\overset{L}{D}_\mu h_{\alpha\nu} = -h^\alpha{}_\nu\overset{L}{D}_\mu h_\alpha{}^\lambda. \tag{6.11}$$

This relation is complementary to the relation (4.78), $\bar{\Gamma}_{\mu\beta}{}^\gamma = h^\gamma{}_\nu\overset{\Gamma}{D}_\mu h_\beta{}^\nu$. Using (6.11), the spin current of the field becomes

$$-\kappa\,\overset{f}{\Sigma}_{\alpha\beta}{}^{\cdot\mu} = 2(S_{\alpha\beta}{}^\mu + h_\alpha{}^\mu S_\beta - h_\beta{}^\mu S_\alpha) = S_{\alpha\beta}{}^\mu, \tag{6.12}$$

in agreement with (3.44), (3.39).

We now calculate the functional derivative of the action with respect to $h_\alpha{}^\mu$. It shows directly that the canonical energy-momentum tensor of the gravitational field coincides with the Einstein tensor,

$$\sqrt{-g}\,\Theta_\mu{}^\alpha = (\delta\mathcal{A}_f/\delta h_\alpha{}^\mu) = \sqrt{-g}\,(h^{\delta\nu}F_{\mu\nu\delta}{}^\alpha - h^\beta{}_\mu F_{\beta\delta}{}^{\delta\alpha})$$

$$= h^{\alpha\nu}\sqrt{-g}\,(R_{\mu\nu} - g_{\mu\nu}R) = \sqrt{-g}\,G_\mu{}^\alpha. \tag{6.13}$$

The use of the field $h_\alpha{}^\mu$ has made it possible to retrieve the Einstein tensor without projecting out the symmetric part of it, as in the previous formulas, (3.28) and (5.51).

6.2. LOCAL TRANSLATIONS

In the literature one often finds the statement that the vierbein field may be considered as a *gauge field of local translations*. In fact, Einstein's transformations

$$x' = x - \xi(x) \tag{6.14}$$

can be considered as local translations and the vierbein field does ensure that the theory is invariant under these, just as any *bona fide* gauge field is supposed to. The covariant derivative

$$D_\alpha \equiv h_\alpha{}^\mu \partial_\mu + \frac{i}{2} A_{\alpha\beta}{}^\gamma \Sigma^\beta{}_\gamma \tag{6.15}$$

may be viewed as a combination of $h_\alpha{}^\mu$ times the translational "matrix" ∂_μ and $(i/2)A_{\alpha\beta}{}^\gamma$ times the Lorentz matrix $\Sigma^\beta{}_\gamma$. This viewpoint becomes most transparent by considering the expression in (5.65), the commutator of two covariant derivatives with respect to the dislocation coordinates,

$$[D_\alpha, D_\beta]\psi = \frac{i}{2} F_{\alpha\beta\gamma}{}^\delta \Sigma^\gamma{}_\delta \psi + i2S_{\alpha\beta}{}^\gamma i D_\gamma \psi. \tag{6.16}$$

Since the factor of $F_{\alpha\beta\gamma}{}^\delta$ is the curl of the gauge field of Lorentz transformations, the factor $2S_{\alpha\beta}{}^\gamma$ of $D_\gamma\psi$ may be considered as the curl of the gauge field of translations. Indeed, if we write $2S_{\alpha\beta}{}^\gamma$ in the form

$$\begin{aligned} 2S_{\alpha\beta}{}^\gamma &= -h^\gamma{}_\nu(h_\alpha{}^\mu \overset{L}{D}_\mu h_\beta{}^\nu - (\alpha \leftrightarrow \beta)) \\ &= h_\alpha{}^\mu h_\beta{}^\nu (\overset{L}{D}_\mu h^\gamma{}_\nu - (\mu \leftrightarrow \nu)), \end{aligned} \tag{6.17}$$

we arrive at the standard form of a curl and the present formulation of gravity of spinning matter can be considered as a gauge theory of both local Lorentz transformations and local translations.

In recent years, this aspect of gravitational theory has received increasing attention, due to the shift in emphasis from geometric principles to gauge principles. There is no space here to go into more details and the interested reader should consult the Notes and References for further reading.

6.3. LOCAL FOUR-FERMION INTERACTION DUE TO TORSION

What additional physics is brought about by torsion? If the field action is of the Einstein-Cartan type (6.8), the consequences are not very dramatic. The field equation (4.69) for the spin density, $\overset{f}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\,\mu} = -\overset{m}{\Sigma}{}^{\nu}{}_{\lambda}{}^{\cdot\,\mu}$, together with (3.44) determines the Palatini tensor (3.39) as being $S_{\mu\nu,\lambda} = -\kappa \overset{m}{\Sigma}_{\mu\nu,\lambda}$ and therefore also the torsion

$$
\begin{aligned}
S_{\mu\nu\lambda} &= (1/2)\left(S_{\mu\nu,\lambda} + \frac{1}{2} g_{\mu\lambda} S_{\nu\varkappa}{}^{\cdot\,\varkappa} - \frac{1}{2} g_{\nu\lambda} S_{\mu\varkappa}{}^{\cdot\,\varkappa} \right) \\
&\equiv (\kappa/2)\left(\overset{m}{\Sigma}_{\mu\nu,\lambda} + \frac{1}{2} g_{\mu\lambda} \overset{m}{\Sigma}_{\nu\varkappa}{}^{\cdot\,\varkappa} - \frac{1}{2} g_{\nu\lambda} \overset{m}{\Sigma}_{\mu\varkappa}{}^{\cdot\,\varkappa} \right) .
\end{aligned} \tag{6.18}
$$

It is obvious that scalar fields which describe spinless particles do not give rise to torsion. A little more surprising is that the same thing holds also for electromagnetic fields even though they describe spin-1 particles. The reason is the absence of the gauge field $A_{\mu\alpha}{}^{\beta}$ in the electromagnetic action,

$$\mathcal{A}_{\rm em} = (-1/4) \int d^4x \sqrt{-g}\, F_{\mu\nu} F^{\mu\nu}, \tag{6.19a}$$

with the field strengths

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{6.19b}$$

which is invariant under the usual electromagnetic gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda \,. \tag{6.20}$$

This invariance forbids replacing the derivatives ∂_μ in $F_{\mu\nu}$ by covariant derivatives D_μ, since this would introduce an additional term

$-2S_{\mu\nu}{}^{\lambda}A_{\lambda}$. Only the covariant derivative $\overset{\{\}}{D}_{\mu}$ is permissible, but the Christoffel symbols are symmetric so they can be omitted, leaving only (6.21) as an object which is gauge invariant under electromagnetic, Einstein, and local Lorentz transformations.

The first non-trivial effects of torsion arise for Dirac fields. The spin density of matter is, from (4.66),

$$\overset{m}{\Sigma}_{\alpha\beta,\gamma} = -(i/2)\,\bar{\psi}[\gamma_{\gamma}, \Sigma_{\alpha\beta}]_{+}\psi, \tag{6.21a}$$

(with $\Sigma_{\alpha\beta} = (i/4)[\gamma_{\alpha}, \gamma_{\beta}]$). This can be written as

$$\overset{m}{\Sigma}_{\alpha\beta,\gamma} = (1/2)\,\bar{\psi}\gamma_{[\alpha}\gamma_{\beta}\gamma_{\gamma]}\psi = (1/2)\,\varepsilon_{\alpha\beta\gamma\lambda}\,\bar{\psi}\gamma^{\lambda}\gamma_5\psi \tag{6.21b}$$

(with $\gamma_5 \equiv (1/4!)\,\varepsilon_{\alpha\beta\gamma\delta}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta}$), where the brackets around the subscripts denote their complete antisymmetrization. Due to antisymmetry, the Palatini tensor (divided by 2), the torsion, and the contortion tensor are all equal to $(\kappa/2)\overset{m}{\Sigma}_{\alpha\beta,\gamma}$

$$(1/2)\,S_{\alpha\beta,\gamma} = S_{\alpha\beta\gamma} = K_{\alpha\beta,\gamma} = (\kappa/2)\,\overset{m}{\Sigma}_{\alpha\beta,\gamma}. \tag{6.22}$$

In Eq. (2.69) we have expressed the curvature tensor in terms of the Riemannian curvature tensor plus the contortion. Two contractions give the corresponding decomposition of the scalar curvature

$$R = \overset{\{\}}{R} + \overset{\{\}}{D}_{\mu}K_{\nu}{}^{\nu\mu} - \overset{\{\}}{D}_{\nu}K_{\mu}{}^{\nu\mu} + (K_{\mu}{}^{\mu\rho}K_{\nu\rho}{}^{\nu} - K_{\nu}{}^{\mu\rho}K_{\mu\rho}{}^{\nu}). \tag{6.23}$$

We integrated over the invariant volume, the $\overset{\{\}}{D}$ K's produce irrelevant surface terms and can be ignored. The Einstein-Cartan action can therefore be separated into a Hilbert-Einstein action

$$\mathscr{A}_f = -(1/2\kappa)\int dx\,\sqrt{-g}\,\overset{\{\}}{R} \tag{6.24}$$

plus a field torsion action

$$\mathscr{A}_{ft} = \int dx\,\sqrt{-g}\,L_{ft}, \tag{6.25}$$

with a Lagraugian

$$L_{ft} = -(1/2\kappa)(K_{\mu}{}^{\mu\rho}K_{\nu\rho}{}^{\nu} - K_{\nu}{}^{\mu\rho}K_{\mu\rho}{}^{\nu}). \tag{6.25'}$$

This can be rearranged to

$$L_{ft} = (1/4\kappa)S_{\mu\nu,\lambda}K^{\lambda\nu\mu}, \tag{6.26}$$

where $S_{\mu\nu,\lambda}$ is the Palatini tensor. As a cross check we differentiate this with respect to $K^{\lambda\nu\mu}$ and obtain

$$\partial L_{ft}/\partial K^{\lambda\nu\mu} = (1/2\kappa)S_{\mu\nu,\lambda}. \tag{6.27}$$

in accordance with (3.17) and (3.44) (the factor 2 comes from the K's inside $S_{\mu\nu}{}^{\lambda}$).

We now add to (6.26) the matter-torsion Lagrangian extracted from the Dirac action (4.59),

$$L_{mt} = (1/2)\overset{m}{\Sigma}_{\mu\nu,\lambda}K^{\lambda\nu\mu}. \tag{6.28}$$

Extremizing the combined torsion Lagrangian $L_t = L_{ft} + L_{mt}$ we recover once more (6.22). Inserting this back into L_t gives, at the extremum, the effective torsion Lagrangian

$$L_t^{\text{eff}} = (\kappa/4)\overset{m}{\Sigma}_{\mu\nu\lambda}K^{\lambda\nu\mu} = (\kappa/8)\overset{m}{\Sigma}_{\mu\nu\lambda}\overset{m}{\Sigma}{}^{\mu\nu\lambda}, \tag{6.29}$$

or explicitly, with (6.21b) [see Hehl and Datta (1970)]

$$L_t^{\text{eff}} = (3\kappa/16)\bar{\psi}\gamma_{\mu}\gamma_5\psi\bar{\psi}\gamma^{\mu}\gamma_5\psi. \tag{6.30}$$

Unfortunately, this interaction is too weak to be detectable by present-day experiments. Moreover, it is not renormalizable, so that it cannot possibly be a fundamental interaction, but at best a phenomenological approximation to some more fundamental theory.

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CHAPTER SEVEN

GEOMETRIC THEORY OF STRESSES AND DEFECTS

After the preparations in the previous chapters it is now easy to construct a geometric theory of stresses and defects. With the defects described by the geometry of an affine space, we have to find an appropriate way of incorporating the correct long-range elastic interactions between the defects into the theory.

7.1. CLASSICAL ELASTICITY

First we look at the simplest case of classical elasticity which involves only the strain tensor u_{ij}. The defects are characterized completely in terms of the plastic strain tensor u^p_{ij}. Recall the partition function of elastic fluctuations in the presence of a given defect configuration u^p_{ij} [Eq. (III.10.32)]

$$Z = \int \mathcal{D}\sigma_{ij} \exp\left\{ -\int d^3x \left[\frac{1}{4\mu T}\left(\sigma_{ij}^2 - \frac{\nu}{1+\nu}\sigma_{ii}^2 \right) + i\sigma_{ij}(\partial_i u_j + \partial_j u_i - 2u^p_{ij}) \right] \right\}. \tag{7.1}$$

Integrating out the displacement field u_i gave the stress conservation law,

$$\partial_i \sigma_{ij} = 0. \tag{7.2}$$

This led to the introduction of a stress gauge field χ_{ij} so that,

$$\sigma_{ij} = \varepsilon_{ik\ell}\varepsilon_{jmn}\partial_n\partial_m\chi_{\ell n}, \tag{7.3}$$

and to a reformulation of (7.1) as a gauge theory [cf. (9.40), Part III]

$$Z = \int \mathscr{D}\chi_{ij}\Phi[\chi_{ij}]\left\{-\int d^3x\left[\frac{1}{4\mu T}\left(\sigma_{ij}^2 - \frac{\nu}{1+\nu}\sigma_{ii}^2\right) + i\chi_{ij}\eta_{ij}\right]\right\}, \tag{7.4}$$

where

$$\eta_{ij} = \varepsilon_{ik\ell}\varepsilon_{jmn}\partial_n\partial_m u_{ij}^p \tag{7.5}$$

was the defect density, which satisfies the same conservation law as the stresses

$$\partial_i\eta_{ij} = 0.$$

We saw in (2.83), that the role of the plastic gauge field u_{ij}^p is played by the metric g_{ij}, apart from a factor 1/2. An elastic distortion of the crystal changes u_{ij}^p by a plastic gauge transformation

$$2u_{ij}^p \to 2u_{ij}^p + \partial_i\xi_j + \partial_j\xi_i. \tag{7.6}$$

An obvious task now is to find the nonlinear geometric generalization of this structure in a Riemannian space: In such a space, the metric changes under coordinate transformations (local translations) as

$$g_{ij} \to g_{ij} + \overset{\{\}}{D}_i\xi_j + \overset{\{\}}{D}_j\xi_i, \tag{7.7}$$

where $\overset{\{\}}{D}_i$ are the covariant derivatives formed with the Christoffel symbols of the metrics. Now, the usual, elastic strains are obtained as

$$u_{ij}^{\rm el} = \partial_i u_j + \partial_j u_i - 2u_{ij}^p. \tag{7.8}$$

Thus they are obtained from the transformation law of the plastic strains $-2u_{ij}^p \to -\partial_i\xi_j - \partial_j\xi_i - 2u_{ij}^p$ by simply replacing on the right-hand side the transformation functions $-\xi_i$ by the total displacement field u_i. The elastic strains are then automatically defect gauge invariant since the transformation functions in (7.6) can be absorbed, by an elastic deformation, into the total displacement field u_i.

$$u_i \rightarrow u_i + \xi_i. \tag{7.9}$$

By analogy we conclude that in the nonlinear theory in affine space, the elastic strain tensor must be given by minus the right-hand side of (7.7) with ξ_i replaced by $-u_i$, i.e.,

$$u_{ij}^{\text{el}} = \overset{\{\}}{D}_i u_j + \overset{\{\}}{D}_i u_j - g_{ij}. \tag{7.10}$$

The generalized partition function should therefore read

$$Z = \int \mathscr{D}\sigma_i^j\, \mathscr{D}u^i \exp\left\{ -\int d^3x \sqrt{g}\, \frac{1}{4\mu T}\left({\sigma^j}_i^2 - \frac{\nu}{1+\nu} {\sigma^i}_j^2 \right) + i\sigma^{ij}(\overset{\{\}}{D}_i u_j + \overset{\{\}}{D}_j u_i - g_{ij}) \Big]\Big] \right\}. \tag{7.11}$$

The factor $\sqrt{g}$ ensures an invariant volume integral under general coordinate transformations.

If we integrate out the u_i field, we obtain the covariant conservation law[a]

$$\overset{\{\}}{D}_i {\sigma^i}_j = 0. \tag{7.12}$$

The generalized defect density is given by the Einstein tensor $\overset{\{\}}{G}_{ij}$ formed with the Christoffel symbols [see Eqs. (2.89), (2.91)]. It satisfies a covariant conservation law of the same form as σ_{ij},

$$\overset{\{\}}{D}_i {G^i}_j = 0. \tag{7.13}$$

In principle, it is possible to consider χ_{ij}, σ_{ij} as the linearized version of another dual geometry, in which χ_{ij} plays the role of the metric, say $\tilde{g}_{ij}$, (called, say, stress metric) and σ_{ij} that of the Einstein tensor $\overset{\tilde{\{\}}}{G}_{ij}$ (called, say, Einstein tensor in stress space). In this way, the stress energy can also be cast into a geometric form. The result would be a geometric version of the double gauge theory (III.9.49) ("double geometry").

[a] It is a useful exercise to verify the invariance of the energy in (7.11) under local translations at this stage. For the transformations of g_{ij}, Eq. 7.7, this is trivial; but also the transformations of $\sqrt{g}\sigma^{ij}$ gives no change.

7.2. SECOND GRADIENT ELASTICITY

If the elastic energy also involves a rotational stiffness, as in Part III Chapter 18, a geometric theory of stresses and defects requires an affine space with curvature and torsion. We take the partition function (18.23) of a given defect distribution β_{ij}^p, ϕ_{ij}^p

$$Z=\int \mathscr{D}\sigma_{ij}\mathscr{D}\tau_{ij}\mathscr{D}u_i\mathscr{D}\omega_j \exp\left\{-\int d^3x\left[\frac{1}{4\mu T}\left(\overset{s}{\sigma}{}_{ij}^2-\frac{\nu}{1+\nu}\overset{s}{\sigma}{}_{ij}^2\right)+\frac{1}{8\mu\ell^2 T}\tau_{ij}^2\right.\right.$$

$$\left.\left.+\,i\sigma_{ij}(\partial_i u_j-\varepsilon_{ijk}\omega_k-\beta_{ij}^p)+i\tau_{ij}(\partial_i\omega_j-\phi_{ij}^p)\right]\right\} \tag{7.14}$$

(where we have assumed $\delta_2=0$, $\delta_1=1$, for simplicity). Integrating out the u_i and ω_i fields gave the stress conservation laws

$$\partial_i\sigma_{ij}=0, \qquad \partial_i\tau_{ij}=-\varepsilon_{jk\ell}\sigma_{k\ell}. \tag{7.15}$$

They are of the same form as those of the defect densities [recall Eqs. (2.68), (2.69), Part III]

$$\alpha_{ij}=\varepsilon_{ik\ell}\partial_k\beta_{\ell j}^p+\delta_{ij}\phi^p_{\ell\ell}-\phi^p_{ji}, \qquad \Theta_{ij}=\varepsilon_{ik\ell}\partial_k\phi^p_{\ell j}, \tag{7.16}$$

which read [see Eqs. (2.45), (2.46)]

$$\partial_i\alpha_{ij}=-\varepsilon_{jk\ell}\Theta_{k\ell}, \qquad \partial_i\Theta_{ij}=0. \tag{7.17}$$

When generalizing the partition function (7.14) to affine space, we want to preserve the structural identity between the conservation laws (7.15) and (7.17). This is achieved by the following partition function,

$$Z=\int \mathscr{D}\sigma^i_\alpha\mathscr{D}\tau_{i\beta}{}^\alpha\mathscr{D}u^\alpha\mathscr{D}\omega_\lambda{}^\beta \exp\left\{-\int d^3\sqrt{g}\left[\frac{1}{4\mu T}\left(\overset{s}{\sigma}{}_i^{j2}-\frac{\nu}{1+\nu}\overset{s}{\sigma}{}_i^{i2}\right)\right.\right.$$

$$\left.+\frac{1}{16\mu\ell^2T}(\tau_{i\alpha}{}^\beta)^2\right]+i\sigma^i_\alpha(D_iu^\alpha-\omega_i{}^\alpha-(A_{\beta i}{}^\alpha-2S_{\beta i}{}^\alpha)u^\beta-h^\alpha{}_i)$$

$$\left.-\frac{i}{2}\tau_{i\beta}{}^\alpha(D_i\omega_\alpha{}^\beta-A_{i\alpha}{}^\beta+D_i(u^\gamma A_{\gamma\alpha}{}^\beta)-u^\gamma F_{i\gamma\alpha}{}^\beta)\right\}. \tag{7.18}$$

Here u^α, ω_α^β are the displacement and rotation fields in non-holonomic coordinates dx^α, and h_i^α, $A_{i\alpha}{}^\beta$ are the gauge fields of local translations and rotations in affine space.

The stress field ${\sigma'}^{i}_{\alpha}$ carries one non-holonomic index α and one Einstein index i. From this we calculate the tensor ${\sigma'}^{i}_{j}$ in the stress energy as

$$ {\sigma'}^{i}_{j} = h^{\alpha}{}_{j}\sigma^{i}_{\alpha}. \tag{7.19} $$

The covariant derivatives D_j are formed using the gauge field of rotations $A_{i\alpha}{}^{\beta}$, for instance,

$$ D_i u^{\alpha} = \partial_i u^{\alpha} + A_{i\beta}{}^{\alpha} u^{\beta}. \tag{7.20} $$

The nonlinear strain tensors in (7.18) are the result of the following consideration: The gauge fields of translations and rotations are the nonlinear generalizations of the plastic dislocation and disclinations β^{p}_{ij} and ϕ^{p}_{ij}. At first sight, the generalization of the elastic strain tensors coupled to the stress tensors in (7.14) seems to be $[\omega_i{}^{\alpha} \equiv h_i{}^{\beta}\omega_{\beta}{}^{\alpha}]$

$$ u^{\mathrm{el}\,\alpha}_{i} = D_i u^{\alpha} - \omega_i{}^{\alpha} - h^{\alpha}{}_{i}, \qquad \omega^{\mathrm{el}\,\beta}_{i\alpha} = D_i \omega^{\beta}_{\alpha} - A_{i\alpha}{}^{\beta}, \tag{7.21} $$

where we have exchanged the vector ω_{γ} by the antisymmetric tensor $\omega_{\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma}\omega_{\gamma}$ and introduced, similarly, $\phi^{p}_{i\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma}\phi^{p}_{i\alpha}$.

Our strain tensors $u^{\mathrm{el}\,\alpha}_{i}$ and $\omega^{\mathrm{el}\,\beta}_{i\alpha}$ differs from this naive choice by the additional nonlinear terms,

$$ -(A_{\beta i}{}^{\alpha} - 2S_{\beta i}{}^{\alpha})u^{\beta} + D_i(u^{\gamma}A_{\gamma\alpha}{}^{\beta}) - u^{\gamma}F_{i\gamma\alpha}{}^{\beta}, \tag{7.22} $$

respectively. Certainly both choices of strain tensors have the same limit of linear elasticity. There is an important geometric reason for our choice. Elastic distortions of the solid must correspond precisely to the local translation and rotations of the coordinate system (which do not change the defect content of the geometry). These were given in (5.43) and (5.46) and read as follows [see also (5.47), (5.48)]

$$ \delta_E x^{\alpha} \equiv x^{\alpha} + \xi^{\alpha}, \qquad \delta_E h^{\alpha}_{i} = D_i\xi^{\alpha} - (A_{\beta i}{}^{\alpha} - 2S_{\beta i}{}^{\alpha})\xi^{\beta}, $$
$$ \delta_E A_{i\alpha}{}^{\beta} = D_i(\xi^{\gamma}A_{\gamma\alpha}{}^{\beta}) - \xi^{\gamma}F_{i\gamma\alpha}{}^{\beta} \tag{7.23} $$

and

$$ \delta_L x^{\alpha} = \Delta\omega^{\alpha}{}_{\beta}x^{\beta}, \qquad \delta_L A_{i\alpha}{}^{\beta} = D_i\Delta\omega_{\alpha}{}^{\beta}, \qquad \delta_L h_{\alpha i} = \Delta\omega_{\alpha}{}^{\beta}h_{\beta i}, \tag{7.24} $$

where ξ^{α} and $\Delta\omega^{\beta}{}_{\alpha}$ and the local displacements and rotations. We have written $\Delta\omega_{\alpha}{}^{\beta}$ instead of $\omega_{\alpha}{}^{\beta}$ to distinguish the local rotations of the coordinates from the total rotational distortions of the solid. Just as in Eq. (7.10), the elastic strain tensors are found from transformation laws (7.23), (7.24), replacing, on the right-hand sides, ξ^α by the displacement fields $-u^\alpha$, and $\Delta\omega_\alpha{}^\beta$ by $-\omega_\alpha{}^\beta$. Then the transformed $-u_i{}^\alpha$, $-A_{i\alpha}{}^\beta$ give precisely the elastic strain tensors of (7.18) [i.e. with the terms (7.22)].

These tensors are manifestly gauge invariant under the transformation (7.23), (7.24) if the total distortions of the crystal u^α, $\omega_\alpha{}^\beta$ are simultaneously changed by

$$u^\alpha \rightarrow u^\alpha + \xi^\alpha, \qquad \omega_\alpha{}^\beta \rightarrow \omega_\alpha{}^\beta + \Delta\omega_\alpha{}^\beta. \tag{7.25}$$

This construction principle has the virtue of maintaining the structural identity between the stress and defect conservation laws (7.15) and (7.17) at the nonlinear level. Indeed, if we integrate out the u^α and $\omega_\alpha{}^\beta$ fields in (7.18), we find the stress conservation laws

$$D_i^*\sigma^i_\alpha = -2S_{\alpha i}{}^\gamma\sigma^i{}_\gamma - \tfrac{1}{2}\tau^i{}_\gamma{}^\beta R_{\alpha i\beta}{}^\gamma, \qquad D_i^*\tau^i{}_\beta{}^\alpha = (\sigma^\alpha{}_\beta - \sigma_\beta{}^\alpha). \tag{7.26}$$

These have exactly the same form as the conservation laws of the defect densities (5.67), (5.66), the dislocation density, α_{ij}, being equal to the Palatini tensor via

$$\mathscr{S}_{ij,k} = \varepsilon_{ij\ell}\,\alpha_{k\ell} \triangleq \tau_{ijk} \tag{7.27}$$

and the disclination density Θ_{ij} to the Einstein tensor [recall (3.56) and (2.93)] via

$$G_{ji} = \Theta_{ij} \triangleq \sigma_{ij}\,. \tag{7.28}$$

There they appeared as a consequence of the first and the Bianchi identity (see Section 2.9). The same conservation laws were obtained for the energy-momentum tensor Θ^i_α and the spin density $\Sigma_\beta{}^{\alpha,i}$ of arbitrary matter moving in the affine space. One merely has to replace

$$\sigma^i{}_\alpha \rightarrow \Theta_\alpha{}^i, \qquad \tau^i{}_\beta{}^\alpha \rightarrow \Sigma_\beta{}^{\alpha,i} \tag{7.29}$$

[recall (5.56), (5.71)]. With the naive choice of the strain tensor in (7.21), this beautiful universality of conservation laws in an affine space would have been lost for the stresses and with it the invariance of (7.18).[b]

This universality is of special importance if one wants to construct the full geometric generalization of the double gauge theory (III.9.49). Due to the conservation laws it is possible to describe the stresses by means of an extra set of translational and rotational gauge fields, say, $\tilde{h}_i^{\alpha}$, $\tilde{A}_{i\alpha}^{\beta}$. As in the Riemannian geometry of defects, discussed of the end of the last section. We may think of these as defining a second geometric description of space, with a metric $\tilde{g}_{ij} = \tilde{h}_i{}^{\alpha}\tilde{h}_{\alpha i}$. The stresses σ_i^{α} and $\tau_{i\beta}{}^{\alpha}$ correspond to the Einstein tensor $\tilde{G}_{\alpha}{}^{i}$ and the Palatini tensor $\tilde{S}_{\beta}{}^{\alpha,}{}_{i}$ within this "stress geometry". The stress-conservation laws are then the first and Bianchi identities in this geometry. This leads to a double geometric formulation of stresses and defects involving h_i^{α}, $A_{i\beta}{}^{\alpha}$, $\tilde{h}_i{}^{\alpha}$, $\tilde{A}_{i\beta}{}^{\alpha}$. We leave it to the reader to write down the explicit partition function.

7.3. SUMMING OVER DEFECT CONFIGURATIONS

The above partition function collects the elastic fluctuations of the solid at a given defect distribution. For the statistical mechanics of the solid we have to sum over all possible defect configurations. In Part III we saw that for a proper inclusion of the defects, the plastic gauge fields had to be discrete-valued, as a reflection of the finite size of the Burgers vectors. If we ignore this important fact, the statistical mechanics of a realistic solid cannot be correctly reproduced. It is conceivable, however, that a nonlinear continuum theory of defects has some means of effectively accounting for the size of the Burgers vectors. We can add, in the energy of (7.11) or (7.18), a core energy of defects: in the first case quadratic in the defect density $\eta_{ij} = \overset{\{\}}{G}_{ij}$; in the second case quadratic in $\alpha_{i\alpha\beta} = S_{\alpha\beta,i}$ and $\Theta_{ij} = G_{ji}$. Then the continuous defect gauge fields will follow nonlinear field equations. Then nonlinearities will, in general, introduce a core region into every defect line, and its radius can set the scale for Burgers' vectors. Thus it is not entirely unphysical if we set up a path integral over all defect configurations in the continuum formulation of the theory. If $\Phi[g_{ij}]$, $\Phi[h_i{}^{\alpha}, A_{j\alpha}{}^{\beta}]$ denote arbitrary gauge fixing functionals, we simply put in front of the partition function (7.11) the measure

$$\int \mathcal{D} g_{ij} \Phi[g_{ij}], \tag{7.30}$$

[b]It is instructive to verify how, after integrating out $u^{\alpha}(x)$, $\omega_{\alpha}{}^{\beta}(x)$ in (7.18), the conservation laws (7.26) ensure the invariance of the energy under Einstein and local Lorentz transformations.

or, in front of (7.18)

$$\int \mathscr{D} h^{\alpha}{}_{i} \int \mathscr{D} A_{j\alpha}{}^{\beta} \Phi[h^{\alpha}{}_{i}, A_{j\alpha}{}^{\beta}]. \tag{7.31}$$

It is an interesting open question whether it will be possible to explain the melting transition on the basis of such a continuum partition function.

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SUMMARY AND OUTLOOK

Let us try to summarize some of the highlights described in this book. After a resumé of the basic field theoretic tools, Part I laid the background for a disorder field theory of ensembles of line-like objects. The statistical mechanics of such objects was shown to be equivalent to the statistical mechanics of a single scalar field which, in this context, was called a *disorder field*. In Parts II and III such fields were used to develop the statistical mechanics of ensembles of vortex and defect lines in superfluids and solids. In the condensed, ordered phase, the quantitative contribution of such lines to the equations of state is very small since they are practically frozen out, and the disorder field has a zero expectation value. Nevertheless, at a certain temperature the lines proliferate due to an excess of their configurational entropy, thereby destroying the order of the system. This is signalled by a non-vanishing expectation values of the disorder fields.

The distribution of the lines in the ensemble is influenced by two types of interactions, one is of short range and of "steric" origin, and the other of long range and "elastic" origin. The former can be described within purely disorder field theory by means of a field self-coupling. By interactions of "elastic" origin we mean, in general, those which are mediated by the long-range excitations of the system. These are usually the Nambu-Goldstone bosons of some spontaneously broken symmetry in the condensed phase. In superfluids, they are the phase fluctuations of the superfluid condensate and are related to the invariance of the condensate energy under overall phase rotations. In solids they are the phonons resulting from the breakdown of translational symmetry. Their long-range properties make it possible to take their conventional description in terms of a phase angle or of a lattice displacement field and reformulate it in terms of gauge fields. The general way in which they enter into the theory went as follows.

For massless fields the energy does not depend on the fields but on the gradients only. The simplest example is superfluid ^{4}He in which the phase fluctuations have, to lowest order, the energy

$$E = \frac{1}{2}\int d^3x(\partial_i \gamma)^2. \tag{1}$$

This can be reexpressed in terms of an auxiliary conjugate field, the supercurrent b_i, as

$$E = \int d^3x\left(\frac{1}{2}b_i^2 - ib_i\partial_i\gamma\right)\cdot \tag{2}$$

Variation with respect to b_i, γ gives the equations of motion,

$$b_i = i\partial_i\gamma. \tag{3a}$$

$$\partial_i b_i = 0. \tag{3b}$$

Equation (3b) for the conjugate field is solved by the introduction of a gauge field $a_i(\mathbf{x})$ so that

$$\mathbf{b} = \partial \times \mathbf{a}. \tag{4}$$

This gauge field is the phototype of what we have called an *elastic gauge field*, or in reference to its significance in solids, a *stress gauge field*.

Equation (4) is invariant under *stress gauge transformations*, i.e., under the gauge transformation

$$a_i(\mathbf{x}) \to a_i(\mathbf{x}) + \partial_i\Lambda(\mathbf{x}). \tag{5}$$

The vortex lines or the defect lines act as sources to the gauge fields with a simple linear coupling between the gauge fields and certain source currents, as in electrodynamics. The gauge invariance of the couplings is guaranteed by vanishing divergences of the currents which amount to conservation laws similar to Kirchhoff's laws in electrodynamics. In the present context, these laws reflect the fact that vortex lines can never end and that defect lines in solids have certain branching rules. The vanishing divergence of the currents allows for the introduction of a second set of gauge fields, the *vortex* or *defect gauge* fields. In the case of solids, they coincide with the plastic distortions that have conventionally been used to describe the plastic properties of materials. This is why we have called them also *plastic gauge fields*.

The plastic gauge fields describe the discrete jumps of the condensate phase or the displacement field across the Volterra surfaces, whose boundaries are the defect lines. The plastic gauge transformations move the Volterra sheets through space, while keeping the boundaries fixed, so that the defect lines are gauge invariant. The fundamental equation which expresses the essence of the defect gauge transformations was then found as follows. A δ-function singular across some Volterra surface S, $\delta_i(\mathbf{x}, S)$ [short notation will be $\delta_i(S)$, see Part II, Eq. (8.20)] changes under a shift of the surface S with a fixed boundary line L as

$$\delta_i(S) \underset{S \to S'}{\to} \delta_i(S') - \partial_i \delta(V) \tag{6}$$

where $\delta_i(V)$ [short for $\delta_i(\mathbf{x}, V)$] is the δ-function on the volume V [defined in Part II, (8.21)] over which the surface has swept. All plastic gauge fields describing defects are characterized by a δ-function on a Volterra surface. For example, the plastic gauge field in the example (1) above is directly $\gamma_i^p = 2\pi\delta_i(S)$. It changes by a pure divergence when this surface is moved.

The gauge invariant defect density is the boundary line. It is obtained by Stokes' theorem when formulated in terms of δ-functions,

$$(\partial \times \delta)_i(S) = \delta_i(L), \tag{7}$$

where $\delta_i(L)$ [short for $\delta_i(\mathbf{x}, L)$] is singular on boundary line. Obviously, (7) is defect gauge invariant under (6). The defect conservation law follows directly from (2),

$$\partial_i \delta_i(L) = 0. \tag{8}$$

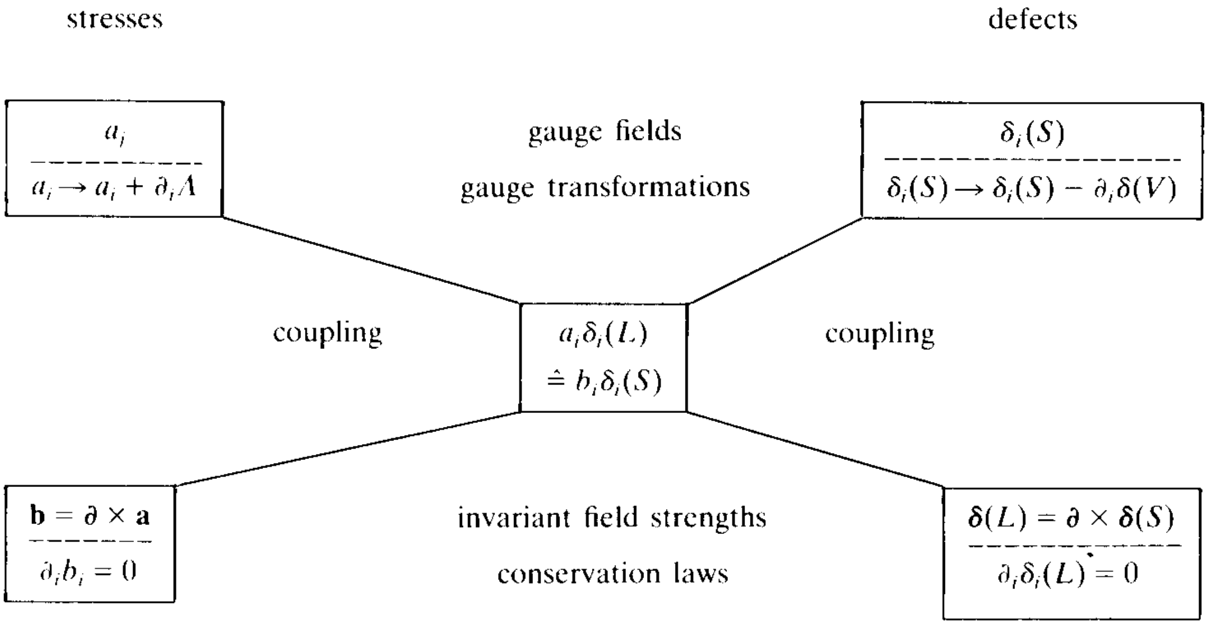

FIG 1. Graphical illustration of the dual relationship between stress and defect gauge structures. Instead of $\delta_i(S)$, $\delta_i(L)$ on the right-hand side we can write the plastic gauge field γ_i^p and the density ℓ_i.

The coupling of the gauge field of stress with the defect density along the line L is given by

$$\int d^3x a_i(\mathbf{x})\delta_i(\mathbf{x}, L). \tag{9}$$

Due to (8) it is automatically invariant under stress gauge transformations (5). Using (7) and (4) we can rewrite this also as

$$\int d^3x b_i(\mathbf{x})\delta_i(\mathbf{x}, S). \tag{10}$$

In the first case, the stress gauge field is coupled locally to the defect gauge invariant source of the defects, in the second the defect gauge field is coupled locally to the stress gauge invariant field $b_i(\mathbf{x})$. This shows that the plastic gauge fields are dual to the elastic ones. The currents of one are coupled to the gauge invariant field strength of the other, and *vice versa.* This dual relationship is illustrated in Fig. 1.

In spite of its symmetry, the duality does not imply a complete structural equivalence. This is due to an important property of the fields, namely, the cyclic property of the phase of the condensate (phase and phase + 2π are indistinguishable) and a similar property of the displacement field (the displacement field is defined modulo lattice vectors). It causes the vortex lines to have a discrete strength (in integer multiples of 2π) and the defect lines to be characterized by discrete Burgers vectors (=lattice vectors). For this reason, the plastic gauge fields *cannot* be assumed to be continuous. They must be discrete-valued, carrying either multiples of 2π in the superfluid or linear combinations of the lattice vectors in solids.

The discrete nature of the plastic fields, which destroys the complete equivalence between elastic and plastic gauge structures, had an important virtue. It made it possible to transform these fields into a *disorder field theory* (with continuous fields) of line-like objects of the same type as had been developed in Part I on general grounds. Their introduction was based on the observation that the correlation function of a massive scalar field,

$$G(\mathbf{x}, \mathbf{x}') = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} \frac{1}{p^2 + m^2},$$

can be written as

$$\int_0^\infty d(\tau/2)\, e^{-m^2(\tau/2)} \int \frac{d^3p}{(2\pi)^3} e^{-\tau(p^2/2)} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} = \int_0^\infty d(\tau/2)\, e^{-m^2\tau/2} \frac{1}{\sqrt{2\pi\tau}^{\,3}} e^{-(1/2)(\mathbf{x}-\mathbf{x}')^2/\tau}.$$

In this form it was seen to describe the probability of a random line of length τ to run from $\mathbf{x}$ to $\mathbf{x}'$, with $e^{-m^2\tau/2}$ describing the distribution of the lengths. The random lines were identified with the vortex or defect lines. Feynman graphs of the disorder field theory became the direct pictures of the vortex or defect lines. Thus we saw that, while the continuum theory of the superfluids and solids is most naturally described by a dual double gauge theory of elastic and plastic distortions, the properly discretized system finds its most convenient field theoretic formulation when its discrete-valued plastic gauge fields are replaced by disorder fields.

Apart from these structural developments, Parts II and III contained detailed statistical investigations of the superfluid-normal and solid-liquid phase transitions viewed as resulting from a proliferation of vortex or defect lines. Graphical low- and high-temperature expansions were developed and detailed comparisons with extensive Monte-Carlo simulations were given.

Part IV, finally, was devoted to the differential geometric aspects of the plastic defect gauge fields. These apply only to the continuum approximation of defects. We began by reviewing the standard Einstein-Cartan theory of gravity in spaces with curvature and torsion and showed that disclination and dislocation densities correspond to the Einstein contraction of the curvature tensor and to the Palatini combination of the torsion tensor, respectively. In general relativity, these describe the canonical energy-momentum tensor and the spin current of the gravitational field. The defect conservation laws were shown to be intimately related to energy- and angular-momentum conservation in gravity. We extended this geometrical description of defects by the appropriate stress interactions. In most straightforward formulation, they play a role similar to the matter fields in general relativity. Gauge transformations of the defects corresponded to Einstein's local translations of the coordinates and to the rotations of the local reference frames in general relativity.

Still in the geometric approach, the particular properties of the couplings between stresses and gauge fields and their defect gauge invariance imply conservation laws for the stress tensors. These are dual and structurally equivalent to the divergence relations for the defects densities. This duality permits the construction of another, dual, differential geometry, corresponding to the gauge fields of stresses, with the geometric objects living in "stress space" rather than in the previous "defect space". Such a geometric formulation could, in principle, be extended beyond the continuum approximation for which the gauge fields of defects must eventually be taken to be discrete-valued and the differential geometric description of defects breaks down (to be replaced by an appropriate disorder field theory). The differential geometry of stresses remains always continuous and applicable also when the geometry of defect becomes discrete.

It is worthwhile displaying the mutually dual gauge structures and the associated differential geometry once more in a unified way. They may be expressed most concisely in the form of what we may call *fundamental phase space identities*. We recall that the first and most important place where such an identity plays a significant role, is in quantum mechanics. There it regulates the fluctuation arena of quantum mechanics and quantum statistics via the relation

$$1 = \int dx'' \langle x', t' | x, t \rangle = \int \mathcal{D} x \int (\mathcal{D} p / 2\pi\hbar) \exp\left\{ i/\hbar \int_t^{t'} dt p\dot{x} \right\}$$

$$= \int (dp''/2\pi\hbar) \langle p', t' | p, t \rangle = \int (\mathcal{D} p/2\pi\hbar) \int \mathcal{D} x \exp\left\{ (-i/\hbar) \int_t^{t'} dt p x \right\} \cdot \quad (11)$$

This is the path integral equivalent of Heisenberg's operator uncertainty relation $[\hat{p},\hat{x}] = -i\hbar$ of quantum physics and specifies the Hilbert space in which the state vectors evolve. It is applicable to arbitrary quantum systems. Whatever the specific dynamics, it is imposed by subtracting, from the exponent in (11), an imaginary time integral of the total energy, $(i/\hbar)\int dt[K(p) + V(x)]$, so that the total exponent becomes $i/\hbar$ times the classical action. Then, by dropping in (11) the left-hand integral $\int dx''$ or $\int dp''/2\pi\hbar$, one obtains the quantum mechanical amplitudes in the x- or p-representations. Furthermore, by placing under the integral a δ-function, $\delta(x - x')$ or $2\pi\hbar\delta(p - p')$, thereby making the paths in the action periodic in $t' - t$, and by continuing $t' - t$ to imaginary $-i\hbar/k_B T$, one obtains the full quantum partition function of the system at temperature T. The phase space identity (11) is therefore truly fundamental in quantum and statistical physics and fully deserves its name.

It is possible to give similar identities also for the two mutually dual gauge field systems of superflow and vortex lines in superfluids, and of stresses and defect lines in solids. We begin with the simpler case of superfluids where the fundamental phase space identity reads

$$1 = \int_{-\infty}^{\infty} \mathscr{D} b_i \int_{-\infty}^{\infty} \mathscr{D}\gamma \int_{-\infty}^{\infty} \mathscr{D}\gamma_i^p \Phi[\gamma_i^p] \exp\left\{ i \int d^3x [b_i(\partial_i\gamma - \gamma_i^p)] \right\}. \tag{12a}$$

Here γ is the phase variable of the superfluid condensate, b_i is the canonically conjugate superfluid current, and γ_i^p is the *plastic deformation* of the phase disortion, which carries the information on the distribution of vortex lines, i.e., the *vortex gauge field*. The associated gauge transformations move Volterra sheets through space at fixed boundaries (which are the vortex lines) and read [compare (6)]

$$\gamma_i^p \to \gamma_i^p + \partial_i N. \tag{13a}$$

They can be absorbed by a shift in the phase variable,

$$\gamma \to \gamma + N. \tag{13b}$$

The functional $\Phi[\gamma_i]$ serves to fix a convenient gauge, for instance $\gamma_3^p = 0$ (axial gauge). In analogy with Eqs. (2)–(3) we see here that integrating out the γ fluctuations leads to the *conservation law of superflow*,

$$\partial_i b_i \equiv 0, \tag{14a}$$

which suggests introducing a *gauge field of superflow*, a_i, via

$$b_i = \varepsilon_{ijk}\partial_j a_k, \tag{14b}$$

with b_i being invariant under the *superflow gauge transformation*,

$$a_i \to a_i + \partial_i \Lambda. \tag{14c}$$

We can therefore go over to a *double gauge* field version of the identity (12a),

$$1 \equiv \int \mathscr{D} a_k \Psi[a_k] \int \mathscr{D} \gamma_i^p \Phi[\gamma_i^p] \exp\left\{-i \int d^3x b_i \gamma_i^p\right\}, \tag{12b}$$

where $\Psi[a_k]$ is a gauge fixing functional of a_k. A partial integration brings (12b) to the form

$$1 = \int \mathscr{D} a_k \Psi[a_k] \int \mathscr{D} \gamma_i^p \Phi[\gamma_i^p] \exp\left\{-i \int dx^3 a_i \ell_i\right\}, \tag{12c}$$

where

$$\ell_i = \varepsilon_{ijk}\partial_i \gamma^p \tag{15a}$$

is the gauge invariant curl of the vortex gauge field. It obviously satisfies the *vortex conservation law*

$$\partial_i \ell_i = 0, \tag{14d}$$

and describes the *vortex density* of the superfluid. We can go one step further and rewrite (12c) in a form dual to (12a), i.e.

$$1 = \int \mathscr{D} a_k \Psi[a_k] \int \mathscr{D} \ell_i \int \mathscr{D} \theta \exp\left\{i \int d^3x (a_i - \partial_i \theta)\ell_i\right\}, \tag{12d}$$

where the auxiliary θ integration enforces the vortex conservation law (14b). The full partition function of the superfluid with vortex lines is obtained by subtracting, in the exponent, $1/k_BT$ times the energy of superflow, the leading term being (with $\beta = 1/k_BT$)

$$\beta E_s = \frac{1}{2}\beta \int d^3 x b_i^2. \tag{15b}$$

In the double gauge version (12b) we might also subtract an extra core energy of vortex lines

$$\beta E_v = -\frac{1}{2}\tilde{\beta} \int d^3 x_i \ell_i^2. \tag{15c}$$

If the field system is placed on a lattice and the vortex gauge field γ_i^p takes only discrete values (≡integer multiples of 2π) the resulting partition function becomes the familiar Villain model of the superfluid phase transition, which, in Part II, was shown to describe correctly the critical regime of superfluid He^4.

We now turn to solids. If the molecules are small and there is little rotational stiffness, the fundamental phase space identity reads

$$1 = \int \mathcal{D}\sigma_{ij} \int \mathcal{D}u_i \int \mathcal{D}u_{ij}^p \Phi[u_{ij}^p] \exp\left\{ i \int d^3 x \sigma_{ij}(\partial_i u_j + \partial_j u_i - 2u_{ij}^p) \right\}, \tag{16a}$$

where σ_{ij} are the stresses, u_i the dislacement fields, and u_{ij}^p is the plastic strain field. The latter is a *defect gauge field* with the *defect gauge transformations*

$$u_{ij}^p \to u_{ij}^p + \partial_i N_j + \partial_j N_i, \qquad u_i \to u_i + N. \tag{17a}$$

Integrating u_i out shows the *stress conservation law*

$$\partial_i \sigma_{ij} = 0. \tag{18a}$$

Thus, there exists a stress gauge field χ_{ij} with

$$\sigma_{ij} = \varepsilon_{ij\ell}\varepsilon_{jmn}\partial_k \partial_m \chi_{\ell n}. \tag{18b}$$

This relation is invariant under the *stress gauge transformations*,

$$\chi_{\ell n} \to \chi_{\ell n} + \partial_\ell \Lambda_n + \partial_n \Lambda_\ell. \tag{18c}$$

The stress gauge field permits rewriting (16a) in the *double gauge* form

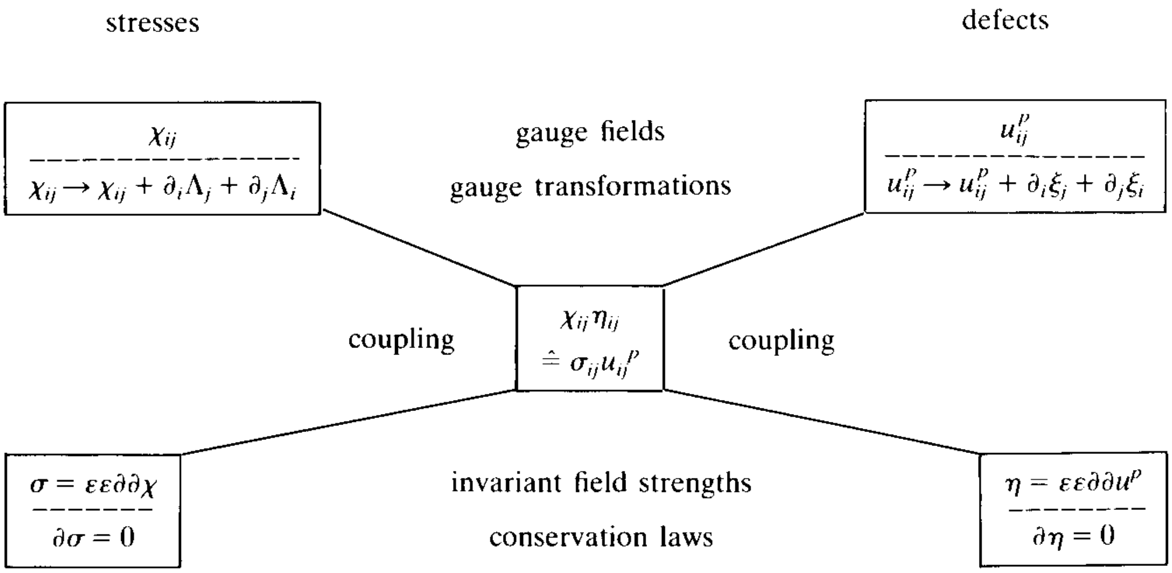

FIG 2. Illustration of stress and defect gauge structures in solids with classical first-gradient elasticity.

$$1 \equiv \int \mathscr{D}\chi_{ij} \Psi[\chi_{ij}] \int \mathscr{D} u^p_{ij} \Phi[u^p_{ij}] \exp\left\{ -i \int d^3x \sigma_{ij} u^p_{ij} \right\}. \tag{16b}$$

A partial integration gives the dual version

$$1 \equiv \int d\chi_{ij} \Psi[\chi_{ij}] \int \mathscr{D} u^p_{ij} \Phi[u^p_{ij}] \exp\left\{ -i \int d^3x \chi_{ij} \eta_{ij} \right\}, \tag{16c}$$

where

$$\eta_{ij} \equiv \varepsilon_{ik\ell} \varepsilon_{jmn} \partial_k \partial_m u^p_{\ell n} \tag{17b}$$

is the defect gauge invariant *defect density*. It satisfies the *defect conservation law*

$$\partial_i \eta_{ij} = 0. \tag{17c}$$

The dual relationship between the two gauge structure is illustrated in Fig. 2.

If we subtract in the exponent the leading stress energy

$$\beta E_s = (1/4\mu k_B T) \int d^3x \left(\sigma_{ij}{}^2 - \frac{\nu}{1+\nu} \sigma_{ii}{}^2 \right) \cdot \tag{18}$$

where μ is the shear modulus and ν the Poisson ratio, the partition function describes ensembles of defects and their proper long-range interactions in the continuum approximation. If desired, we may also introduce extra core energies for the defects up to terms quadratic in η, in analogy with (15c) for vortices. By placing the partition function on a lattice, and letting u^p_{ij} be integer multiples of the lattice spacing, we obtain the simplest model of defect-mediated melting.

If, on the other hand, the molecules in the solid are large, there is rotational stiffness and we have to extend the field variables by the angular degree of freedom, $\omega_k = \frac{1}{2}\varepsilon_{ijk}\partial_i u_j$ and its canonical conjugate τ_{ij}, the torque stress. The fundamental identity now reads

$$1 \equiv \int \mathscr{D}\sigma_{ij} \int \mathscr{D}\tau_{ij} \int \mathscr{D}u_i \int \mathscr{D}\omega_{ij} \int \mathscr{D}\beta^p_{ij} \int \mathscr{D}\phi^p_{ij} \Phi[\beta^p_{ij}, \phi^p_{ij}]$$

$$\exp\left\{ i \int d^3x [\sigma_{ij}(\partial_i u_j - \varepsilon_{ijk}\omega_k - \beta^p_{ij}) + i\tau_{ij}(\partial_i \omega_j - \phi^p_{ij})] \right\} \cdot \tag{19a}$$

where β^p_{ij} and ϕ^p_{ij} are the plastic gauge fields of dislocations and disclinations. The *defect gauge transformations* are

$$\beta^p_{ij} \to \beta^p_{ij} + \partial_i N_j - \varepsilon_{ijk} M_k, \qquad \phi^p_{ij} \to \phi^p_{ij} + \partial_i M_j,$$
$$u_i \to u_i + N_i, \qquad \omega_k \to \omega_k + M_k. \tag{20a}$$

Integrating over u_i and ω_i gives the *stress conservation law*

$$\partial_i \sigma_{ij} = 0, \qquad \partial_i \tau_{ij} = -\varepsilon_{jk\ell}\sigma_{k\ell}. \tag{20b}$$

They are solved in terms of the *stress gauge fields* $A_{\ell j}$, $h_{\ell j}$,

$$\sigma_{ij} = \varepsilon_{jk\ell}\partial_k A_{\ell j}, \qquad \tau_{ij} = \varepsilon_{ik\ell}\partial_k h_{\ell j} + \delta_{ij} A_{\ell\ell} - A_{ji} \tag{21a}$$

with the *stress gauge transformations*,

$$h_{\ell j} \to h_{\ell j} + \partial_\ell \xi_j - \varepsilon_{ljk}\Lambda_k, \qquad A_{\ell j} \to A_{\ell j} + \partial_\ell \Lambda_j. \tag{21b}$$

Using the stress gauge fields, the fundamental identity (19a) takes the *double gauge* form

$$1 = \int \mathcal{D} h_{\ell j} \int \mathcal{D} A_{\ell j} \psi[h_{\ell j}, A_{\ell j}] \int \mathcal{D} \beta^p_{ij} \int \mathcal{D} \phi^p_{ij} \Phi[\beta^p_{ij}, \phi^p_{ij}]$$

$$\exp\left\{ i \int d^3x (\sigma_{ij}\beta^p_{ij} + \tau_{ij}\phi^p_{ij}) \right\}. \tag{19b}$$

A partial integration leads to the alternative dual expression for the exponent,

$$\left\{ i \int d^3x (A_{\ell j}\alpha_{\ell j} + h_{\ell j}\Theta_{\ell j}) \right\}, \tag{19c}$$

where

$$\alpha_{\ell j} = \varepsilon_{\ell k i}\partial_k \beta^p_{ij} + \delta_{\ell j}\phi^p_{kk} - \phi^p_{j\ell}, \qquad \Theta_{\ell j} \equiv \varepsilon_{\ell k i}\partial_k \phi^p_{ij} \tag{22a}$$

are the defect gauge invariant *dislocation* and *disclination densities* with the defect conservation laws

$$\partial_\ell \alpha_{\ell j} = -\varepsilon_{jk\ell}\Theta_{k\ell}, \qquad \partial_\ell \Theta_{\ell j} = 0. \tag{22b}$$

For an illustration of the dual relationship, see Fig. 3. By subtracting, in the exponent, a stress energy (18) plus terms quadratic in τ_{ij}, we obtain the partition function of dislocations and disclinations with their proper long range interactions. If desired we may also subtract extra core energies quadratic in the defect densities. Putting this partition function on a lattice, with discretized plastic gauge fields, has recently explained the two-step melting process in two dimensions at larger angular stiffness.

It is straightforward to write down the fundamental identity within the nonlinear geometric description of dislocations and disclinations developed in Part IV. It reads

$$1 = \int \mathcal{D}\sigma^i_\alpha \int \mathcal{D}\tau^i{}_\beta{}^\alpha \int \mathcal{D}u^\alpha \int \mathcal{D}\omega_\alpha{}^\beta \int \mathcal{D}h^\alpha{}_i \int \mathcal{D}A_{i\alpha}{}^\beta \Phi[h^\alpha{}_i, A_{i\alpha}{}^\beta]$$

$$\times \exp\left\{ i \int d^3x \sqrt{g}\,[\sigma^i{}_\alpha(D_i u^\alpha - \omega_i{}^\alpha - (A_{\beta i}{}^\alpha - 2S_{\beta i}{}^\alpha)u^\beta - h^\alpha{}_i)\right.$$

$$\left. - \tfrac{1}{2}\tau^i{}_\beta{}^\alpha(D_i\omega_\alpha{}^\beta - A_{i\alpha}{}^\beta + D_i(u^\gamma A_{\gamma\alpha}{}^\beta) - u^\gamma F_{i\gamma\alpha}{}^\beta)] \right\}, \tag{23a}$$

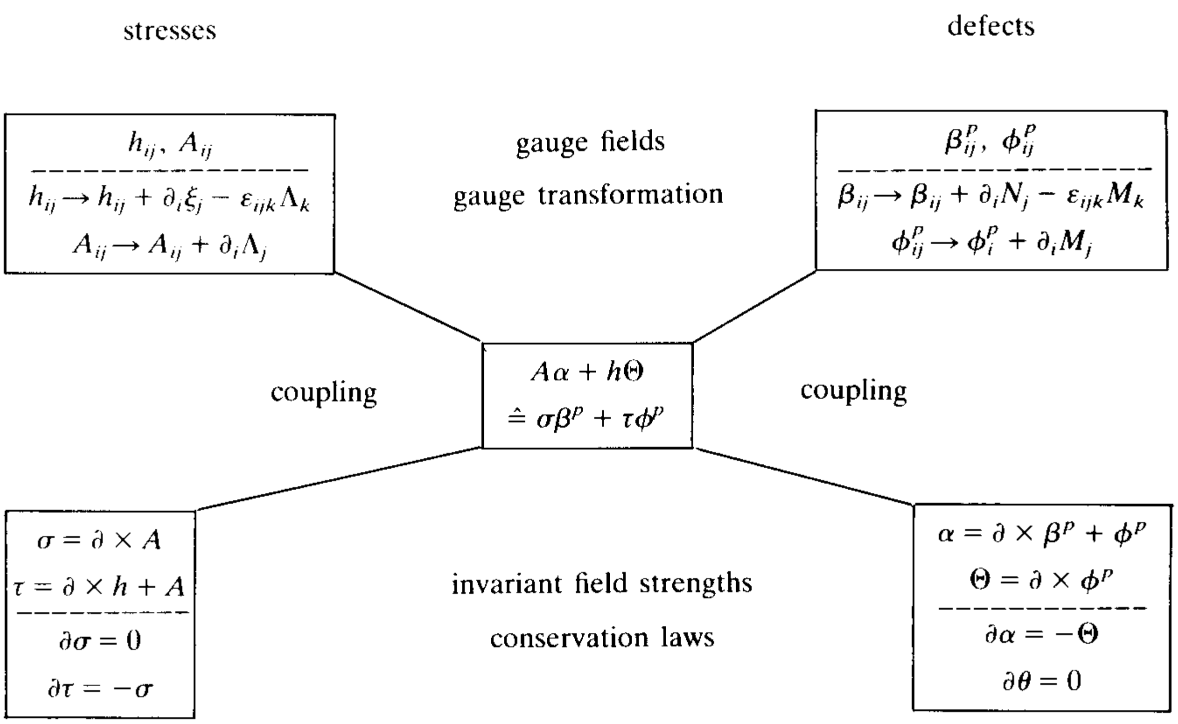

FIG 3. Illustration of stress and defect gauge structures in solids with second-gradient elasticity.

where the conjugate variables $\sigma^i{}_\alpha$ and $\tau^i{}_\beta{}^\alpha$ are again stresses and torque stresses, as in (19a) [which is obviously a linearized version of (23a)]. Integrating out u^α and $\omega_\alpha{}^\beta$ gives the *stress conservation laws* [generalizing (20b)],

$$D_i^*\sigma^i{}_\alpha = -2S_{\alpha i}{}^\gamma \sigma^i{}_\gamma - \tfrac{1}{2}\tau^i{}_\gamma{}^\beta R_{\alpha i\beta}{}^\gamma, \qquad D^*\tau^i{}_\beta{}^\alpha = (\sigma^\alpha{}_\beta - \sigma_\beta{}^\alpha), \quad (24)$$

which have the same form as those of the defect densities $[\alpha_{i\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma}\alpha_{i\gamma}]$

$$D_i^*\Theta^i{}_\alpha = -2S_{i\alpha}{}^\gamma \Theta^i{}_\gamma - \tfrac{1}{2}A^i{}_\gamma{}^\beta R_{\alpha i\beta}{}^\gamma, \qquad D_i^*\alpha^i{}_\alpha{}^\beta = (\Theta^\beta{}_\alpha - \Theta_\alpha{}^\beta) \quad (25)$$

which hold on geometric grounds [cf. (5.67), (5.66)]. The stress gauge transformations

$$\delta_E x^\alpha \equiv x^\alpha + \xi^\alpha, \qquad \delta_E h^\alpha{}_i = D_i\xi^\alpha - (A_{\beta i}{}^\alpha - 2S_{\beta i}{}^\alpha)\xi^\beta,$$
$$\delta_E A_{i\alpha}{}^\beta = D_i(\xi^\gamma A_{\gamma\alpha}{}^\beta) - \xi^\gamma F_{i\gamma\alpha}{}^\beta, \quad (26a)$$

$$\delta_L x^\alpha = \omega^\alpha{}_\beta x^\beta,\ \delta_L A_{i\alpha}{}^\beta = D_i\omega_\alpha{}^\beta,\ \delta_L h_{\alpha i} = \omega_\alpha{}^\beta h_{\beta i}, \quad (26b)$$

are absorbed in a corresponding transformations of the displacement field,

$$\delta_E u^\alpha = \xi^\alpha, \qquad \delta_L u^\alpha = \omega^\alpha{}_\beta u^\beta \tag{27}$$

making (23a) *defect gauge invariant*. The crystal forces are now introduced by subtracting in the exponent of (19a) $1/k_B T$ times an elastic energy,

$$E_{\rm el} \equiv \frac{1}{4\mu} \int d^3 y \sqrt{g} \left[\overset{s}{\sigma}{}^i{}_j{}^2 - \left(\frac{\nu}{1+\nu} \right) \overset{s}{\sigma}{}^i{}_i{}^2 \right] + \frac{1}{16\mu \ell^2} \int d^3 x \sqrt{g} \tau_i{}^\beta{}_\alpha \tau^i{}_\beta{}^\alpha \tag{28}$$

where $\overset{s}{\sigma}{}^i{}_j$ is the symmetrized part of $h^\alpha{}_j \sigma^i{}_\alpha$. We can also add a similar term quadratic in the defect densities $\alpha_{i\alpha\beta}$ and $\Theta_{i\alpha}$ to account for extra core energies. In this way, we obtain a complete non-linear gauge field description of defects with their correct long-range forces.

As mentioned in Chapter 7 of Part IV, it is possible to express the stresses in terms of stress gauge fields $\bar{h}_{\alpha i}$, $\bar{A}_{i\alpha}{}^\beta$ which obviously play the same role for the stresses as $h_{\alpha i}$, $A_{i\alpha}{}^\beta$ do for the defect densities. This would lead to a nonlinear extension of the double gauge theory (19b). Moreover, due to the identical form of defect and stress conservation laws, one may want to consider the gauge theory of stresses as defining the differential geometry of the space, with the defects being the extra matter fields. In this way one would arrive at a "double geometry" of stresses and defects. The interpretation of the stress metric in this formulation is, however, not very transparent.

We hope to have shown that gauge and disorder fields can be a powerful tool in the study of many condensed matter systems containing long-range excitations and line-like defects. The two systems, superfluids and solids, which we have selected for detailed analysis are only two particular examples of a wide variety of possible candidates. Most of the work has been done on these systems. We feel confident that the future will see the emergence of similar detailed applications to many other condensed systems, such as liquid crystals and the condensed states of nuclear matter. The first steps have already been made and the reader is challenged to participate in this exciting development.

NOTE AND REFERENCE

The fundamental identities were developed in
H. Kleinert, *Phys. Lett.* **A130** (1988) 59.
For gauge applications of gauge and disorder fields to other systems such as liquid crystals, pion condensates, and magnetic superconductors, see H. Kleinert, J. de Physique **44** (1983) 353, *Lett. Nuovo Cimento* **34** (1982) 103 and *Phys. Lett.* **90A** (1982) 259 respectively.

INDEX